Prime order derangements in primitive permutation groups

Timothy C. Burness[*]
School of Mathematics
University of Southampton
Southampton SO17 1BJ
UK

Michael Giudici[†]
School of Mathematics and Statistics
The University of Western Australia
35 Stirling Highway
Crawley, WA 6009
Australia

Robert A. Wilson[‡]
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS
UK

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Abstract

Let G be a transitive permutation group on a finite set Ω of size at least 2. An element of G is a derangement if it has no fixed points on Ω. Let r be a prime divisor of $|\Omega|$. We say that G is r-elusive if it does not contain a derangement of order r, and strongly r-elusive if it does not contain one of r-power order. In this note we determine the r-elusive and strongly r-elusive primitive actions of almost simple groups with socle an alternating or sporadic group.

1 Introduction

Let G be a transitive permutation group on a finite set Ω of size $n \geq 2$ with point stabiliser H. By the Orbit-Counting Lemma, G contains an element which acts fixed-point-freely on Ω; such elements are called *derangements*. Equivalently, $x \in G$ is a derangement if and only if the conjugacy class of x in G fails to meet H. The existence of derangements has interesting applications in other areas of mathematics, including number theory, algebraic geometry and topology (see [16, 25], for example).

The study of derangements can be traced back to the early years of probability theory in the 18th century. For example, in 1708 Montmort [20] established that the proportion of derangements in the symmetric group S_n in its natural action on n points is given by the formula

$$\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}$$

[*]t.burness@soton.ac.uk
[†]michael.giudici@uwa.edu.au
[‡]r.a.wilson@qmul.ac.uk

so this proportion tends to e^{-1} as n tends to infinity. More generally, if $\delta(G)$ denotes the proportion of derangements in a transitive group G of degree n then $\delta(G) \geq 1/n$, with equality if and only if G is sharply 2-transitive (see [7]). In recent work, Fulman and Guralnick [11] prove that there is an absolute constant $\epsilon > 0$ such that $\delta(G) > \epsilon$ for any simple group G, confirming a conjecture of Boston and Shalev.

In this paper we are interested in derangements of prime order. By a theorem of Fein, Kantor and Schacher [10], our group G always contains a derangement of prime-power order, but not all such groups contain one of prime order. Following [8], we call a transitive permutation group *elusive* if it does not contain a derangement of prime order; elusive permutation groups have been much studied in recent years, in a number of different contexts (see [8, 14, 15, 33], for example). For instance, the 3-transitive action of the smallest Mathieu group M_{11} on 12 points is elusive since M_{11} has a unique class of involutions, and also a unique class of elements of order 3. Whereas the proof of the existence of a derangement in a transitive permutation group is elementary, the extension in [10] to derangements of prime-power order requires the full force of the Classification of Finite Simple Groups.

A broad class of elusive groups has been classified by Giudici. In [13] he proves that if G is elusive on Ω and contains a transitive minimal normal subgroup then $G = \mathrm{M}_{11} \wr K$ acting with its product action on $\Omega = \Delta^k$ for some $k \geq 1$, where K is a transitive subgroup of S_k and $|\Delta| = 12$. In particular, the aforementioned example of M_{11} on 12 points is the only elusive primitive permutation group which is almost simple. The proof of this result relies on the list of pairs (G, H) given in [18], where G is a simple group and H is a maximal subgroup of G with the property that every prime dividing $|G|$ also divides $|H|$.

The purpose of this paper is to initiate a more quantitative study of derangements of prime order in primitive permutation groups. For example, if G is non-elusive then we would like to determine the primes r for which there exists a derangement of order r. We say that G is *r-elusive* if r is a prime dividing $|\Omega|$ and no derangement of order r exists. In this terminology, G is elusive if and only if it is r-elusive for every prime divisor r of $|\Omega|$. The O'Nan-Scott Theorem essentially reduces the problem of determining the r-elusive actions of primitive groups to the almost simple case, at which point we may appeal to the Classification of Finite Simple Groups and use the wealth of information on the subgroup structure and conjugacy classes of such groups.

In this paper we establish a reduction to the almost simple case (see Theorem 2.1) and we explicitly determine the r-elusive primitive actions of almost simple groups with socle an alternating or sporadic group. Not surprisingly, the situation for groups of Lie type is more complicated. A detailed study of conjugacy classes and derangements of prime order in classical groups is forthcoming in [5].

Theorem 1.1. *Let G be an almost simple primitive permutation group on a set Ω such that the socle of G is either an alternating or a sporadic group. Let r be a prime divisor of $|\Omega|$. Then either G contains a derangement of order r, or (G, Ω, r) is a known exception.*

Corollary 1.2. *Let r be the largest prime divisor of $|\Omega|$. Then either G contains a derangement of order r, or $r = 3$, $G = \mathrm{M}_{11}$ and $|\Omega| = 12$.*

Remark 1.3. For alternating groups, the exceptions referred to in Theorem 1.1 are given explicitly in the statements of Propositions 3.4 – 3.8. Similarly, we refer the reader to Tables 5 and 7 for the list of r-elusive primitive actions of the Baby Monster and Monster groups, respectively, while the exceptions for the other sporadic groups are recorded in Table 2.

For each primitive action of the Baby Monster we also determine the specific conjugacy classes of derangements of prime order (see Table 6). Similarly, for the Monster, Table 8 is complete except for the case $H = 11^2{:}(5 \times 2A_5)$, where we are unable to determine the precise fusion of 3-elements. Here H has a unique class of elements of order 3, which fuse to the Monster class $3B$ or $3C$. A complete list of the conjugacy classes of maximal subgroups of the Monster is not presently available; to date, some 43 classes have been identified, and it is known that any additional maximal subgroup is almost simple with socle $\mathrm{L}_2(13), \mathrm{L}_2(41), \mathrm{U}_3(4), \mathrm{U}_3(8)$ or $\mathrm{Sz}(8)$. Some additional difficulties arise in the analysis of these remaining possibilities and we only provide partial information on the derangements in these cases (see Table 10).

We can extend our analysis to derangements of prime-power order. Let r be a prime divisor of $|\Omega|$. We say that G is *strongly r-elusive* if it does not contain a derangement of r-power order. By the main theorem of [10], if G is transitive then there exists a prime divisor r such that G is not strongly r-elusive. In this paper we determine the strongly r-elusive primitive actions of almost simple groups with an alternating or sporadic socle.

Theorem 1.4. *Let G be an almost simple primitive permutation group on a set Ω such that the socle of G is either an alternating or a sporadic group. Let r be a prime divisor of $|\Omega|$. Then either G contains a derangement of r-power order, or (G, Ω, r) is a known exception.*

Remark 1.5. If the socle of G is an alternating group then the strongly r-elusive examples referred to in Theorem 1.4 are recorded in Proposition 3.9, while the examples with a sporadic socle are listed in Table 11.

In [10], Fein, Kantor and Schacher were motivated by an interesting number-theoretic application. Let $L \supset K$ be fields and let $B(L/K)$ denote the relative Brauer group of L/K; this is the subgroup of the Brauer group of K containing the Brauer classes of finite dimensional central simple K-algebras which are split over L. The main theorem of [10] on the existence of derangements of prime-power order is equivalent to the fact that $B(L/K)$ is infinite for any nontrivial extension of global fields (i.e. a number field, or an algebraic function field in one variable over a finite field).

As explained in [10, Section 3], there is a reduction to the case where L/K is separable, and by a further reduction we may assume $L = K(\alpha)$. Let E be a Galois closure of L/K, let Ω be the set of roots in E of the minimal polynomial of α over K, and set $G = \mathrm{Gal}(E/K)$. Then G acts transitively on Ω and [10, Corollary 3] states that $B(L/K)$ is infinite if and only if G contains a derangement of prime-power order. More precisely, if r is a prime divisor of $|\Omega|$ then the r-torsion subgroup $B(L/K)_r$ of $B(L/K)$ is finite if and only if G is strongly r-elusive. Therefore, by considering the known exceptions in Theorem 1.4, we can identify the primes r for which $B(K(\alpha)/K)_r$ is finite in the case where the Galois group $\mathrm{Gal}(K(\alpha)/K)$ is almost simple with socle an alternating or sporadic group.

Let us make some remarks on the organisation of this paper. In Section 2 we start with a reduction of the general problem to the almost simple case. Next, in Section 3 we prove Theorems 1.1 and 1.4 in the case where the socle of G is an alternating group. Here our analysis depends on whether or not the point stabiliser is primitive; if it is then we can apply some classical results of Jordan, while in the remaining cases we argue directly via the explicit action on subsets or partitions. Finally, in Section 4 we deal with the sporadic groups. If G is not the Baby Monster or the Monster then we can obtain the relevant fusion maps using GAP [12] and known character tables. The two remaining groups are dealt with in Sections 4.2 and 4.3, respectively, where a more delicate analysis is required. We determine the corresponding strongly r-elusive actions in Section 4.4.

Throughout this paper we use the standard Atlas notation, the only difference being that we write $\mathrm{PSp}_n(q)$ and $\mathrm{P}\Omega_n^{\epsilon}(q)$ for simple symplectic and orthogonal groups respectively, rather than $S_n(q)$ and $O_n^{\epsilon}(q)$ adopted in the Atlas.

2 A reduction theorem

Here we use the O'Nan-Scott Theorem to reduce the general problem of determining the r-elusive actions of finite primitive groups to the almost simple case.

Theorem 2.1. *Let G be a primitive permutation group on a finite set Ω, with socle T. Let r be a prime dividing $|\Omega|$. Then one of the following holds:*

(i) G is almost simple;

(ii) T contains a derangement of order r;

(iii) $G \leqslant H \wr S_k$ acting with its product action on $\Omega = \Delta^k$ for some $k \geq 2$, where $H \leqslant \mathrm{Sym}(\Delta)$ is primitive, almost simple and the socle of H is r-elusive.

Proof. Let N be a minimal normal subgroup of G. Then N is transitive on Ω, and $N \cong S^k$ for some simple group S and integer $k \geq 1$. If N is regular then T contains a derangement of order r. Therefore we may assume S is nonabelian and the point stabiliser N_α is non-trivial.

If $k = 1$ then G is almost simple and we are in case (i), so assume $k \geq 2$. Since N is minimal, G acts transitively on the set of k simple direct factors of N. Further, since N is transitive we have $G = G_\alpha N$ and thus G_α also acts transitively on the set of simple factors of N. Therefore, there exists a non-trivial subgroup R of S such that for each coordinate i, the projection $\pi_i(N_\alpha)$ of N_α onto the ith simple factor of N is isomorphic to R. If $R = S$ then [24, p.328, Lemma] implies that there exists a partition $\mathcal{P}$ of $\{1, \ldots, k\}$ such that $N_\alpha = \prod_{P \in \mathcal{P}} D_P$, where $D_P \cong S$ and $\pi_i(D_P) = S$ if $i \in P$ and trivial otherwise. For each $P \in \mathcal{P}$ let N_P be a subgroup given by the direct product of $|P| - 1$ of the simple direct factors of N corresponding to P. Then $\prod_{P \in \mathcal{P}} N_P \leqslant N$ meets N_α trivially and has order $|\Omega|$. Therefore this subgroup is regular and (ii) follows.

Finally, suppose that $R \neq S$. Then [9, Theorem 4.6A] implies that we are in case (iii) and S is the socle of H. If $s \in S$ is a derangement of order r on Δ then $(s, 1, \ldots, 1) \in T$ is a derangement of order r on Ω. □

We note that in case (iii) it is possible that $G \leqslant H \wr S_k$ is r-elusive on Δ^k when $H \leqslant \mathrm{Sym}(\Delta)$ contains a derangement of order r. For example, suppose that H is an almost simple group with socle S such that $H = \langle S, h \rangle$, where S is 2-elusive on Δ and h is a derangement of order 2. (The existence of groups H satisfying these hypotheses will be demonstrated in the next sections – see Proposition 4.4, for example.) Let $G = \langle S^4, (h, h, h, h)\sigma \rangle \leqslant H \wr S_4$, where $\sigma = (1, 2, 3, 4) \in S_4$, and consider the product action of G on Δ^4. There are no derangements in S^4 since S is 2-elusive on Δ. The involutions in $G \setminus S^4$ have the form $(g_1, g_2, g_1^{-1}, g_2^{-1})(1, 3)(2, 4)$ for some $g_1, g_2 \in H$, and such an element fixes the points $(\delta_1, \delta_2, \delta_1^{g_1}, \delta_2^{g_2}) \in \Delta^4$. We conclude that G is 2-elusive.

3 Alternating groups

In this section we prove Theorems 1.1 and 1.4 in the case where the socle of G is an alternating group. We begin by considering derangements of prime order.

