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Abstract. This chapter focuses on the teachers' role in teaching proof and proving in the mathematics classroom. Within an over-arching theme of diversity (of countries, curricula, student age-levels, teachers' knowledge), the chapter presents a review of three carefully-selected theories: the theory of socio-mathematical norms, the theory of teaching with variation, and the theory of instructional exchanges. We argue that each theory starts by abstracting from observations of school mathematics classrooms. Each then uses those observations to probe into the teachers' rationality in order to understand what sustains those classroom contexts and how these might be changed. Here, we relate each theory to relevant research on the role of the teacher in the teaching and learning of proof and proving. Our review offers evidence and support for mathematics educators meeting the challenge of theorising about proof and proving in mathematics classrooms across diverse contexts worldwide.

Chapter 11

Proof, Proving, and Teacher-Student Interaction: Theories and contexts

Keith Jones and Patricio Herbst

1. Introduction

This chapter takes up the challenge of theorising about proof, proving, and teacher-student interactions in mathematics classrooms across diverse contexts around the world. We aim to contribute to what Hanna and de Villiers (2008, p. 331) identify as the need to review "what theoretical frameworks … are helpful in understanding the development of proof" and what Balacheff (2010, p. 133) argues is "the scientific challenge …to better understand the didactical characteristics" of proof and proving. The theme of the chapter is *the role of the teacher* in teaching proof and proving in mathematics, with a particular focus on theories that illuminate *teacher-student interaction* in the context of mathematics teachers' day-to-day instructional practice.

By using phrases like 'teacher-student interaction in the mathematics classroom' and 'the teaching of proof in the context of the day-to-day instructional practice of teachers', we are deliberately choosing to avoid terms such as *pedagogy* or *didactics*. Both terms come with significant theoretical baggage and neither is unproblematic in English. As Hamilton (1999), for example, shows, some Anglo-American usage of the term *pedagogy* mirrors, in many ways, the use of term

K. Jones (✉)
Mathematics and Science Education Research Centre, School of Education,
University of Southampton, Highfield, Southampton, UK
e-mail: d.k.jones@soton.ac.uk
P. Herbst
School of Education, University of Michigan, Ann Arbor, MI, USA
e-mail: pgherbst@umich.edu

didactics in mainland Europe (c.f., Best, 1988; Chevallard, 1999a; Murphy, 2008). The word *instruction*, as used by Cohen, Raudenbush and Ball (2003) to refer to the interactions among teacher-students-content in classroom environments, is probably a better word to designate the locus of the phenomena we target. In focusing on teacher-student interaction, we acknowledge that what learners bring to the classroom (from developmental experiences prior to schooling, to ongoing experiences across varied out-of-school contexts) impacts on such interactions, just as, most certainly, can the diversity of countries, of instructional courses, of student ages, of levels of teacher knowledge, and so on, around the world. Whatever the terminology, our over-arching focus is on *the teacher* – and, in particular, on the teacher's part in the *teacher-student interactions* that occur day-to-day in mathematics classrooms.

In theorising about proof, proving, and teacher-student interaction, we are aware that theories can appear in different guises and operate at different levels and grain sizes. As Silver and Herbst (2007) identify in their analysis, there can be "grand theories", "middle-range theories", and "local theories": where "grand theories" aim at the entirety of phenomena within, say, mathematics education; "middle-range theories" focus on subfields of study; and "local theories" apply to specific phenomena within the field. We also note Kilpatrick's (2010, p. 4) observation: "To call something a theory … is an exceedingly strong claim". It is not our intention to consider whether or not some proposed approach is, or is not, a "theory"; rather, we use the term "theory" as short-hand for 'theoretical framework', 'theoretical perspective', 'theoretical model', or other equivalent terms.

Across all these considerations, we take proof and proving to be “an activity with a social character” (Alibert & Thomas, 1991, p. 216). As such, mathematics classroom communities involve students in communicating their reasoning and in building norms and representations that provide the necessary structures for mathematical proof to have a central presence. Hence, our focus on the role of the teacher in teaching proof and proving in mathematics encompasses the teacher managing the work of proving in the classroom even when proof itself is not the main object of teaching. Clearly, in such situations proofs may be requested, and offered, even when proof itself is not the object of study; such possibilities hinge on customary practices (including matters of language) that the teacher has the responsibility to establish and sustain. Balacheff (1999), Herbst and Balacheff (2009) and Sekiguchi (2006), for examples, have studied these forms of classroom practices, and the role of the teacher in establishing and sustaining the practices.

