ON THE CLASSIFYING SPACE FOR THE FAMILY OF VIRTUALLY CYCLIC SUBGROUPS FOR ELEMENTARY AMENABLE GROUPS

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ABSTRACT. We show that elementary amenable groups, which have a bound on the orders of their finite subgroups, admit a finite dimensional model for $\underline{\underline{E}}G$, the classifying space with virtually cyclic isotropy

1. INTRODUCTION

Classifying spaces with isotropy in a family have been around for a while; most of the research has focussed on $\underline{E}G$, the classifying space with finite isotropy [15–17]. Finiteness conditions for $\underline{E}G$ for elementary amenable groups are very well understood [5, 10]. Finding manageable models for $\underline{\underline{E}}G$ has been shown to be much more elusive. In [8] it was conjectured that the only groups admitting a finite type model for $\underline{\underline{E}}G$ are virtually cyclic, and this was proved for hyperbolic groups. In [9] it was shown that this conjecture also holds for elementary amenable groups. As far as finite dimensional models are concerned, only a little more is known. So far manageable models have been found for crystallographic groups [13], polycyclic-by-finite groups [19] and hyperbolic groups [8]. Adapting the construction of [8], the first author [6] has recently found a nice model for certain HNN-extensions including extensions of the form $A \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts freely on the non-trivial elements of the abelian group A. Utilising this construction we prove:

Main Theorem. *Let G be an elementary amenable group with finite Hirsch length. If G has a bound on the orders of its finite subgroups, then G admits a finite dimensional model for $\underline{\underline{E}}G$.*

The proof of this fact is fairly algebraic by considering finiteness conditions in Bredon cohomology. Bredon cohomology takes the place of ordinary cohomology when studying classifying spaces with isotropy in a family of subgroups. We give a brief introduction into Bredon cohomology in Section 2 and then move on to discussing dimensions in Bredon cohomology for extensions, directed unions and direct products of groups. We also consider the behaviour of the Bredon cohomological dimension when changing the family of subgroups.

Two further crucial ingredients are Hillman–Linnel's and Wehrfritz's [7, 29] characterisation of elementary amenable groups as locally finite-by-soluble-by-finite groups. This allows us to reduce the problem to torsion-free abelian-by-cyclic groups. We show, Proposition 5.4, that a torsion-free abelian-by-cyclic group of finite Hirsch length has finite Bredon-cohomological dimension bounded by a recursively defined integer only depending on the Hirsch length. In Section 6 this result is extended to torsion-free nilpotent-by-abelian groups, which allows us to prove the Main Theorem in Section 7.

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2. Background on Bredon cohomology

In this article a *family* $\mathfrak{F}$ of subgroups of a group G stands for a non-empty set of subgroups of G, which is closed under conjugation and taking finite intersections. Common examples are the *trivial* family of subgroups $\mathfrak{F} = \{1\}$, the family $\mathfrak{F}_{\text{fin}}(G)$ of all finite subgroups of G and the family $\mathfrak{F}_{\text{vc}}(G)$ of all virtually cyclic subgroups of G.

Let $\mathfrak{F}$ be a family of subgroups of G and $K \leq G$ and put

$$\mathfrak{F} \cap K = \{H \cap K \mid H \in \mathfrak{F}\}.$$

In particular $\mathfrak{F} \cap K$ is a family of subgroups of K, if it is non-empty. Now let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be families of subgroups of some groups G_1 and G_2 respectively. Here we put

$$\mathfrak{F}_1 \times \mathfrak{F}_2 = \{H_1 \times H_2 \mid H_1 \in \mathfrak{F}_1 \text{ and } H_2 \in \mathfrak{F}_2\}.$$

This is a family of subgroups of $G_1 \times G_2$. Finally, for any family $\mathfrak{F}$ of subgroups of G we can define its *subgroup completion* $\bar{\mathfrak{F}}$ as

$$\bar{\mathfrak{F}} = \{H \leq K \mid K \in \mathfrak{F}\}.$$

That is, $\bar{\mathfrak{F}}$ is the smallest family of subgroups of G which contains $\mathfrak{F}$ and is closed under forming subgroups.

Given a non-empty G-set X, we denote by $\mathfrak{F}(X)$ the collection of all its isotropy groups. In general this is not a family of subgroups as it may not be closed under finite intersections.

Bredon cohomology has been introduced for finite groups by Bredon [2] and it has been generalised to arbitrary groups by Lück [15]. It is the natural choice for a cohomology theory to study classifying spaces with stabilisers in a prescribed family $\mathfrak{F}$ of subgroups. The reader is referred to Lück's book [15] and the introductory chapters in Mislin's survey [22] for standard facts and definitions. We shall, however include definitions and results on Bredon cohomology needed later.

Given a group G, the *orbit category* $\mathscr{O}G$ is defined as follows: objects are the transitive G-sets G/H with $H \leq G$; the morphisms of $\mathscr{O}G$ are all G-maps $G/H \to G/K$. For a family $\mathfrak{F}$ of subgroups of G, the *orbit category* $\mathscr{O}_{\mathfrak{F}}G$ is the full subcategory of $\mathscr{O}G$, which has as objects the transitive G-sets G/H with $H \in \mathfrak{F}$.

An $\mathscr{O}_{\mathfrak{F}}G$*-module*, or *Bredon module*, is a functor $M\colon\ \mathscr{O}_{\mathfrak{F}}G \to \mathfrak{Ab}$ from the orbit category to the category of abelian groups. If the functor M is contravariant, M is said to be a *right* Bredon module; if M is covariant we call it a *left* Bredon module. A natural transformation $f\colon M \to N$ between two $\mathscr{O}_{\mathfrak{F}}G$-modules of the same variance is called a morphism of $\mathscr{O}_{\mathfrak{F}}G$-modules. If M is a right, respectively left, Bredon module and φ a morphism, then we may abbreviate $M(\varphi)$ by φ^* and φ_* respectively.

The *trivial* $\mathscr{O}_{\mathfrak{F}}G$*-module* is denoted by $\underline{\mathbb{Z}}_{\mathfrak{F}}$. It is given by $\underline{\mathbb{Z}}(G/H) = \mathbb{Z}$ and $\underline{\mathbb{Z}}(\varphi) = \text{id}$ for all objects and morphisms of $\mathscr{O}_{\mathfrak{F}}G$.

The categories of right, respectively left, $\mathscr{O}_{\mathfrak{F}}G$-modules and their morphisms are denoted by Mod-$\mathscr{O}_{\mathfrak{F}}G$ and $\mathscr{O}_{\mathfrak{F}}G$-Mod respectively. These are functor categories and therefore inherit a number of properties from the category $\mathfrak{Ab}$. For example, a sequence $L \to M \to N$ of Bredon modules is exact if and only if when evaluated at every $G/H \in \mathscr{O}_{\mathfrak{F}}G$, we obtain an exact sequence $L(G/H) \to M(G/H) \to N(G/H)$ of abelian groups.

