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R-groups and geometric structure in the representation theory of SL(N)

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Abstract. Let F be a nonarchimedean local field of characteristic zero and let $G = \mathrm{SL}(N) = \mathrm{SL}(N, F)$. This article is devoted to studying the influence of the elliptic representations of $\mathrm{SL}(N)$ on the K-theory. We provide full arithmetic details. This study reveals an intricate geometric structure. One point of interest is that the R-group is realized as an isotropy group. Our results illustrate, in a special case, part (3) of the recent conjecture in [2].

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1. Introduction

Let F be a nonarchimedean local field of characteristic zero and let $G = \mathrm{SL}(N) = \mathrm{SL}(N, F)$. This article is devoted to studying subspaces of the tempered dual of $\mathrm{SL}(N)$ which have an especially intricate geometric structure, and to computing, with full arithmetic details, their K-theory. Our results illustrate, in a special case, part (3) of the recent conjecture in [2].

The subspaces of the tempered dual which are especially interesting for us contain *elliptic* representations. A tempered representation of $\mathrm{SL}(N)$ is *elliptic* if its Harish-Chandra character is not identically zero on the elliptic set.

An element in the discrete series of $\mathrm{SL}(N)$ is an isolated point in the tempered dual of $\mathrm{SL}(N)$ and contributes one generator to K_0 of the reduced C*-algebra of $\mathrm{SL}(N)$.

Now $\mathrm{SL}(N)$ admits elliptic representations which are not discrete series: we investigate, with full arithmetic details, the contribution of the elliptic representations of $\mathrm{SL}(N)$ to the K-theory of the reduced C*-algebra $\mathfrak{A}_N$ of $\mathrm{SL}(N)$.

According to [7], $\mathfrak{A}_N$ is a C*-direct sum of fixed C*-algebras. Among these fixed algebras, we will focus on those whose duals contain elliptic representations. Let n be a divisor of N with $1 \le n \le N$ and suppose that the group $\mathcal{U}_F$ of integer units admits a character of order n. Then the relevant fixed algebras are of the form

$$C(\mathbb{T}^n/\mathbb{T}, \mathfrak{K})^{\mathbb{Z}/n\mathbb{Z}} \subset \mathfrak{A}_N.$$

Here, $\mathfrak{K}$ is the C*-algebra of compact operators on standard Hilbert space, $\mathbb{T}^n/\mathbb{T}$ is the quotient of the compact torus $\mathbb{T}^n$ via the diagonal action of $\mathbb{T}$. The compact group $\mathbb{T}^n/\mathbb{T}$ arises as the maximal compact subgroup of the standard maximal torus of the Langlands dual $\mathrm{PGL}(n, \mathbb{C})$. We prove (Theorem 3.1) that this fixed C*-algebra is strongly Morita equivalent to the crossed product

$$C(\mathbb{T}^n/\mathbb{T}) \rtimes \mathbb{Z}/n\mathbb{Z}.$$

The reduced C*-algebra $\mathfrak{A}_N$ is liminal, and its primitive ideal space is in canonical bijection with the tempered dual of $\mathrm{SL}(N)$. Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual of $\mathrm{SL}(N)$, see [5], 3.1.1, 4.4.1, 18.3.2.

Let $\mathfrak{T}_n$ denote the C*-dual of $C(\mathbb{T}^n/\mathbb{T}, \mathfrak{K})^{\mathbb{Z}/n\mathbb{Z}}$. Then $\mathfrak{T}_n$ is a non-Hausdorff space, and has a very special structure as topological space. When n is a prime number ℓ, then $\mathfrak{T}_\ell$ will contain multiple points. When n is non-prime, $\mathfrak{T}_n$ will contain not only multiple points, but also *multiple subspaces*. This crossed product C*-algebra is a noncommutative unital C*-algebra which fits perfectly into the framework of noncommutative geometry. In the tempered dual of $\mathrm{SL}(N)$, there are connected compact non-Hausdorff spaces, laced with multiple subspaces, and simply described by crossed product C*-algebras.

The K-theory of the fixed C*-algebra is then given by the K-theory of the crossed product C*-algebra. To compute (modulo torsion) the K-theory of this noncommutative C*-algebra, we apply the Chern character for discrete groups [3]. This leads to the cohomology of the *extended quotient* $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$. This in turn leads to a problem in classical algebraic topology, namely the determination of the cyclic invariants in the cohomology of the n-torus.

The ordinary quotient will be denoted by $\mathfrak{X}(n)$:

$$\mathfrak{X}(n) := (\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}).$$

This is a compact connected orbifold. Note that $\mathfrak{X}(1) = pt$. The orbifold $\mathfrak{X}(n, k, \omega)$ which appears in the following theorem is defined in Section 4. The notation is such that $\mathfrak{X}(n, n, 1)$ is the ordinary quotient $\mathfrak{X}(n)$ and each $\mathfrak{X}(n, 1, \omega)$ is a point. The highest common factor of n and k is denoted (n, k).

Theorem 1.1. *The extended quotient $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$ is a disjoint union of compact connected orbifolds:*

$$(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z}) = \bigsqcup \mathfrak{X}(n, k, \omega)$$

The disjoint union is over all $1 \le k \le n$ and all $n/(k, n)$th roots of unity ω in $\mathbb{C}$.

We apply the Chern character for discrete groups [3], and obtain

Theorem 1.2. *The K-theory groups K_0 and K_1 are given by*

$$K_0(C(\mathbb{T}^n/\mathbb{T}),\mathfrak{K})^{\mathbb{Z}/n\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus H^{\mathrm{ev}}(\mathfrak{X}(n,k,\omega);\mathbb{C}),$$
$$K_1(C(\mathbb{T}^n/\mathbb{T}),\mathfrak{K})^{\mathbb{Z}/n\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus H^{\mathrm{odd}}(\mathfrak{X}(n,k,\omega);\mathbb{C}).$$

The direct sums are over all $1 \le k \le n$ and all $n/(k,n)$th roots of unity ω in $\mathbb{C}$.