As Balacheff (2010, p. 116-117) shows, basing classroom practices on "grand theories" such as those of Piaget or Vygotsky has not worked very well. Balacheff argues "The responsibility for all these failures does not belong to the theories which supposedly underlie the educational designs, but to naive or simplifying readers who have assumed that concepts and models from psychology can be freely transferred to education". Balacheff goes on to consider the didactical complexity of learning and teaching mathematical proof by analyzing the gap between knowing mathematics and proving in mathematics. In contrast, our approach in selecting relevant theories to review is to choose ones that represent ongoing and current foci for classroom-based research and, importantly, that start from the abstraction of observations in existing school mathematics classrooms. Using these criteria, we review the *theory of socio-mathematical norms*, the *theory of teaching with variation*, and the *theory of instructional exchanges*. We conclude by giving pointers to future research - both empirical and theoretical - that we hope can advance the field.

2. Teaching proof and proving in diverse contexts

The contexts within which proof and proving are taught around the world vary enormously in terms of curriculum specification, student age-level, teacher knowledge, and so on. In this connection, Stigler and Hiebert (1999) have argued that the teaching of mathematics lessons in different countries follows different lesson scripts. Furthermore, Clarke, Emanuelson, Jablonka and Mok (2006, p. 1) report on “the extent to which students are collaborators with the teacher…. in the development and enactment of patterns of participation that reflect individual, societal and cultural priorities and associated value systems”. Such research recognises the impact that diversity worldwide can have on the form of instructional courses in mathematics, on the student age-levels at which educational ideas of proof and proving are introduced, on the scale and nature of teachers' mathematical knowledge, and so on.

Hoyles (1997) uses the term *curricular shaping* to refer to the ways in which school and curriculum factors influence and shape students’ views of, and competency in, proof and proving in mathematics. Knipping's (2002; 2004) research comparing classroom proof practices in France and Germany stands out as an attempt to understand the role of culture in shaping classroom proof and proving practices. Other studies include the work of Jones and colleagues on the teaching of proof in geometry at the lower secondary school level in the countries of China, Japan and the UK, some of which is summarised in Jones, Kunimune, Kumakura, Matsumoto, Fujita and Ding (2009) and Jones, Zheng and Ding (2009).

Within this diversity in the teaching of proof and proving, we can nevertheless discern some common elements. Proof in elementary school, for example, is generally viewed in terms of informal reasoning and argumentation. In middle school, students continue exploring proof as argumentation while at the same time being exposed to forms of symbolic notation and representation. At the high school level, proofs begin to take on a more formalised character, often (but not always) within topics in geometry – and in some places in a manner commonly called two-column proofs (e.g., Herbst, 2002a; Weiss, Herbst & Chen, 2009). For an international overview of proof and proving across the stages of education, see, for example, Ball, Hoyles, Jahnke and Movshovitz-Hadar (2002).

Given such diversity, building theory that might help us understand and explain the teacher's role in the classroom teaching of mathematical proof and proving is a complex proposition. In this context, in the next section we consider three carefully-selected theories of mathematics teacher-student interaction in more detail, focusing on their relevance to proof and proving.

3. Theories of teacher-student interaction

3.1 *Introduction*

Mathematics education includes a range of theories that in one way or another concern themselves with proof and proving. As Silver and Herbst (2007) note, mathematics education theories can be classified by their 'grain size'. Some are *grand theories*; theories that attempt to organize the whole field, like Chevallard's (1999b) *theorie anthropologique du didactique* within which it would be possible to give an account of proof and the work of the teacher. Others are *local theories*; they take on specific roles articulating the relationships between problems, research, and practice. An example can be found in Martin and Harel's (1989) study of prospective elementary teachers, where the authors theorize about 'inductive verification types' and 'deductive verification types' to design an instrument they use to study participants' views of proof. Yet a third class of theory is what Merton (see Silver and Herbst, 2007) termed a *middle range* theory; this starts from an empirical phenomenon, rather than with broad organizing concepts, and builds up abstract concepts from the phenomenon while accumulating knowledge about the phenomenon through empirical research. Our three examples below – the theory of *socio-mathematical norms*, the theory of *teaching with variation*, and the theory of *instructional exchanges* – are all middle range theories.