Since $\mathfrak{Ab}$ has enough projectives, so does Mod-$\mathcal{O}_{\mathfrak{F}}G$. Hence we can define homology functors in Mod-$\mathcal{O}_{\mathfrak{F}}G$. Denote by $\mathrm{mor}_{\mathfrak{F}}(M,N)$ the set of morphisms between two Bredon modules M and N. Hence the bi-functor

$$\mathrm{mor}_{\mathfrak{F}}(?,??)\colon\ \text{Mod-}\mathcal{O}_{\mathfrak{F}}G\times\text{Mod-}\mathcal{O}_{\mathfrak{F}}G\to\mathfrak{Ab}$$

has derived functors, denoted by $\mathrm{Ext}^*_{\mathfrak{F}}(?,??)$. The categorical tensor product [26, pp. 45ff.] gives rise to a tensor product

$$?\otimes_{\mathfrak{F}}??\colon\ \text{Mod-}\mathcal{O}_{\mathfrak{F}}G\times\mathcal{O}_{\mathfrak{F}}G\text{-Mod}\to\mathfrak{Ab}$$

over the orbit category $\mathcal{O}_{\mathfrak{F}}G$ [15, p. 166]. Its derived functors are denoted by $\mathrm{Tor}^{\mathfrak{F}}_*(?,??)$.
One can also define a tensor product over $\mathbb{Z}$ as follows: For $\mathcal{O}_{\mathfrak{F}}G$-modules M and N, we set $(M\otimes N)(G/K)=M(G/K)\otimes N(G/K)$, see also [15, p. 166].

We shall now describe the basic properties of free, projective and flat $\mathcal{O}_{\mathfrak{F}}G$-modules. Consider the following right Bredon module: $\mathbb{Z}[?,G/K]_G$ with $K\in\mathfrak{F}$. Evaluated at G/H this functor is the free abelian group $\mathbb{Z}[G/H,G/K]_G$ on the set $[G/H,G/K]_G$ of G-maps $G/H\to G/K$. These modules are free, cf. [15, p. 167], and can be viewed as the building blocks of the free right Bredon modules. In general a free object in Mod-$\mathcal{O}_{\mathfrak{F}}G$ is of the form $\mathbb{Z}[?,X]_G$ where X is a G-set with $\mathfrak{F}(X)\subset\mathfrak{F}$. Free left Bredon modules are defined analogously: they are obtained from modules of the form $\mathbb{Z}[G/K,??]_G$, where $K\in\mathfrak{F}$. A projective $\mathcal{O}_{\mathfrak{F}}G$-modules is then defined as a direct summand of a free module.
Note that the construction of free $\mathcal{O}_{\mathfrak{F}}G$-modules is functorial in the second variable. In particular, we have a functor

$$\mathbb{Z}[?,??]_G\colon\ G\text{-}\mathfrak{Set}\to\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$$

defined by $\mathbb{Z}[?,??]_G(X)=\mathbb{Z}[?,X]_G$ for arbitrary G-sets X.
A Bredon module M is *finitely generated,* if it is the homomorphic image of a free Bredon module $\mathbb{Z}[?,X]_G$, where X has only finitely many G-orbits.
A right $\mathcal{O}_{\mathfrak{F}}G$-module M is called *flat* if it is $?\otimes_{\mathfrak{F}}N$-acyclic for every left $\mathcal{O}_{\mathfrak{F}}G$-module N. This is precisely the case if the functor $\mathrm{Tor}^{\mathfrak{F}}_1(M,?)$ is trivial. Flat Bredon modules share many properties with ordinary flat modules. In particular,

Proposition 2.1. [23, Theorem 3.2] *A right $\mathcal{O}_{\mathfrak{F}}G$-module M is flat if and only if it is the filtered colimit of finitely generated free $\mathcal{O}_{\mathfrak{F}}G$-modules.*

Given a covariant functor $F\colon\ \mathcal{O}_{\mathfrak{F}_1}G_1\to\mathcal{O}_{\mathfrak{F}_2}G_2$ between orbit categories, one can now define induction and restriction functors along F, see [15, p. 166]:

$$\begin{aligned}\mathrm{Ind}_F\colon\ \mathcal{O}_{\mathfrak{F}_1}G_1 &\to \mathcal{O}_{\mathfrak{F}_2}G_2\\ M(?) &\mapsto M(?)\otimes_{\mathfrak{F}_1}\mathrm{mor}_{\mathfrak{F}_2}(??,F(?))\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}_F\colon\ \mathcal{O}_{\mathfrak{F}_2}G_2 &\to \mathcal{O}_{\mathfrak{F}_1}G_1\\ M(??) &\mapsto M\circ F(??)\end{aligned}$$

Since these functors are adjoint to each others, Ind_F commutes with arbitrary colimits [20, pp. 118f.] and preserves free and projective Bredon modules [15, p. 169].

Lemma 2.2. *Induction along F preserves flat right Bredon modules.*

Proof. This follows from the fact that both Ind_F and $\operatorname{Tor}_1^{\mathfrak{F}}(?,N)$ commute with filtered colimits and from Proposition 2.1. □

Let $\mathfrak{F} \subset \mathfrak{G}$ be two families of subgroups of a group G, then the inclusion of the respective orbit categories is denoted by:

$$I\colon\ \mathcal{O}_{\mathfrak{F}}G \to \mathcal{O}_{\mathfrak{G}}G$$

Now let K be a subgroup of G such that $\mathfrak{F} \cap K$ is a non-empty subset of $\mathfrak{F}$, then we consider the following functor

$$\begin{aligned} I_K\colon\ \mathcal{O}_{\mathfrak{F}\cap K}K &\to \mathcal{O}_{\mathfrak{F}}G \\ K/H &\mapsto G/H. \end{aligned}$$

Note that for every non-empty K-set X with $\mathfrak{F}(X) \subset \mathfrak{F} \cap K$, the functor I_K can be extended by mapping each K-orbit separately.

Lemma 2.3. [27, Lemma 2.9] *Let K be a subgroup of H such that $\mathfrak{F} \cap K$ is a non-empty subset of $\mathfrak{F}$. Then induction with I_K is an exact functor.*

We conclude this section with a collection of facts considering dimensions both generally and for the family of virtually cyclic subgroups. The *Bredon cohomological dimension* $\operatorname{cd}_{\mathfrak{F}} G$ of a group G with respect to the family $\mathfrak{F}$ of subgroups is the projective dimension $\operatorname{pd}_{\mathfrak{F}} \underline{\mathbb{Z}}$ of the trivial $\mathcal{O}_{\mathfrak{F}}G$-module $\underline{\mathbb{Z}}$. Similarly, the *Bredon homological dimension* $\operatorname{hd}_{\mathfrak{F}} G$ is the flat dimension $\operatorname{fld}_{\mathfrak{F}} \underline{\mathbb{Z}}$ of the trivial $\mathcal{O}_{\mathfrak{F}}G$-module. For $\mathfrak{F} = \mathfrak{F}_{\mathrm{vc}}(G)$ we shall use the following notation: $\underline{\underline{\operatorname{cd}}}\, G = \operatorname{cd}_{\mathfrak{F}} G$ and $\underline{\underline{\operatorname{hd}}}\, G = \operatorname{hd}_{\mathfrak{F}} G$.
The cellular chain complex of a model for $E_{\mathfrak{F}}G$ yields a free resolution of the trivial $\mathcal{O}_{\mathfrak{F}}G$-module $\underline{\mathbb{Z}}$ [15, pp. 151f.]. In particular, this implies that the Bredon geometric dimension $\operatorname{gd}_{\mathfrak{F}} G$, the minimal dimension of a model for $E_{\mathfrak{F}}G$, is an upper bound for $\operatorname{cd}_{\mathfrak{F}} G$. Since projectives are flat this implies that

$$\operatorname{hd}_{\mathfrak{F}} G \le \operatorname{cd}_{\mathfrak{F}} G \le \operatorname{gd}_{\mathfrak{F}} G.$$