For the ordinary quotient $\mathfrak{X}(n)$ we have the following explicit formula (Theorems 6.1 and 6.3). Let $H^\bullet := H^{\mathrm{ev}} \oplus H^{\mathrm{odd}}$ and let ϕ denote the Euler totient.

Theorem 1.3. *Let $\mathfrak{X}(n)$ denote the ordinary quotient $(\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z})$. Then we have*

$$\dim_{\mathbb{C}} H^\bullet(\mathfrak{X}(n);\mathbb{C}) = \frac{1}{2n} \sum_{d|n,\, d \text{ odd}} \phi(d) 2^{n/d}.$$

Theorem 1.1 lends itself to an interpretation in terms of representation theory. When $n = \ell$ a prime number, the elliptic representations of $\mathrm{SL}(\ell)$ are discussed in Section 2. The extended quotient $(\mathbb{T}^\ell/\mathbb{T})/\!/(\mathbb{Z}/\ell Z)$ is the disjoint union of the ordinary quotient $\mathfrak{X}(\ell)$ and $\ell(\ell-1)$ isolated points. We consider the canonical projection π of the extended quotient onto the ordinary quotient:

$$\pi\colon (\mathbb{T}^\ell/\mathbb{T})/\!/(\mathbb{Z}/\ell\mathbb{Z}) \to \mathfrak{X}(\ell).$$

The points $\tau_1,\ldots,\tau_\ell$ constructed in Section 2, are precisely the $\mathbb{Z}/\ell\mathbb{Z}$ fixed points in $\mathbb{T}^\ell/\mathbb{T}$. These are ℓ points of reducibility, each of which admits ℓ elliptic constituents. Note also that, in the canonical projection π, the fibre $\pi^{-1}(\tau_j)$ of each point τ_j contains ℓ points. We may say that the extended quotient encodes, or provides a model of, reducibility. This is a very special case of the recent conjecture in [2].

When n is non-prime, we have points of reducibility, each of which admits elliptic constituents. In addition to the points of reducibility, there is a subspace of reducibility. There are continua of L-packets. Theorem 1.2 describes the contribution, modulo torsion, of all these L-packets to K_0 and K_1.

Let the infinitesimal character of the elliptic representation ϵ be the cuspidal pair (M,σ), where σ is an irreducible cuspidal representation of M with unitary central character. Then ϵ is a constituent of the induced representation $i_{GM}(\sigma)$. Let $\mathfrak{s}$ be the point in the Bernstein spectrum which contains the cuspidal pair (M,σ). To conform to the notation in [2], we will write $E^{\mathfrak{s}} := \mathbb{T}^n/\mathbb{T}$, $W^{\mathfrak{s}} = \mathbb{Z}/n\mathbb{Z}$. The standard projection will be denoted

$$\pi^{\mathfrak{s}}\colon E^{\mathfrak{s}}/\!/W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

The space of tempered representations of G determined by $\mathfrak{s}$ will be denoted $\mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$, and the infinitesimal character will be denoted inf.ch.

Theorem 1.4. *There is a continuous bijection*

$$\mu^{\mathfrak{s}}\colon E^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} \to \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$$

such that

$$\pi^{\mathfrak{s}} = (\mathrm{inf.ch.}) \circ \mu^{\mathfrak{s}}.$$

This confirms, in a special case, part (3) *of the conjecture in* [2].

In Section 2 of this article, we review elliptic representations of the special linear algebraic group $\mathrm{SL}(N, F)$ over a p-adic field F. Section 3 concerns fixed C*-algebras and crossed products. The extended quotient $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$ is computed in Section 4. The formation of the R-groups is described in Section 5. In Section 6 we compute the cyclic invariants in the cohomology of the n-torus.

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2. The elliptic representations of SL(N)

Let F be a nonarchimedean local field of characteristic zero. Let $\boldsymbol{G}$ be a connected reductive linear group over F. Let $G = \boldsymbol{G}(F)$ be the F-rational points of $\boldsymbol{G}$. We say that an element x of G is *elliptic* if its centralizer is compact modulo the center of G. We let G^e denote the set of regular elliptic elements of G.

Let $\mathcal{E}_2(G)$ denote the set of equivalence classes of irreducible discrete series representations of G, and denote by $\mathcal{E}_t(G)$ be the set of equivalence classes of irreducible tempered representations of G. Then $\mathcal{E}_2(G) \subset \mathcal{E}_t(G)$. If $\pi \in \mathcal{E}_t(G)$, then we denote its character by Θ_π. Since Θ_π can be viewed as a locally integrable function, we can consider its restriction to G^e, which we denote by Θ_π^e. We say that π is elliptic if $\Theta_\pi^e \neq 0$. The set of elliptic representations includes the discrete series.