3.1 Prime order derangements

Theorem 3.1. *Let G be an almost simple primitive permutation group on a set Ω such that the socle of G is an alternating group A_n. Let r be a prime divisor of $|\Omega|$. Then either G contains a derangement of order r, or (G, Ω, r) is a known exception.*

The exceptions are recorded explicitly in Propositions 3.4 – 3.8 below. As an easy corollary we obtain the following result (we postpone the proof to the end of this section).

Corollary 3.2. *The following hold:*

(i) G contains a derangement of order r, where r is the largest prime divisor of $|\Omega|$.

(ii) If $|\Omega|$ is even then G is 2-elusive if and only if one of the following holds:

(a) Ω is the set of k-subsets of $\{1,\ldots,n\}$ where k is even, or n is odd, or $G = A_n$ and $n/2$ is odd;

(b) Ω is the set of partitions of $\{1,\ldots,n\}$ into b parts of size a, where $a, b \geq 2$;

(c) $\Omega = G/H$ with $(G,H) = (A_5, D_{10})$, $(\mathrm{PGL}_2(9), D_{20})$, $(\mathrm{M}_{10}, 5{:}4)$ or $(\mathrm{M}_{10}, 3^2{:}Q_8)$.

(iii) If $|\Omega|$ is divisible by an odd prime then G contains a derangement of odd prime order.

For the rest of Section 3.1 we make the following global assumption.

Hypothesis 3.3. *Let G be a primitive permutation group on a set Ω such that the socle of G is A_n, for some $n \geq 5$, and let H be the stabiliser in G of an element of Ω. Note that H is a maximal subgroup of G and $H \neq A_n$. Let r be a prime divisor of $|\Omega|$.*

First we handle the three cases for which G is not isomorphic to A_n or S_n.

Proposition 3.4. *Given Hypothesis 3.3, suppose $G = \mathrm{Aut}(A_6)$, $\mathrm{PGL}_2(9)$ or M_{10}. Then either G contains a derangement of order r, or $r = 2$ and $(G,H) = (\mathrm{PGL}_2(9), D_{20})$, $(\mathrm{M}_{10}, 5{:}4)$ or $(\mathrm{M}_{10}, 3^2{:}Q_8)$.*

Proof. Computation using Magma [2] or GAP [12]. □

For the remainder of this section we may assume $G = A_n$ or S_n. The next proposition deals with the case where H is primitive on $\{1,\ldots,n\}$.

Proposition 3.5. *Given Hypothesis 3.3, suppose $G = A_n$ or S_n and H acts primitively on $\{1,\ldots,n\}$. Then G is r-elusive if and only if $r = 2$ and $(G,H) = (A_5, D_{10})$ or $(A_6, \mathrm{L}_2(5))$.*

Proof. First assume r is odd and let $x \in G$ be an r-cycle. If $r \geq n-2$ then r^2 does not divide $|G|$, so r does not divide $|H|$ and thus x is a derangement. On the other hand, if $r < n-2$ then a theorem of Jordan [17] (also [28, Theorem 13.9]) implies that H does not contain an r-cycle, so once again we conclude that x is a derangement.

Now assume $r = 2$. By the above theorem of Jordan, H does not contain a transposition. In particular, if $G = S_n$ then every transposition in G is a derangement. Finally, suppose $G = A_n$ and let $x \in G$ be a product of two disjoint transpositions. By a theorem of Manning [19] (who attributes it to Jordan), the only primitive groups H of degree n with minimal degree 4 (other than A_n or S_n) are

$$(n,H) = (5, 5{:}4),\ (5, D_{10}),\ (6, \mathrm{PGL}_2(5)),\ (6, \mathrm{L}_2(5)),\ (7, \mathrm{L}_3(2)),\ (8, \mathrm{AGL}_3(2)).$$

The first and third examples are not contained in A_n, while $|\Omega|$ is odd in the last two cases. We conclude that x has fixed points if and only if $(n,H) = (5, D_{10})$ or $(6, \mathrm{L}_2(5))$. □

Next we assume H is transitive but imprimitive on $\{1,\ldots,n\}$, that is, Ω is the set of all partitions of $\{1,\ldots,n\}$ into b parts of size a for suitable integers $a, b \geq 2$.

Proposition 3.6. *Given Hypothesis 3.3, suppose $G = A_n$ or S_n, and Ω is the set of partitions of $\{1,\ldots,n\}$ into b parts of size a with $a, b \geq 2$. Let $a \equiv \ell \pmod r$ and $b \equiv k \pmod r$ with $0 \leq \ell, k < r$. Then G is r-elusive if and only if $r \leq a$ and one of the following holds:*

(i) $\ell = 0$*;*

(ii) $k = 0$ *and* $\ell = 1$*;*

(iii) $\ell, k \neq 0$*,* $k\ell < r$ *and either* $b < r$ *or* $(k+r)\ell \leq ka + r$*.*

Proof. If $r > a$ then r is odd and every r-cycle in G is a derangement. We also observe that if r divides a then any $x \in G$ of order r fixes a partition into parts of size a, where each r-cycle of x is contained in a block of the relevant partition. For the remainder we may assume $r \leq a$ and $\ell > 0$.

Suppose first that r divides b. If $\ell > 1$ then r is odd and there exists $x \in G$ of order r with precisely r fixed points on $\{1, \ldots, n\}$. If x fixes an element of Ω then x must fix at least r blocks; moreover, since $a \equiv \ell \pmod r$ it follows that x fixes at least ℓ points in each fixed block. This contradicts $\ell > 1$ as x only has $r < r\ell$ fixed points. We conclude that x is a derangement.

Next suppose r divides b and $\ell = 1$. Let $x \in G$ have order r. If x has no fixed points on $\{1, \ldots, n\}$ then it fixes a partition in Ω by permuting the blocks in cycles of length r. If x has sr fixed points with $r \leq sr \leq b$ then x fixes a partition with sr fixed blocks, each with precisely one fixed point. If x has sr fixed points with $sr > b$ then x fixes a partition partwise with at least one fixed point in each block. Thus if $(k, \ell) = (0, 1)$ then all elements of order r have fixed points on Ω.

Next we suppose that $\ell, k \neq 0$. If an element of S_n fixes an element of Ω it must fix at least k blocks of the partition and at least ℓ points in each fixed block. Hence it must have at least $k\ell$ fixed points. If $k\ell \geq r$ then r is odd and so G contains permutations with at most $r - 1$ fixed points, and these are fixed point free on Ω. For the remainder we may assume $k\ell < r$. Note that $n \pmod r = k\ell$ and so each element of order r in G has at least $k\ell$ fixed points.

First assume $x \in G$ has order r and exactly $k\ell$ fixed points. Then x fixes a partition by acting semiregularly on $b - k$ blocks and having ℓ fixed points in each of the remaining k fixed blocks. Moreover, any element of order r with at most ka fixed points also fixes such a partition. In particular, if $b < r$ then all elements of order r have fixed points on Ω. Suppose then that $b > r$ and $(k + r)\ell > ka + r$. Then r is odd and G contains an element x with precisely $ka + r$ fixed points. If x fixes a partition in Ω it would fix at least $k + r$ blocks of that partition and hence have at least $(k + r)\ell$ fixed points. As x only has $ka + r < (k + r)\ell$ fixed points, this is a contradiction and thus x is a derangement.

To complete the proof we may assume $b > r$ and $(k + r)\ell \leq ka + r$. Let $x \in S_n$ be an element of order r. It remains to show that x fixes an element of Ω. We have already observed that this holds if x has at most ka fixed points, so we may assume x has $\alpha = (sr + k)a + tr$ fixed points for some $s \geq 0$ and $0 \leq t < a$ with $(s, t) \neq (0, 0)$. When $s = 0$, the condition $(k + r)\ell \leq ka + r$ implies that x fixes a partition with $k + r$ fixed blocks and at least ℓ fixed points in each fixed block. Since $r < a$ we have $(a - \ell) \geq r > \ell$ and so $a > 2\ell$. Thus for $s > 0$ we have $sa > (s + 1)\ell$. It follows that $((s + 1)r + k)\ell < \alpha < ((s + 1)r + k)a$ and so x fixes a partition with $(s + 1)r + k$ fixed blocks and at least ℓ fixed points in each fixed block. This concludes the proof. □

Remark 3.7. In the statement of Proposition 3.6, it is possible that a prime $r \leq a$ satisfying one of the conditions (i) – (iii) does not actually divide $|\Omega|$, in which case G clearly does not contain a derangement of order r. For example, if a is a power of r then it is easy to see that $|\Omega|$ is indivisible by r. Indeed, if $a = r^m$ and n_r denotes the largest power of r dividing

n then $(n!)_r = r^{\alpha_1}$, $((a!)^b)_r = r^{\alpha_2}$ and $(b!)_r = r^{\alpha_3}$, where

$$\alpha_1 = \sum_{i\geq 1} \left\lfloor \frac{ab}{r^i} \right\rfloor = r^{m-1}b + r^{m-2}b + \cdots + rb + b + \left\lfloor \frac{b}{r} \right\rfloor + \left\lfloor \frac{b}{r^2} \right\rfloor + \cdots$$

$$\alpha_2 = b\sum_{i\geq 1} \left\lfloor \frac{a}{r^i} \right\rfloor = b(r^{m-1} + r^{m-2} + \cdots + r + 1)$$

$$\alpha_3 = \sum_{i\geq 1} \left\lfloor \frac{b}{r^i} \right\rfloor .$$

Therefore $|\Omega|_r = r^{\alpha_1-\alpha_2-\alpha_3} = 1$, so $|\Omega|$ is indivisible by r.

To complete the proof of Theorem 3.1 it remains to deal with the case where H is intransitive on $\{1,\ldots,n\}$, that is, Ω is the set of k-element subsets of $\{1,\ldots,n\}$ with $k < n/2$.

Proposition 3.8. *Given Hypothesis 3.3, suppose $G = A_n$ or S_n, and Ω is the set of k-subsets of $\{1,\ldots,n\}$ with $1 \leq k < n/2$. Let $k \equiv j \pmod r$ and $n \equiv i \pmod r$ with $0 \leq i,j < r$.*

(i) If r is odd then G is r-elusive if and only if $r \leq k$ and $i \geq j$.

(ii) G is 2-elusive if and only if k is even, or n is odd, or $G = A_n$ and $n/2$ is odd.

Proof. Note that r divides $n - t$ for some $t = 0,\ldots,k-1$. Assume first that r is odd and $r > k$. Let $x \in A_n$ be a product of $(n-t)/r$ cycles of length r and t fixed points. Since $t < k$ and $r > k$ it follows that x is a derangement.

Now suppose $2 < r \leq k$. If $i < j$ then let $x \in A_n$ be a product of $(n-i)/r$ cycles of length r and i fixed points. Then x is a derangement since any subset fixed setwise by x has size $s'r + i'$ for some $s' \leq (n-i)/r$ and $i' \leq i < j$. Conversely, suppose $i \geq j$ and let $x \in G$ be an element of order r, say x has ℓ cycles of length r and $n - \ell r \geq i \geq j$ fixed points for some integer ℓ, $1 \leq \ell \leq (n-i)/r$. If $\ell r \geq k$ then x fixes the k-set consisting of $(k-j)/r$ of its r-cycles and j fixed points. If $\ell r < k$ then x has at least $n-k > k$ fixed points and hence clearly fixes a k-set.

Finally, let us assume $r = 2$. Clearly, if k is even then any involution of S_n fixes a k-set. Also note that if $x \in S_n$ is an involution with at least one fixed point then x fixes a k-set for any k. In particular, if n is odd then every involution has fixed points. Finally, suppose k is odd and n is even. Let $x \in S_n$ be the product of $n/2$ disjoint transpositions. Then x is a derangement. Note that $x \in A_n$ if and only if $n/2$ is even; if $n/2$ is odd then all involutions of A_n fix a point and hence fix a k-set. □

Proof of Corollary 3.2.

Let r be the largest prime divisor of $|\Omega|$ and consider (i). Note that Hypothesis 3.3 holds. If H is primitive then the result follows at once from Proposition 3.5. Next suppose Ω is the set of partitions of $\{1,\ldots,n\}$ into b parts of size a, where $a,b \geq 2$. Since $a,b \leq n/2$, Bertrand's Postulate implies that $n/2 < r < n$ and the result follows from Proposition 3.6. Finally, suppose Ω is the set of k-subsets of $\{1,\ldots,n\}$, where $1 \leq k < n/2$. Since $n \geq 5$, a theorem of Sylvester [26] implies that $\binom{n}{k}$ is divisible by an odd prime greater than k, so $r > k$ and the result follows from Proposition 3.8. This justifies (i). Parts (ii) and (iii) are clear. Note that the action of A_6 on the set of cosets of $L_2(5)$ is permutationally isomorphic to the action of A_6 on $\{1,\ldots,6\}$.