Furthermore, Lück and Meintrupp gave an upper bound of $\operatorname{gd}_{\mathfrak{F}} G$ in terms of $\operatorname{cd}_{\mathfrak{F}} G$ as follows:

Proposition 2.4. [18, Theorem 0.1 (i)] *Let G be a group. Then*

$$\operatorname{gd}_{\mathfrak{F}} G \le \max(3, \operatorname{cd}_{\mathfrak{F}} G).$$

Hence, as long as $\operatorname{cd}_{\mathfrak{F}} G \ge 3$ or $\operatorname{gd}_{\mathfrak{F}} G \ge 4$, we have equality of these two dimensions.

Now suppose H is a subgroup of G such that $\mathfrak{F} \cap H$ is a non-empty subset of $\mathfrak{F}$. Then

$$\operatorname{gd}_{\mathfrak{F}\cap H} H \le \operatorname{gd}_{\mathfrak{F}} G \quad \text{and} \quad \operatorname{cd}_{\mathfrak{F}\cap H} H \le \operatorname{cd}_{\mathfrak{F}} G.$$

The following result is a consequence of Martínez-Pérez' Lyndon–Hochschild–Serre spectral sequence in Bredon (co)homology [21]. We shall only state the results for the family of virtually cyclic subgroups.

Proposition 2.5. [21, Corollary 5.2] *Let $N \rightarrowtail G \twoheadrightarrow Q$. Assume there exists $n \in \mathbb{N}$ such that $\underline{\underline{\operatorname{cd}}}\, H \le n$ for every $N \le H \le G$ with H/N virtually cyclic. Then*

$$\underline{\underline{\operatorname{cd}}}\, G \le n + \underline{\underline{\operatorname{cd}}}\, Q.$$

A careful inspection of the terms of the spectral sequence [21, Theorem 4.3] yields the following stronger result:

Proposition 2.6. *Let $F \rightarrowtail G \twoheadrightarrow Q$ be a group extension with F finite. Then*

$$\underline{\underline{\mathrm{cd}}}\, G = \underline{\underline{\mathrm{cd}}}\, Q.$$

Proof. The proof is identical to the proof of the corresponding result for the family of all finite subgroups [23, Theorem 5.5]. One checks that the families in question satisfy the conditions of [21, Corollary 4.5]. □

Now suppose G is a finite extension of a group H. Lück has constructed a model for $\underline{\underline{E}}G$ from a model for $\underline{\underline{E}}H$ [16]. This yields the following bound for $\underline{\underline{\mathrm{gd}}}\, G$:

Proposition 2.7. [16, Theorem 2.4] *Let H be a finite index subgroup of G. Then*

$$\underline{\underline{\mathrm{gd}}}\, G \leq |G : H| \cdot \underline{\underline{\mathrm{gd}}}\, H.$$

In particular, $\underline{\underline{\mathrm{gd}}}\, G$ is finite if and only if $\underline{\underline{\mathrm{gd}}}\, H$ is finite.

In light of Proposition 2.5 one needs to understand the behaviour of the Bredon dimensions for the family of virtually cyclic subgroups under extensions with virtually cyclic quotients. This is proving to be difficult. In [6] the second author gave bounds for certain infinite cyclic extensions:

Proposition 2.8. [6, Theorem 15] *Let $G = B \rtimes \mathbb{Z}$ and assume that $\mathbb{Z}$ acts freely via conjugation on the conjugacy classes of non-trivial elements of B. Then*

$$\underline{\underline{\mathrm{gd}}}\, G \leq \underline{\underline{\mathrm{gd}}}\, B + 1.$$

3. DIRECTED UNIONS OF GROUPS

The standard resolution of $\mathbb{Z}$ in classical group cohomology [3, pp. 15f] has been extended to Bredon cohomology for the family $\mathfrak{F}_{\mathrm{fin}}$ of all finite subgroups of a given group G [23]. This construction can be generalised to arbitrary families $\mathfrak{F}$ without any essential changes:
For each $n \in \mathbb{N}$ let Δ_n be the G-set

$$\Delta_n = \{(g_0K_0, \ldots, g_nK_n) \mid g_i \in G \text{ and } K_i \in \mathfrak{F}\}.$$

Since $\mathfrak{F}$ is closed under taking finite intersections it follows that $\mathfrak{F}(\Delta_n) \subset \mathfrak{F}$. For $n \geq 1$ and $0 \leq i \leq n$ we define G-maps $\partial_i \colon \Delta_n \to \Delta_{n-1}$ by

$$\partial_i(g_0K_0, \ldots, g_nK_n) = (g_0K_0, \ldots, \widehat{g_iK_i}, \ldots, g_nK_n)$$

where $(g_0K_0, \ldots, \widehat{g_iK_i}, \ldots, g_nK_n)$ denotes the n-tuple obtained from the $(n+1)$-tuple $(g_0K_0, \ldots, g_nK_n)$ by deleting the i-th component.
Let $\Delta_{-1} = \{*\}$ be the singleton set with trivial G-action. The unique map $\varepsilon \colon \Delta_0 \to \Delta_{-1}$ is obviously G-equivariant. Also note that $\mathbb{Z}[?, \Delta_{-1}]_G = \underline{\mathbb{Z}}$.
We now obtain a resolution of the trivial $\mathcal{O}_{\mathfrak{F}}G$-module $\underline{\mathbb{Z}}_{\mathfrak{F}}$ by right $\mathcal{O}_{\mathfrak{F}}G$-modules

$$\ldots \longrightarrow \mathbb{Z}[?, \Delta_2]_G \xrightarrow{d_2} \mathbb{Z}[?, \Delta_1]_G \xrightarrow{d_1} \mathbb{Z}[?, \Delta_0]_G \overset{\varepsilon^*}{\twoheadrightarrow} \underline{\mathbb{Z}}$$

where

$$d_n = \sum_{i=0}^{n} (-1)^i \partial_i^*.$$

Since $\mathfrak{F}(\Delta_n) \subset \mathfrak{F}$ it follows that this resolution is free and it is called the *standard resolution* of the trivial $\mathscr{O}_{\mathfrak{F}}G$-module $\underline{\mathbb{Z}}$.
There now follows a variation of [23, Theorem 4.2].

