UNIVERSITY OF Southampton
Southampton Statistical Sciences Research Institute

Working Paper M11/02
Methodology

Non-Parametric Bootstrap Mean Squared Error Estimation For M-Quantile Estimators Of Small Area Averages, Quantiles And Poverty Indicators

Stefano Marchetti, Nikos Tzavidis, Monica Pratesi

Abstract

Small area estimation is conventionally concerned with the estimation of small area averages and totals. More recently emphasis has been also placed on the estimation of poverty indicators and of key quantiles of the small area distribution function using robust models for example, the M-quantile small area model (Chambers and Tzavidis, 2006). In parallel to point estimation, Mean Squared Error (MSE) estimation is an equally crucial and challenging task. However, while analytic MSE estimation for small area averages is possible, analytic MSE estimation for quantiles and poverty indicators is extremely difficult. Moreover, one of the main criticisms of the analytic MSE estimator for M-quantile estimates of small area averages proposed by Chambers and Tzavidis (2006) and Chambers et al. (2009) is that it can be unstable when the area-specific sample sizes are small.

Non-parametric Bootstrap Mean Squared Error Estimation for M-quantile Estimators of Small Area Averages, Quantiles and Poverty Indicators

Stefano Marchetti*

Department of Statistics and Mathematics Applied to Economics, University of Pisa, Via Ridolfi, 10 - 56124 Pisa (PI), Italy

Nikos Tzavidis*

Social Statistics and Southampton Statistical Sciences Research Institute, University of Southampton, Highfield, SO17 1BJ, Southampton, UK

Monica Pratesi*

Department of Statistics and Mathematics Applied to Economics, University of Pisa, Via Ridolfi, 10 - 56124 Pisa (PI), Italy

Abstract

Small area estimation is conventionally concerned with the estimation of small area averages and totals. More recently emphasis has been also placed on the estimation of poverty indicators and of key quantiles of the small area distribution function using robust models for example, the M-quantile small area model (Chambers and Tzavidis, 2006). In parallel to point estimation, Mean Squared Error (MSE) estimation is an equally crucial and challenging task. However, while analytic MSE estimation for small area averages is possible, analytic MSE estimation for quantiles and poverty indicators is extremely difficult. Moreover, one of the main criticisms of the analytic MSE estimator for M-quantile estimates of small area averages proposed by Chambers and Tzavidis (2006) and Chambers et al. (2009) is that it can be unstable when the area-specific sample sizes are small.

*Corresponding author

Email addresses: stefano.marchetti@for.unipi.it (Stefano Marchetti), n.tzavidis@soton.ac.uk (Nikos Tzavidis), m.pratesi@ec.unipi.it (Monica Pratesi)

Preprint submitted to Computational Statistics and Data Analysis *December 29, 2010*

We propose a non-parametric bootstrap framework for MSE estimation for small area averages, quantiles and poverty indicators estimated with the M-quantile small area model. Because the small area statistics we consider in this paper can be expressed as functionals of the Chambers-Dunstan estimator of the population distribution function, the proposed non-parametric bootstrap presents an extension of the work by Lombardia et al. (2003). Alternative bootstrap schemes, based on resampling empirical or smoothed residuals, are studied and the asymptotic properties are discussed in the light of the work by Lombardia et al. (2003). Emphasis is also placed on second order properties of MSE estimators with results suggesting that the bootstrap MSE estimator is more stable than corresponding analytic MSE estimators. The proposed bootstrap is evaluated in a series of simulation studies under different parametric assumptions for the model error terms and different scenarios for the area-specific sample and population sizes. We finally present results from the application of the proposed MSE estimator to real income data from the European Survey of Income and Living Conditions (EU-SILC) in Italy and provide information on the availability of R functions that can be used for implementing the proposed estimation procedures in practice.

Keywords: Chambers-Dunstan estimator, Income distribution, Domain estimation, Poverty mapping, Resampling methods, Robust estimation

1. Introduction

Sample surveys provide an effective way of obtaining estimates for population characteristics. Estimation, however, can become difficult when the focus is on domains (areas) with small sample sizes. The term 'small areas' is typically used to describe domains whose sample sizes are not large enough to allow sufficiently precise direct estimation, i.e. estimation based only on the sample data from the domain (Rao, 2003). When direct estimation is not possible, one has to rely upon alternative model-based methods for producing small area estimates. Small area estimation is conventionally concerned with the estimation of small area averages and totals. More recently emphasis has been also placed on the estimation of poverty indicators and of key quantiles of the small area distribution function (Molina et al., 2010) using the M-quantile small area model (Chambers and Tzavidis, 2006).

Estimating the precision of small area estimates is both an important and challenging task. Despite the fact that MSE estimation for M-quantile small area averages has been studied fairly extensively (Chambers and Tzavidis, 2006; Chambers et al., 2009), MSE estimation for more complex small area statistics e.g. for poverty indicators estimated with the M-quantile model is unexplored. What is more, analytic MSE estimation for complex statistics is difficult. For example, all small area statistics we consider in this paper can be expressed as functionals of the population distribution function, which can be consistently estimated by using the Chambers-Dunstan estimator (Chambers and Dunstan, 1986). Although the asymptotic behaviour of this estimator was studied by Chambers and Dunstan (1986) and asymptotic expressions for the bias and the variance were derived by Chambers et al. (1992), the use of these expressions has proven to be impractical. This motivates the work in this paper in which we propose a unified non-parametric bootstrap framework for MSE estimation for small area averages, quantiles and poverty indicators - in particular, for the Head Count Ratio (HCR) and for the Poverty Gap (PG)- estimated with the M-quantile small area model. The proposed bootstrap is based on resampling empirical or smoothed M-quantile model residuals and presents an extension of the work by Lombardia et al. (2003) to small area estimation with the M-quantile model. The choice of a non-parametric bootstrap scheme, instead of a parametric one, is dictated by the fact that the M-quantile small area model does not make explicit parametric assumptions about the model error terms. This is in contrast to the conventional unit level area random effects model which assumes that the unit level and area level error terms are Gaussian. MSE estimation using parametric, instead of non-parametric, bootstrap has been recently used by Sinha and Rao (2009) for estimating the MSE of the Robust Empirical Best Linear Unbiased Predictor (REBLUP) of the small area average and by Molina and Rao (2010) for estimating the MSE of small area poverty indicators estimated by using the Empirical Best Prediction (EBP) approach.

The complexity of the small area target parameters we consider in this paper is only one way of motivating the use of bootstrap. There is one additional reason as to why one may consider using a bootstrap MSE estimator. As we mentioned above, analytic MSE estimation for M-quantile estimates of small area averages has been already proposed. Although this estimator is bias robust against mispecifications of the model assumptions, one of its main criticisms is that it can be unstable when used with small area-specific sample sizes. Second order properties of MSE estimators are, however, also very important. For this reason, a further aim of this paper is to also study the stability of the non-parametric bootstrap MSE estimator and compare this to the stability of corresponding analytic MSE estimators.

The paper is organised as follows. In Section 2 we review the M-quantile small area model and present point estimation for small area averages, poverty indicators and quantiles. Analytic MSE estimation for estimates of small area averages is reviewed. Although the emphasis here is on MSE estimation, rather than on point estimation, we must stress that estimation of poverty indicators under the M-quantile model is presented for the first time in this paper. However, comparisons with alternative poverty estimation approaches -e.g. the EBP method of Molina and Rao (2010)- will be discussed elsewhere. In Section 3 we present the non-parametric bootstrap scheme and provide a sketch of its asymptotic properties. In Section 4 the performance of the proposed MSE estimator is empirically evaluated under different parametric assumptions for the model error terms and for the small area sample and population sizes. For the case of small area averages the bootstrap MSE estimator is also compared to the analytic MSE estimator proposed by Chambers and Tzavidis (2006) and Tzavidis et al. (2010). Using real income data from the EU-SILC survey in Italy, in Section 5 we apply the bootstrap MSE estimator for computing the accuracy of estimates of income averages, income quantiles and poverty indicators for Provinces in Tuscany. Access to software that implements the proposed estimation procedures is important for users of small area estimation methods and Section 6 provides information on the availability of R functions. Finally, in Section 7 we conclude the paper with some final remarks.

2. Small area estimation by using the M-quantile model

In what follows we assume that a vector of p auxiliary variable $\mathbf{x}_{ij}$ is known for each population unit i in small area $j = 1, \ldots, m$ and that values of the variable of interest y are available from a random sample, s, that includes units from all the small areas of interest. We denote the population size, sample size, sampled part of the population and non sampled part of the population in area j respectively by N_j, n_j, s_j and r_j. We assume that the sum over the areas of N_j and n_j is equal to N and n respectively. We further assume that conditional on covariate information for example, design variables, the sampling design is ignorable.

A recently proposed approach to small area estimation is based on the use of a quantile/M-quantile regression model (Chambers and Tzavidis, 2006). The classical regression model summarises the behaviour of the mean of a random variable y at each point in a set of covariates x. Instead, quantile regression summarises the behaviour of different parts (e.g. quantiles) of the conditioned distribution of y at each point in the set of the x's. In the linear case, quantile regression leads to a family of hyper-planes indexed by a real number $q \in (0, 1)$. For a given value of q, the corresponding model shows how the qth quantile of the conditional distribution of y varies with x. For example, for $q = 0.1$ the quantile regression hyperplane separates the lower 10% of the conditional distribution from the remaining 90%.