3.2 Derangements of prime-power order

Here we prove Theorem 1.4 in the case where the socle of G is an alternating group A_n.

Proposition 3.9. *Let G be an almost simple primitive permutation group on a set Ω such that the socle of G is an alternating group A_n. Let r be a prime divisor of $|\Omega|$ and set $m = \lfloor \log_r n \rfloor$. Then G is strongly r-elusive if and only if one of the following holds:*

(i) $(G, H, r) = (A_5, D_{10}, 2)$;

(ii) Ω *is the set of partitions of* $\{1, \ldots, n\}$ *into* b *parts of size* a, *where either*

  (a) $r = 2$, $G = A_n$, $a = 2^{m-1} + 1$ *and* $b = 2$; *or*

  (b) r *is odd,* $a > r^m$, $b \geq 2$ *and* $a \equiv \ell_i \pmod{r^i}$ *with* $0 \leq \ell_i \leq \lfloor (r^i - 1)/b \rfloor$, *for all* $1 \leq i \leq m$.

(iii) $r = 2$, $G = A_n$, $n = 2^m + 1$ *and* Ω *is the set of* 2^{m-1}*-subsets of* $\{1, \ldots, n\}$.

Proof. If $G \neq A_n$ or S_n then we may assume $r = 2$ and (G, H) is one of the three cases listed in Proposition 3.4; it is easy to check that none of these examples are strongly 2-elusive. Similarly, if $(G, H) = (A_6, \mathrm{L}_2(5))$ then G contains derangements of order 4, while $(G, H) = (A_5, D_{10})$ is strongly 2-elusive. In view of Proposition 3.5, for the remainder we may assume H is either transitive and imprimitive on $\{1, \ldots, n\}$, or H is intransitive on $\{1, \ldots, n\}$.

First suppose Ω is the set of partitions of $\{1, \ldots, n\}$ into b parts of size a with $a, b \geq 2$. Note that if a is a power of r then $|\Omega|$ is indivisible by r, so we may assume otherwise (see Remark 3.7). Suppose $r^m > a$. Here an r^m-cycle is a derangement, and G contains such elements unless $G = A_n$ and $r = 2$, so let us assume we are in this latter situation. If $2^{m-1} > a$ then the product of two 2^{m-1}-cycles is a derangement in G, so we may assume $2^{m-1} < a < 2^m$, in which case $b = 2$. Now, if $a \neq 2^{m-1} + 1$ then the product of a 2^m-cycle and a transposition is a derangement. Suppose $a = 2^{m-1} + 1$ and let $x \in G$ be an element of 2-power order. If x has no fixed points on $\{1, \ldots, n\}$ then x fixes an element of Ω by interchanging the two parts of the partition. On the other hand, if x has fixed points (with respect to $\{1, \ldots, n\}$) then all cycles of x have length at most 2^{m-1}, and x fixes an element of Ω by fixing the two parts of the partition. Therefore, G is strongly 2-elusive if we are in case (ii)(a).

Now assume $r^m < a$ (so $b < r$ and r is odd). Write $n = a_m r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0$, where $0 \leq a_i < r$ for all i, and let $g \in G$ be an element with a_i cycles of length r^i for each $0 \leq i \leq m$. Suppose the condition in (ii)(b) holds. Then each a_i is a multiple of b and thus g fixes a partition in Ω (fixing the blocks in the partition setwise). Consequently, every element of r-power order fixes such a partition, so G is strongly r-elusive. Conversely, suppose that the condition in (ii)(b) does not hold, and assume g fixes a partition $\alpha \in \Omega$. Since $b < r$, it follows that the blocks comprising α are fixed setwise by g. Let k be maximal such that $\ell_k > \lfloor (r^k - 1)/b \rfloor$. Then each a_i with $i > k$ is a multiple of b, while a_k is indivisible by b. Now working down from m, if a_m is divisible by b then the a_m cycles of g of length r^m must be evenly distributed amongst the b blocks in α. We continue distributing the r^i-cycles of g in this way until we come to the r^k-cycles. At this stage there are $(a_k r^k + \cdots + a_1 r + a_0)/b$ points left in each block in α, so we can place at most $\lfloor a_k/b \rfloor$ cycles of length r^k in each of the b blocks. However, $b\lfloor a_k/b \rfloor < a_k$, so there is at least one r^k-cycle left over. This is a contradiction, and thus g is a derangement of r-power order.

Finally, let us assume Ω is the set of k-subsets of $\{1, \ldots, n\}$ with $1 \leq k < n/2$. Write $n = a_m r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0$ and $k = b_m r^m + b_{m-1} r^{m-1} + \cdots + b_1 r + b_0$, where $0 \leq a_i, b_i < r$ for all i. Let $k \equiv k_i \pmod{r^i}$ and $n \equiv n_i \pmod{r^i}$ with $0 \leq k_i, n_i < r^i$ for all $i \leq m$. We claim that $k_i > n_i$ for some i. To see this, suppose $k_i \leq n_i$ for all $i \leq m$. Then $b_i \leq a_i$ for all $i \leq m$, and thus $n-k = \sum_{i\geq 0}(a_i - b_i)r^i$ with $0 \leq a_i - b_i < r$ for all i. Therefore

$$\left\lfloor \frac{n}{r^i} \right\rfloor = \sum_{j=i}^{m} a_j r^{j-i}, \quad \left\lfloor \frac{k}{r^i} \right\rfloor = \sum_{j=i}^{m} b_j r^{j-i}, \quad \left\lfloor \frac{n-k}{r^i} \right\rfloor = \sum_{j=i}^{m} (a_j - b_j) r^{j-i}$$

so

$$|\Omega|_r = \left(\frac{n!}{k!(n-k)!} \right)_r = r^{\ell}, \text{ where } \ell = \sum_{i\geq 1} \left(\left\lfloor \frac{n}{r^i} \right\rfloor - \left\lfloor \frac{k}{r^i} \right\rfloor - \left\lfloor \frac{n-k}{r^i} \right\rfloor \right) = 0.$$

This justifies the claim.

Define $g \in S_n$ as above. Let $t \leq m-1$ be maximal such that $b_t > a_t$ and assume g fixes a k-subset $\alpha \in \Omega$. Suppose that the restriction of g to α comprises c_i cycles of length r^i for $t < i \leq m$. Then

$$k \leq c_m r^m + \cdots + c_{t+1} r^{t+1} + a_t r^t + \cdots + a_1 r + a_0 < b_m r^m + \cdots + b_{t+1} r^{t+1} + (a_t + 1) r^t$$

and thus $k < b_m r^m + \cdots + b_1 r + b_0 = k$, which is absurd. Therefore g is a derangement. Now g lies in A_n if and only if r is odd, or $r = 2$ and $\sum_{i\geq 1} a_i$ is even, so we may as well assume $G = A_n$, $r = 2$ and $\sum_{i\geq 1} a_i$ is odd. Let $h \in G$ be an element consisting of $2a_m + a_{m-1}$ cycles of length 2^{m-1} and a_i cycles of length 2^i for each $i \leq m-2$. If $t < m-1$ then the same argument as for g shows that h is a derangement. Suppose then that $t = m-1$. Then $n = 2^m + a_{m-2}2^{m-2} + \cdots + a_1 2 + a_0$, and since $k < n/2$ we have $k = 2^{m-1} + b_{m-2}2^{m-2} + \cdots + 2b_1 + b_0$. Suppose $\sum_{i\geq 1} a_i \geq 3$, so $a_s = 1$ for some $2 \leq s < m-1$. Let $h' \in G$ be a permutation with a_i cycles of length 2^i for all $i \neq s, s-1$, and with $2 + a_{s-1}$ cycles of length 2^{s-1}. Then h' is a derangement as h' contains a 2^m-cycle and $n - 2^m < 2^{m-1} \leq k$. Since $k < n/2$, it remains to consider the case where $n = 2^m + 1$ and $k = 2^{m-1}$. Here every element of A_n of 2-power order preserves a partition of $\{1, \ldots, n\}$ into parts of size 2^{m-1}, 2^{m-1} and 1, so every such element fixes a k-subset. □

4 Sporadic groups

In this final section we deal with the 26 sporadic simple groups and we complete the proof of Theorems 1.1 and 1.4.

Theorem 4.1. *Let G be an almost simple primitive permutation group on a set Ω such that the socle of G is a sporadic group. Let r be a prime divisor of $|\Omega|$. Then either G contains a derangement of order r, or (G, Ω, r) is a known exception. In addition, the cases where G does not contain a derangement of r-power order are also known.*

The r-elusive exceptions are recorded in Tables 5 and 7 for the Baby Monster and Monster, respectively, while the rest can be read off from Tables 2 – 4. The strongly r-elusive examples are listed in Table 11. As an immediate corollary, we get the following result:

Corollary 4.2. *Let G be an almost simple primitive permutation group on a set Ω such that the socle of G is a sporadic group. Let r be the largest prime divisor of $|\Omega|$. Then r is odd and either G contains a derangement of order r, or $r = 3$, $G = \mathrm{M}_{11}$ and $|\Omega| = 12$.*

It is straightforward to extract additional results from Theorem 4.1. For example, one can easily determine the examples for which there exists a derangement of order r for every prime divisor r of $|\Omega|$. Not surprisingly, most of the work in proving Theorem 4.1 involves the Baby Monster $\mathbb{B}$ and the Monster $\mathbb{M}$ sporadic groups, which we consider separately in

Sections 4.2 and 4.3, respectively. As advertised in the Introduction, for $\mathbb{B}$ and $\mathbb{M}$ we also determine the specific conjugacy classes of derangements of prime order (see Tables 6 and 8).

In Sections 4.1 – 4.3 we focus on derangements of prime order. We use very similar methods to determine the strongly r-elusive examples (see Section 4.4 for the details).

4.1 The non-monstrous groups

In this section we establish Theorem 1.1 for the sporadic groups, with the exception of the monstrous groups $\mathbb{B}$ and $\mathbb{M}$. Our main result is the following:

Proposition 4.3. *Let G be a simple sporadic simple group, let H be a maximal subgroup of G and let $\Omega = G/H$. Assume $G \neq \mathbb{B}$ or $\mathbb{M}$, and let r be a prime divisor of $|\Omega|$. Then G is r-elusive if and only if (G, H, r) is listed in Table 2. In particular, G is r-elusive only if $r \leq 7$.*

Let us explain how to read off the r-elusive examples from Table 2. For each group G, a collection of maximal subgroups is defined in Table 1, denoted by $\mathcal{M}(G)$. The numbers appearing in the row of Table 2 corresponding to G refer to the ordered list of subgroups in $\mathcal{M}(G)$. For example, suppose $G = \mathrm{M}_{11}$. Then G is 2-elusive if $H = \mathrm{L}_2(11)$ or S_5, and G is 3-elusive if $H = \mathrm{L}_2(11)$, S_5 or $2S_4$. For any other relevant prime r, G contains derangements of order r.

Next suppose $G = T.2$ with T a sporadic simple group. Let $H \neq T$ be a maximal subgroup of G and set $\Omega = G/H$ and $S = H \cap T$. Let r be a prime divisor of $|\Omega|$ and assume for now that S is a maximal subgroup of T. If r is odd then G is r-elusive if and only if T is r-elusive with respect to the primitive action of T on T/S, so the examples which arise are easily determined from Proposition 4.3. On the other hand, if G is 2-elusive then T is 2-elusive, but the converse does not always hold.

Proposition 4.4. *Suppose $G = T.2$, $S = H \cap T$ is maximal in T and T is 2-elusive on T/S. Then G contains derangements of order 2 if and only if (G, H) is one of the cases listed in Table 3.*

Finally, we deal with the cases where H is a *novelty* subgroup of G, that is, $S = H \cap T$ is not a maximal subgroup of T.

Proposition 4.5. *Suppose $G = T.2$ and $S = H \cap T$ is not maximal in T. Let r be a prime divisor of $|\Omega|$. Then G is r-elusive if and only if (G, H, r) is one of the cases listed in Table 4.*

It is convenient to prove Propositions 4.3 – 4.5 by computational methods, using GAP [12]. In most cases, we may inspect the relevant character tables in the GAP Character Table Library [4], and utilise the stored fusion data therein. This approach is effective unless G is one of the following:

$$\mathrm{HS}.2, \mathrm{He}.2, \mathrm{Fi}_{22}.2, \mathrm{O'N}.2, \mathrm{Fi}_{24}, \mathrm{HN}.2 \qquad (1)$$

In each of these cases the Web Atlas [32] provides an explicit faithful permutation representation of G on $n(G)$ points, where $n(G)$ is defined as follows:

G	HS.2	He.2	$\mathrm{Fi}_{22}.2$	O′N.2	Fi_{24}	HN.2
$n(G)$	100	2058	3510	245520	306936	1140000

For $G = \mathrm{HS}.2$, He.2 or $\mathrm{Fi}_{22}.2$, we may then use the MAGMA command MaximalSubgroups to obtain representatives of the conjugacy classes of maximal subgroups of G, and it is then straightforward to determine the fusion of the relevant H-classes in G. In the remaining cases, explicit generators for H are given in the Web Atlas [32] and we may proceed as before. We leave the details to the reader.