Proposition 3.1. *Let G be a directed union of subgroups G_λ, where $\lambda \in \Lambda$ is some indexing set. Let $\mathfrak{F}$ be a family of subgroups of G. For each $\lambda \in \Lambda$ put $\mathfrak{F}_\lambda = \mathfrak{F} \cap G_\lambda$. Suppose that $\mathfrak{F} = \bigcup_{\lambda\in\Lambda} \mathfrak{F}_\lambda$ and $\mathfrak{F}_\lambda \neq \varnothing$ for every $\lambda \in \Lambda$. Then*

(i) $\operatorname{hd}_{\mathfrak{F}} G = \sup\{\operatorname{hd}_{\mathfrak{F}_\lambda} G_\lambda\}$.
(ii) *If Λ is countable then* $\operatorname{cd}_{\mathfrak{F}} G \leq \sup\{\operatorname{cd}_{\mathfrak{F}_\lambda} G_\lambda\} + 1$.

Corollary 3.2. *Let G and G_λ, $\lambda \in \Lambda$, as in Proposition 3.1. Then*

(i) $\underline{\underline{\operatorname{hd}}}\, G = \sup\{\underline{\underline{\operatorname{hd}}}\, G_\lambda\}$.
(ii) *If Λ is countable then* $\underline{\underline{\operatorname{cd}}}\, G \leq \sup\{\underline{\underline{\operatorname{cd}}}\, G_\lambda\} + 1$.

Proof. This follows from the fact that $\mathfrak{F}_{\mathrm{vc}}(G_\lambda) = \mathfrak{F}_{\mathrm{vc}}(G) \cap G_\lambda$ and that for every finitely generated subgroup H there is a $\lambda \in \Lambda$ such that $H \in G_\lambda$. Now apply Proposition 3.1. $\square$

In particular, Corollary 3.2 (ii) can be applied to countable groups. A countable group is the direct union of its finitely generated subgroups G_λ, $\lambda \in \Lambda$, where Λ is countable. Hence $\underline{\underline{\operatorname{cd}}}\, G \leq \sup\{\underline{\underline{\operatorname{cd}}}\, G_\lambda\} + 1$.

Before we can prove Proposition 3.1, we need the following technical lemma.

Lemma 3.3. (i) *Assume that the G-set X is the direct union of G-sets X_α. Then the homomorphism*

$$\varinjlim \mathbb{Z}[?, X_\alpha]_G \to \mathbb{Z}[?, X]_G \tag{1}$$

induced by the canonical inclusions $\mathbb{Z}[?, X_\alpha]_G \hookrightarrow \mathbb{Z}[?, X]_G$ is an isomorphism.

(ii) *The homomorphism*

$$\varinjlim \mathbb{Z}[?, G/G_\lambda]_G \to \mathbb{Z}[?, G/G]_G \tag{2}$$

induced by the projections $G/G_\lambda \twoheadrightarrow G/G$ is an isomorphism.

(iii) *Let $K \leq G$ such that $\varnothing \neq \mathfrak{F} \cap K \subset \mathfrak{F}$. If X is a K-set with $\mathfrak{F}(X) \subset \mathfrak{F} \cap K$, then*

$$\operatorname{Ind}_{I_K} \mathbb{Z}[?, X]_K \cong \mathbb{Z}[?, I_K(X)]_G,$$

and this isomorphism is natural in X.

Proof. (i) Let $H \in \mathfrak{F}$ and evaluate (1) at G/H. The inclusion $X_\alpha^H \hookrightarrow X^H$ induces a homomorphism

$$\varinjlim \mathbb{Z}[X_\alpha^H] \to \mathbb{Z}[X^H]. \tag{3}$$

$X_\alpha^H = X_\alpha \cap X^H$, implying that $\varinjlim X_\alpha^H = X^H$. Since $\mathbb{Z}[?]$ commutes with colimits it follows that (3) is an isomorphism. Hence (1) is an isomorphism of $\mathscr{O}_{\mathfrak{F}}G$-modules.

(ii) This follows directly from the universal property of a colimit.

(iii) Let R be a complete system of representatives of the orbit space X/K. Then we have the following sequence of isomorphisms of $\mathscr{O}_{\mathfrak{F}}G$-modules:

$$\begin{aligned}
\operatorname{Ind}_{I_K} \mathbb{Z}[?,X]_K &\cong \coprod_{x\in R} \operatorname{Ind}_{I_K} \mathbb{Z}[?,K/K_x]_K \\
&\cong \coprod_{x\in R} \big(\mathbb{Z}[??,K/K_x]_K \otimes_{\mathfrak{F}\cap K} \mathbb{Z}[?,I_K(??)]_G\big) \\
&\cong \coprod_{x\in R} \mathbb{Z}[?,I_K(K/K_x)]_G \\
&\cong \coprod_{x\in R} \mathbb{Z}[?,G/K_x]_G \\
&\cong \mathbb{Z}[?,I_K(X)]_G
\end{aligned}$$

Note that the third isomorphism is a consequence of Yoneda's Lemma and that the composition of these isomorphisms is clearly natural in X. □

Proof of Proposition 3.1. For each $\lambda \in \Lambda$ we have the standard resolution of $\mathscr{O}_{\mathfrak{F}_\lambda} G_\lambda$-modules

$$\ldots \to \mathbb{Z}[?,\Delta_{\lambda,2}]_{G_\lambda} \to \mathbb{Z}[?,\Delta_{\lambda,1}]_{G_\lambda} \to \mathbb{Z}[?,\Delta_{\lambda,0}]_{G_\lambda} \twoheadrightarrow \mathbb{Z}[?]_{G_\lambda}. \tag{4}$$

By Lemma 2.3 the functor $\operatorname{Ind}_{I_{G_\lambda}}$ is exact. Hence for each $\lambda \in \Lambda$ there is an exact sequence of $\mathscr{O}_{\mathfrak{F}}G$-modules:

$$\ldots \to \mathbb{Z}[?,X_{\lambda,2}]_G \to \mathbb{Z}[?,X_{\lambda,1}]_G \to \mathbb{Z}[?,X_{\lambda,0}]_G \twoheadrightarrow \mathbb{Z}[?,G/G_\lambda]_G, \tag{5}$$

where $X_{\lambda,n} = I_{G_\lambda}(\Delta_{\lambda,n})$. Note that the $X_{\lambda,n}$ are G-invariant subsets of Δ_n and that Δ_n is the directed union of the $X_{\lambda,n}$. For each $\lambda \le \mu$ the inclusion $X_{\lambda,n} \hookrightarrow X_{\mu,n}$, $n \ge 0$, induces a homomorphism

$$\eta^\mu_{\lambda,n}\colon \mathbb{Z}[?,X_{\lambda,n}]_G \to \mathbb{Z}[?,X_{\mu,n}]_G.$$

Also, the projection $G/G_\lambda \twoheadrightarrow G/G_\mu$ induces homomorphisms

$$\eta^\mu_{\lambda,-1}\colon \mathbb{Z}[?,G/G_\lambda]_G \to \mathbb{Z}[?,G/G_\mu]_G.$$

Hence we have chain-maps between the corresponding chain complexes (5). These chain complexes together with the chain maps $\eta^\mu_{\lambda,*}$ form a direct limit system indexed by Λ. Lemma 3.3 (i) and (ii) imply that its limit is the sequence

$$\ldots \to \mathbb{Z}[?,\Delta_2]_G \to \mathbb{Z}[?,\Delta_1]_G \to \mathbb{Z}[?,\Delta_0]_G \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}}. \tag{6}$$

Since direct limits preserve exactness [30, p. 57], this sequence is exact.

