UNIVERSITY OF SOUTHAMPTON

SPINORS, EMBEDDINGS AND GRAVITY

by

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A thesis submitted in accordance with the requirements for the degree of Doctor of Philosophy.

To my parents and sister

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UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

MATHEMATICS

Doctor of Philosophy

SPINORS, EMBEDDINGS AND GRAVITY

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This thesis is concerned with the theory of spinors, embeddings and everywhere invariance with applications to general relativity. The approach is entirely geometric with particular emphasis on the use of natural structures. A clear indication of the interaction between the above topics is given; this interaction then sheds light on various aspects of general relativity theory.

The main ideas discussed are:- (i) Spinors, conformal structure and the spacetime projective null bundle framework. (ii) Spaces of embeddings. (iii) Embeddings and spin structure. (iv) Null embeddings and the null limit (a technique for obtaining differential equations on null hypersurfaces). (v) Quasi-local momentum. (vi) The space of metrics, natural group actions and generalized conformal structure. (vii) Everywhere invariance and the invariance equation as a method for obtaining spacetime symmetries.

Three appendices are also provided:- These give comprehensive summaries of the theories of principal bundles, conformal structure and asymptotic simplicity.

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The original ideas in this thesis are indicated in the introduction and are solely the work of the author apart from the following joint work:- The paper, "Momentum and angular momentum in general relativity" (Proc. 5th Int. Conf. Math. Phys., Coimbra, 1987), written jointly with Dr. J. Vickers, forms the basis for some of the discussion in section 3.4, and the paper, "Everywhere invariant spaces of metrics and isometries" (G.R.G. 18, 1093 (1986)), written jointly with Dr. R. d'Inverno and Dr. J. Vickers, provided the stimulus for a large part of the material contained in Chapter Four.

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CHAPTER 0 INTRODUCTION

Since we have provided each of the four principal chapters of this thesis with an individual introduction, this general introduction serves only to indicate the underlying themes and philosophy, to point out the novel ideas and to define notions and notation relevant to the thesis as a whole. The thesis comprises two main parts:- The first consists of Chapters One, Two and Three and discusses spinors and embeddings, whilst Chapter Four constitutes the second part and is concerned with a study of everywhere invariance and spaces of metrics. Although here we have distinguished between the two parts, there exist certain connections between them; firstly because each of the parts is concerned with the application of geometric frameworks to certain aspects of general relativity theory, secondly through our use of infinite dimensional manifolds wherever appropriate throughout the thesis, and finally because of more specific links (indicated at the appropriate points within the text).

The main themes of this thesis are spinors, embeddings, null and conformal structures, metrics and symmetry (manifested in the form of natural and physically important group actions). We show how all of these themes are important, if not essential, to the theory of general relativity. We also indicate the way in which these concepts interact with one another, thereby illuminating certain aspects of general relativity theory. Indeed, all of these themes (and many more!) have been an important ingredient in much of twentieth century theoretical physics, and, if we go by recent trends in the interaction between geometry and physics, it seems that the relationship will become ever more intimate.

Our philosophy, based on the underlying themes mentioned above, is to organize and to geometrize:- On the one hand, we collect together, in a coherent unified fashion, appropriate geometric notions and we demonstrate how these have an impact on gravity theory. On the other hand, we consider various developments within the theory of general relativity in a geometric light. Our aim is to show how these two interacting approaches clarify certain links between geometry and physics, unify ideas within general relativity theory and also lead to new frameworks within which to study physical ideas. Let us now describe the novel aspects of the thesis in more detail:-

Chapter One is concerned with the theory of spinors on manifolds. We develop the theory carefully, emphasizing the necessary geometric structures within a framework of principal fibre bundles. After developing the required background, we show how spinor ideas fit neatly within the theory of general relativity. Much of this material is standard, although our exposition tends to highlight the <u>real</u> differential geometric aspects rather than complex aspects based on algebraic geometry. Certain novel suggestions appear in section 1.4 where we discuss the possibilities for a spinor-metric configuration space. The ideas of this section are based upon the canonical principal (S)O(n)-bundle associated with any (oriented) n-manifold, and this bundle makes several appearances throughout the thesis. Section 1.5 features a self-contained treatment of the relationship between the 2-sphere and the Lorentz group, a relationship which underpins several important ideas in the theory of general relativity. The 2-sphere-Lorentz group interaction is, of course, well known, but, since the 2-sphere (and the Lorentz group!) play several rôles in the thesis and since the interaction is an excellent illustration of the way in which spin and conformal structures come together, we thought it appropriate to include a discussion based upon our own approach (but, at the same time, utilizing standard notation!). The framework considered in section 1.9 is our geometrization of an idea which has appeared in several places in the general relativity literature. This is the idea of constructing a natural 2-sphere bundle over spacetime so as to make use of the 2-sphere-Lorentz group interaction at each spacetime point. This projective null bundle may be regarded as a Lorentzian version of the Penrose twistor space in Riemannian geometry. In addition to pointing out applications of this framework, we show how the idea brings together many of the notions discussed in sections 1.1 to 1.8.

Chapter Two consists of a thorough treatment of embeddings and their use in general relativity theory. In particular, we emphasize infinite-dimensional applications and, in section 2.2, we consider geometric aspects of the structure of spaces of embeddings. In particular, we examine the natural group actions, metrics and connections associated with these spaces. In section 2.3, we describe the interaction of spinors and embeddings. Although this interaction underlies various topics in general relativity theory, we have not seen a general discussion in the literature and therefore we considered it appropriate to include this section. At the end of section 2.3, we show how the general theory is applied to the important cases in four-dimensional Lorentzian geometry. This section also indicates links between other sections of the thesis and places in context various formalisms (for example, GHP) used in general relativity theory.

In Chapter Three, we move on from the non-degenerate embeddings of Chapter Two and we describe various aspects of null embeddings. The main new idea is that of the null limit (see section 3.2) which is a method for obtaining null versions of spacelike equations. In section 3.4, we apply this technique to obtain a very useful spinor null propagation equation which has been used in important work in the area of general relativistic kinematics. The propagation equation turns out to be the null limit of the (Maxwell-)Sen-Witten equation. In order to put the kinematical application in context, we present a thorough and unified review of gravitational momentum - at the asymptotic level in section 3.3 and at the quasi-local level in section 3.4. We indicate several links between the various approaches to this fundamental problem. Another reason for including a discussion of momentum is to provide an important example of the essential use of spinorial concepts within the theory of general relativity.

The subject of Chapter Four is everywhere invariance, and this constitutes the second part of the thesis. The term everywhere invariance is to be understood on two levels:- Firstly, the term refers to a general philosophy of considering natural (usually infinite-dimensional) structures associated with manifolds and related group actions. This is essentially the study of section 4.1 and, to some extent, section 4.6. The second use of the term everywhere invariance refers to a specific concept - a geometrization of the earlier idea of functional form invariance. In sections 4.2 - 4.6 we develop the theory of everywhere invariance and related concepts.

This theory is interesting for two reasons:- Firstly, from a practical viewpoint, it gives a technique for finding the symmetries of a spacetime metric (we illustrate this technique in section 4.5), and secondly on a more abstract level, everywhere invariance involves the study of group actions for which we consider the stabilizers of <u>subsets</u> under the action, rather than of just one element. In section 4.1, we present a survey of the space of metrics on a manifold. Most of the material is standard, but spread out over the literature and therefore we found it useful to bring it together. Two novel aspects of section 4.1 are the suggested use of the canonical $O(n)$-bundle as a means for resolving the singularities in the space of geometries, and also the consideration of the action of subgroups of the automorphism group of the frame bundle on the space of metrics.

In addition to the principal Chapters One, Two, Three and Four, we have also included three appendices, collected together in Chapter Six. The purpose of these appendices is to collect together basic definitions and results of which we have made use throughout the thesis:- Appendix 6.1 consists of a comprehensive summary of the necessary facts from the theory of principal bundles and associated concepts. The second appendix reviews conformal structures - an important ingredient in several of the notions discussed in the main body of the thesis. Note that section 6.2 also includes a list of formulae giving the transformation properties of various useful spinor quantities under a complex conformal deformation of spacetime metric. The third appendix, section 6.3, gives the basic definitions relevant to a study of asymptotically flat spacetimes. We also include a description of semidirect product groups since examples of these arise in several places within the thesis.

Our basic notation is more-or-less standard:- M, N, ... denote (finite-dimensional) manifolds and $C(M,N)$ denotes the manifold of smooth maps from M into N (the superscript ∞ is always omitted since, for us, everything is smooth). The symbol D is almost always used to mean the (Fréchet) derivative of a map between manifolds, whilst ∇ is used for covariant derivatives. The symbols d and δ refer to the exterior derivative and (on a Riemannian manifold) the exterior coderivative respectively. The signature of a spacetime is taken to be -2 (so that the local diagonal form of a spacetime metric is $(+ - - -)$). If G is a Lie group, then the Lie algebra of G, denoted LG, is the space of left-invariant vector fields on G and is naturally isomorphic with T_1G.

Note that we make no attempt to discuss the global analysis underlying the infinite-dimensional spaces which appear in this thesis. Appropriate analytical references are given where appropriate.

For physics, we use geometrized units in which the Newtonian gravitational constant G and the speed of light c are both unity. In these units, any physical quantity with dimension $L^{\alpha}T^{\beta}M^{\gamma}$ in nongeometrized units possesses dimension $L^{\alpha+\beta+\gamma}$, and the corresponding numerical conversion factor required for transforming from nongeometrized to geometrized units is $G^{\gamma}c^{\beta-2\gamma}$. For example; energy, mass and electric charge all have geometrized dimension L, whilst angular momentum has geometrized dimension L^2.

1.0 Introduction - Why Spin?

In this chapter, we introduce and develop the theory of spinors as geometrical objects on a differential manifold. Our aim is to present the material in a real geometric setting as a precursor to using spinor ideas in general relativity, especially in Chapter Three. In this brief introduction, we present an outline of the reasons why spin structure is so important in physics in general, and in general relativity in particular. Many of the points we make in this section will be expanded and clarified in the main body of this chapter.

From a historical point of view, spinors have been associated with much of the theoretical physics developed this century. There has also been an increasing relevance of spinors to "pure mathematics", especially in differential geometry, both real and complex, and in topology during recent years. We indicate some of these more mathematical developments below.

For ease of discussion we (artificially) partition the various interactions between spinor theory and physics into three loose categories, hopefully demonstrating the widespread appearance of spinor ideas in twentieth century physics.

Category 1 may be called 'general relativity - spinors as a tool', category 2 is 'spinors and general relativity - a deeper relationship?', and, finally, category 3 is 'the rest of physics - spinors as matter'.

We begin by considering category 3. This refers to the seminal influence of Dirac [D 6] and followers such as Infeld and Van der

Waerden [I 1], Laporte and Uhlenbeck [L 3], Proca [P 19], Ruse [R 4], Veblen [V 3], and many post-1930 theoretical physicists. These workers successfully developed the idea of spinors, in particular four component or Dirac spinors, as a representation of spin-$\frac{1}{2}$ fields. The study of spinors as representing matter fields and of the fundamental equations of physics which they satisfy has continued to the present day. In recent years fermions of spin-$\frac{3}{2}$ appearing in supertheories have been added to the list of particles having such a spinorial representation. The interaction between spinors and physics, represented by category 3, may be regarded as fundamental in that fermions arising in nature have a natural interpretation as spinor fields, or rather quantized spinor fields, on spacetime.