3.2 *The theory of socio-mathematical norms*

The notion of *socio-mathematical norm* is a component of what Cobb and Bauersfeld (1995) term an "emergent theory" (in that, in coordinating individual and group cognitions within classroom settings, it seeks systematically to combines various "mini-theories"). The theory of *socio-mathematical norms* aims to describe and explain the construction of knowledge in inquiry-based mathematics classrooms (Cobb, Wood, Yackel, & McNeal, 1992) by complementing a constructivist account of how individuals learn with a sociological account of those classrooms where teachers promote learning by inquiry. Taking the notion of *norm* (as conceptualized by Much and Schweder, 1978), Cobb and his colleagues made the observation that students engage in acts of challenge and justification during the process of holding each other accountable for their assertions. The authors proposed that the notion that learners should justify their assertions constituted a social norm in the observed inquiry-based classrooms.

Voigt (1995) and Yackel and Cobb (1996) then argued that teachers, in their role as representatives of the discipline of mathematics in the classroom, could promote *socio-mathematical norms* associated with those social norms. In particular, teachers could promote normative understandings of what counts as an appropriate mathematical justification. In proposing this theory of *socio-mathematical norms*, Yackel and Cobb (1996) provided means to understand how the notion of a proof as an explanation accepted by a community at a given time could result from the interaction and negotiation among individuals who are both adapting their cognitive schemes in the face of perturbations and responding to the values and practices of the discipline of mathematics. Specific studies, such as that by Sekiguchi (2006) have shown how it is possible to track the development of a socio-mathematical norm for what counts as a proof in an inquiry-based classroom.

Martin, McCrone, Bower and Dindyal (2005), in their study of the interplay of teacher and student actions in the teaching and learning of geometric proof, use the notion of socio-mathematical norms to show how the teacher's instructional choices are key to the type of classroom environment that is established and, hence, to students' opportunities to hone their proof and reasoning skills. More specifically, Martin et al. (2005) argue that in order to create a classroom climate in which participating students make conjectures, provide justifications, and build chains of reasoning, the teacher should "engage in dialogue that places responsibility for reasoning on the students, analyze student arguments, and coach students as they reason" (Martin, McCrone, Bower & Dindyal, 2005, p. 95). These instructional choices create a classroom environment in which teacher and students can negotiate socio-mathematical norms such as what counts as an acceptable proof.

This emergent theory with its notion of socio-mathematical norms exemplifies a middle-range theory. In an effort to characterize inquiry-based mathematics classrooms, it uses microanalysis of classroom interactions to track the development of shared norms of classroom mathematics practice. The notion of socio-mathematical norm results from abstracting the directions toward which teachers push classroom norms through social negotiation, not only of what is acceptable mathematical justification but also of other mathematical values.

3.3 *The theory of teaching with variation*

In the 1990s, the theory of *teaching with variation* emerged from two different, though related, academic fields: the work of Gu (1992; 1994) in mathematics education in mainland China, and the work of Marton (Marton, 1981; Marton & Booth, 1997) on phenomenology in Sweden. The meeting of these two ideas in the form of the theory of *teaching with variation* is presented by Gu, Huang, and Marton (2004); see also Ko & Marton (2004, especially pp. 56-62). In this section we review the theory of *teaching with variation* and illustrate how it is beginning to be applied to studies of proof and proving in the mathematics classroom.

Teaching with variation has a long tradition in mainland China. For example, Kangshen, Crossley and Lun (1999), in their presentation of *Jiuzhang Suanshu* or *The Nine Chapters on the Mathematical Art* (a Chinese mathematics corpus compiled by several generations of scholars from the 10th century BCE to the 1st century CE) document the use of methods of varying problems dating back at least 2000 years. In contemporary school classrooms in China, Gu (1992; 1994) conducted a large-scale study that examined how mathematics teachers varied the tasks that they used with their students. At about the same time, Marton and colleagues (Marton & Booth, 1997) were focusing on the variation in ways in which people are capable of experiencing different situations or phenomena.