Let us for the moment and for notational simplicity drop subscript j. Suppose that $(\mathbf{x}_i^T, y_i),\ i = 1, \cdots, n$ denotes the observed values for a random sample consisting of n units, where $\mathbf{x}_i^T$ are row p-vectors of a known design matrix $\mathbf{X}$ and y_i is a scalar response variable corresponding to a realisation of a continuous random variable with unknown continuous cumulative distribution function F. A linear regression model for the qth conditional quantile of y_i given $\mathbf{x}_i$ is

$$Q_{y_i}(q|\mathbf{x}_i) = \mathbf{x}_i{}^T \boldsymbol{\beta}(q).$$

An estimate of the qth regression parameter $\boldsymbol{\beta}(q)$ is obtained by minimizing

$$\sum_{i=1}^{n} \Big\{ |y_i - \mathbf{x}_i^T \boldsymbol{\beta}(q)| \big[(1-q) I(y_i - \mathbf{x}_i^T \boldsymbol{\beta}(q) \le 0) + q I(y_i - \mathbf{x}_i^T \boldsymbol{\beta}(q) > 0) \big] \Big\}.$$

Quantile regression presents a generalization of median regression and expectile regression (Newey and Powell, 1987) a 'quantile-like' generalization of mean regression. M-quantile regression (Breckling and Chambers, 1988) integrates these concepts within a framework defined by a 'quantile-like' generalization of regression based on influence functions (M-regression). The M-quantile of order q for the conditional density of y given the set of covariates x, $f(y|x)$, is defined as the solution $MQ_y(q|x;\psi)$ of the estimating equation $\int \psi_q\{y - MQ_y(q|x;\psi)\} f(y|x) dy = 0$, where ψ_q denotes an asymmetric influence function, which is the derivative of an asymmetric loss function ρ_q. A linear M-quantile regression model y_i given $\mathbf{x}_i$ is one where we assume that

$$MQ_y(q|\mathbf{x}_i;\psi) = \mathbf{x}_i{}^T \boldsymbol{\beta}_\psi(q), \tag{1}$$

and estimates of $\boldsymbol{\beta}_\psi(q)$ are obtained by minimizing

$$\sum_{i=1}^{n} \rho_q\big(y_i - \mathbf{x}_i{}^T \boldsymbol{\beta}_\psi(q)\big). \tag{2}$$

Different regression models can be defined as special cases of (2). In particular, by varying the specifications of the asymmetric loss function ρ_q we obtain the expectile, M-quantile and quantile regression models as special cases. When ρ_q is the square loss function we obtain the linear expectile regression model if $q \neq 0.5$ (Newey and Powell, 1987) and the standard linear regression model if $q = 0.5$. When ρ_q is the loss function described by Koenker and Bassett (1978) we obtain the linear quantile regression. Throughout this paper we will take the linear M-quantile regression model to be defined by when ρ_q is the Huber loss function (Breckling and Chambers, 1988). Setting the first derivative of (2) equal to zero leads to the following estimating equations

$$\sum_{i=1}^{n} \psi_q(r_{iq})\mathbf{x}_i = \mathbf{0},$$

where $r_{iq} = y_i - \mathbf{x}_i^T \boldsymbol{\beta}_\psi(q)$, $\psi_q(r_{iq}) = 2\psi(s^{-1}r_{iq})\{qI(r_{iq} > 0) + (1-q)I(r_{iq} \leq 0)\}$ and $s > 0$ is a suitable estimate of scale. For example, in the case of robust regression, $s = median|r_{iq}|/0.6745$. Since the focus of our paper is on M-type estimation, we use the Huber Proposal 2 influence function, $\psi(u) = uI(-c \leq u \leq c) + c \cdot sgn(u)$. Provided that the tuning constant c is strictly greater than zero, estimates of $\boldsymbol{\beta}_\psi(q)$ are obtained using iterative weighted least squares (IWLS).

2.1. Estimators of small area averages

Chambers and Tzavidis (2006) extended the use of M-quantile regression models to small area estimation. Following their development (see also Kokic et al., 1997; Aragon et al., 2005), these authors characterize the conditional variability across the population of interest by the M-quantile coefficients of the population units. For unit i with values y_i and $\mathbf{x}_i$, this coefficient is the value θ_i such that $MQ_y(\theta_i|\mathbf{x}_i;\psi) = y_i$ The M-quantile coefficients are determined at the population level. Consequently, if a hierarchical structure does explain part of the variability in the population data, then we expect units within clusters (domains) defined by this hierarchy to have similar M-quantile coefficients. When the conditional M-quantiles are assumed to follow the linear model (1), with $\boldsymbol{\beta}_\psi(q)$ a sufficiently smooth function of q, Chambers and Tzavidis (2006) suggested a plug in (naïve) estimator of the average value of y in area j

$$\hat{m}_j^{MQ} = N_j^{-1}\Big[\sum_{i \in s_j} y_i + \sum_{i \in r_j} \mathbf{x}_i^T \hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j)\Big], \quad j = 1, \ldots, m, \tag{3}$$

where $\hat{\theta}_j$ is an estimate of the average value of the M-quantile coefficients of the units in area j. The area-specific M-quantile coefficients, $\hat{\theta}_j$, can be viewed as pseudo-random effects. Empirical work indeed indicates that the area-specific M-quantile coefficients are positively and highly correlated with the estimated random area-specific effects obtained with the nested error regression small area model. Chambers and Tzavidis (2006) also observed that the naïve M-quantile estimator (3) can be biased, especially in the presence of heteroskedastic and/or asymmetric errors. This observation motivated the work in Tzavidis et al. (2010). In particular, these authors proposed a bias adjusted M-quantile estimator for the small area average that is derived by using an estimator of the finite population distribution function such as the Chambers-Dunstan estimator (Chambers and Dunstan, 1986). The Chambers-Dunstan estimator of the small area distribution function is of the form

$$\hat{F}_j^{CD}(t) = N_j^{-1}\Big[\sum_{i\in s_j} I(y_i \le t) + n_j^{-1}\sum_{k\in r_j}\sum_{i\in s_j} I(\mathbf{x}_k^T\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j) + e_i \le t)\Big].$$

Estimates of θ_j and $\boldsymbol{\beta}_\psi(\theta_j)$ are obtained following Chambers and Tzavidis (2006) and $e_i = y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j)$ are model residuals. The M-quantile bias-adjusted estimator of the average of y in small area j is then defined as

$$\begin{aligned}\hat{m}_j^{CD} &= \int_{-\infty}^{+\infty} y \,\mathrm{d}\hat{F}_j^{CD}(y) \\ &= N_j^{-1}\Big[\sum_{i\in s_j} y_i + \sum_{i\in r_j}\hat{y}_i + (1-f_j)\sum_{i\in s_j} e_i\Big].\end{aligned} \tag{4}$$

where $f_j = n_j N_j^{-1}$ is the sampling fraction in area j and $\hat{y}_i = \mathbf{x}_i^T\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j)$, $i \in r_j$. The bias correction in (4) means that this estimator has higher variability than (3). Nevertheless, because of its bias robust properties, (4) is usually preferred, over the naïve M-quantile estimator, in practice. Finally, as we will also see in the next section, by using the Chambers-Dunstan estimator one can define a general framework for small area estimation that extends beyond the estimation of small area averages.

Analytic MSE estimation for M-quantile estimators of small area averages is described in Chambers and Tzavidis (2006) and Chambers et al. (2009). In particular, Chambers et al. (2009) proposed an analytic mean squared error estimator that is a first order approximation to the mean squared error of estimator (4). These authors noted that since an iteratively reweighted least squares algorithm is used to calculate the M-quantile regression fit at $\hat{\theta}_j$,

$$\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j) = (\mathbf{X}_s^T \mathbf{W}_{s_j} \mathbf{X}_s)^{-1} \mathbf{X}_s^T \mathbf{W}_{s_j} \mathbf{y}_s$$

where $\mathbf{X}_s$ and $\mathbf{y}_s$ denote the matrix of sample x values and the vector of sample y values respectively, and $\mathbf{W}_{s_j}$ denotes the diagonal weight matrix of order n that defines the estimator of the M-quantile regression coefficient with $q = \hat{\theta}_j$. It immediately follows that (4) can be written

$$\hat{m}_j^{CD} = \mathbf{w}_{s_j}^T \mathbf{y}_s, \tag{5}$$

where $\mathbf{w}_{s_j} = (w_{ij}) = n_j^{-1}\Delta_{s_j} + (1 - N_j^{-1} n_j)\mathbf{W}_j \mathbf{X}_s (\mathbf{X}_s^T \mathbf{W}_j \mathbf{X}_s)^{-1}(\overline{\mathbf{x}}_{r_j} - \overline{\mathbf{x}}_{s_j})$ with Δ_{sj} denoting the n-vector that 'picks out' the sample units from area j. Here $\overline{\mathbf{x}}_{s_j}$ and $\overline{\mathbf{x}}_{r_j}$ denote the sample and non-sample means of x in area j. Also, these weights are 'locally calibrated' on x since

$$\sum_{i \in s} w_{ij} \mathbf{x}_i = \bar{\mathbf{x}}_{s_j} + (1 - f_j)(\bar{\mathbf{x}}_{r_j} - \bar{\mathbf{x}}_{s_j}) = \bar{\mathbf{x}}_j.$$

A first order approximation to the mean squared error of (5) then treats the weights as fixed and applies standard methods of robust mean squared error estimation for linear estimators of population quantities (Royall and Cumberland, 1978). With this approach, the prediction variance of $\hat{m}_j^{CD}$ is estimated by

$$\widehat{Var}(\hat{m}_j^{CD}) = \sum_{g=1}^{m} \sum_{i \in s_g} \lambda_{ijg} \big(y_i - \mathbf{x}_i \hat{\beta}_\psi(\hat{\theta}_g)\big)^2, \tag{6}$$

where $\lambda_{ijg} = [(w_{ij} - 1)^2 + (n_j - 1)^{-1}(N_j - n_j)]\mathrm{I}(g = j) + w_{ig}^2 \mathrm{I}(g \neq j)$. Empirical studies show that the analytic MSE estimator (6) is bias robust against misspecification of the model (Chambers et al., 2009). However, its main criticism is that it can be unstable especially with small area-specific sample sizes.