Category 3 is concerned with spinor fields propagating on the arena that is spacetime. We now turn to the arena itself and how spinors shape its geometry. Category 1 refers to the use, especially over the past thirty years, of spinors as a tool in general relativity. In particular, we point out the simplifications introduced when structures intrinsic to general relativity are translated into spinor form. For instance conformal and null structures, curvature quantities and Einstein's equations have all enjoyed greater analysis in the spinor setting (see, for example, Penrose and Rindler [P 11], and Geroch et al. [G 7]). The spinors used here are the two component or Weyl spinors, and are introduced initially as a tool, although the physical and geometrical insights aroused by the transition to spinor form already suggests that there is, perhaps, a deeper underlying structure in gravity theory which is related to spinors.

The tensor $\mapsto$ spinor translation used in general relativity rests heavily on the original work of Infeld and Van der Waerden ([I 1], [V 1]), which was developed by other workers such as Bade and Jehle [B 1], Bergmann [B 8], Buchdahl [B 27] and Payne [P 5]. Even Einstein himself realized that spinor-techniques were of use in discussing his theory - see, for example, [E 11]. The geometrical version of the tensor $\mapsto$ spinor translation is introduced in section 1.7.

The 1960's saw a huge unsurge in the popularity of spinor methods in general relativity - inspired by the earlier workers of the 30's, 40's and 50's, and spurred on by the insights of people such as Roger Penrose - see most of the Penrose literature referenced in Chapter 7, but in particular [P 6] and [P 10] for the earliest work by him. The spinor tool was, by now, being applied to basic questions in gravitational theory such as radiation and asymptotic structure. We review some of these developments in Chapter 3 and references to literature may be found there.

Through the work of the 60's and 70's, it became clearer and clearer that spinors were a very powerful technical tool in analyzing the structure of general relativity. The theory of twistors, again initiated by Penrose [P 10], was partly inspired by the obvious importance of two component spinor methods. Twistor theory has lead to further mathematical developments (see, for example, Wells [W 14] as well as being of use in modelling physical phenomena. Whether or not twistors have anything of importance to say about the underlying structure of quantum gravity (as is hoped by certain workers) remains to be seen, but the theory is certainly an important contribution to complex geometry in its own right, as well as shedding light on real geometry and real physics.

We now consider category 2 wherein spinors and spacetime seem to be much more closely linked. Many properties of topology and the geometry defined by Lorentzian metrics in dimension four already contribute to the possibility of spinors in analyzing specetime structure, but since around 1980, these properties, along with certain other observations, have lead to the suspicion that spinors in classical general relativity are not only very useful, but fundamental. In fact, it may be that we shouldn't talk about spinors in general relativity, but about general relativity in spinors.

The main contributions to the suspicion just mentioned are the following four instances: (i) The use of hypersurface Weyl equations in proving the positivity of gravitational mass. This was due to Witten [W 9] for the mass at spacelike infinity, and to Ludvigsen and Vickers, also see Horowitz and Tod [H 14], for the mass at null infinity. Ludvigsen and Vickers have also proved other important physical inequalities using similar methods [L 12]. (ii) The use of spinors in formulating quasi-local definitions of kinematical quantities in general relativity. We refer mainly to the work of Ludwigsen and Vickers [L 10]. These definitions - of mass, momentum and, (unfortunately) to a lesser extent, angular momentum - are apparently fundamentally non-tensorial, and depend upon the use of spinors. Case (iii), which we wish to cite, is along similar lines as (ii), but using a twistorial definition of quasi-local quantities, again in a fundamentally intrinsic way. The spinorial link between (ii) and (iii) is not fully understood, but see Shaw [S 17], and remarks in Chapter 3 of this thesis. (iv) concerns very interesting suggestions of Ashtekar [A 15] in relation to a spinorial formulation of the canonical or 3+1 formalism in general relativity. A possible corollary of Ashtekar's work is the seemingly inseparability of spinors and gravitational energy - $SL(2,\mathbb{C})$ (two component) spinors arise, not because of the Lorentzian signature of space-time, but rather because of a complexification of the group $SU(2)$ associated with a splitting of space and time, as would occur in considerations of energy in a four dimensional setting.

The four cases discussed above are all, a priori, in the context of classical general relativity. It seems possible, however, that the real (or complex!) significance of spinor structure might emerge when the supposed quantum nature of physical reality is taken into account. Ashtekar, in particular, resolves some of the problems encountered previously with a 'quantization' of gravity in his spinorial version of the canonical set-up. In fact his methods lead to a new Hamiltonian in terms of a non-local variable. We have already hinted at the fact that twistor theory is motivated, in part, by a desire for a quantum theory of spacetime, so if this desire is realised, there should be some important rôle for the Penrose, or other, quasi-local quantities to play - perhaps as generators of non-local symmetries in some sense.

Recent developments, such as string theory, which extend or encompass general relativity, also use spin structure in a fundamental way (see, for example, Seiberg and Witten [S 11]), but we do not discuss such matters here. Indeed, our main eventual concern will be the use of spinors (fundamental or otherwise) in classical general relativity, in particular in relation to kinematical quantities. See Chapter Three.

We complete this introduction with a few words concerning the influence of spinors on topology and geometry over recent years. Some of these influences are motivated by physical ideas and some have been of great use in developing physical models of our universe, giving another demonstration of the fruitful interplay between physics and mathematics.

Work of a mathematical nature interacted with the physical use of spinors even in the early days; perhaps <u>especially</u> in the early days, of spin history. The work of Cartan [C 2] was essentially the first treatise on the subject, and influenced other expositions such as those of Brauer and Weyl [B 25], Taub and Veblen [T 2] and Whittacker [W 6]. As differential geometry began to influence mathematics more, so spinors began to reach a wider audience, especially in the 50's and 60's, when the work of Chevalley [C 4], Crumeyrolle [C 14] and Lichnerowicz [L 6] were published.

We mention a few diverse areas of contemporary mathematics where spinor ideas have had an impact. In differential geometry, spinor techniques have been useful in the study of curvature (Lawson and Yau [L 14], elliptic operators (Hitchin [H 10]) and geometric quantization (Blattner and Rawnsley [B 14]). In topology and related areas, there have been applications to K-theory (Atiyah et al. [A 25]), index theory (Baum and Douglas [B 4]) and the spectral theory of Töplitz operators (Boutet de Monvel and Guillemin [B 18]).

We hope that this brief introduction has served to indicate the widespread use of spinors in mathematics and physics, and the bridging rôle they play between these disciplines. Spinors will obviously continue to constitute an important part of general relativity, and the current tantalizing glimpses of the fundamental nature of spinors will hopefully lead to a fuller understanding of the structure of spacetime in the near future.

In the meantime, as we have indicated above, the main aim of this chapter is to explain the theory of spinors necessary for Chapters Two and Three. We also point out several constructions not emphasized in existing literature, in particular in sections 1.4 and 1.9, wherein certain important notions are geometrized.

We will be concerned with Weyl spinors rather than with Dirac spinors, since the former are more important in gravity theory and are also mathematically more basic. Since Dirac or Majorana spinors do not make an appearance in this particular work, we do not give any treatment of Clifford algebras in this chapter. We also avoid the use of complex geometry if real methods suffice, and we make no attempt to discuss the algebraic topological constructions involved in spin structures. Results concerning obstructions and so on will be quoted without proof. In fact, since many of the proofs of results quoted in this chapter are standard, we omit them here.

Chapter One is arranged as follows: In section 1.1, we define spinor structures in both a metric dependent and metric independent manner. Examples of concepts introduced in section 1.1 are given in section 1.2, as well as some extensions of these ideas. For physics (and geometry!), we need covariant derivatives of spinor fields, and these are discussed in section 1.3.

We always regard principal bundles as central and vector bundles as derived, so that covariant derivatives are introduced via connections in principal bundles, though their use will, of course, rely on the Koszul interpretation as a derivation on sections of algebra bundles. The basic results concerning principal bundles which we use in this chapter (and others) are summarized in section 6.1, and we assume that these are known.

Any physical theory requires a configuration space (or at least a phase space), and in section 1.4, we define certain possibilities for such a space for the case of spinor fields. We show the intimate connection with the metrics defining the geometry of the underlying manifold and then, in section 1.6, we demonstrate the inherent difficulties of introducing an appropriate symmetry or invariance group for the configuration space.

Conformal geometry is another important constituent of the theory of general relativity, since the null and causal structure of spacetime are associated with a conformal class of Lorentzian metrics, rather than with just a fixed metric. We develop some relationships between spin and conformal structure in section 1.5. Incidentally, sections 1.5 and 1.6 are both concerned with the relationships between the structure of the space of metrics on a manifold on the one hand and the structure of the spinor configuration space on the other. The ideas of everywhere invariance in the context of the space of metrics (see Chapter Four) may lead to further insight into the structure of the space of spinor fields, and this presents an important avenue for future work.

Since spinors are so useful in physics in general, and in general relativity in particular, it seems natural to assume that the spacetime manifold admits a spin structure, and hence spinors and spinor fields. In section 1.7 we show that spin structures and four dimensional spacetimes fit especially neatly together, and that the requirement of spin structure is a very weak one. Indeed, spin structures will automatically be admitted if other basic physical desires, such as causality, are to be encompassed. Section 1.8 turns around the ideas of section 1.7, and gives a construction of global spacetime geometry starting from a basic assumption of spinors on a four dimensional manifold. The idea that one should regard geometry as derived from a more basic spinor structure is not unattractive, and indicates links with a discrete spacetime structure (see Penrose and MacCallum [P 10]).

We conclude Chapter One by bringing together ideas of earlier sections and demonstrating, in section 1.9, how the very concrete concept of the space or null directions may be analyzed in terms of spinors and conformal structure. The space of null directions may even be used as an arena in which the equations of physics may be formulated. This material is a geometric unification of ideas of earlier workers and may be regarded as a framework for future work.

Chapter One provides a basic account of the spinor ideas we use in the rest of the thesis; it attempts to unify natural notions such as metrics, conformal structures, symmetries, null structure and, of course, spinors and we hope that it goes at least a part of the way towards answering the question 'Why Spin?'.

1.1 Spin Structures and Metrics

Let M be a differential manifold of dimension n. M is usually assumed to be connected, and for simplicity we deal only with the case of orientable manifolds (with a given orientation) in this section. Extensions to the non-orientable case will be briefly discussed in section 1.2. We will be considering metrics on the manifold M, and for definiteness we restrict our attention to Riemannian metrics, i.e. signature zero metrics, in this section. Section 1.2 will give examples of the pseudo-Riemannian case, and all the definitions and results of this section will apply to these metrics. Eventually, of course, we will be mainly concerned with Lorentzian metrics (of signature minus two) on four dimensional manifolds - see sections 1.7, 1.8, 1.9 and Chapters Two and Three. In the Lorentzian case we assume that (M,g) (g the Lorentzian metric under consideration) is not only oriented (a concept which depends only on topology), but also time oriented (a g-dependent concept in general), i.e. we assume that any Lorentzian manifold is spacetime oriented. Whichever signature we use, we will denote the space of metrics of that signature on the manifold M by Met(M). See Chapter Four for more details on the structure of the infinite dimensional manifold Met(M).