In the *theory of teaching with variation* (in Chinese, *bianshi*– see Bao, Huang, Yi & Gu, 2003a; 2003b; 2003c; Sun & Wong, 2005), classroom teaching is seen as aiming to promote learning through the students experiencing two types of variation deemed helpful for meaningful learning of mathematics. Gu et al. (2004) classified the first form of variation as *conceptual variation*, in which the teacher highlights the key features of a mathematical concept by contrasting examples of the concept with counter- or non-examples. The teacher aims thus to provide students with multiple experiences of the selected mathematical concept from different perspectives. The second form of variation, called *procedural variation*, is the process of forming concepts not from different perspectives (as in *conceptual variation*) but through step-by-step changes. An example of *procedural variation* provided by Gu et al. (2004, p. 320-321) concerns the concept of equation. With *procedural variation*, the teacher might begin with examples of representing an unknown by concrete items, such as when solving a problem involving the purchase of three pencils. The next step might be the use of symbols in place of the concrete items. A third step might be fully symbolic.

However, Park and Leung (2006) argue that the terms *conceptual variation* and *procedural variation* may not be the best way of capturing how contemporary mathematics teachers in China promote student learning through teaching with variation. A key reason, even according to Gu et al.'s own definition, is that *procedural variation* is also related to the formation of mathematical concepts for learners. As such, Park and Leung suggest replacing *conceptual variation* with *multi-dimensional variation* (thus capturing teaching through multiple representations) and *procedural variation* with *developmental variation* (since students learn to construct concepts through step-by-step acquisition). Sun (2011) adds that other Chinese researchers use still other terms (e.g., *explicit variation*, *implicit variation*, *form variation*, *solution variation*, and *content variation*).

Whatever the chosen terminology, Gu et al. (2004) helpfully provide a diagram (Fig 11.1) to illustrate how a teacher structures a series of classroom problems through the use of variations. The variations serve as means to connect something the learners know how to solve (the known problem) to something that they are to solve (the unknown problem). Through this way of varying problems in class, ‘‘students’ experience in solving problems is manifested by the richness of varying problems and the variety of transferring strategies’’ (Gu et al., 2004, p. 322).

To date, a number of researchers have used the *theory of teaching with variation* to analyse mathematics teaching. Some have aimed to provide an account of mathematical problem-solving in Chinese mathematics teaching (e.g., Cai & Nie, 2007; Wong, 2002), while others utilise *teaching with variation* when accounting for the classroom teaching of mathematics (e.g., Huang & Li, 2009; Park & Leung, 2006). Some research is beginning to use the theory of variation to research the teaching and learning of proof and proving. For example, Sun (2009) and Sun and Chan (2009) provide reports illustrating that the teaching approach of using 'problem variations with multiple solutions' (where one

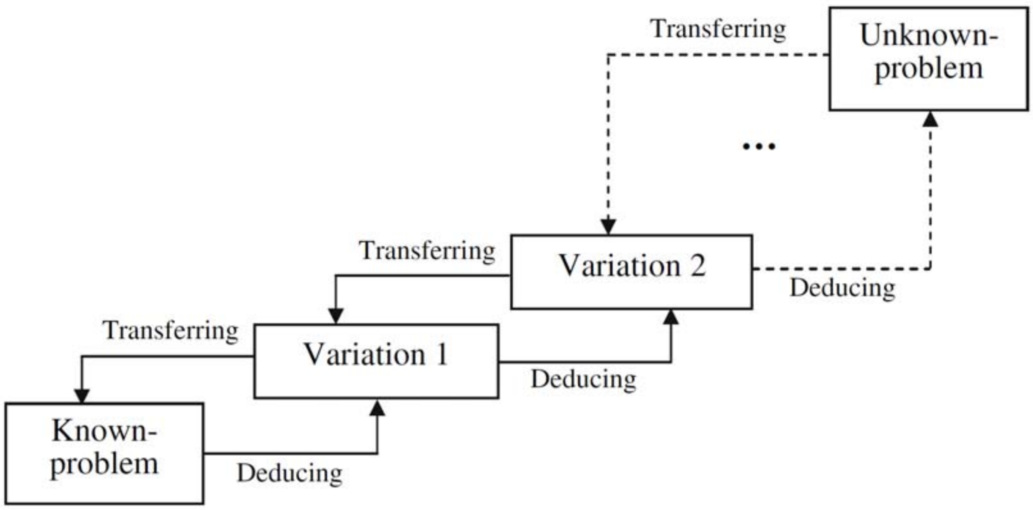

Fig 11.1 Variation for solving problems [source: Gu et al., 2004, p. 322]

problem has a number of solutions) successfully helped older students to reconstruct their own proof solutions by regenerating their past proving experience.