2.2. Estimators of small area poverty indicators and quantiles

Although small area averages are widely used in small area applications, relying only on averages may not be very informative. This is the case for example in economic applications where estimates of average income may not provide an accurate picture of the area wealth due to the high within area inequality. Our goal in this section is to also express quantiles and specific poverty indicators as functionals of the Chambers-Dunstan estimator of the population distribution function.

With regards to the estimation of small area quantiles, an estimate of quantile ϕ for small area j is the value $\hat{q}(j;\phi)$ obtained by a numerical solution to the following estimating equation

$$\int_{-\infty}^{\hat{q}(j;\phi)} \mathrm{d}\hat{F}_j^{CD}(t) = \phi. \tag{7}$$

Estimating poverty indicators at disaggregated geographical levels is also important. In this paper we focus on the estimation of the incidence of poverty or *Head Count Ratio* (HCR) and of the *Poverty Gap* (PG) as defined by Foster et al. (1984). Denoting by t the poverty line, different poverty indicators are defined by using

$$F_{\alpha,i} = \Big(\frac{t-y_i}{t}\Big)^{\alpha} I(y_i \leq t) \quad i = 1, \dots N.$$

The population poverty indicators in small area j, $F_{\alpha,j}$, can then be decomposed as follows,

$$F_{\alpha,j} = N_j^{-1}\Big[\sum_{i \in s_j} F_{\alpha,i} + \sum_{i \in r_j} F_{\alpha,i}\Big].$$

In particular, setting $\alpha = 0$, $F_{0,j}$ defines the HCR while setting $\alpha = 1$ $F_{1,j}$ defines the PG in small area j. Hence, one approach for estimating the HCR in small area j is by using the Chambers-Dunstan estimator of the distribution function and the M-quantile model for predicting for out of sample units as follows,

$$\hat{F}_{0,j} = N_j^{-1}\Big[\sum_{i \in s_j} I(y_i \leq t) + n_j^{-1}\sum_{k \in r_j}\sum_{i \in s_j} I(\mathbf{x}_k^T \hat{\boldsymbol{\beta}}_{\psi}(\hat{\theta}_j) + e_i \leq t)\Big]. \tag{8}$$

Similarly, an estimator of the poverty gap for area j

$$\hat{F}_{1,j} = N_j^{-1}\Big[\sum_{i \in s_j}\Big(\frac{t-y_i}{t}\Big) I(y_i \leq t) + n_j^{-1}\sum_{k \in r_j}\sum_{i \in s_j}\Big(\frac{t - \mathbf{x}_k^T \hat{\boldsymbol{\beta}}_{\psi}(\hat{\theta}_j) - e_i}{t}\Big) I(\mathbf{x}_k^T \hat{\boldsymbol{\beta}}_{\psi}(\hat{\theta}_j) + e_i \leq t)\Big]. \tag{9}$$

In practice the HCR and PG for area j can be estimated by using a Monte Carlo approach. The estimation procedure is as follows:

1 Fit the M-quantile small area model using the sample values $\mathbf{y}_s$ and obtain estimates $\hat{\theta}_j$, $\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j)$, of θ_j and $\boldsymbol{\beta}_\psi(\theta_j)$.

2 Draw an out of sample vector using

$$y_k^* = \mathbf{x}_k^T \hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j) + e_k^*, \quad k \in r_j,$$

where $e_k^*, k \in r_j$ is a vector of size $N_j - n_j$ drawn from the empirical distribution function of the estimated M-quantile model residuals.

3 Repeat the process H times. Each time combine the sample data and out of sample data for estimating $F_{0,j}$ and $F_{1,j}$.

4 Average the results over H simulations.

The M-quantile approach for estimating poverty indicators is similar in spirit to the EBP approach proposed by Molina and Rao (2010). Note for example that $y_k^*, k \in r_j$ is generated using $\mathbf{x}_k^T \hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j)$ i.e. from the conditional M-quantile model, where $\hat{\theta}_j$ plays the role of the area random effects in the M-quantile modelling framework.

3. Non-parametric bootstrap MSE estimation

All small area target parameters we presented in Section 2 have been expressed as functionals of the Chambers-Dunstan estimator of the population distribution function. Unlike MSE estimation for small area averages, analytic MSE estimation for small area poverty indicators and quantiles is complex. In this section we present a nonparametric bootstrap framework for MSE estimation of small area parameters estimated with the M-quantile model and the Chambers-Dunstan estimator.

Let us start with the M-quantile small area model

$$y_{ij} = \mathbf{x}_{ij}^T \boldsymbol{\beta}_\psi(\theta_j) + \varepsilon_{ij}$$

where $\boldsymbol{\beta}_\psi(\theta_j)$ is the unknown vector of M-quantile regression parameters for the unknown area-specific M-quantile coefficient θ_j, and ε_{ij} is the unit level random error term with distribution function G for which no explicit parametric assumptions are being made. Using the sample data we obtain estimates $\hat{\theta}_j$, $\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j)$, of θ_j and $\boldsymbol{\beta}_\psi(\theta_j)$, and estimated model residuals $e_{ij} = y_{ij} - \mathbf{x}_{ij}^T\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j)$. The target is to estimate the small area finite population distribution function, or to be more precise a functional of this distribution function τ, by using the Chambers-Dunstan estimator and the M-quantile small area model,

$$\hat{F}_j^{CD}(t) = N_j^{-1}\Big[\sum_{i\in s_j} I(y_{ij} \le t) + \sum_{k\in r_j} \hat{G}\big(t - \mathbf{x}_{ij}^T\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j)\big)\Big], \tag{10}$$

where $\hat{G}(u)$ is the empirical distribution, $\hat{G}(u) = n_j^{-1}\sum_{i\in s_j} I(e_{ij} \le u)$, of the model residuals e_{ij}. Using (10), we obtain estimates of the small area target parameters we presented in Section 2, which we collectively denote by $\hat{\tau}$.

Given an estimator $\hat{G}_{est}(u)$ of the distribution of the residuals $G(u) = Pr(\varepsilon \le u)$, a bootstrap population, consistent with the M-quantile small area model, $\Omega^* = \{y_{ij}^*, \mathbf{x}_{ij}\}$, can be generated by sampling from $\hat{G}_{est}(u)$ to obtain ε_{ij}^*,

$$y_{ij}^* = \mathbf{x}_{ij}^T\hat{\boldsymbol{\beta}}_\psi(\hat{\theta}_j) + \varepsilon_{ij}^*, \qquad i = 1, \ldots, N_j, \quad j = 1, \ldots, m.$$

For defining $\hat{G}_{est}(u)$ we consider two approaches: (1) sampling from the empirical distribution function of the model residuals or (2) sampling from a smoothed distribution function of the model residuals. For each of the two above mentioned approaches, sampling can be done in two ways namely, by sampling from the distribution of all residuals without conditioning on the small area (unconditional approach) or by sampling from the distribution of the residuals within small area j (conditional approach). The empirical distribution of the residuals for the unconditional approach is

$$\hat{G}_{est}(t) = n^{-1}\sum_{j=1}^{m}\sum_{i\in s_j} I(e_{ij} - \bar{e}_s \le t), \tag{11}$$

where $\bar{e}_s$ is the sample mean of the residuals e_{ij}, while for the conditional approach the empirical distribution is

$$\hat{G}_{j_{est}}(t) = n_j^{-1}\sum_{i\in s_j} I(e_{ij} - \bar{e}_{s_j} \le t),$$

where $\bar{e}_{s_j}$ is the sample mean of the residuals in area j. The corresponding smoothed estimators of the distribution of the residuals for the unconditional and the conditional approaches are respectively

$$\hat{G}_{est} = n^{-1} \sum_{j=1}^{m} \sum_{i \in s_j} K\big(h^{-1}(t - e_{ij} + \bar{e}_s)\big), \qquad (12)$$

and

$$\hat{G}_{j_{est}}(t) = n_j^{-1} \sum_{i \in s_j} K\big(h_j^{-1}(t - e_{ij} + \bar{e}_{s_j})\big),$$

where $h > 0$ (or h_j) is a smoothing parameter and K is the distribution function corresponding to a bounded symmetric kernel density k. Hence, there are four possible approaches for defining ε^*_{ij}. We suggest, however, using the unconditional, empirical or smoothed, approach. The reason is that in applications of small area estimation sampling from the conditional distribution would rely on potentially a very small number of data points which can cause $\hat{G}_{est}(t)$ to be unstable. Let us now define the finite distribution function for the bootstrap population as follows

$$F^*_j(t) = N_j^{-1}\Big[\sum_{i \in s_j} I(y^*_{ij} \le t) + \sum_{i \in r_j} I(y^*_{ij} \le t)\Big].$$

The bootstrap population distribution function can be estimated by selecting a without replacement sample from the bootstrap population and by using the Chambers-Dunstan estimator

$$\hat{F}_j^{*,CD}(t) = N_j^{-1}\Big[\sum_{i \in s_j} I(y^*_{ij} \le t) + \sum_{k \in r_j} \hat{G}^*\big(t - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}^*{}_{\psi}(\hat{\theta}^*{}_j)\big)\Big], \qquad (13)$$

where $\hat{\boldsymbol{\beta}}^*_{\psi}(\hat{\theta}^*_j)$ are bootstrap sample estimates of the M-quantile model parameters and $\hat{G}^* = n_j^{-1} \sum_{i \in s_j} I(e^*_{ij} \le u)$. Using (13) we obtain bootstrap estimates, $\hat{\tau}^*$, of the bootstrap population small area parameters τ^*.