Let $GL^+(M)$ be the principal $GL^+(n,\mathbb{R})$-bundle of oriented frames of the oriented manifold M, so that we may write

$$GL^+(n, \mathbb{R}) \hookrightarrow GL^+(M) \xrightarrow{\pi_M} M \qquad 1.1.1$$

where $GL^+(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A > 0\}$ is the identity component of the group of linear automorphisms of $\mathbb{R}^n$. To be strict we should use the symbol π_M to represent the principal bundle, but it will be convenient to abuse notation and refer to any bundle using its total space. If we wish to be explicit we use an expression of the form 1.1.1. Properties of principal bundles and associated concepts are summarized in section 6.1.

We should note that the frame bundle $GL^+(M)$ is a very distinguished member of the class of principal $GL^+(n,\mathbb{R})$-bundles over the manifold M. In fact $GL^+(M)$ is much more intimately connected with the base M due to the existence of the canonical or soldering 1-form on $GL^+(M)$. The frame bundle may be characterized as the unique principal $GL^+(n,\mathbb{R})$-bundle over a manifold, which possesses a 1-form enjoying the properties of the soldering form (see Appendix 6.1). For this reason, bundles associated with the frame bundle have special properties, for instance the tensor bundles over a manifold are acted upon by the diffeomorphism group in a natural way. The spin bundles we shall define shortly are prolongations of the frame bundle, and they too are more rigidly fixed to the base than other principal Spin(n)-bundles. So, although spinors are not as natural a concept as tensors on a 'bare' manifold M are, they are, at least, bound to the structure of M in an important way. We introduce the spin soldering in section 1.3. The idea of soldering leads, via a connection, to torsion, and although torsion vanishes in classical general relativity, it still plays an important rôle (see Trautman [T 9] and, for an interesting account of spin-torsion interplay, see Rapoport and Sternberg [R 1]).

Now we recall that an SO(n)-structure on M is a reduction of $GL^+(M)$ to a principal SO(n)-subbundle, and that SO(n)-structures are in bijective correspondence with Riemannian metrics on M: For any $g \in Met(M)$, the bundle

$$SO(n) \hookrightarrow SO(M,g) \xrightarrow{\pi_M} M \qquad 1.1.2,$$

of oriented g-orthonormal frames is an $SO(n)$-structure. Conversely, given any $SO(n)$-structure P, there exists a unique $g \in Met(M)$ for which $SO(M,g) = P$.

Let $\Gamma: \widetilde{GL}^+(n,\mathbb{R}) \to GL^+(n,\mathbb{R})$, $\Lambda: Spin(n) \to SO(n)$ be the unique, nontrivial $(n \geq 2)$ double covers of $GL^+(n,\mathbb{R})$, $SO(n)$ respectively (see, for example, Crumeyrolle [C 14]). These covers are both universal for $n \geq 3$, and have the following properties: $Ker\ \Gamma = \mathbb{Z}_2 \leq Cent(\widetilde{GL}^+(n,\mathbb{R}))$, $Ker\ \Lambda = \mathbb{Z}_2 \leq Cent(Spin(n))$, and $\Gamma|Spin(n) = \Lambda$, so that $\Gamma^{-1}(SO(n)) = Spin(n) \leq \widetilde{GL}^+(n,\mathbb{R})$.

We now introduce the notion of spin structure on the manifold M. This is just a prolongation of a frame bundle that agrees with the double covering on each fibre, so that the local group acting at each point in M becomes a double cover of the frame group. In fact we define two notions of spin structure. The first is more universal in that it is metric independent whilst the second is associated with Riemannian structures on M. We demonstrate that the two concepts are closely related and, in fact, that there is essentially a unique notion of spin structure on a manifold. An account of spin structures may be found in many places in the literature, for example Hitchin [H 10], Crumeyrolle [C 14], Dabrowski and Percacci [D 1] and Milnor [M 7].

Definition (1.1)1: A spin structure s on M is a Γ-prolongation $(\widetilde{GL}^+(M), \eta)$ of $GL^+(M)$ to $\widetilde{GL}^+(n,\mathbb{R})$. i.e. $\widetilde{GL}^+(M)$ is a principal $\widetilde{GL}^+(n,\mathbb{R})$-bundle over M, and $\eta: \widetilde{GL}^+(M) \to GL^+(M)$ is a principal bundle homomorphism over the identity, id_M, of M, such that $\eta(\tilde{u}A) = \eta(\tilde{u})\Gamma(A)$, for all $\tilde{u} \in \widetilde{\widetilde{GL}}^+(M)$, $A \in \widetilde{\widetilde{GL}}^+(n,\mathbb{R})$.

Two spin structures s_1, s_2 on M are said to be <u>equivalent</u>, $s_1 \sim s_2$, if the respective prolongations $(\widetilde{\widetilde{GL}}_1^+(M),\eta_1)$, $(\widetilde{\widetilde{GL}}_2^+(M),\eta_2)$ are equivalent Γ-prolongations of $GL^+(M)$. i.e. if there exists a principal $\widetilde{\widetilde{GL}}^+(n,\mathbb{R})$-bundle isomorphism $\tilde{f}\colon \widetilde{\widetilde{GL}}_1^+(M) \longrightarrow \widetilde{\widetilde{GL}}_2^+(M)$ over id_M such that $\eta_2 \circ \tilde{f} = \eta_1$.

Let $\tilde{\Sigma}(M)$ denote the set of all spin structures on M, and let $\Sigma(M) \equiv \tilde{\Sigma}(M)/_\sim$ denote the set of equivalence classes of spin structures on M. Note that $\Sigma(M)$ could be empty since, in general, the fibrewise double coverings will not glue together continuously to form a bundle (i.e. a topological obstruction will exist). If $\Sigma(M) \neq \emptyset$, we say M is <u>spin</u>.

<u>Definition (1.1)2</u>: Let $g \in Met(M)$. A g-spin structure s_g on M is a Λ-prolongation $(\widetilde{\widetilde{SO}}(M,g),\ \eta_g)$ of $SO(M,g)$ to $Spin(n)$.

Two g-spin structures, s_{g_1}, s_{g_2} on M are said to be <u>equivalent</u> $s_{g_1} \sim s_{g_2}$, if the respective prolongations $(\widetilde{\widetilde{SO}}_1(M,g),\ \eta_{g_1})$, $(\widetilde{\widetilde{SO}}_2(M,g),\ \eta_{g_2})$ are equivalent Λ-prolongations of $SO(M,g)$.

Let $\tilde{\Sigma}(M,g)$ denote the set of g-spin structures on M, and let $\Sigma(M,g) \equiv \tilde{\Sigma}(M,g)/\sim$ denote the set of equivalence classes of g-spin structures on M. For $s_g \equiv (\widetilde{\widetilde{SO}}(M,g),\eta_g) \in \tilde{\Sigma}(M,g)$, $\widetilde{\widetilde{SO}}(M,g)$ is called the <u>bundle of (g,s_g)-spin frames</u>.

Given any $g \in Met(M)$, define a map

$$\tilde{r}_g\colon \tilde{\Sigma}(M) \longrightarrow \tilde{\Sigma}(M,g);\quad (\tilde{P},\eta) \longmapsto (\tilde{P}_g,\eta_g) \qquad 1.1.3,$$

(where, for convenience, we write $\tilde{P}$ rather than $\widetilde{\widetilde{GL}}^+(M)$) where $\tilde{P}_g = \eta^{-1}(SO(M,g))$ and $\eta_g = \eta|\tilde{P}_g$.

$(\tilde{P}_g, \eta_g)$ is clearly a g-spin structure on M for each spin structure $s \equiv (\tilde{P},\eta)$ on M, so $\tilde{r}_g$ does indeed map into $\tilde{\Sigma}(M,g)$. In fact it is not hard to show that $\tilde{r}_g$ is a bijection of $\tilde{\Sigma}(M)$ onto $\tilde{\Sigma}(M,g)$, for each $g \in Met(M)$. The inverse map, $\tilde{r}_g^{-1}$: $\tilde{\Sigma}(M,g) \to \tilde{\Sigma}(M)$, may be constructed as follows: Suppose $s_g \equiv (\widetilde{\widetilde{SO}}(M,g),\eta_g) \in \tilde{\Sigma}(M,g)$. $Spin(n) \leq \widetilde{\widetilde{GL}}^+(n,\mathbb{R})$, so there exists the natural left action by group multiplication of $Spin(n)$ on $\widetilde{\widetilde{GL}}^+(n,\mathbb{R})$. Hence we may form the associated bundle $\tilde{P} = \widetilde{\widetilde{SO}}(M,g) \times_{Spin(n)} \widetilde{\widetilde{GL}}^+(n,\mathbb{R})$. Note that $SO(M,g)$ is a reduction of $GL^+(M)$, and so the extension $SO(M,g) \times_{SO(n)} GL^+(n,\mathbb{R})$ may be canonically identified with $GL^+(M)$. Now we may define η: $\tilde{P} \to GL^+(M) \overset{\sim}{=} SO(M,g) \times_{SO(n)} GL^+(n,\mathbb{R})$ by $\eta([(\tilde{u},A)]) = [(\eta_g(\tilde{u}), \Gamma(A))]$, for all $[(\tilde{u},A)] \in \tilde{P}$. The map η is well defined, and it is clear that the map: $(\widetilde{\widetilde{SO}}(M,g),\eta_g) \mapsto (\tilde{P},\eta)$ just constructed is precisely $\tilde{r}_g^{-1} : \tilde{\Sigma}(M,g) \to \tilde{\Sigma}(M)$.

In fact if $s_1 \sim s_2$, then $\tilde{r}_g(s_1) \sim \tilde{r}_g(s_2)$ (similarly if $(s_g)_1 \sim (s_g)_2$ then $\tilde{r}_g^{-1}((s_g)_1) \sim \tilde{r}_g^{-1}((s_g)_2)$), and so $\tilde{r}_g$ projects to a well defined bijection of the quotients by $\sim$:

$$r_g: \Sigma(M) \longrightarrow \Sigma(M,g); \quad [(\tilde{P},\eta)] \longmapsto [(\tilde{P}_g,\eta_g)] \qquad 1.1.4.$$

We summarize the above result: Given any $g \in Met(M)$; $\Sigma(M,g)$, the set of equivalence classes of g-spin structures on M, is in bijective correspondence with $\Sigma(M)$, the set of equivalence classes of spin structures on M. In particular, card $(\Sigma(M,g'))$ = card$(\Sigma(M,g))$ for all metrics g, g', and M is spin, i.e. $\Sigma(M) \neq \phi$, if and only if $\Sigma(M,g) \neq \phi$ (any $g \in Met(M)$).

Therefore, the topological obstruction to the prolongation of $GL^+(M)$ to $\widetilde{\widetilde{GL}}^+(n,\mathbb{R})$ is precisely the same as the obstruction to prolonging $SO(M,g)$ to $Spin(n)$ (for any $g \in Met(M)$). It is well known that this obstruction is the second Stiefel-Whitney class, $w_2(TM) \in H^2(M;\mathbb{Z}_2)$, of the tangent bundle, TM, of M (See, for example, Milnor [M 7]). If $w_2(TM)$ vanishes, then, for any $g \in Met(M)$, there will exist at least one g-spin structure on M, and hence at least one spin structure on M. In general there will exist inequivalent (g-) spin structures if M is spin.