Here is a classical example where elliptic representations occur [1]. We consider the group $\mathrm{SL}(\ell, F)$ with ℓ a prime not equal to the residual characteristic of F. Let K/F be a cyclic of order ℓ extension of F. The reciprocity law in local class field theory is an isomorphism

$$F^\times / N_{K/F}\, K^\times \cong \Gamma(K/F) = \mathbb{Z}/\ell\mathbb{Z},$$

where $\Gamma(K/F)$ is the Galois group of K over F. Let now $\mu_\ell(\mathbb{C})$ be the group of ℓth roots of unity in $\mathbb{C}$. A choice of isomorphism $\mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell(\mathbb{C})$ then produces a character κ of $F^\times$ of order ℓ as follows:

$$\kappa\colon F^\times \to F^\times / N_{K/F}\, K^\times \cong \mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell(\mathbb{C}).$$

Let B be the standard Borel subgroup of $\mathrm{SL}(\ell)$, let T be the standard maximal torus, and let $B = T \cdot N$ be its Levi decomposition. Let τ be the character of T defined by

$$\tau := 1 \otimes \kappa \otimes \cdots \otimes \kappa^{\ell-1}$$

and let

$$\pi(\tau) := \mathrm{Ind}_B^G(\tau \otimes 1)$$

be the unitarily induced representation of $\mathrm{SL}(\ell)$.

Now $\pi(\tau)$ is a representation in the minimal unitary principal series of $\mathrm{SL}(\ell)$. It has ℓ distinct irreducible elliptic components and the Galois group $\Gamma(K/F)$ acts simply transitively on the set of irreducible components. The set of irreducible components of $\pi(\tau)$ is an L-packet.

Let

$$\pi(\tau) = \pi_1 \oplus \cdots \oplus \pi_\ell$$

be the ℓ components of $\pi(\tau)$. The character Θ of $\pi(\tau)$, as character of a principal series representation, *vanishes on the elliptic set*. The character Θ_1 of π_1 on the elliptic set is therefore *cancelled out* by the sum $\Theta_2 + \cdots + \Theta_\ell$ of the characters of the relatives $\pi_2, \ldots, \pi_\ell$ of π_1.

Let ω denote an ℓth root of unity in $\mathbb{C}$. All the ℓth roots are allowed, including $\omega = 1$. In the definition of τ, we now replace κ by $\kappa \otimes \omega^{\mathrm{val}}$. This will create ℓ characters, which we will denote by $\tau_1, \ldots, \tau_\ell$, where $\tau_1 = \tau$. For each of these characters, the R-group is given as follows:

$$R(\tau_j) = \mathbb{Z}/\ell\mathbb{Z}$$

for all $1 \le j \le \ell$, and the induced representation $\pi(\tau_j)$ admits ℓ elliptic constituents.

If $P = MU$ is a standard parabolic subgroup of G then $i_{GM}(\sigma)$ will denote the induced representation $\mathrm{Ind}_{MU}^G(\sigma \otimes 1)$ (normalized induction). The R-group attached to σ will be denoted $R(\sigma)$.

Let $P = MU$ be the standard parabolic subgroup of $G := \mathrm{SL}(N, F)$ described as follows. Let $N = mn$, let $\widetilde{M}$ be the Levi subgroup $\mathrm{GL}(m)^n \subset \mathrm{GL}(N, F)$ and let $M = \widetilde{M} \cap \mathrm{SL}(N, F)$.

We will use the framework, notation and main result in [6]. Let $\sigma \in \mathcal{E}_2(M)$ and let $\pi_\sigma \in \mathcal{E}_2(\widetilde{M})$ with $\pi_\sigma | M \supset \sigma$. Let $W(M) := N_G(M)/M$ denote the Weyl group of M, so that $W(M)$ is the symmetric group on n letters. Let

$$\bar{L}(\pi_\sigma) := \{\eta \in \widehat{F^\times} \mid \pi_\sigma \otimes \eta \simeq w\pi_\sigma \text{ for some } w \in W\},$$

$$X(\pi_\sigma) := \{\eta \in \widehat{F^\times} \mid \pi_\sigma \otimes \eta \simeq \pi_\sigma\}.$$

By [6], Theorem 2.4, the R-group of σ is given by

$$R(\sigma) \simeq \bar{L}(\pi_\sigma)/X(\pi_\sigma).$$

We follow [6], Theorem 3.4. Let η be a smooth character of $F^\times$ such that $\eta^n \in X(\pi_1)$ and $\eta^j \notin X(\pi_1)$ for $1 \le j \le n-1$. Set

$$\pi_\sigma \simeq \pi_1 \otimes \eta\pi_1 \otimes \eta^2\pi_1 \otimes \cdots \otimes \eta^{n-1}\pi_1, \quad \pi_\sigma|M \supset \sigma, \tag{1}$$

with $\pi_1 \in \mathcal{E}_2(\mathrm{GL}(m))$, $\eta\pi_1 := (\eta \circ \det) \otimes \pi_1$. Then we have

$$\bar{L}(\pi_\sigma)/X(\pi_\sigma) = \langle \eta \rangle$$

and so $R(\sigma) \simeq \mathbb{Z}/n\mathbb{Z}$. The elliptic representations are the constituents of $i_{GM}(\sigma)$ with π_σ as in equation (1).

3. Fixed algebras and crossed products

Let M denote the Levi subgroup which occurs in Section 2. Denote by $\Psi^1(M)$ the group of unramified unitary characters of M. Now $M \subset \mathrm{SL}(N, F)$ comprises blocks $x_1, \ldots, x_n$ with $x_i \in \mathrm{GL}(m, F)$ and $\prod \det(x_i) = 1$. Each unramified unitary character $\psi \in \Psi^1(M)$ can be expressed as

$$\psi\colon\ \mathrm{diag}(x_1, \ldots, x_n) \to \prod_{j=1}^{n} z_j^{\mathrm{val}(\det x_j)},$$

with $z_1, z_2, \ldots, z_n \in \mathbb{T}$, i.e., $|z_i| = 1$. Such unramified unitary characters ψ correspond to coordinates $(z_1 : z_2 : \cdots : z_n)$ with each $z_i \in \mathbb{T}$. Since

$$\prod_{i=1}^{n} (zz_i)^{\mathrm{val}(\det x_i)} = \prod_{i=1}^{n} z_i^{\mathrm{val}(\det x_i)}$$

we have *homogeneous* coordinates. We have the isomorphism

$$\Psi^1(M) \cong \{(z_1 : z_2 : \cdots : z_n) \mid |z_i| = 1,\ 1 \le i \le n\} = \mathbb{T}^n/\mathbb{T}.$$

If M is the standard maximal torus T of $\mathrm{SL}(N)$ then $\Psi^1(T)$ is the maximal *compact* torus in the dual torus

$$T^\vee \subset G^\vee = \mathrm{PGL}(N, \mathbb{C}),$$

where $G^\vee$ is the Langlands dual group.