The steps of the bootstrap procedure are as follows: starting from sample s, selected from a finite population Ω without replacement, we fit the M-quantile small area model and obtain estimates of θ_j and $\boldsymbol{\beta}_{\psi}(\theta_j)$ which are used to compute the model residuals. We then generate B bootstrap populations, Ω^{*b}, using one of the previously described methods for estimating the distribution of the residuals, $G(u)$. From each bootstrap population, Ω^{*b}, we select L bootstrap samples using simple random sampling within the small areas and without replacement in a way such that $n^*_j = n_j$. Using the bootstrap samples we obtain estimates of τ. Bootstrap estimators of the bias and variance of the estimated target small area parameter, $\hat{\tau}$, derived from the distribution function in area j are defined respectively by

$$\widehat{\text{Bias}}(\hat{\tau}_j) = B^{-1}L^{-1}\sum_{b=1}^{B}\sum_{l=1}^{L}\left(\hat{\tau}_j^{*bl} - \tau_j^{*b}\right),$$
$$\widehat{\text{Var}}(\hat{\tau}_j) = B^{-1}L^{-1}\sum_{b=1}^{B}\sum_{l=1}^{L}\left(\hat{\tau}_j^{*bl} - \bar{\hat{\tau}}_j^{*bl}\right)^2,$$

where τ_j^{*b} is the small area parameter of the bth bootstrap population, $\hat{\tau}_j^{*bl}$ is the small area parameter estimated by using (13) with the lth sample of the bth bootstrap population and $\bar{\hat{\tau}}_j^{*bl} = L^{-1}\sum_{l=1}^{L}\hat{\tau}_j^{*bl}$. The bootstrap MSE estimator of the estimated small area target parameter is then defined as

$$\widehat{MSE}\,(\hat{\tau}_j) = \widehat{\text{Var}}(\hat{\tau}_j) + \widehat{\text{Bias}}(\hat{\tau}_j)^2. \tag{14}$$

3.1. A note on asymptotic properties

The asymptotic properties of the smoothed bootstrap method, under a linear model, have been studied by Lombardia et al. (2003). Here we comment on the validity of the assumptions by Lombardia et al. (2003) under the M-quantile model. To start with, we note that the superpopulation model assumed by Lombardia et al. (2003) is a special case of the linear M-quantile model when a squared loss function is used in (2) and $q = 0.5$ (see Breckling and Chambers, 1988; Newey and Powell, 1987). Under this model and using the assumptions on page 371 of their paper Lombardia et al. (2003) showed that the smoothed bootstrap estimator $\hat{F}^{*,CD}(t)$ is consistent, in that its behaviour relative to the smoothed bootstrap population distribution function $F^*(t)$ is identical to the relationship between $\hat{F}^{CD}(t)$ and the corresponding population distribution function $F(t)$. The asymptotic behaviour of the latter was studied by Chambers et al. (1992) under assumptions relating to the superpopulation model density and the asymptotic behavior of the sampling fraction (H1-H3 on page 371 of Lombardia et al. (2003)). Moreover, Lombardia et al. (2003) show that the smoothed bootstrap estimator is asymptotically normally distributed. The assumptions made by Lombardia et al. (2003) relate to the kernel function, the bandwidth parameter and the density g of G. In our case the assumptions about the kernel density

k and the bandwidth parameter h (K1 and K2 on page 371 of Lombardia et al. (2003)) hold and in our empirical evaluations we use the same kernel function and bandwidth selection method as those used by Lombardia et al. (2003). In addition, the assumptions about the superpopulation model and the asymptotic behavior of the sampling fraction (H1 to H4 on page 371) are reasonable assumptions also under the M-quantile linear model. Finally, conditional on the small areas the assumption of independence of the errors is also reasonable.

4. Empirical evaluations

In this section we use model-based Monte-Carlo simulations to empirically evaluate the performance of the bootstrap MSE estimator (14) when used to estimate the MSE of the M-quantile estimators of (a) the small area average (4), (b) the small area quantile (7), (c) the head count ratio (HCR) (8) and (d) the poverty gap (PG) (9). Moreover, since analytic MSE estimation for M-quantile estimates of small area averages is possible, the proposed bootstrap MSE estimator is also contrasted to the corresponding analytic MSE estimator (6) both in terms of bias and stability. The behaviour of the alternative MSE estimators is assessed under two different parametric assumptions for the model error terms namely, Normal and Chi-square errors, and two scenarios for the area-specific sample and population sizes. Finally, we also present results on how well estimators of small area averages, quantiles and poverty indicators estimate the corresponding population parameters.

In what follows subscript j identifies small areas, $j = 1, \ldots, m$ and subscript i identifies units in a given area, $i = 1, \ldots, n_j$. Population data $\Omega = (x, y)$ in $m = 30$ small areas are generated under two parametric scenarios for the model error terms. Population data under the first parametric scenario were generated by using a unit level area random effects model with normally distributed random area effects and unit level errors as follows

$$y_{ij} = 3000 - 150 * x_{ij} + \gamma_j + \varepsilon_{ij},$$

where $\gamma_j \sim N(0, 200^2)$, $\varepsilon_{ij} \sim N(0, 800^2)$, $x_{ij} \sim N(\mu_j, 1)$, $\mu_j \sim U[4, 10]$ and μ_j was held fixed over simulations. Similarly, under the second parametric scenario population data were generated using

$$y_{ij} = 11 - x_{ij} + \gamma_j + \varepsilon_{ij},$$

where now $\gamma_j \sim \chi^2(1)$, $\varepsilon_{ij} \sim \chi^2(6)$ and x_{ij} was generated as in the first scenario but with $\mu_j \sim U[8, 11]$.

For each Monte Carlo simulation a within small areas random sample is selected from the corresponding generated population. Two scenarios for the population and sample sizes are investigated. Under the first scenario (denoted in the tables of results by $\lambda = 0$) the total population size is $N = 8400$ with small area-specific population sizes ranging between $150 \leq N_j \leq 440$. The total sample size is $n = 840$ and the area-specific sample sizes are ranging between $15 \leq n_j \leq 44$. Under the second scenario (denoted in the tables of results by $\lambda = 1$) the total population size is $N = 2820$ with area-specific population sizes ranging between $50 \leq N_j \leq 150$ and the total sample size is $n = 282$ with area-specific sample sizes ranging between $5 \leq n_j \leq 15$.

Using the sample data we obtain point estimates of small area averages with (4), of the 0.25, 0.50 and 0.75 percentiles of the distribution of y with (7) and of the HCR and PG with (8) and (9) respectively. For small area averages MSE estimation is performed using both the analytic MSE (6) and the bootstrap MSE estimator (14). For estimators of small area percentiles and poverty indicators MSE estimation is performed using the bootstrap MSE estimator (14). We run in total $H = 500$ Monte-Carlo simulations. For bootstrap MSE estimation we used one bootstrap population ($B = 1$) from which we drew 400 bootstrap samples ($L = 400$). Because the evaluation of the bootstrap MSE estimator is taking place within a Monte-Carlo framework, the generation of a new Monte-Carlo population and of a new bootstrap population in each iteration is imitating the generation of many bootstrap populations. For the bootstrap MSE estimation we used the unconditional approach with both the empirical (11) and smoothed (12) versions of the error distribution. For the smoothed case, we use the Epanechnikov kernel density, $k(u) = (3/4)(1 - u^2)I(|u| < 1)$, where the smoothing parameter h in (12) was chosen so that it minimizes the cross-validation criterion suggested by Bowman et al. (1998). That is, h was chosen in order to minimize

$$CV(h) = n^{-1} \sum_{j=1}^{m} \sum_{i \in s_j} \int \left[I(e_{ij} - \bar{e}_s) \leq t - G_{-i}(t) \right]^2 \mathrm{d}t,$$

where $G_{-i}(t)$ is the version of $G(t)$ that omits sample unit i (Li and Racine (2007), section 1.5). To compute the smoothing parameter h in (12) we used the `np` package (Hayfield and Racine, 2008) in the R environment (R

Development Core Team, 2010).

Denoting by τ_j the true and unknown parameter and by $\hat{\tau}_j$ the corresponding estimate, the performance of MSE estimators is evaluated using the relative bias and Root MSE (RMSE) defined by

$$RBIAS(\hat{\tau}_j) = H^{-1}\sum_{h=1}^{H}\Big(\frac{\hat{\tau}_{j,h}-\tau_{j,h}}{\tau_{j,h}}\Big)$$

$$RMSE(\hat{\tau}_j) = \Big[H^{-1}\sum_{h=1}^{H}(\hat{\tau}_{j,h}-\tau_{j,h})^2\Big]^{1/2}$$

Finally, coverage rates of 95% confidence intervals constructed by using the bootstrap MSE estimator are computed. Although the detailed results of coverage rates are not reported in the tables of results, we do provide summary results of coverage rates in our commentary.