Denote by $K_{\lambda,n}$ the n-th kernel of the sequence (4). As before, $\operatorname{Ind}_{G_\lambda}(K_{\lambda,n})$ is the n-th kernel in (5) and the chain maps $\eta^\mu_{\lambda,n}$ yield an inverse limit system. Since taking direct limits preserves exactness we get that

$$K_n = \varinjlim(\operatorname{Ind}_{G_\lambda} K_{\lambda,n})$$

is the n-th kernel of (6).

Now suppose that there exists a $n \in \mathbb{N}$ such that $\operatorname{hd}_{\mathfrak{F}_\lambda} G_\lambda \le n$ for all $\lambda \in \Lambda$. In particular, all $K_{\lambda,n}$ are flat. Now Lemma 2.2 implies that $\operatorname{Ind}_{G_\lambda} K_{\lambda,n}$ all are flat. Since $\operatorname{Tor}_1^{\mathfrak{F}}(?,M)$ commutes with direct limits it follows that K_n is a flat $\mathscr{O}_{\mathfrak{F}}G$-module. In particular, $\operatorname{hd}_{\mathfrak{F}} G \le n$ proving (i).
The proof of (ii) is analogous. Apply [23, Lemma 3.4], which states that a countable colimit of projective $\mathscr{O}_{\mathfrak{F}}G$-modules has projective dimension ≤ 1. □

Lemma 3.4. *Let A be a countable abelian group with finite Hirsch length $h(A)$. Then*

$$\underline{\underline{\mathrm{cd}}}A \leq h(A)+2.$$

Proof. Write A as the countable direct union $A = \varinjlim A_\lambda$ of its finitely generated subgroups A_λ. [19, Theorem 5.13] implies $\underline{\underline{\mathrm{gd}}}A_\lambda \leq h(A_\lambda)+1$, and hence $\underline{\underline{\mathrm{cd}}}A_\lambda \leq h(A_\lambda)+1 \leq h(A)+1$. Thus, by Corollary 3.2, $\underline{\underline{\mathrm{cd}}}A \leq h(A)+2$ as required. □

4. Change of family and direct products of groups

The following result is the algebraic counterpart to [19, Proposition 5.1 (i)]. Although we only state and prove it for Bredon cohomology, an analogous statement also holds for Bredon homology.

Proposition 4.1. *Let G be a group and $\mathfrak{F}$ and $\mathfrak{G}$ two families of subgroups of G such that $\mathfrak{F} \subset \mathfrak{G}$ and that, for every $K \in \mathfrak{G}$, $\mathfrak{F} \cap K$ is a non-empty subset of $\mathfrak{F}$. Suppose there exists $k \geq 0$ such that, for every $K \in \mathfrak{G}$, $\mathrm{cd}_{\mathfrak{F}\cap K} K \leq k$. Then*

$$\mathrm{cd}_{\mathfrak{F}} G \leq \mathrm{cd}_{\mathfrak{G}} G + k.$$

Proof. Let $I\colon \mathcal{O}_{\mathfrak{F}}G \hookrightarrow \mathcal{O}_{\mathfrak{G}}G$ be the inclusion functor. We begin by showing that for every projective $\mathcal{O}_{\mathfrak{G}}G$-module P, $\mathrm{pd}_{\mathfrak{F}}(\mathrm{Res}_I P) \leq k$. Since restriction is an exact additive functor, it suffices to prove this claim for $P = \mathbb{Z}[?, G/K]_G$ where $K \in \mathfrak{G}$. Since $\mathrm{cd}_{\mathfrak{F}\cap K} K \leq k$ there exists a projective resolution

$$0 \to P_k \to \ldots \to P_0 \to \underline{\mathbb{Z}}_{\mathfrak{F}\cap K} \to 0$$

of the trivial $\mathcal{O}_{\mathfrak{F}\cap K}K$-module $\underline{\mathbb{Z}}_{\mathfrak{F}\cap K}$. Since induction with I_K is exact, see Lemma 2.3, and preserves projectives, we obtain a projective resolution

$$0 \to \mathrm{Ind}_{I_K} P_k \to \ldots \to \mathrm{Ind}_{I_K} P_0 \to \mathbb{Z}[?, G/K]_G \to 0$$

of length k of the $\mathcal{O}_{\mathfrak{F}}G$-module $\mathbb{Z}[?, G/K]_G$. However, by [27, Lemma 2.7], $\mathbb{Z}[?, G/K]_G \cong \mathrm{Res}_I P$, implying $\mathrm{pd}_{\mathfrak{F}}(\mathrm{Res}_I P) \leq k$ as claimed.

Now $\mathrm{cd}_{\mathfrak{G}} G = n$. Then there exists a projective resolution

$$0 \to P_n \to \ldots \to P_0 \to \underline{\mathbb{Z}}_{\mathfrak{G}} \to 0$$

of the trivial $\mathcal{O}_{\mathfrak{G}}G$-module $\underline{\mathbb{Z}}_{\mathfrak{G}}$. Upon restriction we obtain a resolution

$$0 \to \mathrm{Res}_I P_n \to \ldots \to \mathrm{Res}_I P_0 \to \underline{\mathbb{Z}}_{\mathfrak{F}} \to 0 \tag{7}$$

of the trivial $\mathcal{O}_{\mathfrak{F}}G$-module $\underline{\mathbb{Z}}_{\mathfrak{F}}$ by $\mathcal{O}_{\mathfrak{F}}G$-modules of projective dimension at most k. The result now follows by a dimension shifting argument. □

Proposition 4.2. *Let G_1 and G_2 be groups and let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be subgroup-closed families of subgroups of G_1 and G_2 respectively. Let $G = G_1 \times G_2$ and $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2$ and take $\mathfrak{G} \subset \bar{\mathfrak{F}}$ to be a subgroup-closed family of subgroups of G. Assume that there exists $k \in \mathbb{N}$ such that $\mathrm{cd}_{\mathfrak{G}\cap K} K \leq k$ for every $K \in \mathfrak{F}$. Then*

$$\mathrm{cd}_{\mathfrak{G}} G \leq \mathrm{cd}_{\mathfrak{F}_1} G_1 + \mathrm{cd}_{\mathfrak{F}_2} G_2 + k.$$

Similar results for the families $\mathfrak{F}_1 = \mathfrak{F}_2 = \mathfrak{F}_{\mathrm{fin}}$ and G-CW-complexes have been obtained in [14, 25].