G	$\mathcal{M}(G)$
M_{11}	$\mathrm{L}_2(11), S_5, 2S_4$
M_{12}	$A_6.2^2, 2\times S_5, A_4\times S_3$
M_{22}	$\mathrm{L}_3(4), A_7, 2^4{:}S_5, 2^3{:}\mathrm{L}_3(2), \mathrm{M}_{10}, \mathrm{L}_2(11)$
M_{23}	A_8, M_{11}
M_{24}	$\mathrm{M}_{22}{:}2, \mathrm{M}_{12}{:}2, 2^6{:}(\mathrm{L}_3(2)\times S_3)$
HS	$\mathrm{U}_3(5){:}2, \mathrm{L}_3(4){:}2_1, S_8, 2^4.S_6, 4^3{:}\mathrm{L}_3(2), 4.2^4.S_5, 2\times A_6.2^2, 5{:}4\times A_5$
J_2	$3.A_6.2, 2^{2+4}{:}(3\times S_3), A_4\times A_5, A_5\times D_{10}, \mathrm{L}_3(2){:}2, 5^2{:}D_{12}$
Co_1	$3.\mathrm{Suz}{:}2, 2^{1+8}.\Omega_8^+(2), \mathrm{U}_6(2){:}S_3, (A_4\times G_2(4)){:}2, 2^{4+12}.(S_3\times 3.S_6),$ $3^2.\mathrm{U}_4(3).D_8, (A_5\times \mathrm{J}_2){:}2, (A_6\times \mathrm{U}_3(3)).2, 3^{3+4}.2.(S_4\times S_4), A_9\times S_3,$ $(A_7\times \mathrm{L}_2(7)){:}2, (D_{10}\times(A_5\times A_5).2).2, 5^3{:}(4\times A_5).2$
Co_2	$\mathrm{U}_6(2){:}2, 2^{1+8}{:}\mathrm{Sp}_6(2), \mathrm{HS}{:}2, (2^4\times 2^{1+6}).A_8, \mathrm{U}_4(3){:}D_8,$ $2^{4+10}.(S_5\times S_3), 3^{1+4}.2^{1+4}.S_5$
Co_3	$\mathrm{McL}.2, \mathrm{HS}, \mathrm{U}_4(3).2^2, 3^5{:}(2\times \mathrm{M}_{11}), \mathrm{U}_3(5){:}S_3, 3^{1+4}{:}4.S_6, \mathrm{L}_3(4).D_{12},$ $2\times \mathrm{M}_{12}, [2^{10}.3^3], S_3\times \mathrm{L}_2(8){:}3, A_4\times S_5$
McL	$\mathrm{U}_3(5), 3^{1+4}{:}2.S_5, 3^4{:}\mathrm{M}_{10}, 2.A_8, \mathrm{M}_{11}, 5^{1+2}{:}3{:}8$
Suz	$G_2(4), 3.\mathrm{U}_4(3){:}2, \mathrm{J}_2{:}2, 2^{4+6}{:}3.A_6, (A_4\times \mathrm{L}_3(4)){:}2, \mathrm{M}_{12}{:}2,$ $3^{2+4}{:}2.(A_4\times 2^2).2, (A_6\times A_5).2, (A_6\times 3^2{:}4).2, \mathrm{L}_3(2){:}2$
He	$\mathrm{Sp}_4(4){:}2, 2^2.\mathrm{L}_3(4).S_3, 2^6{:}3.S_6, 3.S_7, 7^{1+2}{:}(3\times S_3), S_4\times \mathrm{L}_3(2),$ $7{:}3\times \mathrm{L}_3(2), 5^2{:}4.A_4$
HN	$A_{12}, 2.\mathrm{HS}.2, \mathrm{U}_3(8){:}3, 2^{1+8}.(A_5\times A_5).2, (D_{10}\times \mathrm{U}_3(5)).2, 5^{1+4}.2^{1+4}.5.4,$ $2^6.\mathrm{U}_4(2), (A_6\times A_6).D_8, 2^{3+2+6}.(3\times \mathrm{L}_3(2)), \mathrm{M}_{12}{:}2, 3^4{:}2.(A_4\times A_4).4$
Th	${}^3D_4(2){:}3, 2^5.\mathrm{L}_5(2), 2^{1+8}.A_9, \mathrm{U}_3(8){:}6, (3\times G_2(3)){:}2, 3.3^2.3.(3\times 3^2).3^2{:}2S_4,$ $3^2.3^3.3^2.3^2{:}2S_4, 3^5{:}2.S_6, 5^{1+2}{:}4S_4, 5^2{:}\mathrm{GL}_2(5), 7^2{:}(3\times 2S_4), \mathrm{L}_2(19){:}2, \mathrm{L}_3(3),$ $\mathrm{M}_{10}, 31{:}15, S_5$
Fi_{22}	$2.\mathrm{U}_6(2), \Omega_7(3), \Omega_8^+(2){:}S_3, 2^{10}{:}\mathrm{M}_{22}, 2^6{:}\mathrm{Sp}_6(2), (2\times 2^{1+8}){:}(\mathrm{U}_4(2){:}2),$ $\mathrm{U}_4(3){:}2\times S_3, 2^{5+8}{:}(S_3\times A_6), 3^{1+6}{:}2^{3+4}{:}3^2{:}2, S_{10}, \mathrm{M}_{12}$
Fi_{23}	$2.\mathrm{Fi}_{22}, \mathrm{P}\Omega_8^+(3){:}S_3, 2^2.\mathrm{U}_6(2).2, \mathrm{Sp}_8(2), \Omega_7(3)\times S_3, 2^{11}.\mathrm{M}_{23}, 3^{1+8}.2^{1+6}.3^{1+2}.2S_4,$ $S_{12}, (2^2\times 2^{1+8}).(3\times \mathrm{U}_4(2)).2, 2^{6+8}{:}(A_7\times S_3), \mathrm{Sp}_6(2)\times S_4$
Fi_{24}'	$\mathrm{Fi}_{23}, 2.\mathrm{Fi}_{22}{:}2, (3\times \mathrm{P}\Omega_8^+(3){:}3){:}2, \Omega_{10}^-(2), 3^7.\Omega_7(3), 3^{1+10}{:}\mathrm{U}_5(2){:}2, 2^{11}.\mathrm{M}_{24},$ $2^2.\mathrm{U}_6(2){:}S_3, 2^{1+12}{:}3_1.\mathrm{U}_4(3).2, [3^{13}]{:}(\mathrm{L}_3(3)\times 2), 3^{2+4+8}.(A_5\times 2A_4).2,$ $(A_4\times \Omega_8^+(2){:}3){:}2, \mathrm{He}{:}2, 2^{3+12}.(\mathrm{L}_3(2)\times A_6), 2^{6+8}.(S_3\times A_8), (G_2(3)\times 3^2{:}2).2,$ $(A_9\times A_5){:}2, \mathrm{L}_2(8){:}3\times A_6, A_7\times 7{:}6$
J_1	$\mathrm{L}_2(11), 19{:}6, 11{:}10, D_6\times D_{10}, 7{:}6$
O′N	$\mathrm{L}_3(7){:}2, \mathrm{J}_1, 4_2.\mathrm{L}_3(4){:}2_1, (3^2{:}4\times A_6).2, 3^4{:}2^{1+4}.D_{10}, \mathrm{L}_2(31), 4^3.\mathrm{L}_3(2), \mathrm{M}_{11}, A_7$
J_3	$\mathrm{L}_2(16){:}2, \mathrm{L}_2(19), 2^4{:}(3\times A_5), \mathrm{L}_2(17), (3\times A_6){:}2_2, 3^{2+1+2}{:}8, 2^{2+4}{:}(3\times S_3)$
Ru	$(2^2\times \mathrm{Sz}(8)){:}3, 2^{3+8}{:}\mathrm{L}_3(2), \mathrm{U}_3(5){:}2, 2^{1+4+6}.S_5, \mathrm{L}_2(25).2^2, A_8, \mathrm{L}_2(29),$ $5^2{:}4{:}S_5, 3.A_6.2^2, 5^{1+2}{:}[2^5], \mathrm{L}_2(13){:}2, A_6.2^2, 5{:}4\times A_5$
J_4	$2^{10}{:}\mathrm{L}_5(2), 2^{3+12}.(S_5\times \mathrm{L}_3(2)), \mathrm{U}_3(11){:}2, \mathrm{M}_{22}{:}2, 11^{1+2}{:}(5\times 2S_4), \mathrm{L}_2(32){:}5,$ $\mathrm{L}_2(23){:}2, 37{:}12$
Ly	$G_2(5), 3.\mathrm{McL}{:}2, 5^3.\mathrm{L}_3(5), 2.A_{11}, 5^{1+4}{:}4.S_6, 3^5{:}(2\times \mathrm{M}_{11}), 3^{2+4}{:}2.A_5.D_8,$ $67{:}22, 37{:}18$

Table 1: Some maximal subgroups of sporadic simple groups

G	$r = 2$	3	5	7
M_{11}	$1, 2$	$1, 2, 3$		
M_{12}	$1, 2, 3$	3		
M_{22}	$1, 2, 4, 5, 6$	$3, 4, 6$		
M_{23}	$1, 2$			
M_{24}	$1, 2$	3		
HS	$1, 2, 3, 4, 7, 8$	$5, 6, 8$		
J_2	$1, 3, 4, 5, 6$	$2, 3$		
Co_1	$1, 4, 6, 7, 8, 9, 10, 11, 12, 13$	$1, 3, 5, 6, 8, 10$	$2, 7, 12$	
Co_2	$1, 3, 4, 5, 7$	$2, 4, 6$		
Co_3	$1, 2, 3, 4, 5, 6, 7, 8, 10, 11$	$5, 9, 10$		
McL	$1, 2, 3, 5, 6$	4		
Suz	$1, 2, 3, 5, 6, 7, 8, 9, 10$	$4, 9$		
He	$1, 2, 4, 6, 8$	$5, 6, 7$	$2, 3, 4$	
HN	$1, 2, 5, 6, 7, 8, 10, 11$	$1, 3, 4, 7, 8, 9, 10$	5	
Th	$1, 4, 5, 6, 7, 8, 9, 10,$ $11, 12, 13, 14, 16$	$1, 3, 4, 5, 8$	$2, 3, 8, 12, 14, 15, 16$	$2, 3, 4, 5$
Fi_{22}	$1, 2, 3, 5, 7, 9, 10$	$3, 7, 8$	$1, 2, 4, 5, 6, 7, 8, 11$	
Fi_{23}	$2, 4, 5, 7, 8, 11$	$1, 5, 9, 10, 11$	$3, 5, 6, 9, 10, 11$	
Fi_{24}'	$1, 2, 3, 4, 5, 6, 8, 10, 11,$ $12, 13, 16, 17, 18, 19$	$3, 8, 9, 12, 16, 18$	$5, 6, 7, 8, 9, 11,$ $14, 15, 18, 19$	19
J_1	$1, 2, 3, 4, 5$			
O′N	$1, 2, 4, 5, 6, 8, 9$	$1, 2, 3, 6, 7, 8, 9$		
J_3	$1, 2, 3, 4, 5, 6$	$3, 5, 7$		
Ru	$1, 5, 9, 10, 11, 12, 13$	$1, 2, 3, 4, 5, 6, 7, 8,$ $11, 12, 13$	13	
J_4	$1, 3, 5, 7$	$1, 2, 3, 4, 5, 6, 7, 8$		
Ly	$1, 3, 5, 6, 7, 8, 9$	$1, 4, 5$	$2, 4$	

Table 2: Some r-elusive simple sporadic groups

G	H
$M_{22}.2$	$L_3(4).2, L_2(11).2$
$J_2.2$	$(A_5 \times D_{10}).2, 5^2{:}(4 \times S_3)$
$J_3.2$	$L_2(16).4$
Suz.2	$G_2(4).2$
He.2	$Sp_4(4){:}4, 5^2{:}4.S_4$
O′N.2	$3^4{:}2^{1+4}.D_{10}.2$