4.1. Results for small area averages

Table 1 presents the results for MSE estimation of M-quantile small area averages obtained with (4), under the two parametric scenarios and the two scenarios for the area-specific sample and population sizes, using the analytic MSE estimator (6) and the bootstrap MSE estimator (14). For bootstrap estimation we used the smoothed unconditional approach for estimating the distribution of the residuals. Results from the implementation of the empirical unconditional approach have been also produced but in the economy of space are not reported here. The table reports the distribution over areas of the empirical, Monte Carlo RMSE, the estimated RMSE, the relative bias (%) of the estimated RMSE and the RMSE of the RMSE estimators, which is used for assessing the stability of the bootstrap and analytic MSE estimators.

These results suggest that for all scenarios we studied the analytic and the bootstrap MSE estimators track very well the empirical MSE and have on average reasonably low relative bias. However, the bootstrap MSE estimator appears to be notably more stable. In particular, the RMSE of the bootstrap MSE estimator is approximately half that of the analytic estimator (scenario with $\lambda = 0$) and differences become more pronounced for the smaller area sample sizes (scenario $\lambda = 1$). Therefore, there is evidence to suggest that the bootstrap MSE estimator is more stable than the analytic MSE estimator

Averages, Smoothed Approach	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Normal scenario, $\lambda = 0$						
True	113.4	125.8	147.6	147.9	168.6	189.5
Estimated(Analytic)	113.4	125.5	140.4	146.1	165.1	190.3
Estimated(Bootstrap)	112.6	125.4	140.0	147.5	167.9	193.8
Rel. Bias(%)(Analytic)	−10.08	−3.35	−1.627	−0.988	0.594	8.59
Rel. Bias(%)(Bootstrap)	−10.15	−2.709	−0.908	−0.189	1.802	11.19
RMSE(Analytic)	11.23	15.69	21.540	22.420	28.400	39.82
RMSE(Bootstrap)	6.858	8.278	10.700	11.040	12.280	20.16
Normal scenario, $\lambda = 1$						
True	177.6	219.5	249.4	255.6	291.1	333.3
Estimated(Analytic)	188.3	206.5	232.5	237.2	262.6	298.3
Estimated(Bootstrap)	186.5	208.9	236.1	245.3	273.4	324.1
Rel. Bias(%)(Analytic)	−16.29	−10.06	−6.24	−6.53	−4.65	6.05
Rel. Bias(%)(Bootstrap)	−10.26	−5.78	−4.52	−3.78	−2.06	5.03
RMSE(Analytic)	36.10	45.98	60.510	67.160	84.570	117.700
RMSE(Bootstrap)	14.76	21.460	27.650	28.170	34.050	45.850
χ^2 scenario, $\lambda = 0$						
True	0.506	0.564	0.645	0.644	0.718	0.845
Estimated(Analytic)	0.484	0.542	0.614	0.634	0.717	0.822
Estimated(Bootstrap)	0.488	0.542	0.607	0.639	0.728	0.841
Rel. Bias(%)(Analytic)	−11.24	−6.043	−1.665	−1.505	2.587	9.16
Rel. Bias(%)(Bootstrap)	−8.323	−4.866	−0.323	−0.818	2.448	12.22
RMSE(Analytic)	0.0804	0.0972	0.1246	0.1346	0.1707	0.2162
RMSE(Bootstrap)	0.0353	0.0483	0.0587	0.0605	0.0714	0.1028
χ^2 scenario, $\lambda = 1$						
True	0.752	0.936	1.059	1.098	1.271	1.494
Estimated(Analytic)	0.789	0.890	1.005	1.019	1.122	1.269
Estimated(Bootstrap)	0.812	0.915	1.025	1.069	1.192	1.415
Rel. Bias(%)(Analytic)	−15.04	−9.198	−7.173	−6.531	−3.994	7.424
Rel. Bias(%)(Bootstrap)	−7.241	−4.934	−1.836	−2.478	−0.773	7.953
RMSE(Analytic)	0.2156	0.2642	0.328	0.3693	0.4524	0.6686
RMSE(Bootstrap)	0.0846	0.1089	0.1344	0.1482	0.1781	0.2729

Table 1: Distribution over areas and simulations of the empirical, Monte Carlo RMSE, of the estimated RMSE (analytic and bootstrap), of the relative bias (%) of the RMSE estimators and of the RMSE of the RMSE estimators for M-quantile estimators of small area averages. Bootstrap results are produced using the unconditional smoothed approach.

proposed by Chambers and Tzavidis (2006) and hence it should be preferred in practical applications. The results using the empirical distribution, instead of the smoothed distribution of the residuals, are consistent with the results we present here. Figure 1 present averages, over simulations, of true and estimated, using (4), small area means. In the economy of space we present results only for the χ^2 scenario and the Normal scenario with the smaller area sample sizes ($\lambda = 1$). These results show that estimates of small area averages are close to population values. The results for other scenarios are consistent with the ones we present here. Results for coverage rates of 95% confidence intervals for estimates of small area averages constructed by using the bootstrap MSE estimator are on average close to 95% for both parametric scenarios and for both scenarios of the area sample and population sizes.

4.2. Results for small area poverty indicators and percentiles

Tables 2 and 3 present results on the performance of the bootstrap MSE estimator (14) when used to estimate the MSE of estimates of HCR and PG obtained with (8) and (9) respectively. Bootstrap MSE estimation is implemented using the smoothed unconditional approach for estimating the distribution of the residuals. Results from the implementation of the empirical unconditional approach have been also produced but in the economy of space are not reported here. The tables report the distribution over areas of the empirical, Monte Carlo RMSE, the estimated RMSE, the relative bias of the bootstrap MSE and the RMSE of the RMSE estimator, which is used for assessing the stability of the bootstrap MSE estimator.

From tables 2 and 3 we see that the estimated RMSE for HCR and PG tracks well the entire distribution of the empirical RMSE, both for the Normal and χ^2 scenarios. For the normal scenario, these results also show evidence of substantial relative bias ranging on average between (-16% , - 7.6%), which may be due to the large values of the error variance components we used for generating the Monte-Carlo population creating some instability when estimating an indicator. For the chi-square scenario the relative bias is substantially lower ranging on average between (-6.9% , -0.19%). In any case the results on the relative bias must be interpreted with care since the values of the MSEs are small and hence even small differences will result in substantial relative bias. This is the case even with values that agree up to the second decimal place. As expected, the variability of the RMSE estimator is greater when the sample size is smaller, however, given the decrease in the area sample sizes (see scenario $\lambda = 1$) the stability of the MSE estimator

HCR, Smoothed Approach	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Normal scenario, $\lambda = 0$						
True	0.023	0.034	0.036	0.041	0.051	0.064
Estimated	0.018	0.027	0.032	0.035	0.043	0.062
Rel. Bias(%)	−24.320	−18.080	−14.130	−14.220	−10.850	−3.635
RMSE	0.009	0.012	0.0129	0.013	0.0141	0.0173
Normal scenario, $\lambda = 1$						
True	0.032	0.050	0.058	0.063	0.076	0.104
Estimated	0.027	0.046	0.053	0.059	0.073	0.102
Rel. Bias(%)	−20.700	−11.680	−7.849	−7.665	−2.307	5.204
RMSE	0.0136	0.019	0.0217	0.0217	0.0243	0.0293
χ^2 scenario, $\lambda = 0$						
True	0.053	0.057	0.062	0.063	0.069	0.079
Estimated	0.050	0.052	0.056	0.059	0.065	0.076
Rel. Bias(%)	−19.650	−8.425	−7.186	−6.936	−3.731	1.599
RMSE	0.0135	0.0148	0.0158	0.0163	0.0173	0.0223
χ^2 scenario, $\lambda = 1$						
True	0.076	0.084	0.093	0.096	0.107	0.127
Estimated	0.078	0.085	0.094	0.096	0.107	0.126
Rel. Bias(%)	−5.036	−1.139	0.121	0.195	2.190	5.811
RMSE	0.0146	0.0165	0.0181	0.0188	0.0199	0.0288

Table 2: Distribution over areas and simulations of the empirical, Monte Carlo RMSE, of the bootstrap RMSE and of the relative Bias (%) and RMSE of the bootstrap RMSE estimator for the HCR. Results are produced using the unconditional smoothed approach.