Corollary 4.3. *Let $G = G_1 \times G_2$. Then $\underline{\underline{\mathrm{cd}}}G \leq \underline{\underline{\mathrm{cd}}}G_1 + \underline{\underline{\mathrm{cd}}}G_2 + 3$.*

Proof. Every $K \in \mathfrak{F}_{\mathrm{vc}}(G_1) \times \mathfrak{F}_{\mathrm{vc}}(G_2)$ is virtually polycyclic of $\operatorname{vcd} K \leq 2$. Thus $\underline{\underline{\mathrm{cd}}}\, K \leq \underline{\underline{\mathrm{gd}}}\, K \leq 3$ by [19, Theorem 5.13] and the result follows from Proposition 4.2. $\square$

Remark 4.4. Suppose $G = G_1 \times G_2$ and $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2$ are as in Proposition 4.2. Let X_i and Y_i be G_i-sets and let $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ be G-sets with the obvious G-action. We denote by $p_i\colon G \to G_i$ the canonical projections. Since $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are assumed to be closed under forming subgroups it follows that for every $H \in \bar{\bar{\mathfrak{F}}}$, $p_1(H) \times p_2(H) \in \mathfrak{F}$. Hence the homomorphism

$$f\colon \mathbb{Z}[?,X]_G \to \mathbb{Z}[?,Y]_G$$

of right $\mathcal{O}_{\mathfrak{F}}G$-modules extends to a homomorphism $f\colon \mathbb{Z}[?,X]_G \to \mathbb{Z}[?,Y]_G$ of $\mathcal{O}_{\bar{\bar{\mathfrak{F}}}}G$-modules as follows: for every $H \in \bar{\bar{\mathfrak{F}}}$ let $f_H = f_{p_1(H)\times p_2(H)}$.

Also note that the natural projections $p_i : G \to G_i$ give rise to functors

$$p_i\colon\ \mathcal{O}_{\mathfrak{F}}G \to \mathcal{O}_{\mathfrak{F}_i}G_i\,.$$

Proof of Proposition 4.2. Let $P_* \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$ and $Q_* \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_2}$ be free resolutions. Hence there exist G_1-sets X_i and G_2-sets Y_j such that $P_i = \mathbb{Z}[?,X_i]_{G_1}$ and $Q_j = \mathbb{Z}[?,Y_j]_{G_2}$. Let $P'_* = \operatorname{Res}_{p_1} P_*$ and $Q'_* = \operatorname{Res}_{p_2} Q_*$. These $\mathcal{O}_{\mathfrak{F}}G$-modules are of the form

$$P'_i = \mathbb{Z}[?,X_i]_G \qquad \text{and} \qquad Q'_j = \mathbb{Z}[?,Y_i]_G,$$

where the action of G on X_i and Y_i is given by $gx = p_1(g)x$ and $gy = p_2(g)y$ respectively. For each $i,j \in \mathbb{N}$ we have an identification of $\mathcal{O}_{\mathfrak{F}}G$-modules

$$P'_i \otimes Q'_j = \mathbb{Z}[?,X_i \times Y_j]_G.$$

Here G acts diagonally on $X_i \times Y_j$.
This gives rise to a double complex in the usual way, see for example [30, pp. 58f.]. Denote by C_k its total complex:

$$C_k = \coprod_{i=0}^{k} \mathbb{Z}[?,X_i \times Y_{k-i}]_G.$$

The augmentation maps $\varepsilon_1\colon P_0 \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$ and $\varepsilon_2\colon Q_0 \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$ induce an augmentation map $\varepsilon\colon C_0 \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}}$. Altogether we obtain a resolution

$$\ldots \to C_2 \to C_1 \to C_0 \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}} \tag{8}$$

of the trivial $\mathcal{O}_{\mathfrak{F}}G$-module by free $\mathcal{O}_{\mathfrak{F}}G$-modules.
Now the free $\mathcal{O}_{\mathfrak{F}}G$-modules C_k are also free $\mathcal{O}_{\bar{\bar{\mathfrak{F}}}}G$-modules. Since the families $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are assumed to be subgroup closed we can extend, using Remark 4.4, every morphism in the sequence (8) to a morphism of the corresponding $\mathcal{O}_{\bar{\bar{\mathfrak{F}}}}G$-modules. It follows that we obtain a resolution

$$\ldots \to C_2 \to C_1 \to C_0 \twoheadrightarrow \underline{\mathbb{Z}}_{\bar{\bar{\mathfrak{F}}}} \tag{9}$$

of the trivial $\mathcal{O}_{\bar{\bar{\mathfrak{F}}}}G$-module by free $\mathcal{O}_{\bar{\bar{\mathfrak{F}}}}G$-modules.
Now assume that $m = \operatorname{cd}_{\mathfrak{F}_1} G_1$ and $n = \operatorname{cd}_{\mathfrak{F}_2} G_2$. Then it follows from an Eilenberg Swindle argument that there are free resolutions $P_* \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$ and $Q_* \twoheadrightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_2}$ as above, of lengths m and n respectively. This implies that $C_k = 0$ for all $k > m+n$. In particular

$$\operatorname{cd}_{\bar{\bar{\mathfrak{F}}}} G \leq \operatorname{cd}_{\mathfrak{F}_1} G_1 + \operatorname{cd}_{\mathfrak{F}_2} G_2.$$

Let $K \in \bar{\bar{\mathfrak{F}}}$. Then $K \leq K_1 \times K_2$ for some $K_1 \times K_2 \in \mathfrak{F}$. Since $\mathfrak{G}$ is assumed to be closed under forming subgroups it follows that $\varnothing \neq \mathfrak{G} \cap K \subset \mathfrak{G} \cap (K_1 \times K_2)$.

Therefore we have $\mathrm{cd}_{\mathfrak{G}\cap K} K \leq \mathrm{cd}_{\mathfrak{G}\cap(K_1\times K_2)}(K_1 \times K_2)$. By assumption the latter is bounded by k. Thus we have $\mathrm{cd}_{\mathfrak{G}}\, G \leq \mathrm{cd}_{\bar{\mathfrak{F}}}\, G + k$ by Proposition 4.1 and the claim of the proposition follows. □

Remark 4.5. Note that the special case of Corollary 4.3 follows almost immediately by applying Martínez-Pérez' spectral sequence Proposition 2.5 twice, but we have included the above for its generality and for being rather elementary. The alternative argument is as follows: Consider $G = G_1 \times G_2$ as an extension

$$G_1 \rightarrowtail G \twoheadrightarrow G_2$$

By Proposition 2.5 we have $\underline{\underline{\mathrm{cd}}}\, G \leq m + \underline{\underline{\mathrm{cd}}}\, G_2$ where m is the supremum of $\underline{\underline{\mathrm{cd}}}\, H$, and H ranges over all $H \leq G$ such that $G_1 \leq G$ and H/G_1 is virtually cyclic. But these $H \leq G$ are of the form $H = G_1 \times V$ with V a virtually cyclic subgroup of G_2. This gives rise to an extension

$$V \rightarrowtail H \twoheadrightarrow G_1.$$

Applying Proposition 2.5 again yields that $\underline{\underline{\mathrm{cd}}}\, H \leq n + \underline{\underline{\mathrm{cd}}}\, G_1$ where n is the supremum of $\underline{\underline{\mathrm{cd}}}\, L$, and L ranges over all $L \leq H$ with $V \leq L$ and L/V is virtually cyclic. These L are of the form $V \times W$ with W a virtually cyclic subgroup of G_1. Thus

$$\underline{\underline{\mathrm{cd}}}\, G \leq k + \underline{\underline{\mathrm{cd}}}\, G_1 + \underline{\underline{\mathrm{cd}}}\, G_2$$

where k is the supremum of $\underline{\underline{\mathrm{cd}}}(V_1 \times V_2)$ where V_1 and V_2 range over all virtually cyclic subgroups of G_1 and G_2 respectively. In particular, $k \leq 3$ by [19, Theorem 5.13].