It is known that $H^1(M;\mathbb{Z}_2)$ has a natural free transitive action on $\Sigma(M,g)$ (for any $g \in Met(M)$), and hence on $\Sigma(M)$, so that $\Sigma(M)$ is an affine space for $H^1(M;\mathbb{Z}_2)$. The action: $H^1(M;\mathbb{Z}_2) \times \Sigma(M) \to \Sigma(M)$ may be constructed using a Steenrodesque argument using transition functions and a representation of cohomology classes as Čech-cocycles. Fixing an arbitrary $[(\tilde{P}_o,\eta_o)] \in \Sigma(M)$, we obtain a bijection:

$$H^1(M;\mathbb{Z}_2) \longrightarrow \Sigma(M); \quad \alpha \mapsto \alpha\cdot[(P_o,\eta_o)] \qquad 1.1.5,$$

and so the different equivalence classes of spin structures are parameterized (after choosing an arbitrary origin) by the elements of $H^1(M;\mathbb{Z}_2)$.

Note that, a priori, given two inequivalent spin structures $(\tilde{P}_1,\eta_1)$, $(\tilde{P}_2,\eta_2)$, the inequivalence could be due to the fact that the principal $\widetilde{GL}^+(n,\mathbb{R})$-bundles, $\tilde{P}_1$, $\tilde{P}_2$ belong to different isomorphism classes, or, given $P_1 \cong P_2$, η_1 and η_2 might still be inequivalent maps. We will remark on the possible physical significance of such inequivalences in section 1.3.

We have shown that, for any $g \in Met(M)$, g-spin structures are in bijective correspondence with spin structures. In other words, there is only one notion of spin structure on a manifold M. The inter-relationship between spin structures and metrics will be explored in greater depth in sections 1.4, 1.5 and 1.6, but we make one remark now concerning this inter-relationship:

Let $\tilde{\Sigma}\mathrm{Met}(M) = \{(g,s_g)\colon\ s_g$ is a g-spin structure, $g \in \mathrm{Met}(M)\}$.

$\tilde{\Sigma}\mathrm{Met}(M)$ is a bundle over $\mathrm{Met}(M)$ with projection; $(g,s_g) \longrightarrow g$, and the fibre over the metric g is precisely $\tilde{\Sigma}(M,g)$. Now define $\tilde{r}\colon \mathrm{Met}(M) \times \tilde{\Sigma}(M) \longrightarrow \tilde{\Sigma}\mathrm{Met}(M)$; $(g,s) \longmapsto (g,\tilde{r}_g(s))$, where $\tilde{r}_g$ is defined as in equation 1.1.3. Then we have, for each spin structure s on M, a section $\tilde{r}^s$ of $\tilde{\Sigma}\mathrm{Met}(M)$, given by:

$$\tilde{r}^s\ :\ \mathrm{Met}(M) \longrightarrow \tilde{\Sigma}\mathrm{Met}(M); \quad g \longmapsto \tilde{r}(g,s) \qquad 1.1.6$$

for each metric g on M.

This trivial construction will simplify some of this discussion in section 1.4.

In this section, we have introduced the notion of a spin structure on a manifold M and the equivalent notion of a g-spin structure. These two ideas will form the basis of our constructions of spin objects for use in physics and geometry, namely spinors, spinor fields and spin connections. Recall that a spin structure contains two pieces of data; a principal bundle and also a bundle map. The former will be used to construct associated bundles, in particular vector bundles of spinors and thence spinor fields by taking sections. The bundle map part of a spin structure will be used to prolong linear connections on the manifold M, the resulting connections in the bundle of spin frames being the so-called spin connections, essential for constructing spinor differential equations. We will discuss fields and connections in section 1.3, but first, we give some examples illustrating the ideas of this section. We also extend the definitions to show how non-orientable manifolds may be treated, and we introduce the case of Lorentzian metrics.

1.2 Examples and Extensions

1.2.1: If the manifold M is parallelizable (i.e. if $GL^{+}(M)$ is trivializable, so that it admits a global section), then M is spin. Obviously, in this case, there is no obstruction to prolonging the frame bundle to a principal $\widetilde{GL}^{+}(n,\mathbb{R})$-bundle. The converse to this is not true in general, as examples given below will illustrate, so spinor fields, for instance, may exist without the need for a global frame field.

1.2.2: If M is a compact, oriented manifold of dimension ≤ 3, then M is spin. Indeed, in the cases $\dim M = 1,3$, M is parallelizable and hence spin by 1.2.1. In dimension two, the Stiefel-Whitney class is just the Euler class modulo two, but the Euler class $\chi = 2(1-g)$, where g is the genus of M, so again the obstruction to spin vanishes. In fact, there are 4^{g} inequivalent spin structures on a two dimensional oriented compact manifold, so for the two sphere S^2, for example, there is a unique (up to equivalence) spin structure.

In dimension four, the above result is not valid, e.g. $\mathbb{CP}^2$ (two dimensional complex projective space) does not admit a spin structure, since $w_2(T\mathbb{CP}^2) \neq 0$. $\mathbb{CP}^2$ is of interest to gravity theory since it represents a gravitational instanton.

1.2.3: We now consider an example which will be of utmost importance to Chapters Two and Three, and indeed, to general relativity in general; the case of non-compact Lorentzian four-manifolds. Suppose (M,g) is a non-compact, spacetime oriented (see section 1.7), four dimensional Lorentzian manifold, i.e. a possible model for spacetime. Then M admits a g-spin structure if and only if M is parallelizable. This result is proved in Geroch [G 2], and in a more general setting in Parker [P 2]. For example, any globally hyperbolic Lorentzian manifold of dimension four will always admit a spin structure and, since asymptotically simple and empty spacetimes (see section 6.3 and Chapter 3) are globally hyperbolic, spinors will exist in important physical situations.

Another important feature of non-compact Lorentzian four-manifolds is that any principal $SL(2,\mathbb{C})$ ($\equiv$ spin(1,3); see section 1.7)-bundle is necessarily trivial (see Isham, [I 18]) and so the information concerning inequivalent spin structures is carried by the bundle map part of the spin structure.

We note that there do exist non-compact Lorentzian four-manifolds which are not spin. Of course, such manifolds must necessarily be non-parallelizable. For example, (see Plymen [P 14]), let $M = \mathbb{CP}^2 - \{*\}$ so that M is a non-compact (real) four-manifold. Since M is non-compact it admits a Lorentzian metric g, and, since M is simply connected, (M,g) is spacetime orientable. It can be shown that $w_2(TM) \neq 0$ so M is not spin. In fact, M does admit countably many inequivalent spin^C-structures (see 1.2.6 for a definition of a spin^C-structure), so a certain kind of spinor structure does exist on M.

We shall usually assume that spacetime is non-compact (see section 1.7), but compact spacetimes sometimes provide interesting examples. For completeness, we mention briefly the case of spin structures on compact spacetimes. We refer the reader to Whiston [W 5] for more (topological) details:

Let (M,g) be a spacetime oriented Lorentzian four-manifold which is spin. If M is non-compact, then we have noted above that M is parallelizable, but this is not necessarily the case if M is compact. Indeed, by a theorem of Hirzebruch and Hopf, a compact spin spacetime is parallelizable if and only if its Pontrjagin number is zero. An example of a compact, non-parallelizable, spin spacetime (M,g) may be constructed as follows: Let $\mathbb{Z}_2$ act continuously on the four-torus $\mathbb{T}^4 \equiv (S^1)^4$ by conjugation in each S^1-factor. The sixteen singularities in the resulting quotient space may be smoothed out to form the Kummer surface K. In order to introduce a Lorentzian metric, the Euler number, $e(K) = 24$, must be killed off, and this may be achieved by performing twelve spherical modifications. The resulting manifold M then admits a Lorentzian metric g. (M,g) now provides an example of a compact spacetime which is spin but not parallelizable (since the Pontrjagin number of M is non-zero).

1.2.4: Let $M = S^n$, the n-sphere, and let $\mathrm{can} \in \mathrm{Met}(S^n)$ be the standard Riemannian metric on S^n, induced by the round embedding of S^n in $\mathbb{R}^{n+1}$. Then the bundle of oriented can-orthonormal frames is given by

$$SO(n) \hookrightarrow SO(S^n,\mathrm{can}) \rightarrow S^n \qquad 1.2.1,$$

where $SO(S^n, can) \cong SO(n+1)$, and S^n is obtained as the (Riemannian) homogeneous space $SO(n+1)/_{SO(n)}$.

Now we consider spin structures on spheres. For details of constructions, see, for example, Dambrowski and Trautman [D $\mathcal{Z}$]). The case $n = 1$ is exceptional in that S^1 admits two inequivalent spin structures, whereas for $n \geqslant 2$, there is a unique spin structure. The case $n = 1$ is as follows: $S^1 \cong SO(2) \cong U(1) \cong Spin(2)$, and $Spin(1) \cong \mathbb{Z}_2$. The two spin structures are given by

$$s_1: \quad \mathbb{Z}_2 \hookrightarrow U(1) \times \mathbb{Z}_2 \xrightarrow{pr_1} U(1), \quad U(1) \times \mathbb{Z}_2 \xrightarrow[\eta_1]{pr_1} U(1) \qquad 1.2.2,$$

$$s_2: \quad \mathbb{Z}_2 \hookrightarrow U(1) \xrightarrow{\text{square}} U(1), \qquad U(1) \xrightarrow[\eta_2]{\text{square}} U(1) \qquad 1.2.3.$$

For $n \geqslant 2$, the unique can-spin structure on S^n is:

$$Spin(n) \hookrightarrow Spin(n+1) \to S^n, \qquad Spin(n+1) \xrightarrow[\eta]{\Lambda} SO(n+1) \qquad 1,2,4,$$

using obvious notation.

<u>1.2.5</u>: For completeness, we shall now make several remarks concerning non-orientable manifolds, although below we shall only consider orientable (or rather spacetime orientable in the case of general relativity) manifolds. Recall that in the case of a non-orientable manifold equipped with a metric, a reduction of the bundle of orthonormal frames to the group $O(n)$ is possible, but not to $SO(n)$ in general. For a notion of spin structure, therefore, we must consider coverings of $O(n)$. Note that $O(n)$ (or its indefinite analogue $O(p,q)$) must be used if we wish

consider reflections (or space reflections and time reversals), even in the case of an orientable (or spacetime orientable) manifold.

We recall that $SO(n)$ admits a unique, non-trivial (for $n \geq 2$), double cover (universal for $n \geq 3$) by $Spin(n)$. However, the full orthogonal group $O(n)$ has, in general several equivalent double coverings (see, for example, Whiston [W 4]). A simple example is the case $n = 1$; $O(1) \cong \mathbb{Z}_2$, $SO(1) \cong 1$, $Spin(1) \cong \mathbb{Z}_2$, and we have the unique (trivial) double cover: $Spin(1) \xrightarrow{\Lambda} SO(1)$. The orthogonal group has two inequivalent coverings given by $\Lambda^+: \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$, $\Lambda^-: \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$, and, in fact, for any n, there exist two such inequivalent double coverings of $O(n)$. The corresponding covering groups are known as $Pin^+(n)$, $Pin^-(n)$, so that we have $\Lambda^+: Pin^+(n) \longrightarrow O(n)$, $\Lambda^-: Pin^-(n) \longrightarrow O(n)$ as the two double coverings of $O(n)$ (see Atiyah et al. [A 25]). $Spin(n)$ may be obtained from either of $Pin^{\pm}(n)$ by taking the identity component. The covering $\Lambda: Spin(n) \longrightarrow SO(n)$ is then just $\Lambda^{\pm}|Spin(n)$.