Table 3: Some sporadic groups with involutory derangements

G	H	r
He.2	$(S_5 \times S_5).2$	2
$\mathrm{Fi}_{22}.2$	$3^5{:}(2 \times \mathrm{U}_4(2){:}2)$	$2, 5$
O′N.2	$7^{1+2}{:}(3 \times D_{16})$	$2, 3$
	31:30	3
	$A_6{:}2$	$2, 3$
	$\mathrm{L}_2(7){:}2$	$2, 3$

Table 4: Some r-elusive sporadic groups arising from novelties

4.2 The Baby Monster

Let $G = \mathbb{B}$ be the Baby Monster sporadic group and consider the action of G on $\Omega = G/H$, where H is a maximal subgroup of G. Let r be a prime divisor of $|\Omega|$ and observe that G contains a derangement of order r if r^2 does not divide $|G|$. It follows that G is r-elusive only if $r \leq 7$. Now G has a unique class of elements of order 7, so all such elements have fixed points if $|H|$ is divisible by 7. For $r \leq 5$, the relevant classes in G are the following:

$$2A, 2B, 2C, 2D,\ 3A, 3B,\ 5A, 5B.$$

The classification of the maximal subgroups of the Baby Monster was completed in [29]; in total, there are 30 conjugacy classes of maximal subgroups, and they are listed in [29, Table I]. The two main results in this section are the following:

Proposition 4.6. *Let r be a prime divisor of $|\Omega|$ and $|H|$. Then G is r-elusive if and only if $r \leq 7$ and (H, r) is one of the cases listed in Table 5.*

Proposition 4.7. *Suppose G contains a derangement of prime order $r \leq 5$, where r divides $|\Omega|$ and $|H|$. Then the G-classes of derangements of order r are listed in Table 6.*

First suppose H is one of the following subgroups:

$$\begin{array}{c} 2.{}^2E_6(2){:}2, \mathrm{Fi}_{23}, \mathrm{Th}, S_3 \times \mathrm{Fi}_{22}{:}2, \mathrm{HN}{:}2, \mathrm{P}\Omega_8^+(3){:}S_4, 3^{1+8}.2^{1+6}.\mathrm{U}_4(2).2 \\ 5{:}4 \times \mathrm{HS}{:}2, S_4 \times {}^2F_4(2), S_5 \times \mathrm{M}_{22}{:}2, 5^2{:}4S_4 \times S_5, \mathrm{L}_2(31), \mathrm{L}_2(17){:}2 \end{array} \qquad (2)$$

Then the character table of H (and of course G also) is available in the GAP Character Table Library [4], together with precise fusion information. Similarly, the character tables of the subgroups

$$(2^2 \times F_4(2)).2, (3^2{:}D_8 \times \mathrm{U}_4(3).2.2).2, (S_6 \times \mathrm{L}_3(4){:}2).2, \mathrm{L}_2(49).2_3$$

are known. Indeed, the character tables of $(2^2 \times F_4(2)).2$ and $\mathrm{L}_2(49).2_3$ are available directly in the GAP Character Table Library, while the other character tables can be constructed using the Magma implementation of the algorithm of Unger [27], and a suitable permutation representation of the subgroup (see [6, Proposition 3.3]). The fusion of H-classes in G can now be computed with the aid of GAP. In all of these cases, the desired results follow at once from the relevant fusion maps. Finally, if H is one of the following 2-local subgroups

$$2^{9+16}.\mathrm{Sp}_8(2), 2^{2+10+20}.(\mathrm{M}_{22}{:}2 \times S_3), [2^{30}].\mathrm{L}_5(2), [2^{35}].(S_5 \times \mathrm{L}_3(2))$$

then the character table of H is known and we can use character restriction to obtain precise fusion information (see [6, Proposition 3.4]).

In view of [29, Table I], it remains to deal with the following 8 subgroups:

$$\mathcal{A} = \left\{\mathrm{L}_2(11){:}2, \mathrm{L}_3(3), \mathrm{M}_{11}, (S_6 \times S_6).4, 5^{1+4}.2^{1+4}.A_5.4, 5^3.\mathrm{L}_3(5), [3^{11}].(S_4 \times 2S_4), 2^{1+22}.\mathrm{Co}_2\right\}$$

H	r		
$2.{}^2E_6(2){:}2$	$2,3$	HN:2	$3,7$
$2^{1+22}.\mathrm{Co}_2$	$3,5,7$	$\mathrm{P}\Omega_8^+(3){:}S_4$	$2,7$
Fi_{23}	7	$3^{1+8}.2^{1+6}.\mathrm{U}_4(2).2$	2
$2^{9+16}.\mathrm{Sp}_8(2)$	$3,5,7$	$(3^2{:}D_8\times \mathrm{U}_4(3).2.2).2$	$2,3,7$
Th	3	$5{:}4\times \mathrm{HS}{:}2$	$2,5,7$
$(2^2\times F_4(2)){:}2$	$2,3$	$S_4\times {}^2F_4(2)$	$2,3$
$2^{2+10+20}.(\mathrm{M}_{22}{:}2\times S_3)$	$3,7$	$S_5\times \mathrm{M}_{22}{:}2$	$2,3,5,7$
$[2^{30}].\mathrm{L}_5(2)$	$2,3,7$	$(S_6\times \mathrm{L}_3(4){:}2).2$	$2,3,5,7$
$S_3\times \mathrm{Fi}_{22}{:}2$	$2,3,7$	$(S_6\times S_6).4$	$2,3,5$
$[2^{35}].(S_5\times \mathrm{L}_3(2))$	$3,7$	$5^2{:}4S_4\times S_5$	$2,3,5$

Table 5: The r-elusive actions of $\mathbb{B}$

H	$r=2$	3	5
$2.{}^2E_6(2){:}2$			$5B$
Fi_{23}	$2C$		$5B$
Th	$2A,2B,2C$		$5A$
$(2^2\times F_4(2)){:}2$			$5B$
$2^{2+10+20}.(\mathrm{M}_{22}{:}2\times S_3)$			$5B$
$[2^{30}].\mathrm{L}_5(2)$			$5A$
$S_3\times \mathrm{Fi}_{22}{:}2$			$5B$
$[2^{35}].(S_5\times \mathrm{L}_3(2))$			$5B$
HN:2	$2A$		
$\mathrm{P}\Omega_8^+(3){:}S_4$			$5B$
$3^{1+8}.2^{1+6}.\mathrm{U}_4(2).2$			$5A$
$(3^2{:}D_8\times \mathrm{U}_4(3).2.2).2$			$5B$
$5{:}4\times \mathrm{HS}{:}2$		$3B$	
$S_4\times {}^2F_4(2)$			$5B$
$[3^{11}].(S_4\times 2S_4)$	$2B$		
$5^3.\mathrm{L}_3(5)$	$2A,2B,2C$	$3A$	
$5^{1+4}.2^{1+4}.A_5.4$	$2A$	$3A$	
$\mathrm{L}_2(49).2_3$	$2A,2B,2C$	$3A$	$5A$
$\mathrm{L}_2(31)$	$2A,2B,2C$	$3A$	$5A$
M_{11}	$2A,2B,2C$	$3A$	$5A$
$\mathrm{L}_3(3)$	$2A,2B,2C$	$3A$	
$\mathrm{L}_2(17){:}2$	$2A,2B,2C$	$3A$	
$\mathrm{L}_2(11){:}2$	$2A,2B,2C$	$3A$	$5A$

Table 6: Some derangements of order r in $\mathbb{B}$

Lemma 4.8. *Propositions 4.6 and 4.7 hold when $H \in \mathcal{A}$.*

Proof. Let π be the set of common prime divisors of $|H|$ and $|\Omega|$. If $\pi = \{r_1, \ldots, r_m\}$ then set $\kappa = [k_1, \ldots, k_m]$, where k_i is the number of H-classes of elements of order r_i.

First suppose $H = \mathrm{L}_2(11){:}2$, so $\pi = \{2, 3, 5\}$ and $\kappa = [2, 1, 2]$. By [29, Section 4], H contains $2D$, $3B$ and $5B$-elements. Using this, and computing possible fusions in GAP, we deduce that all elements in H of order 2 are in $2D$ and all 5-elements are in $5B$. Similarly, if $H = \mathrm{L}_3(3)$ then $\pi = \{2, 3\}$, $\kappa = [1, 2]$ and H contains $2D$-elements, but it does not meet $3A$ (see [29, Section 9]). For $H = \mathrm{M}_{11}$ we have $\pi = \{2, 3, 5\}$, $\kappa = [1, 1, 1]$ and [29, Section 5] reveals that H contains $2D$, $3B$ and $5B$-elements.

Next consider $H = (S_6 \times S_6).4$. Here $\pi = \{2, 3, 5\}$ and H is the unique subgroup of index two in $\mathrm{Aut}(S_6 \times S_6) = (S_6 \times S_6).D_8$ which maps onto the cyclic subgroup of D_8 of order 4. The two S_6 factors contain $5A$-elements, since $C_G(5B)$ does not contain S_6. Hence they also contain $3A$-elements. Also, [29] says there is a diagonal S_6 which contains $3B$ and $5B$ elements. By looking at the subgroup $D_{10} \times S_6.2$ in $D_{10} \times \mathrm{L}_3(4).2.2$ inside $D_{10} \times \mathrm{HS}{:}2$ we see that the involutions in $S_6.2$ fuse to classes $2A$, $2C$ and $2D$ in $\mathrm{L}_3(4).2^2$, which fuse to classes $2A$, $2C$ and $2D$ in HS:2. These correspond to elements of classes $10B$, $10A$ and $10E$ respectively in the Baby Monster, which power to $2B$, $2A$ and $2D$. Finally, according to [21, Table 4], the normaliser in the Monster of the $A_5 \times A_5$ in here is $\frac{1}{4}(D_8 \times S_5 \times S_5)$, which means it contains some elements of order 4 which square to the central involution of $2.\mathbb{B}$. Therefore H contains $2C$-elements.

Now suppose $H = 5^{1+4}.2^{1+4}.A_5.4$, in which case $\pi = \{2, 3\}$. If $x \in H$ has order 3 then x centralises a $5B$-element, so x is in class $3B$ and there are no $3A$-elements in H. Similarly, if $x \in H$ has order 2, then x either centralises or inverts a $5B$-element. In the former case it is in class $2B$ or $2D$, while in the latter case we count the dihedral groups using the structure constants. We find that $\xi(2A, 2A, 5B) = 0$ and $\xi(2C, 2C, 5B) \neq 0$, so x can lie in class $2C$ but cannot lie in class $2A$. Thus of the involution classes only the $2A$-class fails to meet H.

If $H = 2^{1+22}.\mathrm{Co}_2$ then $\pi = \{3, 5, 7\}$, $H = C_G(2B)$ and the character table of G indicates that a $2B$-element commutes with elements from all of the classes $3A$, $3B$, $5A$, $5B$, $7A$ and $7B$. Therefore there are no derangements of order 3, 5 or 7 in this case.

Next we turn to $H = 5^3.\mathrm{L}_3(5)$. Here $\pi = \{2, 3\}$ and $\kappa = [1, 1]$, so G clearly contains derangements of order 2 and 3. Since $\mathrm{L}_3(5)$ contains a unique class of involutions and a unique class of elements of order 3, the same is true in $5^3.\mathrm{L}_3(5)$, and both classes can be seen in the 5-element centraliser $5^3.5^2.\mathrm{SL}_2(5) = 5^{1+4}.\mathrm{SL}_2(5)$. In particular the elements of order 3 commute with $5B$-elements, so are in class $3B$. Looking in the full 5-centraliser $5^{1+4}.2^{1+4}.A_5$ we see that the involutions have centraliser $5 \times 2^{1+4}.A_5$ of order 9600, so correspond to elements of class $10F$ in the Baby Monster, which power into class $2D$.