5. INFINITE CYCLIC EXTENSIONS OF ABELIAN GROUPS

Lemma 5.1. *Let G be a torsion-free abelian-by-(infinite cyclic) group, i.e. there is a short exact sequence*

$$A \rightarrowtail G \twoheadrightarrow \langle t \rangle$$

with A abelian and $\langle t \rangle \cong \mathbb{Z}$. Consider the subgroup $\bar{A} = \{a \in A \mid a^t = a\}$. Then $G/\bar{A}$ is torsion-free.

Proof. $\bar{A}$ is obviously a central subgroup in G and we have a short exact sequence

$$A/\bar{A} \rightarrowtail G/\bar{A} \overset{\pi}{\twoheadrightarrow} \langle t \rangle.$$

To prove the claim it suffices to show that $A/\bar{A}$ is torsion-free.
Suppose there is an $a \in A$ such that $\pi(a)^n = \pi(a^n) = 1$. This implies that $a^n \in \bar{A}$ and hence $(a^n)^t = a^n$. Since A is abelian, we have $(a^t a^{-1})^n = 1$. But A is torsion-free and hence $a^t = a$. This implies $a \in \bar{A}$. □

In a torsion-free abelian-by-(infinite cyclic) group, the generator t of the infinite cyclic group acts by automorphisms on the abelian group A. As we will see in Proposition 5.4, there is no problem if t acts trivially or freely on the non-trivial elements of A. The main problem arises when t acts by a finite order automorphism. But the following folklore version of Selberg's Lemma tells us that the order of this automorphism has a bound only depending on A. We shall state the Lemma as a special case of [28, Theorem T1]:

Lemma 5.2. [28] *Let Γ be a group of automorphisms of a torsion-free abelian group A of finite Hirsch length. Then there exists an integer $m(A)$ such that every periodic subgroup of Γ has order at most $m(A)$.*

For our purpose we need the following consequence of Selberg's Lemma, which is probably known. We include it for completeness.

Lemma 5.3. *Let A be a torsion-free abelian group with finite Hirsch length $h(A)$. Then there exists an integer $\nu = \nu(A)$ which depends only on $h(A)$ such that for any automorphism t of A the finite orbits of elements in A under the action of t have length at most ν.*

Proof. Let $n = h(A)$. Then $A \otimes \mathbb{Q} \cong \mathbb{Q}^n$ and A can be viewed as an additive subgroup of $\mathbb{Q}^n$ by $a \mapsto a \otimes 1$. The automorphism t of A extends to an automorphism $t \otimes \mathrm{id}\colon \mathbb{Q}^n \to \mathbb{Q}^n$ of $\mathbb{Q}$-vector spaces, which we denote by φ.
Let $U = \{a \in A \mid \varphi^k(a) = a \text{ for some } 0 \neq k \in \mathbb{Z}\}$. Then U is a φ-invariant subspace of $\mathbb{Q}^n$. It has a complement V in $\mathbb{Q}^n$ and there exists a unique linear map $\psi\colon \mathbb{Q}^n \to \mathbb{Q}^n$ which agrees with φ on U and which is the identity on V. Then ψ is an isomorphism which is periodic by construction.
By Selberg's Lemma there exists a number $\nu(n)$ such that every periodic automorphism of $\mathbb{Q}^n$ has order at most $\nu(n)$. Therefore $\psi^m = \mathrm{id}$ for some $1 \leq m \leq \nu(n)$. In particular we have that $\varphi^m(a) = a$ for each $a \in U \cap A$. □

Proposition 5.4. *Let A be a torsion-free abelian group of finite Hisch length $h(A)$. There exists a recursively defined integer $f(h(A))$ depending only on the Hirsch length of A such that for every infinite cyclic extension $G = A \rtimes \langle t \rangle$ we have*

$$\underline{\underline{\mathrm{cd}}}\, G \leq f(h(A)).$$

Proof. We prove the proposition by induction on the Hirsch length $h(A) = n$. Since A is torsion-free, $h(A) = 0$ implies that A is trivial. In this case G is infinite cyclic and therefore $f(0) = 0$.
Now suppose $n \geq 1$ and assume that the statement is true for all torsion-free abelian groups B with $h(B) < n$. Let A be a torsion-free abelian group with $h(A) = n$ and let $\bar{A}$ be as in Lemma 5.1. Then precisely one of the following three cases occures.

(1) $\{1\} \neq \bar{A}$: As in the proof of Lemma 5.1 we have a short exact sequence

$$A/\bar{A} \rightarrowtail G/\bar{A} \twoheadrightarrow \langle t \rangle$$

with $A/\bar{A}$ torsion-free. Since A is torsion-free and $\bar{A} \neq \{1\}$ it follows that $h(\bar{A}) \geq 1$ and thus $h(A/\bar{A}) < h(A)$. Then

$$\underline{\underline{\mathrm{cd}}}(G/\bar{A}) \leq f(h(A) - 1),$$

by induction.
Consider the short exact sequence

$$\bar{A} \rightarrowtail G \rightarrowtail G/\bar{A}.$$

We use Proposition 2.5 to find a bound for $\underline{\underline{\mathrm{cd}}}\, G$. Let S be a subgroup of G such that $\bar{A} \leq S$ and $S/\bar{A}$ is virtually cyclic. Since $G/\bar{A}$ is torsion free, $S/\bar{A} = \langle s \rangle$ is infinite cyclic. Then the fact that $\bar{A}$ is a central subgroup of G implies that

$$S = \bar{A} \times \langle s \rangle.$$

In particular S is countable abelian with finite Hirsch length $h(S) = h(\bar{A}) + 1$. Therefore we have $\underline{\underline{\mathrm{cd}}}\, S \leq h(\bar{A}) + 3$ by Lemma 3.4. Hence Proposition 2.5 gives

$$\begin{aligned} \underline{\underline{\mathrm{cd}}}\, G &\leq h(\bar{A}) + 3 + \underline{\underline{\mathrm{cd}}}\, G/\bar{A} \\ &\leq h(A) + 3 + f(h(A) - 1). \end{aligned}$$