A $Pin^{\pm}$-structure on M is then defined in a way analogous to that of a Spin structure in section 1.1, i.e. as a prolongation of the bundle of g-orthogonal frames, $O(M,g)$, ($g \in Met(M)$) to the group $Pin^{\pm}(n)$.

The topological obstructions to the existence of $Pin^{\pm}$-structures are different from each other. In some cases one exists whereas the other doesn't (see, for example, Dabrowski and Trautman [D 2]).

The number of distinct notions of (s)pin-structures (corresponding to the number of inequivalent double coverings of the corresponding orthogonal group) increases if we consider the case of indefinite metrics. For example, in the Lorentzian case, there are eight inequivalent double coverings of $O(1,3)$. These coverings correspond to the various combinations of signs for P^2, T^2, $(PT)^2$ where $P(T)$ is a choice of one of the two spin transformations in the fibre above space reflection (time reversal) in $O(1,3)$. Each covering will give rise to a different notion of "(s)pin" structure on a spacetime manifold, and hence, if such a structure exists, to different notions of spinors, spinor fields and spin connections. An example of such a construction, where the manifold is taken to be $M = \text{Möb} \times \mathbb{R}^2$ ("space" $\cong$ Möb $\times \mathbb{R}$, Möb is the Möbius band), may be found on p. 421 in Choquet-Bruhat et al. [C 5]. See also Whiston [W 4].

1.2.6: Returning now to the case of orientable manifolds, we mention certain generalizations of the notion of spin structure. A generalization of the definition given in section 1.1 may be required for various reasons; perhaps because the underlying manifold is not spin, although 1.2.3 indicates that in cases of interest in general relativity, there will be a spin structure. Recall, however, that one considers occasionally compact four-manifolds (often S^4 in particle physics) rather than imposing boundary conditions on fields propagating on non-compact space-time, or in a Euclideanization procedure. In the compact case, a spin structure need not exist (Cf. 1.2.2). Another reason for generalizing spin structures is in order to incorporate extra structure into the theory - perhaps additional physical fields whose spin transformation group is different from Spin(n).

A generalization which has appeared in the literature (see Isham [I 8], Isham and Avis [A29], Hawking and Pope [H 6]) is that of enlarging the group Spin(n) to a group of the form $G = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} H$, where H is a group whose centre contains $\mathbb{Z}_2$. An important example is the so called spin^C structure where H is taken to be S^1 - this may be taken as the appropriate structure in electromagnetism where one has charged spinor fields coupling to a connection in a principal S^1-bundle.

Extending the spin group in the way just described can ensure that a particular manifold M admits a generalized structure, even though M is not spin in the sense of 1.1. For example, an analogue of the result stated in 1.2.2 is that any compact, oriented manifold of dimension $\leqslant 4$ admits a spin^C structure (Whitney's theorem). This result is not true for dimension > 4, e.g.: ${}^{SU(3)}/SO(3)$, $\mathbb{CP}^2 \times_{\mathbb{Z}_2} S^1$ do not admit spin^C structures (see, for example, Killingback and Rees [K 4]), so a further enlargement of the spin group may be necessary to remove the obstruction.

More details concerning the above examples may be found in the references cited. From now on we deal only with the spin structures as defined in 1.1, although the conformal spin structure which we discuss in section 1.5 may be regarded as a slight generalization. We will be concerned with orientable manifolds and, when we discuss spacetimes, Lorentzian metrics. Embeddings into a spacetime may induce on their domain a metric of positive (or negative)-definite signature, so the Riemannian case is not unimportant in general relativity, especially when induced spin structures and so on, are considered (see Chapters 2, 3).

Armed with a supply of examples convincing us that the concepts introduced in section 1.1 are not empty, we move onto the notion of spinor fields and spin connections.

1.3 Connections and Fields

Let M be an oriented n-dimensional spin manifold and let g be a metric on M. We choose an equivalence class $[(\widetilde{SO}(M,g),\eta_g)]$ of g-spin structures on M and, for ease of exposition, we work with an arbitrary representative g-spin structure $s_g \equiv (\widetilde{SO}(M,g),\eta_g) \in \tilde{\Sigma}(M,g)$. s_g corresponds, via $\tilde{r}_g$, to a unique spin structure on M. The purpose of this section is to consider bundles associated with the principal Spin(n) bundle $\widetilde{SO}(M,g)$, and to construct connections in $\widetilde{SO}(M,g)$ using η_g. Choosing a different representative of $[(\widetilde{SO}(M,g),\eta_g)]$ will lead to isomorphic associated bundles and equivalent connections. On the other hand, different elements of $\Sigma(M,g)$ will give rise to inequivalent bundles and connections, and this may have repercussions on any physical situation being described (See Avis and Isham [A 28], Isham [I 8]).

We remark on the use of elements of $\Sigma(M,g)$ rather than elements of $\Sigma(M)$. Since $\Sigma(M,g)$ is in bijective correspondence with $\Sigma(M)$, we could start with a representative spin structure $s \equiv (\widetilde{GL}^+(M),\eta)$ of an element of $\Sigma(M)$, and then consider the bundles associated with $\widetilde{GL}^+(M)$ and connections in $\widetilde{GL}^+(M)$. However, to define spinor fields representing useful geometrical and physical quantities, we require that the fields transform under representations of Spin(n) (the double cover of the physically

significant rotation group $SO(n)$) rather than of $\tilde{\tilde{GL}}^+(n,\mathbb{R})$.
In other words, a given metric $g \in Met(M)$ is needed to act qua Higgs field, and to break the symmetry by reducing the structure group down from $GL^+(n,\mathbb{R})$ to $SO(n)$.

Given any metric g, then we must just consider the g-spin structure $\tilde{r}_g(s)$, corresponding to any spin structure s on M, so, from a mathematical point of view, considering g-spin structures only involves no loss of generality. We may also wish to consider the coupled configuration space of metrics and spinor fields (see section 1.4), and then we must take into account the fact that each metric gives rise to a distinct space of spinor fields. These distinct spaces fit together as fibres of an infinite dimensional vector bundle over $Met(M)$ (or algebra bundle if we consider sums over all representations). There is also the possibility of taking into account the fact that there may exist inequivalent spin structures, and then the metric-spinor field configuration space should be extended to a metric-spin structure space. We investigate these configuration spaces in more detail in section 1.4, but, for the moment, let us return to a fixed metric g and a fixed g-spin structure s_g.

We have at our disposal a principal $Spin(n)$-bundle $\tilde{\tilde{SO}}(M,g)$ together with a principal bundle homomorphism $\eta_g\colon \tilde{\tilde{SO}}(M,g) \to SO(M,g)$, such that $\eta_g(\tilde{u}A) = \eta_g(\tilde{u})\Lambda(A)$, for all $\tilde{u} \in \tilde{\tilde{SO}}(M,g)$, $A \in Spin(n)$. Suppose, now, that $\tilde{\rho} \in Hom(Spin(n), Diff(V))$ is a left action of $Spin(n)$ on a manifold V. We may form the associated bundle $\tilde{\tilde{SO}}(M,g) \times_{Spin(n)} V$ with typical fibre V. In particular, given any $\rho \in Hom(SO(n), Diff(V))$, we have the lifted left action $\tilde{\rho} \equiv \rho\circ\Lambda \in Hom(Spin(n), Diff(V))$, and the corresponding bundle isomorphism (over id_M):

$$\eta_g(\rho) : \widetilde{\widetilde{SO}}(M,g) \times_{Spin(n)} V \longrightarrow SO(M,g) \times_{SO(n)} V$$

defined by $\eta_g(\rho)([(\tilde{u},\xi)]) = [(\eta_g(\tilde{u}),\xi)]$ 1.3.1,

for all $[(\tilde{u},\xi)] \in \widetilde{\widetilde{SO}}(M,g) \times_{Spin(n)} V$. Of course, in general, there will exist actions of Spin(n) which do not arise as lifts of SO(n)-actions. Indeed, those actions, in particular representations, of Spin(n) which are not equivalent to lifts of SO(n)-actions, are one of the reasons for the introduction of spin structures in the first place. e.g. : the classification, using spin, of particles transforming under irreducible representations of the Poincaré group in particle physics (See Wigner [W 19]).

We now restrict to the case where V is a vector space (often finite dimensional) and $\tilde{\rho} \in \mathrm{Hom}(Spin(n), GL(V))$ is a representation of Spin(n) on V. In this case, the associated bundle $\widetilde{\widetilde{SO}}(M,g) \times_{Spin(n)} V$ is a vector bundle over the manifold M.

Definition (1.3)1: Let $s_g \equiv (\widetilde{\widetilde{SO}}(M,g),\eta_g)$ be a g-spin structure on the manifold M, and let $\tilde{\rho}$ be a representation of Spin(n) on the vector space V. Then a spinor of type $(s_g,\tilde{\rho}, V)$ is any element of the associated vector bundle $\widetilde{\widetilde{SO}}(M,g) \times_{Spin(n)} V$, and a spinor field of type $(s_g,\tilde{\rho},V)$ is any element of $\Gamma(\widetilde{\widetilde{SO}}(M,g) \times_{Spin(n)} V)$.

Typical examples of representations $\tilde{\rho}$ of Spin(n) arise as representations of the Clifford algebra of $(\mathbb{R}^n$, can) or, as described above, lifts of representations of SO(n). In the latter case, suppose $\rho \in \mathrm{Hom}(SO(n), GL(V))$, then we have an isomorphism of vector bundles:

$$\eta_g(\rho) : \quad \tilde{\tilde{SO}}(M,g) \times_{Spin(n)} V \longrightarrow SO(M,g) \times_{SO(n)} V \qquad 1.3.2,$$

where $\eta_g(\rho)$ is defined as in equation 1.3.1. This isomorphism will be important in section 1.7. For example, let ρ be the defining representation of $SO(n)$ on $\mathbb{R}^n$. Then we have the vector bundle isomorphism:

$$\eta_g(\rho) : \quad \tilde{\tilde{SO}}(M,g) \times_{Spin(n)} \mathbb{R}^n \longrightarrow TM \qquad 1.3.3,$$

since, in this instance, the vector bundle associated, via ρ, to the bundle of g-orthnormal frames $SO(M,g)$ is precisely the tangent bundle, TM. An analogue of equation 1.3.3 will form part of the Infeld-Van der Waerden isomorphism in section 1.7.

Before defining spin connections and associated covariant derivatives, we make two remarks: Firstly, using the fact (see section 6.1) that there is a bijective correspondence between $\Gamma(\tilde{\tilde{SO}}(M,g) \times_{Spin(n)} V)$ and $C_{Spin(n)}(\tilde{\tilde{SO}}(M,g), V)$, we may regard a spinor field as an equivariant map from the bundle of (g,s_g)-spin frames into the vector space V (this map just associates with each spinor field the components of the field at a point with respect to a particular spin frame at that point). Secondly, once we have spinor bundles, we may consider k-forms on the manifold M which take their values in such bundles, i.e. spinor valued differential forms on M. Such vector bundle valued forms may be used in the formulation of definitions of quasi-local momenta in general relativity (see Chapter 3).