Ding and Jones (2009) and Jones, Zheng and Ding (2009) report on the instructional practices of a sample of expert teachers of geometry at Grade 8 (pupils aged 13-14) in Shanghai, China. From an analysis of the data collected, the research found that two factors characterise the instructional strategies used by the teachers to help their students to understand the discovery function of proof in geometry: the *variation of mathematical problems*, and the *variation of teaching questions*. In the variation of mathematical problems, the teachers started by guiding their students to understand the principles of a '*problem to find*' in order to begin engaging them in seeking the logical connections to the principal parts of a '*problem to prove*'. The data also provided evidence of the variation of teacher questions, in which the teacher used questions both to encourage students to formulate plausible reasons for the properties and relations of a chosen geometric figure and to increase students' awareness of the discovery function of deductive proof.

While such studies provide a start, we need more empirical data on using the theory of teaching with variation. Researchers such as Mok, Cai and Fong Fung (2008) have raised the issue that while students might master the target mathematical ideas being taught, teaching with variation can mean that they miss opportunities for independent exploration. When the difference between one variation and another is rather small, the students have little room to think independently. Thus, teaching with variation does not *necessarily* lead to the full development of mathematical competency (c.f. Huang, Mok & Leung, 2006).

Similarly, the type of engagement the teacher creates in the lesson may fail to foster students' higher-order thinking in terms of proof and proving. We need further research on instructional designs that use teaching with variation to develop the appropriate range of mathematical skills and approaches. One such avenue for research is on how teachers can provide for students' mathematical exploration in a way that supports proof and proving while not limiting the students' thinking by making the variations of problems too small.

3.4 *The theory of instructional exchanges*

The *theory of instructional exchanges* proposed by Herbst, and based on prior work by Brousseau (1997) and by Doyle (1988) on the study of classrooms, is a descriptive theory of the role of the teacher in classroom instruction. It is another example of a "middle range theory" (Silver & Herbst, 2007) in that it does not attempt to account for all practices related to mathematical thinking, learning, and teaching but rather concentrates on understanding the phenomena associated with the teaching and learning of prescribed knowledge in school classrooms as they exist (i.e. not only of 'inquiry-type' mathematics classrooms). It proposes that mathematics instruction proceeds as a sequence of exchanges or transactions between, on the one hand, the moment-to-moment, possibly interactive, work that students do with their teacher and, on the other hand, the discrete claims a teacher can lay on what has been accomplished.

Central to this theory of instructional exchanges is the notion of *didactical contract* (Brousseau, 1997): the hypothesis that a bond exists that makes teacher and students mutually responsible vis-à-vis their relationships with knowledge; in particular, a contract that makes the teacher responsible for attending not only to the students as learners of mathematics but also to mathematics as the discipline that needs to be represented to be learned. Particular classrooms may have specific customary ways of negotiating and enacting that contract and these may vary quite a bit, but in general these various ways will always amount to establishing the teacher's accountability not only to the students but also to the discipline of mathematics.

A second, related hypothesis that is helpful when analysing the teacher's instructional work derives from the observation that classroom activity takes place over multiple timescales. For example, while meaningful classroom interactions (e.g., utterances) can be detected at a timescale of the fraction of a second, progress in the syllabus and consequent examinations take place over a larger timescale of weeks and months. Thus, the second key hypothesis is that the work of the teacher includes managing activities and objects within two different timescales: the work done moment-to-moment (at the scale of the utterance) and the mathematical objects of knowledge that exist at the larger scales of the week, month, or year-long curriculum (Lemke, 2000, p. 277). In other words, the teacher needs to operate symbolic transactions or exchanges between activities in one timescale and objects of knowledge in the other: moment-to-moment activities serve the teacher to deploy or instantiate large-scale mathematical objects of knowledge; reciprocally, objects of knowledge serve to account for the moment-to-moment activities.