(2) $\{1\} = \bar{A}$ *but there is an element* $1 \neq a \in A$ *and a positive integer* m *such that* $a^{t^m} = a$*:* Lemma 5.3 implies that m is bounded by a number ν which depends only on $n = h(A)$. Let $r_n = \operatorname{lcm}(1, \dots, \nu)$ and set $A_0 = A$ and $t_0 = t^{r_n}$. Then $G_0 = A_0 \rtimes \langle t_0 \rangle$ is a subgroup of G with index $|G : G_0| = r_n$ and $\bar{A}_0 \neq \{1\}$. Thus Case (1) applies to G_0. Then

$$\begin{aligned}
\underline{\underline{\mathrm{cd}}}\, G &\leq \underline{\underline{\mathrm{gd}}}\, G \\
&\leq r_n \, \underline{\underline{\mathrm{gd}}}\, G_0 && \text{(Proposition 2.7)} \\
&\leq r_n \max(3, \underline{\underline{\mathrm{cd}}}\, G_0) && \text{(Proposition 2.4)} \\
&\leq r_n (h(A) + 3 + f(h(A) - 1)). && \text{(Case (1))}
\end{aligned}$$

(3) $\{1\} = \bar{A}$ *and for all* $1 \neq a \in A$ *and all* $m \neq 0$ *we have* $a^{t^m} \neq a$*:* Then Proposition 2.8 applies to G and it follows that $\underline{\underline{\mathrm{gd}}}\, G \leq \underline{\underline{\mathrm{gd}}} A + 1$. Thus

$$\underline{\underline{\mathrm{cd}}}\, G \leq \underline{\underline{\mathrm{gd}}}\, G \leq \underline{\underline{\mathrm{gd}}} A + 1 \leq h(A) + 3$$

where the last inequality is due to Theorems 4.3 and 5.13 in [19].

Therefore, if we recursively define $f(n) = r_n(n + 3 + f(n-1))$, then

$$\underline{\underline{\mathrm{cd}}}\, G \leq f(n)$$

in all three cases. □

6. Nilpotent-by-abelian groups

For any group G we denote its centre by $Z(G)$.

Lemma 6.1. *Let* G *be a group such that there is a short exact sequence*

$$N \rightarrowtail G \twoheadrightarrow Q,$$

where N *is torsion-free nilpotent and* Q *is torsion-free. Then* $G/Z(N)$ *is torsion-free.*

Proof. Since $Z(N)$ is normal in G we have a short exact sequence

$$N/Z(N) \rightarrowtail G/Z(N) \twoheadrightarrow Q$$

and it suffices to show that $N/Z(N)$ is torsion-free. Since $Z(N)$ is torsion-free, a Theorem of Mal'cev [24, 5.2.19] implies that every upper central factor of N is also torsion-free. Hence, in particular $Z(N/Z(N)) = Z_2(N)/Z(N)$ is torsion-free. As $N/Z(N)$ is nilpotent, applying Mal'cev's result to $N/Z(N)$ yields the result. □

Theorem 6.2. *Let* G *be a group such that there is a short exact sequence*

$$N \rightarrowtail G \twoheadrightarrow Q,$$

where N *is torsion-free nilpotent and* Q *is torsion-free abelian. Assume that the Hirsch length of* G *is finite. Then* $\underline{\underline{\mathrm{cd}}}\, G < \infty$.

Proof. By [1] and Corollary 3.2 we might assume that G is finitely generated. Q is finitely generated abelian of finite Hirsch length $h(Q)$. [19] implies that $\underline{\underline{\mathrm{cd}}}\, Q \leq h(Q) + 1$. To apply Proposition 2.5 we need to consider all infinite cyclic extensions $H = N \rtimes \langle t \rangle$ and show that there is a positive integer M such that $\underline{\underline{\mathrm{cd}}}(N \rtimes \langle t \rangle) \leq M$. We prove this by induction on the nilpotency length c of N.
If $c = 1$ we are in the situation of Proposition 5.4 and $M \leq f(h(N))$.
Now suppose $c > 1$ and for all nilpotent groups N_1 with nilpotency length $< c$, we have an integer M_1, depending only on the nilpotency class and the Hirsch

length of N_1, such that $\underline{\underline{\mathrm{cd}}}(N_1 \rtimes \langle t \rangle) \leq M_1$. N is nilpotent so $Z(N) \neq 1$ and $H/Z(N)$ is (torsion-free nilpotent)-by-(infinite cyclic) and $c(N/Z(N)) < c$, see Lemma 6.1. Hence, by induction, $\underline{\underline{\mathrm{cd}}}(H/Z(N)) \leq M_1$. Furthermore, we can apply Proposition 5.4 to every infinite cyclic extension $T = Z(N) \rtimes \langle t \rangle$ of $Z(N)$ which gives $\underline{\underline{\mathrm{cd}}}\, T \leq f(Z(N))$. Now apply Martínez-Pérez' spectral sequence to the short exact sequence

$$Z(N) \rightarrowtail H \twoheadrightarrow H/Z(N)$$

to give

$$\underline{\underline{\mathrm{cd}}}\, H = \underline{\underline{\mathrm{cd}}}(N \rtimes \langle t \rangle) \leq f(Z(N)) + M_1 = M.$$

Since both, $f(Z(N))$ and M_1 are independent of the choice of cyclic extension H of N, so is M. Another application of the spectral sequence yields

$$\underline{\underline{\mathrm{cd}}}\, G \leq M + \underline{\underline{\mathrm{cd}}}\, Q \leq M + h(Q) + 1. \qquad \square$$

7. Proof of the Main Theorem

The proof is now an easy application of a theorem by Hillman and Linnell [7]. We shall refer to an alternative proof of their theorem, see points (f) and (g) in Wehrfritz [29], whose statement is better suited to our purpose. For any group G we denote by $\tau(G)$ its unique maximal normal locally finite subgroup.

Theorem 7.1. [7, 29] *Let G be an elementary amenable group of finite Hirsch length h. Then there is an integer-valued function $j(h)$ of h only such that G has characteristic subgroups $\tau(G) \leq N \leq M$, with $N/\tau(G)$ torsion-free nilpotent, M/N free abelian of finite rank, and $|G : M|$ at most $j(h)$.*

Proof of the Main Theorem. Since G has a bound on the orders of the finite subgroups, $\tau(G)$ is finite. An application of Proposition 2.6 allows us to assume that $\tau(G) = \{1\}$ and hence that G is virtually torsion-free. Using Proposition 2.7 we can assume that G is torsion-free nilpotent-by-abelian. Hence we can apply Theorem 6.2. $\square$

Remark 7.2. To remove the condition that there is a bound on the orders of the finite subgroups, one needs to understand virtually cyclic extensions of large locally finite groups. This would allow us to apply Martínez-Pérez' spectral sequence as before. Since $\tau(G)$ is locally finite, every virtually cyclic subgroup is, in fact, finite and hence $\underline{\underline{E}}(\tau(G)) = \underline{E}(\tau(G))$ and these are well understood [4], yet finding a classifying space for the family of virtually cyclic subgroups of virtually cyclic extensions of $\tau(G)$ so far eludes us.

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