In order to write down spinor differential equations as are used in particle physics (for example the Dirac and Weyl equations, and also the wave equations of supertheories) and in general relativity (the spinor versions of Einstein's equations, for instance, as well as the spinor propagation/initial value equations used in positivity proofs and in quasi-local kinematics - see Chapter 3), we need the notion of covariant derivatives of spinor fields. These arise from a connection in the bundle of spin frames, as we now indicate.

Consider the principal bundle homomorphism $\eta_g: \widetilde{\widetilde{SO}}(M,g) \to SO(M,g)$, arising from $s_g \in \tilde{\Sigma}(M,g)$. This homomorphism is two-to-one on each fibre but $D\eta_g: T\widetilde{\widetilde{SO}}(M,g) \longrightarrow TSO(M,g)$ restricts to an isomorphism of each tangent space of $\widetilde{\widetilde{SO}}(M,g)$ onto the corresponding tangent space of $SO(M,g)$. The Levi-Cività connection, $\omega_g \in \mathrm{Conn}(SO(M,g))$, of g gives rise to a distribution on $SO(M,g)$ which may be pulled back using $D\eta_g$ to a distribution on $\widetilde{\widetilde{SO}}(M,g)$. Since η_g is a homomorphism of principal bundles, the induced distribution on $\widetilde{\widetilde{SO}}(M,g)$ will define a connection in $\widetilde{\widetilde{SO}}(M,g)$. A more useful construction of this induced connection is in terms of the connection forms (see section 6.1 for more details concerning induced connections):

Let $\Lambda_* = D\Lambda(e): L(\mathrm{Spin}(n)) \longrightarrow L(SO(n))$ be the Lie algebra isomorphism induced by the covering $\Lambda: \mathrm{Spin}(n) \longrightarrow SO(n)$. Then the connection form $\tilde{\omega} \equiv \tilde{\omega}(s_g)$ corresponding to the distribution constructed above may be written

$$\tilde{\omega} = \Lambda_*^{-1} \circ \eta_g^* \omega_g \qquad 1.3.4,$$

i.e. for all $\tilde{u} \in \widetilde{\widetilde{SO}}(M,g)$ and $\tilde{v} \in T_{\tilde{u}}\widetilde{\widetilde{SO}}(M,g)$, we have

$$\tilde{\omega}(\tilde{u}).\tilde{v} = \Lambda_*^{-1}(\omega_g(\eta_g(\tilde{u})).D\eta_g(\tilde{u}).\tilde{v}) \qquad 1.3.5.$$

Definition (1.3)2: Let $\tilde{\omega} \in \mathrm{Conn}(\tilde{\tilde{SO}}(M,g))$ be as just constructed. Then $\tilde{\omega}$ is called the spin connection associated to the g-spin structure s_g on M.

Given $\tilde{\omega}$, and a representation $\tilde{\rho}$ of $\mathrm{Spin}(n)$ on a vector space V, we may define various derivations as in section 6.1: Let E be any vector bundle over the manifold M, and let $\Omega^k(E) \equiv \Gamma(\Lambda^k TM \otimes E)$ be the space of k-forms on M taking their values in the vector bundle E. Here we take $E = \tilde{\tilde{SO}}(M,g) \times_{\mathrm{Spin}(n)} V$, and, using $\tilde{\omega}$, we obtain the exterior covariant derivatives:

$$d^{\tilde{\omega}} : \Omega^k(E) \longrightarrow \Omega^{k+1}(E) \qquad 1.3.5,$$

for $k \geqslant 0$, and, in the case $k = 0$:

$$\nabla^{\tilde{\omega}} : \Gamma(E) \longrightarrow \Omega^1(E) \qquad 1.3.6,$$

so that $\nabla^{\tilde{\omega}}$ is just the covariant derivative on spinor fields of type $(s_g, \tilde{\rho}, V)$.

Using again the results summarized in section 6.1, we have an analogue of equation 1.3.4 relating the curvature forms $\Omega = d^{\omega_g}\omega_g$, $\tilde{\Omega} = d^{\tilde{\omega}}\tilde{\omega}$ of the connections ω_g, $\tilde{\omega}$ respectively (Note that the $d^{\tilde{\omega}}$ appearing here is not the same as those in equation 1.3.5, although there does exist a relation between the two - see section 6.1):

$$\tilde{\Omega} = \Lambda_*^{-1} \circ \eta_g^* \,\Omega \qquad 1.3.7.$$

We now have enough machinery to construct spinor differential equations, but we conclude this section by remarking on the notion of spin soldering, referred to earlier in this chapter.

There exists on $\tilde{\tilde{SO}}(M,g)$ another vector valued form, independent of connections in $SO(M,g)$. Recall that on the frame bundle of any manifold is defined the canonical $\mathbb{R}^n$- valued 1-form or soldering form θ (see section 6.1). The form θ restricts to $GL^+(M)$, and thence to $SO(M,g)$, for any $g \in Met(M)$. We denote the restricted soldering form by the same letter θ, so that $\theta \in \Omega^1(SO(M,g), \mathbb{R}^n)$ is a tensorial 1-form of type $(\rho^{(1,0)}, \mathbb{R}^n)$, where $\rho^{(1,0)}$ is the defining representation of $SO(n)$ on $\mathbb{R}^n$. θ is defined by:

$$\theta(u).v \quad = \quad \kappa_u^{-1}(D\pi_M(u).v) \qquad 1.3.8,$$

for all $v \in T_u SO(M,g)$, $u \in SO(M,g)$, and where $\kappa_u : \mathbb{R}^n \longrightarrow T_{\pi_M(u)}M$ is the usual isomorphism of vector spaces corresponding to the g-orthonormal frame u.

Given the g-spin structure $s_g = (\widetilde{\widetilde{SO}}(M,g), \eta_g)$, we may lift θ to an $\mathbb{R}^n$-valued 1-form $\tilde{\theta} \equiv \tilde{\theta}(s_g) = \eta_g^* \theta$ on $\widetilde{\widetilde{SO}}(M,g)$. Since θ vanishes on vertical vectors in $TSO(M,g)$, and since η_g is a principal bundle homomorphism whose derivative restricts to an isomorphism on fibres of $T\widetilde{\widetilde{SO}}(M,g)$, we see that θ also vanishes on vertical vectors (in $T\widetilde{\widetilde{SO}}(M,g)$), and also that $\tilde{\theta}$ is equivariant:

$$\tilde{R}_A^* \tilde{\theta} \quad = \quad \tilde{\rho}(A^{-1}).\tilde{\theta} \qquad 1.3.9,$$

for all $A \in Spin(n)$, and where $\tilde{\rho} = \rho \circ \Lambda \in Hom(Spin(n), GL(n,\mathbb{R}))$, and $\tilde{R}$ is the right action of $Spin(n)$ on $\widetilde{\widetilde{SO}}(M,g)$. i.e. $\tilde{\theta} \in \Omega^1(\widetilde{\widetilde{SO}}(M,g), \mathbb{R}^n)$ is a tensorial 1-form of type $(\tilde{\rho}, \mathbb{R}^n)$.

<u>Definition (1.3)3</u>: $\tilde{\theta}$, as just constructed, is called the <u>spin soldering</u> form associated to the g-spin structure s_g on M.

As with any tensorial 1-form on a principal bundle, $\tilde{\theta}$ may be regarded as 1-form on M taking its values in the appropriate associated vector bundle. In this case, the vector bundle is $\widetilde{SO}(M,g) \times_{Spin(n)} \mathbb{R}^n \cong TM$ by equation 1.3.3, so that $\tilde{\theta} \in \Gamma(T^*M \otimes TM) \cong \Gamma(End(TM))$. In fact, since θ corresponds to the identity endomorphism of TM under this identification, so does $\tilde{\theta}$, but note that there is an additional step in the identification of $\tilde{\theta}$, namely that which uses equation 1.3.3, and hence $\eta_g(\rho)$ which depends on the spin structure $s_g = (\widetilde{SO}(M,g), \eta_g)$. In other words, $\tilde{\theta}$ may be regarded as the identity endomorphism of TM, but only after a spin structure-dependent identification.

Soldering and connections give rise to torsion forms. We have that the torsion, $\Theta^{\omega_g} \equiv d^{\omega_g}\theta$, vanishes identically because ω_g is the Levi-Cività connection of g. Let hor, $\widetilde{hor}$ be the horizontal projections on TSO(M,g), $T\widetilde{SO}(M,g)$ corresponding to the connections ω_g, $\tilde{\omega}$ respectively. Then it is easily shown that:

$$\widetilde{hor} \circ \eta_g^* = \eta_g^* \circ hor \qquad 1.3.10,$$

so that $\tilde{\Theta} \equiv d^{\tilde{\omega}}\tilde{\theta} = (\widetilde{hor} \circ d)(\eta_g^*\theta) = \widetilde{hor}(\eta_g^* d\theta) = \eta_g^*(d^{\omega}\theta) = \eta_g^*\,\Theta = 0$, so that $\tilde{\omega}$ has vanishing "torsion" also. So the "spin torsion" associated with a spin connection as given by definition (1.3)2 vanishes identically, but the fact that it exists is important, just as the Levi-Cività torsion is important - fluctuations within Conn(SO(M,g)) about ω_g will introduce torsion $\neq 0$.

We complete this section with some remarks on the use of spinor differential equations. In Chapter Three, we examine certain spinor differential equations, in particular on embedded submanifolds of a spacetime, and we shall introduce the required explicit formulae involving spin connections when we need them. Spinor differential equations, in addition, are used extensively in geometry (see the references cited in section 1.0). Of particular importance is the Dirac operator (associated with the Dirac representation of the Clifford algebra $(\mathbb{R}^n, \mathrm{can})$ and with the Levi-Cività connection of a metric), and corresponding Weitzenböck type formulae for the "Dirac Laplacian". The use of one such formula is an essential ingredient in the Witten proof of the positivity of the ADM mass in general relativity. See Chapter Three and also Witten [W 9], Parker and Taubes [P 4]. The Dirac operator is also very important in string theory, see for example, Mikkelson [M 11].

The spin connections defined in this section depend both on the metric and the particular spin structure used (The fact that we have used a particular representative of the equivalence class of spin structures $[s_g] \in \Sigma(M,g)$ is unimportant because, as we have remarked above, equivalent spin structures give rise to equivalent connections). In the next section, we examine more closely the inter-relationship between metrics and spinor fields, an inter-relationship that is important in any dynamical theory of metrics and fields.

1.4 Configurations

In this section we attempt to examine more closely the coupled configuration space of metrics and spinors. We commence with a definition of the canonical principal $SO(n)$-bundle of a manifold.

Definition (1.4)1: Let M be an oriented manifold of dimension n. Then the canonical principal $SO(n)$-bundle of M is given by:

$$SO(n) \hookrightarrow SO(M) \xrightarrow{\sigma_M} Met(M) \times M \qquad 1.4.1,$$

where $SO(M) = \{(g,u) \in Met(M) \times GL^+(M): \ u \in SO(M,g)\}$, and $\sigma_M(g,u) = (g,\pi_M(u))$, for all $(g,u) \in SO(M)$.