Finally, let us assume $H = [3^{11}].(S_4 \times 2.S_4)$. Here $\pi = \{2\}$ and H is the normaliser of an elementary abelian 3^2. This 3^2 group embeds in the Monster, where its normaliser is of shape $3^{2+5+10}.(\mathrm{M}_{11} \times 2S_4)$, and we obtain the subgroup of the Baby Monster by centralising and then factoring out an involution in the M_{11}. In particular, H contains a subgroup $S_4 \times 3^2{:}2S_4$, in which the S_4 factor contains elements of classes $2A$ and $2C$ in the Baby Monster. But there is a unique conjugacy class of S_4 of this type in the Baby Monster, and such an S_4 has normaliser $S_4 \times {}^2F_4(2)$. Now the class fusion from $3^2{:}2S_4$ via $\mathrm{L}_3(3)$ and ${}^2F_4(2)$ to $F_4(2)$ reveals that the involutions are in $F_4(2)$-class $2D$. Now this class and all the diagonal classes of involutions fuse to $2C$ or $2D$ in the Baby Monster. □

4.3 The Monster

Let $G = \mathbb{M}$ be the Monster sporadic group, let H be a maximal subgroup of G and consider the action of G on $\Omega = G/H$. Let r be a prime divisor of $|\Omega|$ and note that G is r-elusive only

H	r		
$2.\mathbb{B}$	$2,5,11$	$(A_5 \times A_{12})$:2	$2,5,11$
$2^{1+24}.\mathrm{Co}_1$	$3,5,7,11$	$5^{3+3}.(2 \times \mathrm{L}_3(5))$	2
$3.\mathrm{Fi}_{24}$	$2,3,7,11$	$(A_6 \times A_6 \times A_6).(2 \times S_4)$	$2,3,5$
$2^2.{}^2E_6(2)$:S_3	$2,3,11$	$(A_5 \times \mathrm{U}_3(8)$:$3_1)$:2	$2,3$
$2^{10+16}.\Omega_{10}^+(2)$	5	5^{2+2+4}:$(S_3 \times \mathrm{GL}_2(5))$	2
$2^{2+11+22}.(\mathrm{M}_{24} \times S_3)$	$3,11$	$(\mathrm{L}_3(2) \times \mathrm{Sp}_4(4)$:2).2	2
$3^{1+12}.2\mathrm{Suz}.2$	$2,5,11$	7^{1+4}:$(3 \times 2S_7)$	3
$2^{5+10+20}.(S_3 \times \mathrm{L}_5(2))$	3	$(5^2$:$[2^4] \times \mathrm{U}_3(5)).S_3$	$2,3,5$
$S_3 \times \mathrm{Th}$	$2,3$	$(\mathrm{L}_2(11) \times \mathrm{M}_{12})$:2	$2,5$
$2^{3+6+12+18}.(\mathrm{L}_3(2) \times 3S_6)$	3	$(A_7 \times (A_5 \times A_5)$:$2^2)$:2	$2,5$
$3^8.\mathrm{P}\Omega_8^-(3).2_3$	2	5^4:$(3 \times 2\mathrm{L}_2(25))$:$2_2$	5
$(D_{10} \times \mathrm{HN}).2$	$2,5,11$	$\mathrm{M}_{11} \times A_6.2^2$	$2,5,11$
$(3^2$:$2 \times \mathrm{P}\Omega_8^+(3)).S_4$	$2,3$	$(S_5 \times S_5 \times S_5)$:$S_3$	$2,3,5$
$3^{2+5+10}.(\mathrm{M}_{11} \times 2S_4)$	$2,11$	$(\mathrm{L}_2(11) \times \mathrm{L}_2(11))$:4	$2,5$
$3^{3+2+6+6}$:$(\mathrm{L}_3(3) \times \mathrm{SD}_{16})$	2	$(7^2$:$(3 \times 2A_4) \times \mathrm{L}_2(7)).2$	$2,3,7$
5^{1+6}:$2\mathrm{J}_2$:4	2	$(13$:$6 \times \mathrm{L}_3(3)).2$	$2,3,13$
$(7$:$3 \times \mathrm{He})$:2	$2,3,7$	11^2:$(5 \times 2A_5)$	5

Table 7: Some r-elusive actions of $\mathbb{M}$

if $r \leq 13$. Now G has a unique class of elements of order 11, so all such elements have fixed points if 11 divides $|H|$. For $r \in \{2,3,5,7,13\}$, the relevant classes in G are the following:

$$2A, 2B,\ 3A, 3B, 3C,\ 5A, 5B,\ 7A, 7B,\ 13A, 13B$$

Recall that there are currently 43 known conjugacy classes of maximal subgroups of the Monster, and any additional maximal subgroup is almost simple with socle $\mathrm{L}_2(13)$, $\mathrm{L}_2(41)$, $\mathrm{U}_3(4)$, $\mathrm{U}_3(8)$ or $\mathrm{Sz}(8)$ (see [3, Section 1] and [22]). Let $\mathcal{K}$ be a set of representatives for the known maximal subgroups of G and let $\mathcal{U}$ be the additional set of undetermined maximal subgroups (up to conjugacy). For the primitive actions of the Monster corresponding to subgroups in $\mathcal{K}$ we prove the following results:

Proposition 4.9. *Let $H \in \mathcal{K}$ and let r be a prime divisor of $|\Omega|$ and $|H|$. Then G is r-elusive if and only if $r \leq 13$ and (H, r) is one of the cases listed in Table 7.*

Proposition 4.10. *Suppose G contains a derangement of prime order $r \leq 11$, where $H \in \mathcal{K}$ and r divides both $|\Omega|$ and $|H|$. Then the G-classes of derangements of order r are listed in Table 8.*

Remark 4.11. As remarked in the Introduction, Table 8 gives complete information on the classes of derangements of prime order, except for the case $H = 11^2$:$(5 \times 2A_5)$, where we are unable to determine the precise fusion of 3-elements. Here H has a unique class of such elements, which fuse to the Monster class $3B$ or $3C$.

We begin by considering the 43 known conjugacy classes of maximal subgroups in $\mathcal{K}$, which are conveniently listed in the Web Atlas. It is helpful to partition $\mathcal{K}$ into 5 subsets, $\mathcal{K}_i$ for $1 \leq i \leq 5$ as given in Table 9, which we deal with in turn. We define π and κ as in the proof of Lemma 4.8.

Lemma 4.12. *Propositions 4.9 and 4.10 hold for $H \in \mathcal{K}_1$.*

H	$r=2$	3	5	7	13
$2.\mathbb{B}$		$3C$		$7B$	$13B$
$2^{1+24}.\mathrm{Co}_1$					$13A$
$3.\mathrm{Fi}_{24}$			$5B$		$13B$
$2^2.{}^2E_6(2){:}S_3$			$5B$	$7B$	$13B$
$2^{10+16}.\Omega_{10}^+(2)$		$3C$		$7B$	
$2^{2+11+22}.(\mathrm{M}_{24}\times S_3)$			$5B$	$7B$	
$3^{1+12}.2\mathrm{Suz}.2$				$7A$	$13A$
$2^{5+10+20}.(S_3\times \mathrm{L}_5(2))$			$5A$	$7B$	
$S_3\times\mathrm{Th}$			$5A$	$7B$	$13B$
$2^{3+6+12+18}.(\mathrm{L}_3(2)\times 3S_6)$			$5B$	$7B$	
$3^8.\mathrm{P}\Omega_8^-(3).2_3$			$5B$	$7B$	$13B$
$(D_{10}\times\mathrm{HN}).2$		$3C$		$7B$	
$(3^2{:}2\times\mathrm{P}\Omega_8^+(3)).S_4$			$5B$	$7B$	$13B$
$3^{2+5+10}.(\mathrm{M}_{11}\times 2S_4)$			$5B$		
$3^{3+2+6+6}{:}(\mathrm{L}_3(3)\times\mathrm{SD}_{16})$					$13B$
$5^{1+6}{:}2\mathrm{J}_2{:}4$		$3A$		$7A$	
$(7{:}3\times\mathrm{He}){:}2$			$5B$		
$(A_5\times A_{12}){:}2$		$3C$		$7B$	
$5^{3+3}.(2\times\mathrm{L}_3(5))$		$3A, 3C$			
$(A_5\times\mathrm{U}_3(8){:}3_1){:}2$			$5B$	$7B$	
$5^{2+2+4}{:}(S_3\times\mathrm{GL}_2(5))$		$3A$			
$(\mathrm{L}_3(2)\times\mathrm{Sp}_4(4){:}2).2$		$3C$	$5B$	$7B$	
$7^{1+4}{:}(3\times 2S_7)$	$2A$		$5A$		
$(5^2{:}[2^4]\times\mathrm{U}_3(5)).S_3$				$7B$	
$(\mathrm{L}_2(11)\times\mathrm{M}_{12}){:}2$		$3C$			
$(A_7\times(A_5\times A_5){:}2^2){:}2$		$3C$		$7B$	
$5^4{:}(3\times 2\mathrm{L}_2(25)){:}2_2$	$2A$	$3A$			$13A$
$7^{2+1+2}{:}\mathrm{GL}_2(7)$	$2A$	$3B$			
$\mathrm{M}_{11}\times A_6.2^2$		$3C$			
$(\mathrm{L}_2(11)\times\mathrm{L}_2(11)){:}4$		$3C$			
$13^2{:}2\mathrm{L}_2(13).4$	$2A$	$3A, 3B$		$7A$	
$13^{1+2}{:}(3\times 4S_4)$	$2A$	$3A$			
$\mathrm{L}_2(71)$	$2A$	$3A, 3C$	$5A$	$7A$	
$\mathrm{L}_2(59)$	$2A$	$3A, 3C$	$5A$		
$11^2{:}(5\times 2A_5)$	$2A$	$3A, ?$			
$\mathrm{L}_2(29){:}2$	$2A$	$3A, 3C$	$5A$	$7A$	
$7^2{:}\mathrm{SL}_2(7)$	$2A$	$3A, 3B$		$7A$	
$\mathrm{L}_2(19){:}2$	$2A$	$3A, 3C$	$5A$		
$41{:}40$	$2A$		$5A$		

Table 8: Some derangements of order r in $\mathbb{M}$

$\mathcal{K}_1$	$2.\mathbb{B}, 2^{1+24}.\mathrm{Co}_1, 3.\mathrm{Fi}_{24}, 2^2.{}^2E_6(2).S_3, 13^{1+2}{:}(3 \times 4S_4), 3^{1+12}.2\mathrm{Suz}.2$
$\mathcal{K}_2$	$(A_5 \times A_{12}){:}2, (A_6 \times A_6 \times A_6).(2 \times S_4), (A_5 \times \mathrm{U}_3(8){:}3_1){:}2, 5^{2+2+4}{:}(S_3 \times \mathrm{GL}_2(5)),$ $(\mathrm{L}_3(2) \times \mathrm{Sp}_4(4){:}2).2, (5^2{:}[2^4] \times \mathrm{U}_3(5)).S_3, (\mathrm{L}_2(11) \times \mathrm{M}_{12}){:}2, (A_7 \times (A_5 \times A_5){:}2^2){:}2,$ $5^4{:}(3 \times 2\mathrm{L}_2(25)){:}2_2, 7^{2+1+2}{:}\mathrm{GL}_2(7), \mathrm{M}_{11} \times A_6.2^2, (S_5 \times S_5 \times S_5){:}S_3,$ $(\mathrm{L}_2(11) \times \mathrm{L}_2(11)){:}4, 13^2{:}2\mathrm{L}_2(13).4, (7^2{:}(3 \times 2A_4) \times \mathrm{L}_2(7)).2, (13{:}6 \times \mathrm{L}_3(3)).2,$
$\mathcal{K}_3$	$41{:}40, \mathrm{L}_2(19){:}2, 7^2{:}\mathrm{SL}_2(7), \mathrm{L}_2(29){:}2, 11^2{:}(5 \times 2A_5), \mathrm{L}_2(59), \mathrm{L}_2(71)$
$\mathcal{K}_4$	$7^{1+4}{:}(3 \times 2S_7), 5^{3+3}.(2 \times \mathrm{L}_3(5)), (7{:}3 \times \mathrm{He}){:}2, 5^{1+6}.2\mathrm{J}_2{:}4, 3^{3+2+6+6}{:}(\mathrm{L}_3(3) \times \mathrm{SD}_{16}),$ $3^{2+5+10}{:}(\mathrm{M}_{11} \times 4S_4), (3^2{:}2 \times \mathrm{P}\Omega_8^+(3)).S_4, 3^8.\mathrm{P}\Omega_8^-(3).2_3$
$\mathcal{K}_5$	$(D_{10} \times \mathrm{HN}).2, 2^{3+6+12+18}.(\mathrm{L}_3(2) \times 3S_6), S_3 \times \mathrm{Th}, 2^{5+10+20}.(S_3 \times \mathrm{L}_5(2)),$ $2^{2+11+22}.(\mathrm{M}_{24} \times S_3), 2^{10+16}.\Omega_{10}^+(2)$

Table 9: The $\mathcal{K}_i$ collections

Proof. The character table (and class fusions) of $H = 3^{1+12}.2\mathrm{Suz}.2$ has been calculated by Barraclough and Wilson (see [1, §4.1]), and the result quickly follows. In each of the other cases, the character table of H is available in the GAP Character Table Library [4], together with complete fusion information, and once again the result follows. ☐

Lemma 4.13. *Propositions 4.9 and 4.10 hold for $H \in \mathcal{K}_2$.*

Proof. In each of these cases the Web Atlas [32] provides a faithful permutation representation of H (see [3] for more details) and using GAP we can construct the character table of H and determine a short list of possibilities for the fusion of H-classes in G (via the command PossibleClassFusions). This information is sufficient to completely determine the relevant classes of derangements. ☐

Lemma 4.14. *Propositions 4.9 and 4.10 hold for $H \in \mathcal{K}_3$.*

Proof. Here we proceed as in the proof of the previous lemma, but in these cases some additional reasoning is required in order to determine the precise list of derangements.