PG, Smoothed Approach	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Normal scenario, $\lambda = 0$						
True	0.008	0.013	0.015	0.018	0.024	0.038
Estimated	0.006	0.011	0.012	0.015	0.019	0.034
Rel. Bias(%)	−28.200	−20.360	−16.610	−16.990	−13.580	−7.395
RMSE	0.0034	0.0054	0.006	0.0066	0.0078	0.0115
Normal scenario, $\lambda = 1$						
True	0.011	0.020	0.024	0.028	0.036	0.058
Estimated	0.009	0.017	0.021	0.025	0.032	0.056
Rel. Bias(%)	−22.200	−15.000	−11.310	−12.060	−9.022	−1.492
RMSE	0.0051	0.0093	0.0111	0.0116	0.0149	0.0176
χ^2 scenario, $\lambda = 0$						
True	0.049	0.056	0.062	0.062	0.067	0.079
Estimated	0.046	0.052	0.056	0.058	0.065	0.075
Rel. Bias(%)	−16.430	−7.964	−5.567	−5.587	−2.205	2.723
RMSE	0.0128	0.0137	0.0143	0.0148	0.0153	0.0214
χ^2 scenario, $\lambda = 1$						
True	0.073	0.081	0.093	0.094	0.103	0.120
Estimated	0.071	0.082	0.093	0.095	0.106	0.124
Rel. Bias(%)	−5.595	−1.016	0.072	0.261	2.476	5.046
RMSE	0.0146	0.0168	0.0188	0.0195	0.0214	0.0305

Table 3: Distribution over areas and simulations of the empirical, Monte Carlo RMSE, of the bootstrap RMSE and of the relative Bias (%) and RMSE of the bootstrap RMSE estimator for the PG. Results are produced using the unconditional smoothed approach.

remains satisfactory. These stability results will be only effectively evaluated when compared to alternative MSE estimators of the HCR and PG. Currently, the only alternative available is the parametric bootstrap proposed by Molina and Rao (2010). Figure 1 present averages, over simulations, of true and estimated, using (8) and (9), small area HCRs and PGs. In the economy of space we present results only for the χ^2 scenario and the Normal senario with the smaller area sample sizes ($\lambda = 1$). These results show that estimates of small area HCRs and PGs are close to population values. The results for other scenarios are consistent with the ones we present here.

In tables 4, 5 and 6 we present MSE estimation results for the percentiles of y and more specifically for $q = 0.25, 0.5, 0.75$ estimated with (7). The tables report the distribution over areas of the empirical, Monte Carlo RMSE, the estimated RMSE, the relative bias of the bootstrap RMSE and the RMSE of the RMSE estimators, which is used for assessing the stability of the bootstrap MSE estimator. Bootstrap MSE estimation is implemented using the smoothed unconditional approach for estimating the distribution of the residuals. Results from the implementation of the empirical unconditional approach have been also produced but in the economy of space are not reported here. The bootstrap MSE estimator tracks well the distribution of the empirical MSE of the three percentiles under both parametric scenarios and both scenarios for the small area sample sizes. Some underestimation is present but in terms of percentage relative bias this underestimation is not excessive. Figure 2 presents averages, over simulations, of true and estimated, using (7), small area percentile estimates of $q = 0.25, 0.5, 0.75$. In the economy of space we present results only for the χ^2 and Normal scenario with the smaller area sample sizes ($\lambda = 1$). These results show that estimates of small area percentiles are close to population values. The results for other scenarios are consistent with the ones we present here.

Coverage rates of 95% confidence intervals for estimates of small area quantiles constructed by using the bootstrap MSE estimator range on average between 93% to 95% for both parametric scenarios and for both scenarios of the area sample and population sizes. The coverage rates of 95% confidence intervals for estimates of small area poverty indicators (HCR and PG) constructed by using the bootstrap MSE estimator range on average between 90% to 94% for both parametric scenarios and for both scenarios of the area sample and population sizes.

The results we presented in the section indicate that the bootstrap MSE can be reliably used for estimating the MSE of M-quantile small area av-

Q25, Smoothed Approach	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Normal scenario, $\lambda = 0$						
True	154.20	172.90	195.6	201.8	233.3	261.0
Estimated	151.50	169.10	189.8	198.6	226.9	260.4
Rel. Bias(%)	−6.191	−3.562	−1.256	−1.605	−0.075	2.771
RMSE	15.120	19.320	22.450	23.610	28.320	32.170
Normal scenario, $\lambda = 1$						
True	247.6	288.1	340.7	344.0	388.6	447.5
Estimated	254.5	280.9	317.9	330.7	368.9	442.4
Rel. Bias(%)	−9.946	−6.122	−4.661	−3.707	−1.499	2.793
RMSE	37.230	45.250	54.150	55.190	64.000	78.640
χ^2 scenario, $\lambda = 0$						
True	0.435	0.496	0.582	0.591	0.669	0.770
Estimated	0.448	0.500	0.560	0.589	0.673	0.772
Rel. Bias(%)	−9.965	−2.371	−0.116	−0.076	2.061	7.073
RMSE	0.0406	0.0497	0.0593	0.0632	0.0778	0.0909
χ^2 scenario, $\lambda = 1$						
True	0.757	0.881	1.035	1.037	1.168	1.440
Estimated	0.754	0.850	0.969	1.007	1.121	1.350
Rel. Bias(%)	−8.593	−5.135	−1.655	−2.714	−0.933	4.078
RMSE	0.0892	0.1176	0.1537	0.1478	0.1736	0.2227

Table 4: Distribution over areas and simulations of the empirical, Monte Carlo RMSE, of the bootstrap RMSE and of the relative Bias (%) and the RMSE of the bootstrap RMSE estimator for the 0.25 percentile. Results are produced using the unconditional smoothed approach.

Q50, Smoothed Approach	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Normal scenario, $\lambda = 0$						
True	135.90	157.40	182.80	184.20	212.8	242.4
Estimated	137.20	152.70	170.90	179.10	203.5	234.6
Rel. Bias(%)	−8.635	−4.863	−3.489	−2.743	−0.416	4.99
RMSE	12.970	16.240	19.980	20.380	24.270	27.85
Normal scenario, $\lambda = 1$						
True	232.0	266.2	307.3	313.9	351.2	414.7
Estimated	221.3	246.1	277.7	290.5	324.4	383.1
Rel. Bias(%)	−13.48	−9.774	−7.326	−7.272	−5.344	1.019
RMSE	31.35	38.810	47.620	48.710	56.740	72.920
χ^2 scenario, $\lambda = 0$						
True	0.546	0.626	0.716	0.730	0.827	0.958
Estimated	0.541	0.602	0.675	0.707	0.805	0.929
Rel. Bias(%)	−10.210	−4.987	−3.212	−3.069	−1.707	4.202
RMSE	0.0474	0.0598	0.0802	0.0804	0.0916	0.1129
χ^2 scenario, $\lambda = 1$						
True	0.926	1.071	1.234	1.259	1.397	1.680
Estimated	0.879	0.973	1.113	1.155	1.286	1.544
Rel. Bias(%)	−13.71	−9.968	−8.144	−8.165	−6.568	−2.795
RMSE	0.1239	0.1632	0.194	0.2015	0.2317	0.2935

Table 5: Distribution over areas and simulations of the empirical, Monte Carlo RMSE, of the bootstrap RMSE and of the relative Bias (%) and the RMSE of the bootstrap RMSE estimator for the 0.5 percentile. Results are produced using the unconditional smoothed approach.

Q75, Smoothed Approach	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Normal scenario, $\lambda = 0$						
True	151.60	171.90	200.8	204.7	235.3	268.4
Estimated	150.70	168.00	186.5	197.3	224.8	260.4
Rel. Bias(%)	−11.44	−6.018	−2.583	−3.553	−1.398	1.712
RMSE	15.64	19.210	25.540	24.560	28.830	32.750
Normal scenario, $\lambda = 1$						
True	270.4	299.3	335.6	348.1	393.9	473.6
Estimated	249.3	278.9	312.9	326.6	367.8	432.6
Rel. Bias(%)	−11.62	−8.069	−6.231	−6.132	−4.508	0.226
RMSE	38.45	46.160	52.640	55.770	64.340	86.540
χ^2 scenario, $\lambda = 0$						
True	0.802	0.880	1.001	1.024	1.180	1.322
Estimated	0.768	0.846	0.956	1.001	1.137	1.317
Rel. Bias(%)	−9.388	−4.126	−3.067	−2.239	0.081	5.385
RMSE	0.0936	0.1029	0.1224	0.1291	0.1519	0.1864
χ^2 scenario, $\lambda = 1$						
True	1.262	1.482	1.659	1.681	1.873	2.187
Estimated	1.246	1.366	1.522	1.573	1.748	2.007
Rel. Bias(%)	−11.13	−8.644	−7.011	−6.308	−4.609	2.358
RMSE	0.1942	0.2372	0.2691	0.2713	0.3092	0.3556

Table 6: Distribution over areas and simulations of the empirical, Monte Carlo RMSE, of the bootstrap RMSE and of the relative Bias (%) and the RMSE of the bootstrap RMSE estimator for the 0.75 percentile. Results are produced using the unconditional smoothed approach.

erages, percentiles and poverty indicators. One way of potentially improving the performance of the bootstrap MSE estimator is by generating more than one bootstrap population. Generating more than one bootstrap population within a Monte-Carlo simulation study, however, significantly increases the computational effort. Having said this, when the proposed bootstrap MSE estimator is used in applications with real data we suggest generating $B \in [50, 100]$ bootstrap populations.