Let $\sigma, \pi_\sigma, \pi_1$ be as in equation (1). Let g be the order of the group of unramified characters χ of $F^\times$ such that $(\chi \circ \det) \otimes \pi_1 \simeq \pi_1$. Now let

$$E := \{\psi \otimes \sigma \mid \psi \in \Psi^1(M)\}.$$

The base point $\sigma \in E$ determines a homeomorpism

$$E \simeq \mathbb{T}^n/\mathbb{T}, \quad (z_1^{\mathrm{val}\circ\det} \otimes \cdots \otimes z_n^{\mathrm{val}\circ\det}) \otimes \sigma \mapsto (z_1^g : \cdots : z_n^g).$$

From this point onwards, we will require that the *restriction of η to the group $\mathcal{U}_F$ of integer units is of order n.* Let $W(M)$ denote the Weyl group of M and let $W(M,E)$ be the subgroup of $W(M)$ which leaves E globally invariant. Then we have $W(M,E) = W(\sigma) = R(\sigma) = \mathbb{Z}/n\mathbb{Z}$.

Let $\mathfrak{K} = \mathfrak{K}(H)$ denote the C*-algebra of compact operators on the standard Hilbert space H. Let $\mathfrak{a}(w,\lambda)$ denote normalized intertwining operators. The fixed C*-algebra $C(E,\mathfrak{K})^{W(M,E)}$ is given by

$$\{f \in C(E,\mathfrak{K}) \mid f(w\lambda) = \mathfrak{a}(w,\lambda\tau) f(\lambda) \mathfrak{a}(w,\lambda\tau)^{-1},\ w \in W(M,E)\}.$$

This fixed C*-algebra is a C*-direct summand of the reduced C*-algebra $\mathfrak{A}_N$ of SL(N), see [7].

Theorem 3.1. *Let $G = \mathrm{SL}(N,F)$, and M be a Levi subgroup consisting of n blocks of the same size m. Let $\sigma \in \mathcal{E}_2(M)$. Assume that the induced representation $i_{GM}(\sigma)$ has elliptic constituents, then the fixed C*-algebra $C(E,\mathfrak{K})^{W(M,E)}$ is strongly Morita equivalent to the crossed product C*-algebra $C(E) \rtimes \mathbb{Z}/n\mathbb{Z}$.*

Proof. For the commuting algebra of $i_{MG}(\sigma)$, we have [12]

$$\mathrm{End}_G((i_{MG}(\sigma)) = \mathbb{C}[R(\sigma)].$$

Let w_0 be a generator of $R(\sigma)$, then the normalized intertwining operator $\mathfrak{a}(w_0,\sigma)$ is a unitary operator of order n. By the spectral theorem for unitary operators, we have

$$\mathfrak{a}(w_0,\sigma) = \sum_{j=0}^{n-1} \omega^j E_j$$

where $\omega = \exp(2\pi i/n)$ and E_j are the projections onto the irreducible subspaces of the induced representation $i_{MG}(\sigma)$. The unitary representation

$$R(\sigma) \to U(H), \quad w \mapsto \mathfrak{a}(w,\sigma)$$

contains each character of $R(\sigma)$ countably many times. Therefore condition (***) in [10], p. 301, is satisfied. The condition (**) in [10], p. 300, is trivially satisfied since $W(\sigma) = R(\sigma)$.

We have $W(\sigma) = \mathbb{Z}/n\mathbb{Z}$. Then a subgroup $W(\rho)$ of order d is given by $W(\rho) = k\mathbb{Z} \mod n$ with $dk = n$. In that case, we have

$$\mathfrak{a}(w_0,\sigma)|_{W(\rho)} = \sum_{j=0}^{n-1} \omega^{kj} E_j.$$

We compare the two unitary representations

$$\phi_1 : W(\rho) \to U(H), \quad w \mapsto \mathfrak{a}(w,\sigma)|_{W(\rho)},$$

$$\phi_2\colon W(\rho) \to U(H), \quad w \mapsto \mathfrak{a}(w,\rho).$$

Each representation contains every character of $W(\rho)$. They are *quasi-equivalent* as in [10]. Choose an increasing sequence (e_n) of finite-rank projections in $\mathcal{L}(H)$ which converge strongly to I and commute with each projection E_j. The compressions of ϕ_1, ϕ_2 to $e_n H$ remain quasi-equivalent. Condition (*) in [10], p. 299, is satisfied.

All three conditions of [10], Theorem 2.13, are satisfied. We therefore have a strong Morita equivalence

$$(C(E) \otimes \mathfrak{K})^{W(M,E)} \simeq C(E) \rtimes R(\sigma) = \mathbb{C}(E) \rtimes \mathbb{Z}/n\mathbb{Z}. \qquad \square$$

We will need a special case of the Chern character for discrete groups [3].