First consider $H = 41{:}40$. Here $\pi = \{2, 5\}$, $\kappa = [1, 4]$ and the possible fusion information provided by GAP implies that all involutions are in $2B$. Arguing as in [22], we see that all 5-elements are in $5B$. (Note that in [21, §6] there is a purported, but incorrect, proof that they are in $5A$.) Similarly, if $H = \mathrm{L}_2(19){:}2$ then $\pi = \{2, 3, 5\}$, $\kappa = [2, 1, 2]$ and we see that all involutions are in $2B$ and all 3-elements are in $3B$. According to [21, Table 5], all 5-elements are in $5B$.

Next suppose $H = 7^2{:}\mathrm{SL}_2(7)$, so $\pi = \{2, 3, 7\}$ and $\kappa = [1, 1, 9]$. By [30], all 7-elements are in $7B$. The involutions invert 7-elements, so are in $2B$. Now it is shown in [21] that the only 3-elements which properly normalise a $7B$ are in class $3C$. If $H = \mathrm{L}_2(29){:}2$ then $\pi = \{2, 3, 5, 7\}$ and all involutions are in $2B$, since they invert elements of order 15. According to [21, Table 5], all 5-elements are in $5B$, while [21, §6] indicates that all 3-elements are in $3B$ and all 7-elements are in $7B$.

Now turn to $H = \mathrm{L}_2(59)$. Here we have $\pi = \{2, 3, 5\}$, $\kappa = [1, 1, 2]$ and by considering the possible fusion maps we see that all involutions are in $2B$. Further, according to [21, Table 5], all 5-elements are in $5B$, while all 3-elements are in $3B$ (see [21, §6]). Similarly, if $H = \mathrm{L}_2(71)$ then $\pi = \{2, 3, 5, 7\}$, $\kappa = [1, 1, 2, 3]$ and the partial fusion information obtained via the character table of H implies that all involutions are in $2B$, and all 3-elements are in $3B$. According to [21, Table 5], all 5-elements are in $5B$, while all 7-elements are in $7B$ (see [21, §6]).

Finally, let us assume $H = 11^2{:}(5 \times 2A_5)$. Here $\pi = \{2, 3, 5\}$, $\kappa = [1, 1, 14]$ and the possible fusion maps indicate that all involutions are in $2B$ and there are no derangements of order 5. We also deduce that the 3-elements are in $3B$ or $3C$, but we are unable to determine the precise fusion. □

Lemma 4.15. *Propositions 4.9 and 4.10 hold for $H \in \mathcal{K}_4$.*

Proof. In each case, the Web Atlas [32] (see also [3]) provides an explicit faithful permutation representation of H and we can use this to compute the number of conjugacy classes of order r in H (or classes of subgroups of order r). In a few cases, we can also construct the character table of H, but we were unable to compute possible class fusions.

First take $H = 7^{1+4}{:}(3 \times 2S_7)$, so $\pi = \{2, 3, 5\}$ and $\kappa = [2, 8, 1]$. Since a $7B$-element does not commute with any $5A$-element, the $2S_7$ contains $5B$-elements and thus elements in $5A$ are derangements. Now an involution which centralises or inverts a $7B$-element must be in class $2B$, so H has no $2A$ elements. The central 3 in $3 \times 2S_7$ is in $3C$, since it properly normalises a $7B$-element. Finally, using the embedding of $2S_7$ in $2^{1+8}.A_9$ inside the Thompson group we see that the $2S_7$ contains elements of both classes $3A$ and $3B$, so G is 3-elusive.

Next suppose $H = 5^{3+3}.(2 \times \mathrm{L}_3(5))$. Here $\pi = \{2, 3\}$ and $\kappa = [3, 1]$. By considering the maximal subgroup $5^3.\mathrm{L}_3(5)$ of $\mathbb{B}$ (see Section 4.2), we deduce that the 3-elements are in $\mathbb{B}$-class $3B$ and thus G-class $3B$. Again using $5^3.\mathrm{L}_3(5)$ in $\mathbb{B}$, the involutions are in $\mathbb{B}$-class $2D$ as we saw above. Therefore H contains elements of $\mathbb{M}$-class $2B$, as well as the central $2A$-element of $2 \times 5^3.\mathrm{L}_3(5)$.

If $H = (7{:}3 \times \mathrm{He}){:}2$ then $\pi = \{2, 3, 5, 7\}$ and $\kappa = [3, 8, 1, 9]$. Here the elements of order 5 are $5A$-elements, since they commute with the central $7A$-element of H'. By [30], H contains both $7A$ and $7B$ elements. By looking at powers of elements of order 14 we see that He already contains both $2A$ and $2B$ elements. In the inclusion of He in Fi_{24}, the 3-elements fuse to Fi_{24}-classes $3C$ and $3E$, which lift to all three $\mathbb{M}$-classes $3A$, $3B$, $3C$ in $3.\mathrm{Fi}_{24}$, so G is 3-elusive.

Next we turn to $H = 5^{1+6}.2\mathrm{J}_2{:}4$, where $\pi = \{2, 3, 7\}$ and $\kappa = [4, 2, 1]$. As stated in [23, §2.2], the G-classes of elements of order 3 which commute with a $5B$-element are $3B$ and $3C$, so there are no $3A$-elements in H. Since elements of both classes $2A$ and $2B$ commute with a $5B$-element, there are no derangements of order 2. The elements of order 7 which commute with a $5B$-element are in class $7B$, so the $7A$-elements are derangements.

Now assume $H = 3^{3+2+6+6}{:}(\mathrm{L}_3(3) \times \mathrm{SD}_{16})$, so $\pi = \{2, 13\}$ and $\kappa = [5, 4]$. We also note that H contains a unique class of subgroups of order 13, so G contains derangements of this order. Now H contains elements of order 104, which power to $13A$-elements and $2A$-elements. It also has the same Sylow 2-subgroup as $H = 3^{2+5+10}{:}(\mathrm{M}_{11} \times 2S_4)$, which contains elements of order 44 powering to $2B$ elements. Therefore both of these subgroups contain $2A$ and $2B$ elements. For the latter subgroup we have $\pi = \{2, 5, 11\}$, $\kappa = [5, 1, 2]$ and since the only class of $11{:}5$ in the Monster which contains $5B$-elements has centraliser of order 5, it follows that H contains elements of $5A$, so $5B$-elements are derangements.

The penultimate subgroup in $\mathcal{K}_4$ is $H = (3^2{:}2 \times \mathrm{P}\Omega_8^+(3)).S_4$. Here $\pi = \{2, 3, 5, 7, 13\}$ and $\kappa = [7, 17, 1, 1, 2]$. Using GAP, there is a unique fusion map for a subgroup $\mathrm{P}\Omega_8^+(3)$ of the Monster and we deduce that such a subgroup contains $3A$, $3B$, $5A$, $7A$ and $13A$ elements. Now the normaliser of a subgroup of order 13 in H is $3^2{:}2S_4 \times 13{:}6$, which contains elements of classes $3C$, $2A$ (centralising the 13-element) and $2B$ (inverting it), whence $5B$, $7B$ and $13B$ are the only relevant derangements.

Finally, let us assume $H = 3^8.\mathrm{P}\Omega_8^-(3).2_3$, where we have $\pi = \{2,5,7,13\}$ and $\kappa = [3,1,1,2]$. The elements of orders 5 and 7 lie in subgroups $(3^2{:}D_8 \times 2.\mathrm{U}_4(3).2.2).2$ inside the double cover of the Baby Monster, so are in classes $5A$ and $7A$ respectively. Now $\mathrm{P}\Omega_8^-(3)$ contains $2 \times \mathrm{L}_4(3)$, so the central involution in this must lift to a $2A$ element in H, and the 13-elements commute with this so are in class $13A$. It also contains $\mathrm{L}_2(81)$, whose involutions lift to $2B$-elements in H. □

Lemma 4.16. *Propositions 4.9 and 4.10 hold for $H \in \mathcal{K}_5$.*

Proof. First consider $H = (D_{10} \times \mathrm{HN}).2$. Here $\pi = \{2,3,5,7,11\}$ and using the Atlas character table, we see that the subgroup HN of H contains $2A$, $2B$, $3A$, $3B$, $5A$, $5B$ and $7A$ elements, but no $3C$ or $7B$ elements. The result follows.

Next suppose $H = 2^{3+6+12+18}.(\mathrm{L}_3(2) \times 3S_6)$, so $\pi = \{3,5,7\}$ and H has a unique class of subgroups of order 5 or 7. Since H contains $7 \times 2^6.3S_6$ the 7-elements are in class $7A$, and thus the 5-elements are in class $5A$. It now follows that H has the same Sylow 3-subgroup as the maximal subgroup $(7{:}3 \times \mathrm{He}){:}2$, so the proof of Lemma 4.15 indicates that there are no derangements of order 3.

If $H = S_3 \times \mathrm{Th}$ then $\pi = \{2,3,5,7,13\}$ and $\kappa = [3,7,1,1,1]$. Using the Atlas character table, we deduce that every copy of Th in G contains $5B$, $7A$ and $13A$ elements. It is well-known that the S_3 factor contains $2A$-elements and $3C$-elements. Hence Th contains $2B$-elements. Fusion via the Baby Monster shows that Th also contains $3A$ and $3B$ elements, so the relevant derangements here are $5A$, $7B$ and $13B$ elements.

Now consider $H = 2^{5+10+20}.(S_3 \times \mathrm{L}_5(2))$. Here $\pi = \{3,5,7\}$ and H has a unique class of subgroups of order 5 or 7. The subgroup $S_3 \times 2^5.\mathrm{L}_5(2)$ of H lies in $S_3 \times \mathrm{Th}$, so contains elements of class $3C$, $5B$ and $7A$. Moreover, its Sylow 3-subgroup lies in $S_3 \times 2^{1+8}.A_8$, from which we see that it has only elements of Th-classes $3A$ and $3C$. But these fuse to $3A$ and $3B$ respectively in G, so all three 3-classes are in H.

Similarly, if $H = 2^{2+11+22}.(\mathrm{M}_{24} \times S_3)$ then $\pi = \{3,5,7,11\}$ and H has a unique class of subgroups of order 5 or 7. Since there is a subgroup $2^{11}.\mathrm{M}_{24}$ of H inside $3\mathrm{Fi}_{24}$ we see that the 5-elements are in class $5A$ and the 7-elements are in class $7A$. Moreover, H has the same Sylow 3-subgroup as the maximal subgroups $H = 2^{3+6+12+18}.(\mathrm{L}_3(2) \times 3S_6)$ and $(7{:}3 \times \mathrm{He}){:}2$, so the above argument reveals that there are no derangements of order 3.

Finally, let us assume $H = 2^{10+16}.\Omega_{10}^+(2)$. Here $\pi = \{3,5,7\}$ and H has a unique class of subgroups of order 7. The full Sylow 3-, 5-, and 7-subgroups of H lie in $2^{10+16}.\mathrm{Sp}_8(2)$ which is inside $2.\mathbb{B}$, so we know that it contains elements from every Baby Monster class of elements of order $3,5,7$. Therefore it contains elements of Monster classes $3A$, $3B$, $5A$, $5B$ and $7A$, but does not contain elements of classes $3C$ or $7B$. □

To complete the proof of Theorem 1.1, it remains to deal with the additional collection $\mathcal{U}$ of undetermined maximal subgroups of the Monster. Some extra difficulties arise here and consequently we only provide partial information on the classes of derangements.

Proposition 4.17. *Let $H \in \mathcal{U}$ and let r be a prime divisor of $|\Omega|$ and $|H|$. Then G contains a derangement of order r, unless possibly $r = 3$ and H has socle $\mathrm{U}_3(8)$. In addition, some partial information on the classes of derangements in G is recorded in Table 10.*

Proof. Let H_0 denote the socle of H. Then by [3, Section 1], as corrected in [22], H_0 is one of the five groups

$$\mathrm{L}_2(13), \mathrm{L}_2(41), \mathrm{U}_3(4), \mathrm{U}_3(8), \mathrm{Sz}(8).$$

First assume $H_0 = \mathrm{L}_2(13)$, so $\pi = \{2,3,7,13\}$. By calculating all possible class fusions from the character table of H we deduce that all involutions in H are of type $2B$. In addition,

H_0	$r = 2$	3	5	7	13
$\mathrm{L}_2(13)$	$2A$	$3A, ?$		$7A$	?
$\mathrm{L}_2(41)$	$2A$	$3A, 3C$	$5A$	$7A$	
$\mathrm{U}_3(4)$	$2A$	$3A, 3B$	$5A$		?
$\mathrm{U}_3(8)$	$2A$	?		$7B$	
$\mathrm{Sz}(8)$	$2A$	$3A, ?$	$5A$	?	?