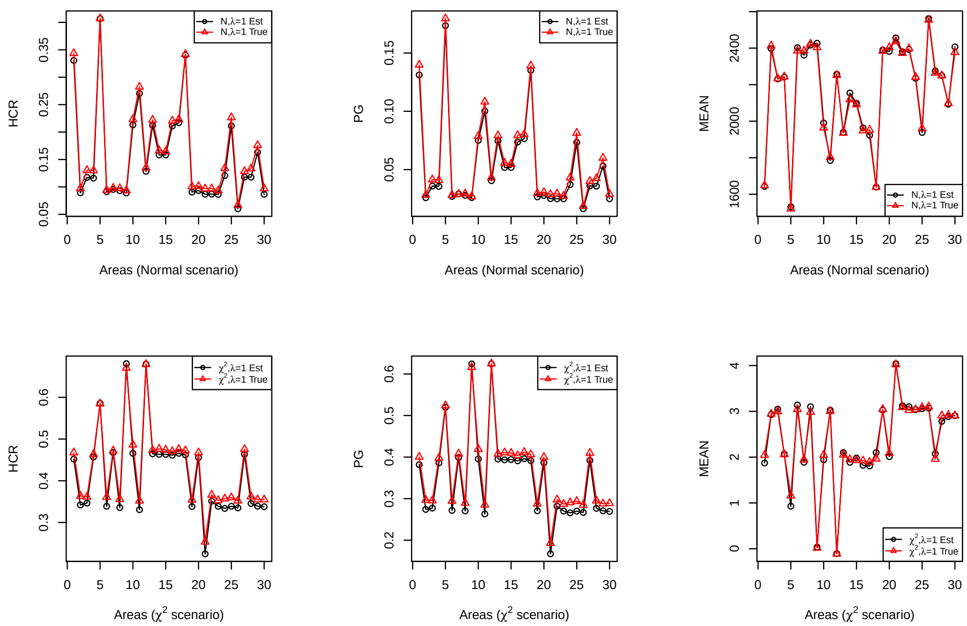


Figure 1: Point estimates and true values for small area averages, HCRs and PGs. The first row of plots refers to the Normal scenario with sample size $\lambda = 1$, the second row refers to χ^2 scenario with sample size $\lambda = 1$.

5. An Application: Estimating the income distribution and poverty indicators for provinces in Tuscany

The aim of this section is to provide a picture of the economic conditions in Tuscan provinces. This is achieved by computing province-specific estimates of average equivalised income, of key percentiles of the income distribution

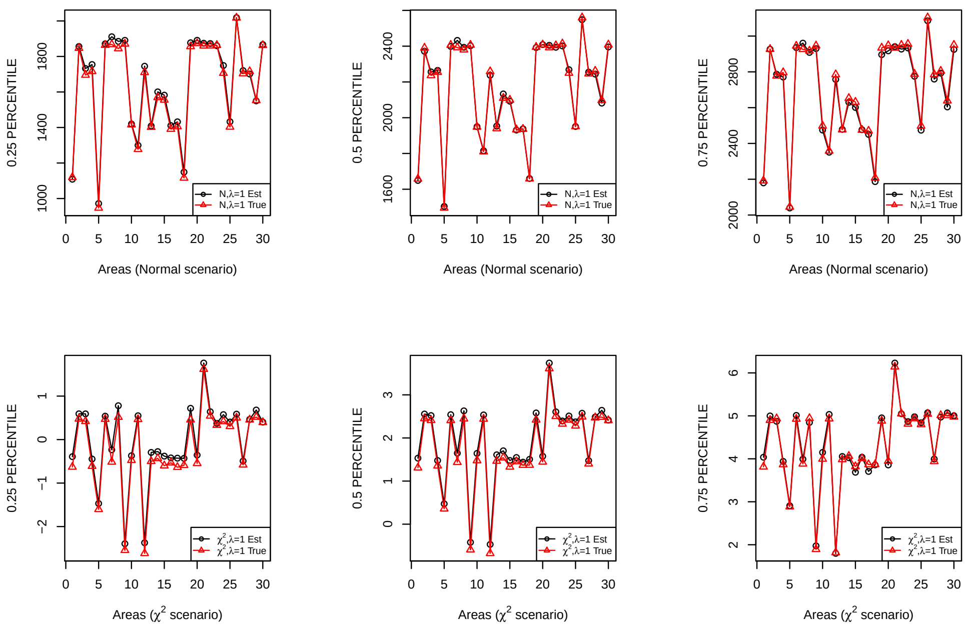


Figure 2: Point estimates and true values for small area percentiles ($q = 0.25, 0.5, 0.75$). The first row of plots refers to the Normal scenario with sample size $\lambda = 1$, the second row refers to χ^2 scenario with sample size $\lambda = 1$.

function (25th, 50th, 75th) and of two poverty indicators namely, the HCR and PG as well as corresponding MSE estimates. Small area estimation is performed by using data from the 2006 European Survey on Income and Living Conditions (EU-SILC) in Italy and the 2001 Census microdata for the region of Tuscany. Provinces within regions are unplanned domains and the sample sizes for provinces in Tuscany range from 59 households in the Grosseto province to 445 households in the Florence Province with an average sample size of 149 households (median 123 households). The population of households in the different provinces, using 2001 Census data, ranges from 80,810 households in the province of Massa-Carrara to 376,300 in the province of Florence with the total number of households in Tuscany being 1,388,252.

We start by first building a small area working model that is estimated using the EU-SILC survey data. The response variable is the household equivalized income. The explanatory variables we considered are those that are common to the survey and Census datasets. This includes explanatory variables that relate to the head of the household namely, gender, age, occupational status and years in education, and explanatory variables that relate to the household namely, the ownership status of the house and the number of household members. Fitting a two-level (households within provinces) random effects model using the above explanatory variables and performing residual analysis reveals departures from the assumed normality of the level 1 and level 2 error terms. For this reason, we decided to use an oultier robust model, in this case the M-quantile small area model (see Section 2). A logarithmic transformation of income is not easily applicable since there are negative income values in our survey data. Small area estimates of average household income are derived using (4). Small area estimates of the three income percentiles are derived by using (7). Finally, estimates of HCR and PG are obtained by using (8) and (9) respectively. For estimating the poverty indicators the poverty line is computed at regional level as 60% of the median household equivalised income. Corresponding estimates of the MSE are obtained by using the bootstrap MSE estimator (14), which is implemented by generating $B = 50$ bootstrap populations and selecting $L = 100$ bootstrap samples from each bootstrap population.

The results are summarized in tables 7 and 8 and in figures 3 and 4, which present point estimates and corresponding estimates of RMSE. A clear picture about the wealth of Tuscan provinces emerges with the provinces of Siena and Florence being the wealthiest and the provinces of Massa-Carrara and Lucca the least wealthy (darker colors indicate higher wealth

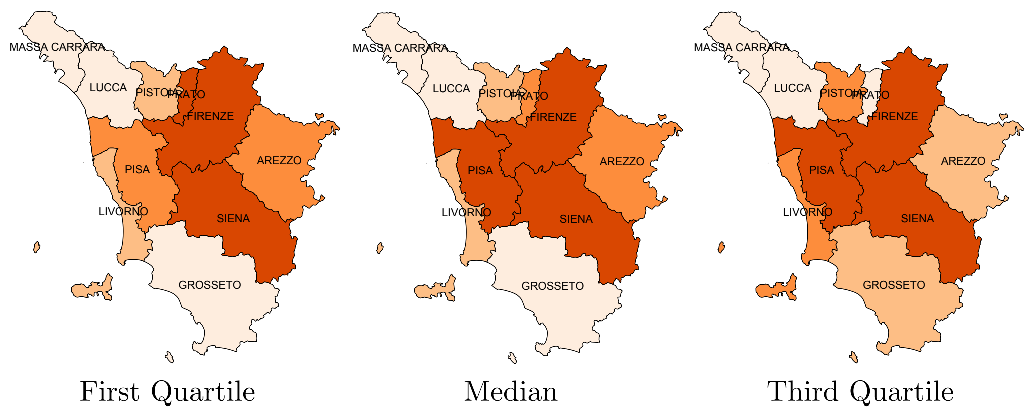
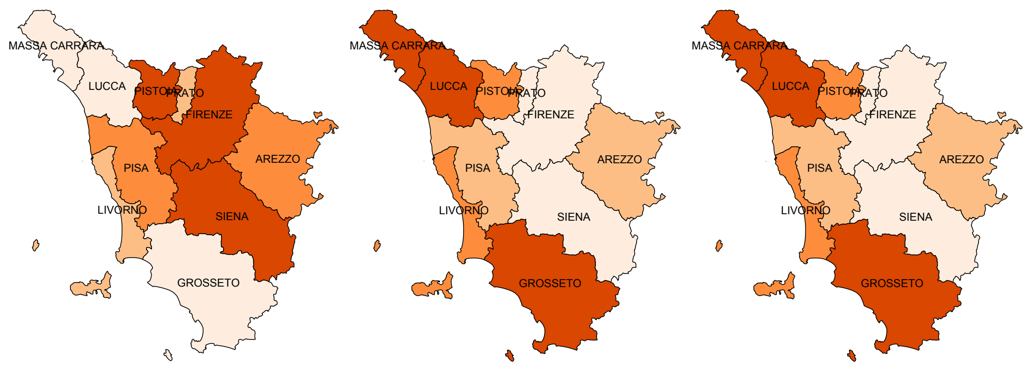


Figure 3: Estimates of the income percentiles for provinces in Tuscany

and lower poverty). In particular, the provinces of Massa-Carrara and Lucca have clearly the lowest average equivalised household income whereas the provinces of Siena and Firenze the highest. Massa-Carrara and Lucca also have the highest number of households below the poverty line whereas Siena and Firenze the lowest and this picture remains the same when we look at the spatial distribution of PG. Examining the percentiles of the province-specific income distributions we note that estimates of average income are higher than estimates of median income, which highlights the right asymmetry of the income distributions. Using the percentile estimates of income we can draw some further conclusions. Looking at the average income and the HCR we note that the province of Grosseto is among the least wealthy Tuscan provinces. However, when examining the estimate of the third quartile for Grosseto we note that this is similar to the estimate of the third quartile of Arezzo, which is one of the wealthiest provinces. This indicates the presence of inequality in Grosseto. Some evidence of inequality also exists for the provinces of Livorno and Pisa.