There exists a natural free right action of $SO(n)$ on $SO(M)$ defined by $((g,u),a) \mapsto (g,ua)$ for all $(g,u) \in SO(M)$, $a \in SO(n)$, and it is easy to see that σ_M, as just defined, is the corresponding quotient map making equation 1.4.1 a principal $SO(n)$-fibration.

Let pr_1: $Met(M) \times M \to Met(M)$ be projection onto the first factor. Then $(pr_1 \circ \sigma_M)^{-1}(g) \cong SO(M,g)$, for each metric g on M. i.e. the fibre above g in $pr_1 \circ \sigma_M$: $SO(M) \to Met(M)$ is precisely the bundle of oriented g-orthonormal frames.

Now let us assume M is spin and let $s \in \tilde{\Sigma}(M)$ be a spin structure on M. s is a Γ-prolongation $(\widetilde{\widetilde{GL}}^+(M),\eta)$ of $GL^+(M)$ to the group $\widetilde{\widetilde{GL}}^+(n,\mathbb{R})$, and we may now define a Λ-prolongation of $SO(M)$ to $Spin(n)$ as follows:

Let $\widetilde{\widetilde{SO}}(M,s) = \{(g,\tilde{u}) \in Met(M) \times \widetilde{\widetilde{GL}}^+(M): \ \tilde{u} \in \eta^{-1}(SO(M,g))\}$, and define $\eta(s)$: $\widetilde{\widetilde{SO}}(M,s) \to SO(M)$; $(g,\tilde{u}) \mapsto (g,\eta(\tilde{u}))$, for each $(g,\tilde{u}) \in \widetilde{\widetilde{SO}}(M,s)$. Now let $\tilde{\sigma}_M(s)$; $\widetilde{\widetilde{SO}}(M,s) \to Met(M) \times M$; $(g,\tilde{u}) \mapsto (g,\tilde{\pi}_M(\tilde{u}))$, where $\tilde{\pi}_M$: $\widetilde{\widetilde{GL}}^+(M) \to M$ is projection.

It is clear that

$$\mathrm{Spin}(n) \hookrightarrow \widetilde{\widetilde{SO}}(M,s) \xrightarrow{\tilde{\sigma}_M(s)} \mathrm{Met}(m)\times M \qquad 1.4.2,$$

is a principal Spin(n)-bundle, and that $(\widetilde{\widetilde{SO}}(M,s),\eta(s))$ is a Λ-prolongation of SO(M) to Spin(n).

Choosing an equivalent spin structure s' on M leads to a prolongation $(\widetilde{\widetilde{SO}}(M,s'),\eta(s'))$ which is equivalent, qua prolongation, to $(\widetilde{\widetilde{SO}}(M,s),\eta(s))$. Therefore, each $\alpha \in \Sigma(M)$ gives rise to an equivalence class $F(\alpha) = [(\widetilde{\widetilde{SO}}(M,s),\eta(s))]$ (any $s \in \alpha$) of Λ-prolongations of SO(M) to Spin(n).

Now fix the spin structure s (representing $\alpha \in \Sigma(M)$ - any other choice of representative leads to equivalent structures in what follows, so that all the constructions are natural). We now define the projection:

$$p(s) = pr_1 \circ \tilde{\sigma}_M(s)\colon \widetilde{\widetilde{SO}}(M,s) \to \mathrm{Met}(M) \qquad 1.4.3,$$

so that, for each metric g, $p(s)^{-1}(g)$ is just the bundle of $(g, \tilde{r}_g(s))$-spin frames (see the remark following definition (1.1)2 regarding the notion of (g, s_g)-spin frames for a g-spin structure s_g). The principal Spin(n)-bundle $\widetilde{\widetilde{SO}}(M,s)$ may be thus regarded as a kind of universal s-spin bundle in the sense that it contains all the spin frames (coming from s) for all the metrics on M.

Now suppose $\tilde{\rho} \in \mathrm{Hom}(\mathrm{Spin}(n), GL(V))$ is a representation of Spin(n) on the vector space V. We may define the associated vector bundle $\widetilde{\widetilde{SO}}(M,s) \times_{\mathrm{Spin}(n)} V$, so that, for each $g \in \mathrm{Met}(M)$, the bundle of spinors of type $(\tilde{r}_g(s), \tilde{\rho}, V)$ is precisely the fibre $(pr_1 \circ \tilde{\rho}_s)^{-1}(g)$, where $\tilde{\rho}_s : \widetilde{SO}(M,s) \times_{Spin(n)} V \to Met(M) \times M$ is projection.

A similar construction using the sum over representations of Spin(n) will lead to an algebra bundle over $Met(M) \times M$ which, when pr_1-projected onto $Met(M)$, has, as fibre over $g \in Met(M)$, the spinor algebra bundle associated to g and s.

For a fixed representation $\tilde{\rho}$ of Spin(n) on V, the associated vector bundle may be therefore regarded as a coupled configuration space of metrics and type-$(s, \tilde{\rho}, V)$ spinors. To obtain a configuration space of metrics and spinor <u>fields</u>, we consider the (infinite dimensional) vector bundle

$$E(M, s, \tilde{\rho}) \longrightarrow Met(M) \qquad 1.4.4,$$

where $E(M, s, \tilde{\rho}) = \{(g,\psi) : g \in Met(M) \text{ and } \psi \in \Gamma((pr_1 \circ \tilde{\rho}_s)^{-1}(g))\}$, and projection is just projection onto the first factor. The fibre of $E(M, s, \tilde{\rho})$ over a particular metric g is then the $C(M)$-module of spinor fields of type $(\tilde{r}_g(s), \tilde{\rho}, V)$.

Choosing another representative spin structure s' will yield an isomorphic vector bundle $E(M,s', \tilde{\rho})$ and hence an equivalent metric-spinor field configuration space. We write $E(M, \alpha, \tilde{\rho})$ for the vector bundle isomorphism class of metric-spinor field configuration spaces $E(M,s, \tilde{\rho})$, where s is a representative of $\alpha \in \Sigma(M)$. There are as many $E(M, \alpha, \tilde{\rho})$ as there are elements in $H^1(M;\mathbb{Z}_2)$, as was discussed in section 1.1. Obviously, an analogous construction gives rise to an isomorphism class of algebra bundles $A(M,\alpha)$ – the fibre of a particular representative $A(M,s)$ in this case being the algebras of spinor fields for the metrics on M arising from the spin structure s.

Returning now to, say, $E(M,s,\tilde{\rho})$, we remark that, in general, there is no natural way of identifying the fibres of $E(M,s,\tilde{\rho})$, i.e. we can't identify the spaces of spinor fields arising from different metrics. In certain cases, however, an identification of these spaces may be made, and we discuss two such possibilities in sections 1.5 and 1.6. In section 1.5 we restrict our attention to a subspace of $Met(M)$ consisting of a conformal class C of metrics, and then there is an association of the spinor fields arising from different metrics in the conformal class C. Indeed, we may construct a spin conformal structure which depends on C only, and not on a particular choice of representative metric. In section 1.6, we discuss the action of the diffeomorphism group on spinor fields. The diffeomorphism group action may be interpreted as an identification of the spaces of spinor fields arising from the subspace of metrics corresponding to an orbit of a particular metric on M, in other words, to a geometry on M.

A further enlargement of, say $E(M,\alpha,\tilde{\rho})$ may be made in order to take into account the possible existence of inequivalent spin structures on M (depending, of course, on the cohomology group $H^1(M;\mathbb{Z}_2)$). We may introduce the metric-spin structure-spinor field configuration space $\tilde{E}(M,\tilde{\rho}) = \{(g,s,\psi)\colon g \in Met(M),$ $s \in \tilde{\Sigma}(M)$ and ψ is a spinor field of type $(\tilde{r}_g(s), \tilde{\rho}, V)\}$. Quotienting $\tilde{E}(M,\tilde{\rho})$ by the equivalence relation $\sim$ on spin structures s on M yields a space $E(M,\tilde{\rho})$ equipped with obvious projections onto $Met(M)$ and onto $\Sigma(M)$. Since we shall always be concerned with a particular representative spin structure s of a fixed $\alpha \in \Sigma(M)$, we shall pursue the study of $E(M,\tilde{\rho})$ no further here. Obviously, the metric-spin structure-spinor field space $E(M,\tilde{\rho})$ would be important in a dynamical theory in which a transition between inequivalent spin structures could take place. In such a theory, an integration over $E(M,\tilde{\rho})$ would be performed in any path integral approach, and such an integration would involve a summation over inequivalent spin structures, as well as over spinor fields and metrics.

This section has introduced possibilities for a configuration space in the theory of spinors and we have indicated some of the complications inherent in building up a space of coupled metrics and spinor fields. In sections 1.5, 1.6 we investigate two ways in which structures on the space of metrics influence the coupled configuration space. There is obvious scope for further exploration of the way in which the structure of $Met(M)$, some aspects of which are discussed in Chapter Four, interacts with that of such spaces as $E(M,\alpha,\tilde{\rho})$.

The diffeomorphism group action of section 1.6 is actually an action on $E(M,\tilde{\rho})$ (some $\tilde{\rho}$), or on $E(M,\alpha,\tilde{\rho})$ if our attention is restricted to diffeomorphisms leaving invariant the spin structure. Before considering the diffeomorphism group, we turn to a study of spinors and conformal structure - an interaction which makes several appearances in this thesis.

1.5 Spinors and Conformal Structure

The reasons for this section are two-fold. Firstly, as we remarked in section 1.4, in general there is no way of identifying the spinor fields associated with different metrics, but if the metrics are members of some parameterized subspace of $Met(M)$, as in Chapter Four, an identification may sometimes be made. A simple example of such a parameterized subspace is provided by a conformal class.

A second reason for being interested in spinors and conformal structure is that the interaction between the two is important in various aspects of general relativity. One important area of general relativity where spinors and conformal structure come together is that of null structure. Null structure is very closely linked with conformal structure (see section 6.2), and, as we shall see in Chapter Three, there exist very natural and important spinor propagation equations on null hypersurfaces.

A particular example we shall give in this section, namely the spin conformal structure on S^2, is precisely the source of the notion of spin and conformally weighted functions, used in general relativity. Another example of how spin structure and conformal structure come together in a very physical situation is given in section 1.9.

We use the notation of section 6.2 throughout this section. Alternative approaches to this topic may be found in Huggett and Tod [H 16], Penrose [P 6], Penrose and Rindler [P 11], and Plymen and Westbury [P 15]. More geometrical interactions are discussed in Hijazi [H 9] and Branson [B 23].

The natural setting for spinors and conformal structure is a spin conformal structure and we discuss this idea shortly. Firstly though, following the spirit of section 1.4, we may consider a more naïve identification of fibres in $E(M,s,\tilde{\rho})$ over metrics belonging to a given $C \in \mathrm{Con}(M)$ (see, for example, Hijazi [H 9] for an application of these ideas to the Dirac operator, and section 6.2 for computations showing how the spin connections corresponding to metrics in the conformal class are related).