Theorem 3.2. *We have an isomorphism*

$$K_i(C(E) \rtimes \mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{j \in \mathbb{N}} H^{2j+i}(E/\!/(\mathbb{Z}/n\mathbb{Z}); \mathbb{C})$$

with $i = 0, 1$, *where* $E/\!/(\mathbb{Z}/n\mathbb{Z})$ *denotes the extended quotient of* E *by* $\mathbb{Z}/n\mathbb{Z}$.

When N is a prime number ℓ, this result already appeared in [8], [10].

4. The formation of the fixed sets

Extended quotients were introduced by Baum and Connes [3] in the context of the Chern character for discrete groups. Extended quotients were used in [9], [8] in the context of the reduced group C*-algebras of $\mathrm{GL}(N)$ and $\mathrm{SL}(\ell)$ where ℓ is prime. The results in this section extend results in [8], [10].

Definition 4.1. Let X be a compact Hausdorff topological space. Let Γ be a finite *abelian* group acting on X by a (left) continuous action. Let

$$\widetilde{\mathrm{X}} = \{(x, \gamma) \in \mathrm{X} \times \Gamma \mid \gamma x = x\}$$

with the group action on $\widetilde{\mathrm{X}}$ given by

$$g \cdot (x, \gamma) = (gx, \gamma)$$

for $g \in \Gamma$. Then the *extended quotient* is given by

$$\mathrm{X}/\!/\Gamma := \widetilde{\mathrm{X}}/\Gamma = \bigsqcup_{\gamma \in \Gamma} \mathrm{X}^{\gamma}/\Gamma$$

where X^{γ} is the γ-fixed set.

The extended quotient will always contain the ordinary quotient. The standard projection $\pi\colon X/\!/\Gamma \to X/\Gamma$ is induced by the map $(x,\gamma) \mapsto x$. We note the following elementary fact, which will be useful later (in Lemma 5.2): let $y = \Gamma x$ be a point in X/Γ. Then the cardinality of the pre-image $\pi^{-1}y$ is equal to the order of the isotropy group Γ_x:

$$|\pi^{-1}y| = |\Gamma_x|.$$

We will write X $= E = \mathbb{T}^n/\mathbb{T}$, where $\mathbb{T}$ acts diagonally on $\mathbb{T}^n$, i.e.,

$$t(t_1, t_2, \ldots, t_n) = (tt_1, tt_2, \ldots, tt_n), \quad t, t_i \in \mathbb{T}.$$

We have the action of the finite group $\Gamma = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{T}^n/\mathbb{T}$ given by cyclic permutation. The two actions of $\mathbb{T}$ and of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{T}^n$ commute. We will write (k,n) for the highest common factor of k and n.

Theorem 4.2. *The extended quotient $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$ is a disjoint union of compact connected orbifolds:*

$$(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z}) \simeq \bigsqcup_{\substack{1\le k\le n \\ \omega^{n/(k,n)}=1}} \mathfrak{X}(n,k,\omega).$$

Here ω is a $n/(k,n)$th root of unity in $\mathbb{C}$.

Proof. Let γ be the standard n-cycle defined by $\gamma(i) = i+1 \mod n$. Then γ^k is the product of n/d cycles of order $d = n/(n,k)$. Let ω be a dth root of unity in $\mathbb{C}$. All dth roots of unity are allowed, including $\omega = 1$. The element $t(\omega) = t(\omega; z_1, \ldots, z_n) \in \mathbb{T}^n$ is defined by imposing the relations

$$z_{i+k} = \omega^{-1} z_i,$$

all suffices mod n. This condition allows n/d of the complex numbers $z_1, \ldots, z_n$ to vary freely, subject only to the condition that each z_j has modulus 1. The crucial point is that

$$\gamma^k \cdot t(\omega) = \omega t(\omega)$$

Then ω determines a γ^k-fixed set in $\mathbb{T}^n/\mathbb{T}$, namely the set $\mathfrak{Y}(n,k,\omega)$ of all cosets $t(\omega)\cdot\mathbb{T}$. The set $\mathfrak{Y}(n,k,\omega)$ is an $(n/d-1)$-dimensional subspace of fixed points.

Note that $\mathfrak{Y}(n,k,\omega)$, as a coset of the closed subgroup $\mathfrak{Y}(n,k,1)$ in the compact Lie group E, is homeomorphic (by translation in E) to $\mathfrak{Y}(n,k,1)$. The translation is by the element $t(\omega : 1, \ldots, 1)$. If ω_1, ω_2 are distinct dth roots of unity, then $\mathfrak{Y}(n,k,\omega_1), \mathfrak{Y}(n,k,\omega_2)$ are disjoint.

We define the quotient space

$$\mathfrak{X}(n,k,\omega) := \mathfrak{Y}(n,k,\omega)/(\mathbb{Z}/n\mathbb{Z})$$

and apply Definition 4.1. □

When $k = n$, we must have $\omega = 1$. In that case, the orbifold is the ordinary quotient: $\mathfrak{X}(n, n, 1) = \mathfrak{X}(n)$.

Let $(n, k) = 1$. The number of such k in $1 \le k \le n$ is $\phi(n)$. In this case, ω is an nth root of unity and $\mathfrak{X}(n, k, \omega)$ is a point. There are n such roots of unity in $\mathbb{C}$. Therefore, the extended quotient $(\mathbb{T}^n/\mathbb{T})/\!/(\mathbb{Z}/n\mathbb{Z})$ always contains $\phi(n)n$ isolated points.

Theorem 1.1 is a consequence of Theorems 3.1, 3.2 and 4.2. If, in Theorem 1.1, we take n to be a prime number ℓ, then we recover the following result in [8], p. 30: the extended quotient $(\mathbb{T}^\ell/\mathbb{T})/\!/(\mathbb{Z}/\ell\mathbb{Z})$ is the disjoint union of the ordinary quotient $\mathfrak{X}(\ell)$ and $(\ell - 1)\ell$ points.