Herbst (2003; 2006) has proposed two basic ways in which that exchange can be facilitated. One, “negotiation of task”, describes how a teacher needs to handle 'novel' tasks, ones that are completely new to the students. In these tasks the teacher needs to engage students in identifying, perhaps deciding upon, how the didactical contract applies to the task at hand. In particular, the negotiation includes identifying what aspects of the task embody the target knowledge and what aspects of students' work on that task attest that they are learning the knowledge or know it already.

The second way in which that exchange is facilitated is by 'default to an instructional situation', by framing the exchange according to norms that have framed other exchanges (possibly set up previously through negotiation). In this case, the work done is not one of identifying the mathematics in the task as much as identifying the situation, or cueing into the situation, by acting in compliance with the norms that constitute the situation. Thus, the situation frames that exchange, saving the effort of having to negotiate what needs to be done and what is at stake.

Negotiation of task, and default to a situation, are two 'ideal types' (in the Weberian sense) of teacher-student interaction about content. In practice, there would always be some amount of default and some amount of negotiation of how to handle breaches to the default situation. Nevertheless, this theorization helps describe how regularities in interaction about content structure much of the workings of the didactical contract. More importantly, the hypothesis explains that novelty is constructed against a background of customary situations; specifically, that novel interaction is constructed by negotiating how to handle a breach in a customary situation.

Some of the tasks in which students might engage, and which (according to the second hypothesis above) the teacher needs to exchange for items of knowledge, involve mathematical moves like those identified by Lakatos (1976) as part of the method of proofs and refutations. Those operations could include deriving a logical consequence from a given statement; proposing a statement whose logical consequence is a given statement; reducing a given problem into smaller problems whose solutions logically entail that problem’s solution; bringing new, warranted mathematical objects to a problem in order to translate or reduce the problem; translating strings of symbols into other, equivalent, strings of symbols; operating on one set of objects as if they behaved like other similar set of objects, and so forth. Hence, the mathematical work of proving involves a host of actions that students could perform as transient moves when working on tasks, and for which, the theory anticipates, a teacher might need to find exchange values within the elements of the target knowledge.

In the US high school curriculum, as well as in other countries, proof has traditionally appeared as an element of target knowledge in the context of the study of Euclidean geometry (González & Herbst, 2006; Herbst, 2002b). Teachers of geometry create work contexts in which students have the chance to experience, learn, and demonstrate knowledge of 'proof'. The notion of an instructional situation as a 'frame' (a set of norms regulating who does what and when) for the exchange between work done and knowledge transacted was initially exemplified in what Herbst and associates called the 'doing proofs' situation (Herbst & Brach, 2006; Herbst, Chen, Weiss, González et al., 2009). That work of modelling classroom interaction as a system of norms produced the observation that many of the operations in the work of proving (e.g., those listed in the previous paragraph) are not accommodated in classroom work contexts where knowledge of proof is exchanged. In other words, 'doing proofs' has become a stable work context where students can learn some of the work of proving but this, at the same time, excludes other important mathematical actions of proving, perhaps by exporting them to other instructional situations where they are disconnected from the functions of proof in the discipline of mathematics.

An important question is whether the practical rationality (Herbst & Chazan, 2003) that underpins the teacher's work contains resources that could be used to give value to classroom work that embodies the different functions of proof in mathematics (which contain all the actions that constitute the work of proving). To study that rationality, Herbst and Chazan, and their associates (see http://grip.umich.edu/themat) have created classroom scenarios (complete with animated cartoon characters) depicting mathematical work that create contexts for the work of proving; the latter is sometimes explicitly executed and other times glaringly absent. The researchers have used those animations to engage groups of geometry teachers in conversations about instruction. They have found that, as a group, teachers have resources to justify positive appraisals of certain elements of the work of proving: the use of an unproven conjecture as a premise in proving a target conclusion; the identification of new mathematical concepts and their properties from objects introduced and observations made in justifying a construction; the deductive derivation of a conditional statement connecting two concomitant facts about a diagram; the prediction of an empirical fact by operating algebraically with symbols representing the quantities to be measured; the breaking up of a complicated proof problem into smaller problems (lemmas); the application of a specific proving technique (e.g., reduction to a previously proven case); and the establishment of equivalence relationships among a set of concomitantly true statements.