6. R functions for point and MSE estimation

R functions that implement small area estimation with the M-quantile model are available upon request from the authors. In particular, function *mq.sae* produces M-quantile estimates of small area averages using (4) and MSE estimation using the analytic MSE estimator (6). Function *mq.sae.quant* produces M-quantile estimates of the small area quantiles of the distribution



Mean HCR PG

Figure 4: Estimates of the average income, Head Count Ratio and Poverty Gap for provinces in Tuscany

PROVINCE	MEAN	RMSE	HCR	RMSE	PG	RMSE
MASSA	14128.26	664.84	0.280	0.039	0.117	0.022
LUCCA	15867.69	766.81	0.239	0.026	0.094	0.015
PISTOIA	18980.76	1119.33	0.195	0.019	0.073	0.011
FIRENZE	19184.92	498.35	0.166	0.012	0.061	0.007
LIVORNO	17875.01	919.41	0.193	0.020	0.075	0.012
PISA	18550.16	876.38	0.175	0.018	0.065	0.010
AREZZO	18665.97	1014.42	0.182	0.018	0.068	0.010
SIENA	20228.98	1113.91	0.161	0.023	0.060	0.012
GROSSETO	16152.47	1151.83	0.231	0.029	0.093	0.019
PRATO	17702.87	632.74	0.172	0.021	0.062	0.011

Table 7: Point estimates and corresponding RRMSEs of small area averages, HCRs and PGs for Provinces in Tuscany

PROVINCE	Q1	RMSE	Q2	RMSE	Q3	RMSE
MASSA	8837.42	712.33	13498.69	831.75	18528.25	1164.96
LUCCA	9715.74	640.71	14733.29	690.70	20650.60	1087.89
PISTOIA	11412.26	669.47	16124.78	685.77	22243.96	1017.31
FIRENZE	12628.02	335.54	17364.81	377.37	23328.47	547.68
LIVORNO	11338.34	610.02	16662.83	701.85	22991.71	983.31
PISA	11571.98	618.04	17161.42	681.67	23867.96	989.89
AREZZO	12205.01	578.72	16724.22	638.74	22100.72	949.08
SIENA	12639.00	662.31	18373.53	703.94	25471.22	1087.76
GROSSETO	9924.80	924.38	15456.41	1016.58	22069.22	1483.27
PRATO	12779.53	669.54	16968.74	708.72	21796.88	1101.34

Table 8: Point estimates and corresponding RRMSEs of small area quartiles for Provinces in Tuscany

of y using (7) and bootstrap MSE estimation using MSE estimator (14). Function *mq.sae.poverty* produces M-quantile estimates of the small area HCR and PG using respectively (8) and (9) and bootstrap MSE estimation using MSE estimator (14). Options for using empirical or smoothed, conditional and unconditional residuals for generating the bootstrap population are available. The details of each function are provided in the appendix at the end of the paper.

7. Conclusions

In this paper we propose the use of non-parametric bootstrap for estimating the MSE for small area averages, quantiles and poverty indicators estimated with the M-quantile model and the Chambers-Dunstan estimator. Given that analytic MSE estimation for quantiles and poverty indicators is difficult, the proposed MSE estimator provides one practical approach for MSE estimation of complex small area statistics. As illustrated in the empirical section, the proposed bootstrap MSE estimator approximates well the 'true' MSE error of the target parameters. In addition, these results show that bootstrap MSE estimation is notably more stable than corresponding analytic estimation. The practical implementation of the estimation procedures we describe in this paper is assisted by the availability of R functions. In work we will be conducting in the near future we aim to implement the bootstrap MSE estimator for estimating the accuracy of quantiles estimates of the income distribution function and of poverty indicators for UK Local Authority Districts using data from the UK Family Resources Survey and UK Census micro-data.

Acknowledgements

This work is financially supported by the European Project SAMPLE "Small Area Methods for Poverty and Living Condition Estimates", funded by the European Commission's 7th Framework Programme (www.sample-project.eu).

Appendix A. Specifications of R functions

Appendix A.1. Estimation of small area means with mq.sae

- Required Packages: *MASS*
- *mq.sae(y,x,regioncode.s,m,p,x.outs,regioncode.r,tol.value,maxit.value,k.value)*

The function provides estimates of small area averages using (3) and (4) and corresponding analytic estimates of MSE using (6). Key arguments required are the response variable y, the matrix of covariates x, covariate information for out of sample areas $x.outs$, the number of small areas m and values relating to the convergence of the algorithm.

Appendix A.2. Estimation of small area quantiles with mq.sae.quant

- Required Packages: *MASS* and *np*
- *mq.sae.quant(q,y,x,x.outs,regioncode.s,regioncode.r,MSE,B,R,method,maxit)*

The function provides estimates of small area quantiles using (7) and corresponding bootstrap estimates of MSE using (14). Key arguments required are the response variable y, the matrix of covariates x, covariate information for out of sample areas $x.outs$ and values relating to the convergence of the algorithm. If $MSE = TRUE$ bootstrap MSE estimates are produced. B denotes the number of bootstrap populations and R denotes the number of bootstrap samples from each bootstrap population. Finally *method* defines the type of residuals used for generating the bootstrap population: 'su' (smooth unconditional),'eu' (emprirical unconditional),'sc' (smooth conditional),'ec' (empirical unconditional). The default is set to 'eu', which is computationally faster.

Appendix A.3. Estimation of small area poverty indicators with mq.sae.poverty

- Required Packages: *MASS* and *np*
- *mq.sae.poverty(y,x,x.outs,regioncode.s,regioncode.r,L,MSE,B,R,method)*

The function provides estimates of small area HCRs and PGs using (8) and (9) and corresponding bootstrap estimates of MSE using (14). Key arguments required are the response variable y, the matrix of covariates x, covariate information for out of sample areas $x.outs$ and values relating to the convergence of the algorithm. L specifies the number of Monte Carlo runs for estimating HCR and PG using the estimation method in Section 2.2. If $MSE = TRUE$ bootstrap MSE estimates are produced. B denotes the number of bootstrap populations and R denotes the number of bootstrap samples from each bootstrap population. Finally *method* defines the type of residuals used for generating the bootstrap population: 'su' (smooth unconditional),'eu' (emprirical unconditional),'sc' (smooth conditional),'ec' (empirical unconditional). The default is set to 'eu', which is computationally faster.

Aragon, Y., Casanova, S., Chambers, R., Leoconte, E., 2005. Conditional ordering using nonparametric expectiles. Journal of Official Statistics 21 (4).

Bowman, A., Hall, P., Prvan, T., 1998. Bandwidth selection for the smoothing of distribution functions. Biometrika 85, 799–808.

Breckling, J., Chambers, R., 1988. M-quantiles. Biometrika 75 (4), 761–771.

Chambers, R., Chandra, H., Tzavidis, N., 2009. On robust mean squared error estimation for linear predictors for domains. Working Paper 11, University of Wollongong, Australia.

Chambers, R., Dorfman, A., Peter, H., 1992. Properties of estimators of the finite population distribution function. Biometrika 79 (3), 577–582.

Chambers, R., Dunstan, 1986. Estimating distribution function from survey data. Biometrika 73, 597–604.

Chambers, R., Tzavidis, N., 2006. M-quantile models for small area estimation. Biometrika 93 (2), 255–68.

Foster, J., Greer, J., Thorbecke, E., 1984. A class of decomposable poverty measures. Econometrica 52, 761–766.

Hayfield, T., Racine, J. S., 2008. Nonparametric econometrics: The np package. Journal of Statistical Software 27 (5).
URL http://www.jstatsoft.org/v27/i05/

Koenker, R., Bassett, G., 1978. Regression quantiles. Econometrica 46, 33–50.

Kokic, P., Chambers, R., Breckling, J., Beare, S., 1997. A measure of production performance. Journal of Business and Economic Statistics 15 (4), 445–51.

Li, Q., Racine, J., 2007. Nonparametric econometric: theory and practice. Princeton University Press.

Lombardia, M., Gonzalez-Manteiga, W., Prada-Sanchez, J., 2003. Bootstrapping the chambers-dunstan estimate of finite population distribution function. Journal of Statistical Planning and Inference 116, 367–388.

Molina, I., Morales, D., Pratesi, M., Tzavidis, N. (Eds.), 2010. Final small area estimation development and simulation results. No. Deliverable D12 and D16 - S.A.M.P.L.E. project. European Union - 7th Framework Programme.

Molina, I., Rao, J., 2010. Small area estimation of poverty indicators. The Canadian Journal of Statistics.

Newey, W., Powell, J., 1987. Asymmetric least squares estimation and testing. Econometrica 55 (4), 819–47.

R Development Core Team, 2010. R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria.

Rao, J., 2003. Small Area Estimation. New York:Wiley.

Royall, R., Cumberland, W., 1978. Variance estimation in finite population sampling. Journal of the American Statistical Association 73, 351–58.

Sinha, S., Rao, J., 2009. Robust small area estimation. The Canadian Journal of Statistics 37 (3), 381–399.

Tzavidis, N., Marchetti, S., Chambers, R., 2010. Robust estimation of small area means and quantiles. Australian and New Zeland Journal of Statistics 52 (2), 167–186.