5. The formation of the R-groups

We continue with the notation of Section 3. Let σ, π_σ, π_1, η be as in equation (1). The n-tuple $t := (z_1, \ldots, z_n) \in \mathbb{T}^n$ determines an element $[t] \in E$. We can interpret $[t]$ as the unramified character

$$\chi_t := (z_1^{\mathrm{val}\circ\det}, \ldots, z_n^{\mathrm{val}\circ\det}).$$

Let $\Gamma = \mathbb{Z}/n\mathbb{Z}$, and let $\Gamma_{[t]}$ denote the isotropy subgroup of Γ.

Lemma 5.1. *The isotropy subgroup $\Gamma_{[t]}$ is isomorphic to the R-group of $\chi_t \otimes \sigma$:*

$$\Gamma_{[t]} \simeq R(\chi_t \otimes \sigma).$$

Proof. Let the order of $\Gamma_{[t]}$ be d. Then d is a divisor of n. Let γ be a generator of $\Gamma_{[t]}$. Then γ is a product of n/d disjoint d-cycles, as in Section 4. We must have $t = t(\omega)$ with ω a dth root of unity in $\mathbb{C}$. Note that $\gamma \cdot t(\omega) = \omega t(\omega)$. Then we have

$$\begin{aligned} R(\chi_t \otimes \sigma) &= \bar{L}(\chi_t \otimes \pi_\sigma)/X(\chi_t \otimes \pi_\sigma) \\ &= \{\alpha \in \widehat{F^\times} \mid w\pi_\sigma \simeq \pi_\sigma \otimes \alpha \text{ for some } w \text{ in } W\}/X(\chi_t \otimes \pi_\sigma) \\ &= \langle \omega^{\mathrm{val}\circ\det} \otimes \eta^{n/d} \rangle \\ &= \mathbb{Z}/d\mathbb{Z} \\ &= \Gamma_{[t]} \end{aligned}$$

since, modulo $X(\chi_t \otimes \pi_\sigma)$, the character $\eta^{n/d}$ has order d. □

Lemma 5.2. *In the standard projection $p\colon E/\!/\Gamma \to E/\Gamma$, the cardinality of the fibre of $[t]$ is the order of the R-group of $\chi_t \otimes \sigma$.*

Proof. This follows from Lemma 5.1. □

We will assume that σ is a *cuspidal* representation of M with unitary central character. Let $\mathfrak{s}$ be the point in the Bernstein spectrum of SL(N) which contains the cuspidal pair (M, σ). To conform to the notation in [2], we will write $E^{\mathfrak{s}} := \mathbb{T}^n/\mathbb{T}$, $W^{\mathfrak{s}} = \mathbb{Z}/n\mathbb{Z}$. The standard projection will be denoted

$$\pi^{\mathfrak{s}} \colon E^{\mathfrak{s}} /\!/ W^{\mathfrak{s}} \to E^{\mathfrak{s}}/W^{\mathfrak{s}}.$$

The space of tempered representations of G determined by $\mathfrak{s}$ will be denoted by $\mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}}$, and the infinitesimal character will be denoted $inf.ch.$

Theorem 5.3. *We have a commutative diagram*

$$\begin{tikzcd} E /\!/ W^{\mathfrak{s}} \arrow[r, "\mu^{\mathfrak{s}}"] \arrow[d, "\pi^{\mathfrak{s}}"'] & \mathrm{Irr}^{\mathrm{temp}}(G)^{\mathfrak{s}} \arrow[d, "\mathrm{inf.ch.}"] \\ E/W^{\mathfrak{s}} \arrow[r] & E/W^{\mathfrak{s}} \end{tikzcd}$$

in which the map $\mu^{\mathfrak{s}}$ is a continuous bijection. This confirms, in a special case, part (3) *of the conjecture in* [2].

Proof. We have

$$\mathbb{C}[R(\sigma)] \simeq \mathrm{End}_G(i_{GM}(\sigma)).$$

This implies that the characters of the cyclic group $R(\sigma)$ parametrize the irreducible constituents of $i_{GM}(\sigma)$. This leads to a labelling of the irreducible constituents of $i_{GM}(\sigma)$, which we will write as $i_{GM}(\sigma : r)$ with $0 \le r < n$.

The map $\mu^{\mathfrak{s}}$ is defined as follows:

$$\mu^{\mathfrak{s}} \colon (t, \gamma^{rd}) \mapsto i_{GM}(\chi_t \otimes \sigma : r).$$

We now apply Lemma 5.2.

Theorem 3.2 in [7] relates the natural topology on the Harish-Chandra parameter space to the Jacobson topology on the tempered dual of a reductive p-adic group. As a consequence, the map $\mu^{\mathfrak{s}}$ is continuous. □

6. Cyclic invariants

We will consider the map

$$\alpha \colon \mathbb{T}^n \to (\mathbb{T}^n/\mathbb{T}) \times \mathbb{T}, \quad (t_1, \dots, t_n) \mapsto ((t_1 : \cdots : t_n), t_1 t_2 \dots t_n),$$

where $(t_1 : \cdots : t_n)$ is the image of $(t_1, \dots, t_n)$ via the map $\mathbb{T}^n \to \mathbb{T}^n/\mathbb{T}$. The map α is a homomorphism of Lie groups. The kernel of this map is

$$\mathcal{G}_n := \{\omega I_n \mid \omega^n = 1\}.$$

We therefore have the isomorphism of compact connected Lie groups:

$$\mathbb{T}^n/\mathcal{G}_n \cong (\mathbb{T}^n/\mathbb{T}) \times \mathbb{T}. \tag{2}$$