Herbst, Miyakawa and Chazan (2010) have proposed that teachers might use the various functions of mathematical proof documented in the literature (e.g., verification, explanation, discovery, communication, systematization, development of an empirical theory, and container of techniques) (de Villiers, 1990; Hanna & Barbeau, 2008; Hanna & Jahnke, 1996) to attach contractual value to actions like those listed above. There remain two questions; whether classroom exchanges are possible (manageable) between these actions and the elements of currency; and whether the exchanges can be contained within instances of the 'doing proofs' situation or otherwise whether they require more explicit negotiations of the didactical contract. The theory of instructional exchanges thus illustrates another middle range theory that starts from abstracting from observations in mathematics classrooms where there has been no special instructional intervention (in other words, *intact* mathematics classrooms) and uses those observations to probe into how teachers manage and sustain those work contexts and also how these might be changed.

4. Directions for future research

The development of each of the three theories above began with abstraction from observations of mathematics classrooms. Simon (1987, p. 371) wrote that pedagogy (or didactics) is:

> the integration in practice of particular curriculum content and design, classroom strategies and techniques, and evaluation, purpose and methods. All of these aspects of educational practice come together in the realities of what happens in classrooms. Together they organize a view of how a teacher's work within an institutional context specifies a particular version of what knowledge is of most worth, what it means to know something, and how we might construct representations of ourselves, others and our physical and social environment.

This passage, famously taken up by McLaren (1998, p. 165), returns us not only to the complexity of developing theory about the role of the teacher in the teaching and learning of proof and proving, but also the diversity of contexts within which proof and proving are taught around the world – for example, in terms of curriculum specification, student age-level, teacher knowledge and so on.

Pollard (2010, p. 5) offers the representation in Fig. 11.2 (slightly amended here) as a way of capturing teacher-student interaction as a science, a craft and an art. This representation might point to a way to take into account the complexity and diversity of classroom teaching strategies when "All of these aspects of educational practice come together in the realities of what happens in classrooms" Simon (1987, p. 371).

Of the three theories reviewed here, the theory of teaching with variation (Gu et al., 2004) appears closer to "craft" (the "craft" vertex of the triangle in Fig. 11.2) than the other two, in that teaching with variation entails teacher mastery of an appropriate repertoire of classroom teaching skills and processes. In Cobb and colleagues' (Cobb & Bauersfeld, 1995; Cobb et al. 1992) account of teaching in different mathematical traditions and its use of the idea of socio-mathematical norms to examine the work that a mathematics teacher does in an inquiry-based approach to teaching, this encompasses a responsive and creative capacity, a way in which the teacher responds both to mathematical demands and to students' cognitive demands at the same time. As such, the theory of socio-mathematical norms might fit with the "art" vertex of the triangle (Fig. 11.2). The theory of instructional exchanges, with its pretence of universality (to describe all kinds of

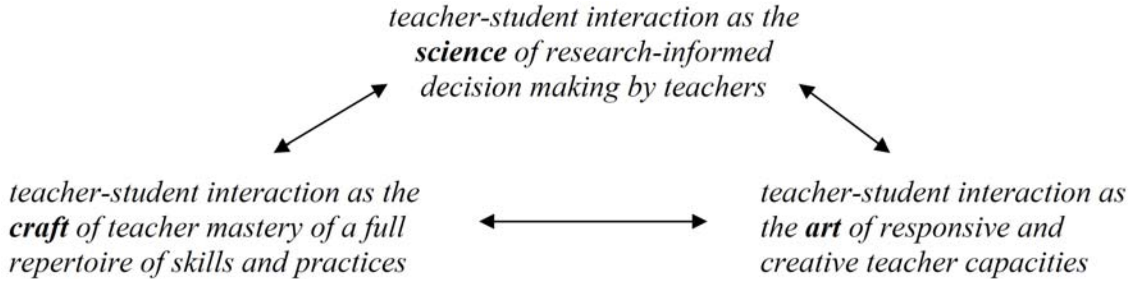

Fig 11.2. Teaching as a science, a craft and an art [adapted from Pollard, 2010, p. 5]

teaching and to focus on concepts that are general enough to describe all observations) may come closer to the "science" vertex.

All three theories are *middle range theories*; and it remains an open question whether a *grand theory* of the teacher's role in teaching proof and proving in mathematics is a reasonable longer-term goal, especially in terms of accounting for the nature of teacher-student interaction. Pollard's (2010, p. 5) representation (Fig. 11.2) may provide some ideas towards a way of encompassing all the complexity and diversity of classroom proof teaching.