Table 10: Derangements for subgroups in $\mathcal{U}$

we observe that there are derangements of order 3, 7 and 13; in fact, there are two classes of derangements of order 3. Using [21] we get that the 7-elements are in class $7B$, and the 3-elements in either $3B$ or $3C$, but we are unable to determine the precise class here.

If $H_0 = \mathrm{L}_2(41)$, then the class fusion is determined in [22], and H contains elements in classes $2B$, $3B$, $5B$ and $7B$.

Next suppose $H_0 = \mathrm{U}_3(4)$. Here $\pi = \{2, 3, 5, 13\}$ and as before we deduce that all involutions are of type $2B$, there are two classes of derangements of order 3, and a class of derangements of order 13. From [21, Table 5] we see that the A_5 in H cannot contain $5A$-elements. From [21, Table 3] there is a unique conjugacy class of subgroups $5 \times A_5$ in the Monster in which the A_5 contains $5B$-elements, and in this case the other class of 5-elements also fuses to $5B$. Hence H contains $5B$-elements, but not $5A$-elements. We also see from the same source that the elements of order 3 are in class $3C$. At this stage we are unable to determine whether the 13-elements lie in $13A$ or $13B$.

Now consider $H_0 = \mathrm{U}_3(8)$. Here $\pi = \{2, 3, 7\}$ and the possible class fusion information reveals that all involutions in H are of type $2B$. Furthermore, using GAP we deduce that $H \neq \mathrm{U}_3(8).3_2$, $\mathrm{U}_3(8).3^2$, $\mathrm{U}_3(8).S_3$ or $\mathrm{U}_3(8).(S_3 \times 3)$ (there are no possible class fusions in each of these cases). According to [21], the subgroup $3 \times \mathrm{L}_2(8)$ of H contains an $\mathrm{L}_2(8)$ of type $(2B, 3B, 7A)$, which rules out the possibility that $H = \mathrm{U}_3(8).3_3$ since GAP tells us that such a subgroup would contain $7B$-elements. Therefore we have $H_0 \leqslant H \leqslant \mathrm{U}_3(8){:}6$ and all 7-elements lie in the Monster class $7A$. The remaining elements of order 3 in H_0 centralise the $\mathrm{L}_2(8)$, which contains $7A$-elements, so are either $3A$ or $3C$ elements. We are unable to determine this precisely.

Finally, let us assume $H_0 = \mathrm{Sz}(8)$, so $\pi = \{2, 3, 5, 7, 13\}$ (in fact, if $H = H_0$ then $3 \notin \pi$ since $|\mathrm{Sz}(8)|$ is not divisible by 3). In the usual manner, using GAP we observe that all involutions in H are of type $2B$. Moreover, there are derangements of order r for all $r \in \pi$, and all 3-elements in H (if there are any) lie in $3B$ or $3C$, so there are at least two classes of derangements of order 3, one of which is $3A$. Finally, according to [21], H does not contain $5A$-elements. We are unable at present to determine the precise fusion of elements of order 3, 5 and 7, as indicated in Table 10. □

This completes the proof of Theorem 1.1.

4.4 Derangements of prime-power order

In this final section we complete the proof of Theorem 1.4. Let r be a prime divisor of $|\Omega|$ and recall that G is said to be *strongly r-elusive* if it does not contain a derangement of r-power order. If there are no elements of order r^2 in G then the above analysis applies, so we will assume otherwise. Our main result is the following:

Proposition 4.18. *Let G be an almost simple sporadic group, let H be a maximal subgroup of G and let $\Omega = G/H$. Let r be a prime divisor of $|\Omega|$ and assume G contains elements of order r^2. Then G is strongly r-elusive if and only if (G, H, r) is listed in Table 11.*

G	H	r
M_{22}	M_{10}	2
M_{23}	M_{11}	2
M_{24}	$\mathrm{M}_{12}.2$	2
J_2	$3.A_6.2, \mathrm{L}_3(2){:}2$	2
Co_1	$3.\mathrm{Suz}{:}2, \mathrm{U}_6(2){:}S_3, 3^2.\mathrm{U}_4(3).D_8, A_9 \times S_3$	3
Co_2	$2^{1+8}{:}\mathrm{Sp}_6(2)$	3
Co_3	$S_3 \times \mathrm{L}_2(8){:}3$	3
	$\mathrm{McL}.2, \mathrm{U}_4(3).2^2, 3^{1+4}{:}4.S_6, \mathrm{L}_3(4).D_{12}, 2 \times \mathrm{M}_{12}$	2
McL	$\mathrm{U}_3(5), 3^{1+4}{:}2.S_5, 3^4{:}\mathrm{M}_{10}, \mathrm{M}_{11}, 5^{1+2}{:}3{:}8$	2
Suz	$(A_4 \times \mathrm{L}_3(4)){:}2$	2
He	$\mathrm{Sp}_4(4){:}2, 2^2.\mathrm{L}_3(4).S_3$	2
HN	$A_{12}, \mathrm{U}_3(8){:}3, 2^6.\mathrm{U}_4(2)$	3
	$5^{1+4}.2^{1+4}.5.4, (A_6 \times A_6).D_8, 3^4{:}2.(A_4 \times A_4).4$	2
HN.2	$S_{12}, \mathrm{U}_3(8){:}6, 2^6.\mathrm{U}_4(2).2$	3
Fi_{23}	$3^{1+8}.2^{1+6}.3^{1+2}.2S_4$	2
Fi_{24}'	$2.\mathrm{Fi}_{22}{:}2$	2
Fi_{24}	$(2 \times 2.\mathrm{Fi}_{22}){:}2$	2
J_3	$\mathrm{L}_2(17), (3 \times A_6){:}2_2, 3^{2+1+2}{:}8$	2
Ly	$2.A_{11}$	3
	$G_2(5), 5^{1+4}{:}4.S_6, 3^5{:}(2 \times \mathrm{M}_{11}), 3^{2+4}{:}2.A_5.D_8$	2
$\mathbb{B}$	Th	3
	$(2^2 \times F_4(2)){:}2$	2
$\mathbb{M}$	$2.\mathbb{B}$	$2, 5$
	$(D_{10} \times \mathrm{HN}).2$	5
	$3.\mathrm{Fi}_{24}, S_3 \times \mathrm{Th}$	3

Table 11: Some strongly r-elusive actions of sporadic groups

Proof. First assume $G \neq \mathbb{B}, \mathbb{M}$. If G is not one of the groups in (1) then the character tables of G and H are stored in the GAP Character Table Library [4], together with the relevant fusion maps, so these cases are straightforward. For the 6 remaining non-monstrous groups we proceed as before. First, using an explicit faithful permutation representation of G provided in the Web Atlas [32], we construct G as a permutation group in Magma. Next we construct the relevant maximal subgroups H of G (via the command MaximalSubgroups, or by using explicit generators for H listed in the Web Atlas). For each relevant prime r we can find representatives of the conjugacy classes in H containing elements of r-power order and it is straightforward to determine the fusion of these H-classes in G (by considering the cycle-structure of class representatives with respect to the underlying permutation representation of G, for example).

Next suppose $G = \mathbb{B}$. Here the relevant primes are $2, 3$ and 5, and the G-classes of interest are labeled as follows:

$$4A\text{–}4J,\ 8A\text{–}8N,\ 16A\text{–}16H,\ 32A\text{–}32D,\ 9A, 9B,\ 27A,\ 25A.$$

The relevant maximal subgroups are recorded in Table 5, and we consider each in turn. If H is one of the subgroups listed in (2) then the fusion map on H-classes is stored in the GAP Character Table Library and the result quickly follows. Similarly, if $H = (2^2 \times F_4(2)){:}2$, $(3^2{:}D_8 \times \mathrm{U}_4(3).2.2).2$, $(S_6 \times \mathrm{L}_3(4)){:}2$ or $[2^{30}].\mathrm{L}_5(2)$ then the character table of H is known and we can compute class fusions using GAP. (More precisely, for $H = (2^2 \times F_4(2)){:}2$ we calculate that there are 64 possible class fusion maps; in each case, H meets every G-class of elements of 2-power order, so G is strongly 2-elusive.) In each of the remaining cases it is easy to see that H does not contain any elements of order r^ℓ, where r^ℓ is the maximal order of an r-element of G.

Finally, let us assume $G = \mathbb{M}$. The relevant primes are $2, 3$ and 5, and the G-classes of interest are the following:

$$4A\text{–}4D,\ 8A\text{–}8F,\ 16A\text{–}16C,\ 32A, 32B,\ 9A, 9B,\ 27A, 27B,\ 25A.$$

Define the subgroup collections $\mathcal{K}$ and $\mathcal{U}$ as in Section 4.3. If $H \in \mathcal{U}$ then we may assume $r = 3$ and H has socle $\mathrm{U}_3(8)$. However, $\mathrm{Aut}(\mathrm{U}_3(8))$ does not contain any elements of order 27, so G is not strongly 3-elusive. For the remainder, we may assume $H \in \mathcal{K}$ is one of the cases listed in Table 7.

If $H = 2.\mathbb{B}$, $3.\mathrm{Fi}_{24}$ or $2^2.{}^2E_6(2){:}S_3$ then the corresponding fusion map is stored in the GAP Character Table Library; we find that the action corresponding to $H = 2.\mathbb{B}$ is strongly 2- and 5-elusive, while G is strongly 3-elusive when $H = 3.\mathrm{Fi}_{24}$. Next suppose $H = S_3 \times \mathrm{Th}$, so $r = 2$ or 3. Clearly, H does not contain any elements of order 32, so G is not strongly 2-elusive. Now H has 6 classes of elements of order 27; using GAP we calculate that there are 2 possible fusion maps from $S_3 \times \mathrm{Th}$ to G, and in each case 3 of the 6 classes containing 27-elements fuse to each of $27A$ and $27B$ in G. Therefore G contains no derangements of order 27, so G is strongly 3-elusive.

Next consider $H = (D_{10} \times \mathrm{HN}).2$, in which case the relevant primes are $r = 2, 5$. Now H contains elements of order 25, so G is strongly 5-elusive (since there is a unique G-class of such elements). However, H contains no elements of order 32, so G is not strongly 2-elusive.

Next suppose $H = (3^2{:}2 \times \mathrm{P}\Omega_8^+(3)).S_4$, so $r = 2, 3$. Here G is not strongly 2-elusive because H does not contain elements of order 32, so assume $r = 3$. The Web Atlas provides a faithful permutation representation of H on 3369 points and using this we calculate that H has two classes of elements of order 27. Moreover, there exists $x \in H$ of order 27 with $|C_H(x)| = 162$, so x is a $27A$-element (since $|C_{\mathbb{M}}(27A)| = 486$ and $|C_{\mathbb{M}}(27B)| = 243$). Next observe that all elements of 3-power order in H lie in a subgroup $(3^2{:}2 \times \mathrm{P}\Omega_8^+(3)).S_3$, which is a maximal subgroup of $3.\mathrm{Fi}_{24}$, so we have a chain

$$(3^2{:}2 \times \mathrm{P}\Omega_8^+(3)).S_3 < 3.\mathrm{Fi}_{24} < G$$

of maximal subgroups. We have already observed that $3.\mathrm{Fi}_{24}$ contains $27A$- and $27B$-elements (indeed, $(G, 3.\mathrm{Fi}_{24})$ is strongly 3-elusive), so we need to determine the fusion of 27-elements in $(3^2{:}2 \times \mathrm{P}\Omega_8^+(3)).S_3 < 3.\mathrm{Fi}_{24}$. If we consider the corresponding maximal subgroup $S_3 \times \mathrm{P}\Omega_8^+(3){:}S_3 < \mathrm{Fi}_{24}$, then with the aid of Magma it is easy to check that all 27-elements in $S_3 \times \mathrm{P}\Omega_8^+(3){:}S_3$ are in the $27A$-class of Fi_{24}. Consequently, every 27-element in $(3^2{:}2 \times \mathrm{P}\Omega_8^+(3)).S_3$ is in the $27A$-class of $3.\mathrm{Fi}_{24}$ and we conclude that there are no $27B$-elements in H. In particular, G is not strongly 3-elusive.

All of the remaining subgroups are easy to deal with. In most of these cases, a permutation representation of H is provided in the Web Atlas and using Magma it is easy to compute the conjugacy classes of H. In this way we easily deduce that H does not contain an element of order r^ℓ, where r^ℓ is defined as before, hence G is not strongly r-elusive. □

This completes the proof of Theorem 1.4.

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