This isomorphism is equivariant with respect to the $\mathbb{Z}/n\mathbb{Z}$-action, and we infer that

$$(\mathbb{T}^n/\mathcal{G}_n)/(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}) \times \mathbb{T}. \tag{3}$$

Theorem 6.1. *Let $H^\bullet(-;\mathbb{C})$ denote the total cohomology group. We have*

$$\dim_{\mathbb{C}} H^\bullet(\mathfrak{X}(n);\mathbb{C}) = \tfrac{1}{2} \cdot \dim_{\mathbb{C}} H^\bullet(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}}.$$

Proof. The cohomology of the orbit space is given by the fixed set of the cohomology of the original space [4], Corollary 2.3, p. 38. We have

$$H^j(\mathbb{T}^n/\mathcal{G}_n;\mathbb{C}) \cong H^j(\mathbb{T}^n;\mathbb{C})^{\mathcal{G}_n} \cong H^j(\mathbb{T}^n;\mathbb{C}) \tag{4}$$

since the action of $\mathcal{G}_n$ on $\mathbb{T}^n$ is homotopic to the identity. We spell this out. Let $z := (z_1,\dots,z_n)$ and define $H(z,t) = \omega^t \cdot z = (\omega^t z_1,\dots,\omega^t z_n)$. Then $H(z,0) = z$, $H(z,1) = \omega \cdot z$. Also, H is equivariant with respect to the permutation action of $\mathbb{Z}/n\mathbb{Z}$. That is to say, if $\epsilon \in \mathbb{Z}/n\mathbb{Z}$ then $H(\epsilon \cdot z,t) = \epsilon \cdot H(z,t)$. This allows us to proceed as follows:

$$\begin{aligned} H^j(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} &\cong H^j(\mathbb{T}^n/\mathcal{G}_n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \\ &\cong H^j((\mathbb{T}^n/\mathbb{T}) \times \mathbb{T};\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \\ &\cong H^j((\mathbb{T}^n/\mathbb{T})/(\mathbb{Z}/n\mathbb{Z}) \times \mathbb{T};\mathbb{C}). \end{aligned} \tag{5}$$

We apply the Künneth theorem in cohomology (there is no torsion):

$$(H^j(\mathbb{T}^n;\mathbb{C}))^{\mathbb{Z}/n\mathbb{Z}} \cong H^j(\mathfrak{X}(n);\mathbb{C}) \oplus H^{j-1}(\mathfrak{X}(n);\mathbb{C}) \quad \text{with } 0 < j \le n,$$
$$(H^n(\mathbb{T}^n;\mathbb{C}))^{\mathbb{Z}/n\mathbb{Z}} \simeq H^{n-1}(\mathfrak{X}(n);\mathbb{C}), \quad H^0(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} \cong H^0(\mathfrak{X}(n);\mathbb{C}) \simeq \mathbb{C},$$
$$H^{\mathrm{ev}}(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} = H^\bullet(\mathfrak{X}(n);\mathbb{C}), \quad H^{\mathrm{odd}}(\mathbb{T}^n;\mathbb{C})^{\mathbb{Z}/n\mathbb{Z}} = H^\bullet(\mathfrak{X}(n);\mathbb{C}).$$

□

We now have to find the cyclic invariants in $H^\bullet(\mathbb{T}^n;\mathbb{C})$. The cohomology ring $H^\bullet(\mathbb{T}^n,\mathbb{C})$ is the exterior algebra $\bigwedge V$ of a complex n-dimensional vector space V, as can be seen by considering differential forms $d\theta_1 \wedge \cdots \wedge d\theta_r$. The vector space V admits a basis $\alpha_1 = d\theta_1,\dots,\alpha_n = d\theta_n$. The action of $\mathbb{Z}/n\mathbb{Z}$ on $\bigwedge V$ is induced by permuting the elements $\alpha_1,\dots,\alpha_n$, i.e., by the regular representation ρ of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. This representation of $\mathbb{Z}/n\mathbb{Z}$ on $\bigwedge V$ will be denoted $\bigwedge \rho$. The dimension of the space of cyclic invariants in $H^\bullet(\mathbb{T}^n,\mathbb{C})$ is equal to the multiplicity of the unit representation 1 in $\bigwedge \rho$. To determine this, we use the theory of group characters.

Lemma 6.2. *The dimension of the subspace of cyclic invariants is given by*

$$(\chi_{\wedge\rho}, 1) = \tfrac{1}{n}(\chi_{\wedge\rho}(0) + \chi_{\wedge\rho}(1) + \cdots + \chi_{\wedge\rho}(n-1)).$$

Proof. This is a standard result in the theory of group characters [11]. $\square$

Theorem 6.3. *The dimension of the space of cyclic invariants in $H^\bullet(\mathbb{T}^n, \mathbb{C})$ is given by the formula*

$$g(n) := \frac{1}{n} \sum_{d|n,\, d\ odd} \phi(d) 2^{n/d}$$

Proof. We note first that

$$\chi_{\wedge\rho}(0) = \text{Trace } 1_{\wedge V} = \dim_{\mathbb{C}} \bigwedge V = 2^n.$$

To evaluate the remaining terms, we need to recall the definition of the elementary symmetric functions e_j:

$$\prod_{j=1}^{n} (\lambda - \alpha_j) = \lambda^n - \lambda^{n-1} e_1 + \lambda^{n-2} e_2 - \cdots + (-1)^n e_n.$$