It is worth reflecting on Sfard's (2002) caution about the over-proliferation of theories. Prolific theorizing may signify a "young and healthy scientific discipline" (p. 253), but, in contrast, it may mean that "theories are not being sufficiently examined, tested, refined and expanded" (op cit.). Sfard elaborates that "one of the trademarks of a mature science is that it strives for unity; that it directs its collective thought toward unifying theories and frameworks" (op cit.), at the same time noting that this is "neither a quick nor an easy process" (op cit.). As to directions for future research on proof, proving, and teacher-student interaction in the mathematics classroom, we list Sfard's challenges for the mathematics education community as ones which might inform further research work:

- "To carry out research studies within frameworks determined by existing theories with the intention to establish the range of applicability or validity or usefulness of these theories.
- To carry out comparative surveys of several theories, in particular of theories that purport to provide frameworks for dealing with the same or related areas, topics and questions.
- To compare the terminologies used by different theories in order to identify cases where different terms are used for essentially the same idea or where the same term is used to designate ideas that are essentially different.
- To attempt to see the common ideas between different theories and work toward their partial unification; this might be particularly promising in cases where the theories deal with different but closely related issues or areas" (op cit.).

Rising to Sfard's challenge, we suggest that one goal for further research in the field of research on the teaching and learning of mathematical proof and proving is to probe the existing theories, perhaps by focussing on what each allows us to accomplish as far as describing, explaining and reconciling novel phenomena in the mathematics classroom. The methodologies for such studies might adopt approaches reviewed by Herbst and Chazan (2009) and might look to using Pollard's (2010, p. 5) representation of teacher-student interaction as a science, a craft, and an art (Fig. 11.2).

5. Concluding comments

At the start of this chapter, we choose to avoid using terms such as *pedagogy* or *didactics*, instead using phrases such as *teacher-student interaction in the mathematics classroom* and *the teaching of proof in the context of teachers' day-to-day instructional practice*. In the US, at least, the promotion of the term *instruction* (following Cohen, Raudenbush & Ball, 2003) has had the good effect of getting people to see that the interactions that mathematics educators need to examine are ternary (teacher-student-content) rather than binary (teacher-student). In that sense, the term *instruction* has been able to achieve what *didactics* (at least in the Anglo-American world) has not. However, a lingering problem is that 'instruction' can conjure up notions of giving orders. In this sense, rather than ternary (teacher-student-content), or even binary (teacher-student), 'instruction' might evoke the idea of the teacher unilaterally issuing orders.

In another starting point to this chapter, we recognized the diversity of countries worldwide and the impact that this has on forms of instructional courses, on the student age-level at which educational ideas are introduced, on teacher knowledge, and so on. Our prior comments about terms like *pedagogy*, *didactics* and *instruction* reminds us of the influence of language on the ways in which people express themselves. Linguists predominantly think that the fundaments of language are somehow encoded in human genes and are, as such, the same across the human species. From such a perspective, all languages share the same *Universal Grammar*, the same underlying concepts, the same degree of systemic complexity, and so on. The resulting conclusion is that the influence of one's mother tongue on the way one thinks is negligible or trivial. Yet recent work (e.g., Deutscher, 2010) is challenging this conclusion, arguing that cultural differences are reflected in language in profound ways, and that emerging evidence indicates that mother-tongue can affect how individuals in different cultural settings think and perceive the world (concurring with the longstanding view of some anthropologists of language).

How such cultural influences might impact on collective work towards a theory of the role of the teacher in teaching proof and proving in mathematics (possible *grander* than the *middle range* theories covered in this chapter), especially in terms of teacher-student interaction, remains to be seen. As Stylianou, Blanton and Knuth (2009, p. 5-6) point out, to date there have not been enough research studies "focused on the teaching of proof in the context of teachers' day-to-day instructional practice". More is currently known about the *learning* of proof (e.g., Harel & Sowder, 1998; 2007); the *teaching* of proof warrants equally close attention (e.g., Harel & Rabin, 2010a; 2010b). Our review of a carefully-selected trio of theoretical frameworks is offered as support for further theorising about teaching proof and proving in mathematics classrooms worldwide.

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