If we need to mark the dependence on $\alpha_1, \ldots, \alpha_n$ we will write $e_j = e_j(\alpha_1, \ldots, \alpha_n)$. Set $\alpha_j = \omega^{j-1}, \omega = \exp(2\pi i/n)$. Then we get

$$\lambda^n - 1 = \prod_{j=1}^{n} (\lambda - \alpha_j) = \lambda^n - \lambda^{n-1} e_1 + \lambda^{n-2} e_2 - \cdots + (-1)^n e_n.$$

Let $d|n$, let ζ be a *primitive* dth root of unity. Let $\alpha_j = \zeta^{j-1}$. We have

$$(\lambda^d - 1)^{n/d} = (\lambda^d - 1) \ldots (\lambda^d - 1) = \prod_{j=1}^{n} (\lambda - \alpha_j). \tag{6}$$

Set $\lambda = -1$. If d is even, we obtain

$$0 = 1 + e_1(1, \zeta, \zeta^2, \ldots) + e_2(1, \zeta, \zeta^2, \ldots) + \cdots + e_n(1, \zeta, \zeta^2, \ldots). \tag{7}$$

If d is odd, we obtain

$$2^{n/d} = 1 + e_1(1, \zeta, \zeta^2, \ldots) + e_2(1, \zeta, \zeta^2, \ldots) + \cdots + e_n(1, \zeta, \zeta^2, \ldots). \tag{8}$$

We observe that the regular representation ρ of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of the characters $m \mapsto \omega^{rm}$ with $0 \leq r \leq n$. This direct sum decomposition allows us to choose a basis $v_1, \ldots, v_n$ in V such that the representation $\bigwedge \rho$ is diagonalized by the wedge products $v_{j_1} \wedge \cdots \wedge v_{j_l}$. This in turn allows us to compute the character of $\bigwedge \rho$ in terms of the elementary symmetric functions $e_1, \ldots, e_n$.

With $\zeta = \omega^r$ as above, we have

$$\chi_{\wedge\rho}(r) = 1 + e_1(1, \zeta, \zeta^2, \dots) + e_2(1, \zeta, \zeta^2, \dots) + \dots + e_n(1, \zeta, \zeta^2, \dots).$$

We now sum the values of the character $\chi_{\wedge\rho}$. Let $d := n/(r, n)$. Then ζ is a primitive dth root of unity. If d is even then $\chi_{\wedge\rho}(r) = 0$. If d is odd, then $\chi_{\wedge\rho}(r) = 2^{n/d}$. There are $\phi(d)$ such terms. So we have

$$\chi_{\wedge\rho}(0) + \chi_{\wedge\rho}(1) + \dots + \chi_{\wedge\rho}(n-1) = \sum_{d|n,\, d \text{ odd}} \phi(d) 2^{n/d}. \tag{9}$$

We now apply Lemma 6.2. □

The sequence $n \mapsto g(n)/2$, $n = 1, 2, 3, 4, \dots,$ is

1, 1, 2, 2, 4, 6, 10, 16, 30, 52, 94, 172, 316, 586, 1096, 2048, 3856, 7286, ...

as in http://www.research.att.com/~njas/sequences/A000016. Thanks to Kasper Andersen for alerting us to this web site.

References

[1] M. Assem, The Fourier transform and some character formulae for p-adic SL_l, l a prime. *Amer. J. Math.* **116** (1994), 1433–1467. Zbl 0837.20051 MR 1305872 268

[2] A.-M. Aubert, P. Baum, and R. Plymen, Geometric structure in the representation theory of p-adic groups. *C. R. Math. Acad. Sci. Paris* **345** (2007), 573–578. Zbl 1128.22009 MR 2374467 265, 267, 268, 275

[3] P. Baum and A. Connes, Chern character for discrete groups. In *A fête of topology*, Academic Press, Boston 1988, 163–232. Zbl 0656.55005 MR 0928402 266, 272

[4] A. Borel, *Seminar on transformation groups*. Ann. of Math. Stud. 46, Princeton University Press, Princeton, N.J., 1960. Zbl 0091.37202 MR 0116341 276

[5] J. Dixmier, *C*-algebras*. North-Holland Publishing Co., North-Holland Math. Library 15, Amsterdam 1977. Zbl 0372.46058 MR 0458185 266

[6] D. Goldberg, R-groups and elliptic representations for SL_n. *Pacific J. Math.* **165** (1994), 77–92. Zbl 0855.22016 MR 1285565 269, 270

[7] R. J. Plymen, Reduced C^*-algebra for reductive p-adic groups. *J. Funct. Anal.* **88** (1990), 251–266. Zbl 0718.22003 MR 1038441 265, 271, 275

[8] R. J. Plymen, Elliptic representations and K-theory for $\mathrm{SL}(l)$. *Houston J. Math.* **18** (1992), 25–32. Zbl 0757.46060 MR 1159436 272, 274

[9] R. J. Plymen, Reduced C^*-algebra of the p-adic group $\mathrm{GL}(n)$ II. *J. Funct. Anal.* **196** (2002), 119–134. Zbl 1014.22014 MR 1941993 272

[10] R. J. Plymen and C. W. Leung, Arithmetic aspect of operator algebras. *Compositio Math.* **77** (1991), 293–311. Zbl 0843.22011 MR 1092771 271, 272

[11] J.-P. Serre, *Linear representations of finite groups*. Grad. Texts in Math. 42, Springer-Verlag, New York 1977. Zbl 0355.20006 MR 0450380 277

[12] A. J. Silberger, The Knapp-Stein dimension theorem for p-adic groups. *Proc. Amer. Math. Soc.* **68** (1978), 243–246; Correction *ibid.* **76** (1979), 169–170. Zbl 0348.22007 MR 0492091 Zbl 0415.22020 MR 0534411 271

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