Suppose M is a spin manifold, g is a metric on M and $s_g = (\widetilde{\widetilde{SO}}(M,g), \eta_g)$ is a g-spin structure on M. Let $f \in C^+(M)$ and fg the corresponding conformally related metric. The function f gives rise to an isomorphism c_f of principal $SO(n)$-bundles, given by:

$$c_f\colon\ SO(M,g) \longrightarrow SO(M,fg);\quad u \longmapsto \{f(\pi_M(u))^{-\frac{1}{2}} e_a\} \qquad 1.5.1,$$

for all $u = \{e_a\} \in SO(M,g)$. We recall (equation 1.3.2) that, for any $\rho \in \mathrm{Hom}(SO(n), GL(V))$, V any vector space, there exists an isomorphism $\eta_g(\rho)\colon\ \widetilde{\widetilde{SO}}(M,g) \times_{\mathrm{Spin}(n)} V \longrightarrow SO(M,g) \times_{SO(n)} V$ of vector bundles. Here, $\mathrm{Spin}(n)$ acts on V via $\tilde{\rho} \equiv \rho \circ \Lambda$, as before. We therefore have an isomorphism (in fact an isometry) of $(SO(n)$-$)$ vector bundles

$$c_f(\rho) \circ \eta_g(\rho)\colon\ \widetilde{\widetilde{SO}}(M,g) \times_{\mathrm{Spin}(n)} V \longrightarrow SO(M,fg) \times_{SO(n)} V \qquad 1.5.2,$$

using equations 1.3.2 and 1.5.1.

We now let $\widetilde{\widetilde{SO}}(M,fg)$ be the unique principal $\mathrm{Spin}(n)$-bundle over M, such that $SO(M,fg) \times_{SO(n)} V$ is the vector bundle associated with $\widetilde{\widetilde{SO}}(M,fg)$ via the representation $\tilde{\rho}$ of $\mathrm{Spin}(n)$ on V. In other words, $\widetilde{\widetilde{SO}}(M,fg)$ is defined to be the bundle of $\mathrm{Spin}(n)$-frames for the vector bundle $SO(M,fg) \times_{SO(n)} V$. The identification of $\widetilde{\widetilde{SO}}(M,fg) \times_{\mathrm{Spin}(n)} V$ with $SO(M,fg) \times_{SO(n)} V$ used to define $SO(M,fg)$ gives rise to a homomorphism:

$$\eta_{fg}\ :\ \widetilde{\widetilde{SO}}(M,fg) \longrightarrow SO(M,g) \qquad 1.5.3,$$

of principal bundles over id_M, and then we may define a unique lift

$\tilde{c}_f$ of the principal SO(n)-bundle isomorphism c_f, so that $\tilde{c}_f$ is an isomorphism of principal Spin(n)-bundles

$$\tilde{c}_f : \widetilde{\widetilde{SO}}(M,g) \longrightarrow \widetilde{\widetilde{SO}}(M,fg) \qquad 1.5.4,$$

such that $c_f \circ \eta_g = \eta_{fg} \circ \tilde{c}_f$.

Using the fact that c_f gives rise to isometries of vector bundles associated with $SO(M,g)$, it is easily seen that $(\widetilde{\widetilde{SO}}(M,fg),\ \eta_{fg})$ is a fg-spin structure on M.

We summarize the above construction: Given any g-spin structure $s_g = (\widetilde{\widetilde{SO}}(M,g),\ \eta_g)$ for some metric g on the spin manifold M, there is a natural fg-spin structure $s_{fg} = (\widetilde{\widetilde{SO}}(M,fg),\ \eta_{fg})$ for any metric fg in the conformal class C_g defined by g. Moreover, there is a corresponding identification of spinors (and hence spinor fields) of type $(s_g, \tilde{\rho}, V)$ with spinors (spinor fields) of type $(s_{fg}, \tilde{\rho}, V)$, for any $\tilde{\rho} \in \mathrm{Hom}(\mathrm{Spin}(n), GL(V))$. This identification is such that any spinor (field) associated with fg has the same components (in V) with respect to any $(s_{fg}$, fg)-spin frame as the corresponding spinor (field) associated with g has with respect to the corresponding (s_g, g)-spin frame. Corresponding identifications of connections and so on may similarly be made, but it is more natural to consider such questions within the framework of spin conformal structures as we now indicate:

Suppose $C \in \mathrm{Con}(M)$ is a conformal structure on the oriented manifold M. Then C is equivalent to a reduction CO(M,C) of the bundle $GL^+(M)$ of oriented frames of M to the subgroup $CO(n) \cong SO(n) \times \mathbb{R}^+$ of $GL^+(n, \mathbb{R})$. We recall that any representation $\hat{\rho}$ of CO(n) on a vector space V is of the form $\hat{\rho}(a,r) = r^w \rho(a)$, for all $(a,r) \in CO(n)$, where $w \in \mathbb{C}$ (or $\mathbb{R}$ if V is real) is called the conformal weight of the representation $\hat{\rho}$, and ρ is some representation of the group $SO(n)$ on V.

There exists a trivial extension of the double covering Λ: $Spin(n) \to SO(n)$ to a non-trivial $(n \geq 2)$, double covering (universal for $n \geq 3$) $\hat{\Lambda}$ of $CO(n)$ given by:

$$\hat{\Lambda} : \widetilde{\widetilde{CO}}(n) \to CO(n); \quad (A,r) \mapsto (\Lambda(A), r) \qquad 1.5.5,$$

for all $(A,r) \in \widetilde{\widetilde{CO}}(n) \cong Spin(n) \times \mathbb{R}^+$.

We may now define a C-spin structure s_C on M in an obvious manner:

Definition (1.5)1: Let M be an oriented manifold of dimension n and let $C \in Con(M)$. Then a C-spin structure s_C is a $\hat{\Lambda}$-prolongation $(\widetilde{\widetilde{CO}}(M,C), \eta_C)$ of $CO(M,C)$ to the group $\widetilde{\widetilde{CO}}(n)$.

We denote the set of C-spin structures on M by $\tilde{\Sigma}(M,C)$ and the set of equivalence classes by $\Sigma(M,C)$, as in section 1.1. The obstruction to defining a C-spin structure on M is identical to that for a spin structure or a g-spin structure, namely the second Stiefel-Whitney class $w_2(TM)$. We have analogous maps to those in section 1.1: $\tilde{\Sigma}(M,C) \leftrightarrow \tilde{\Sigma}(M)$, $\Sigma(M,C) \leftrightarrow \Sigma(M)$, for any $C \in Con(M)$ and we may perform similar constructions to those in sections 1.3 and 1.4 for g-spin structures.

Any representation of $\widetilde{\widetilde{CO}}(n)$ is of the form:

$$\overset{\Delta}{\rho}(A,r) = r^w \tilde{\rho}(A) \qquad 1.5.6,$$

for all $(A,r) \in \widetilde{\widetilde{CO}}(n)$, where $w \in \mathbb{C}$ (or $\mathbb{R}$ if the representation is real) is called the conformal weight of the representation $\overset{\Delta}{\rho}$, and $\tilde{\rho}$ is some representation of $Spin(n)$. A differential form on M with values in the vector bundle $\widetilde{\widetilde{CO}}(M,C) \times_{\widetilde{\widetilde{CO}}(n)} V$, associated to $\widetilde{\widetilde{CO}}(M,C)$ via the representation $\overset{\Delta}{\rho} \in Hom(\widetilde{\widetilde{CO}}(n), GL(V))$, is said to have conformal weight w. In particular for the case of 0-forms on M with values in $\widetilde{\widetilde{CO}}(M,C) \times_{\widetilde{\widetilde{CO}}(n)} V$, we have the notion of a spinor field of type $(s_C, \overset{\Delta}{\rho}, V)$; we say such a spinor field has conformal weight w and spin (representation) $\tilde{\rho}$.

<u>Definition (1.5)2</u>: Let $s_C = (\widetilde{\widetilde{CO}}(M,C), \eta_C)$ be a C-spin structure on the oriented manifold M where C is some conformal structure on M. Let $\overset{\Delta}{\rho} \in Hom(\widetilde{\widetilde{CO}}(n), GL(V))$ have conformal weight $w \in \mathbb{C}$ and spin (representation) $\tilde{\rho} \in Hom(Spin(n), GL(V))$. A <u>spinor</u> of type $(s_C, \overset{\Delta}{\rho}, V)$, i.e. an element of $\widetilde{\widetilde{CO}}(M,C) \times_{\widetilde{\widetilde{CO}}(n)} V$, is said to have <u>conformal weight</u> w and <u>spin</u> (representation) $\tilde{\rho}$. Similarly for <u>spinor fields</u>, $\psi \in \Gamma(\widetilde{\widetilde{CO}}(M,C) \times_{\widetilde{\widetilde{CO}}(n)} V)$.

If the spin transformation properties of a particular spinor field are known, then the conformal weight w may often be assigned using geometrical or physical considerations: For example, a Dirac spinor field has dimension $(Length)^{-3/2}$, and so such a field is assigned conformal weight $w = -3/2$ (see Penrose and Rindler [P 11] and Audretsch et al. [A 27] for more discussion of such matters).

The definition of spin conformal connection follows that of a spin connection in section 1.3. We have

$$\hat{\Lambda}_* \equiv D\hat{\Lambda}(e,1): L(\widetilde{\widetilde{CO}}(n)) \longrightarrow L(CO(n)) \qquad 1.5.7,$$

is an isomorphism of the Lie algebra $L(\widetilde{\widetilde{CO}}(n)) \cong L(Spin(n)) \oplus \mathbb{R}$ onto the Lie algebra $L(CO(n)) \cong L(SO(n)) \oplus \mathbb{R}$, so given any connection ω_C in $CO(M,C)$, we may define a connection $\tilde{\omega} \equiv \tilde{\omega}(s_C, C)$ in $\widetilde{\widetilde{CO}}(M,C)$ by

$$\tilde{\omega} = \hat{\Lambda}_*^{-1} \circ \eta_C^* \overset{\vee}{\omega}_C \qquad 1.5.8.$$

$\tilde{\omega}$ is uniquely defined by the requirement that $D\eta_C$ maps $\tilde{\omega}$-horizontal subspaces onto ω_C horizontal subspaces (see sections 1.3 and 6.1). $\tilde{\omega}$ is called a <u>spin conformal connection</u>.

The bundle of oriented C-frames is a subbundle of $GL^+(M)$ and so has induced on it a soldering form θ, and hence the notion of (conformal) torsion $\Theta^{\omega_C} \equiv d^{\omega_C}\theta$, associated with any connection ω_C in $CO(M,C)$. If Θ^{ω_C} vanishes, we say that ω_C is a <u>Weyl connection</u>. Similarly, we have a spin conformal soldering on $\widetilde{CO}(M,C)$ with associated spin conformal torsion $\Theta^{\tilde{\omega}} \equiv d^{\tilde{\omega}}\,\tilde{\omega}$.

Since $L(CO(n)) \cong L(SO(n)) \oplus \mathbb{R}$, we may write any connection ω_C in $CO(M,C)$ in the form:

$$\omega_C = \overset{\vee}{\omega}_C \oplus \phi_C.1 \qquad 1.5.9,$$

where $\overset{\vee}{\omega}_C$ is a 1-form on $CO(M,C)$ taking its values in $L(SO(n))$, and ϕ_C is an $\mathbb{R}$-valued 1-form on $CO(M,C)$. We have a similar splitting for the form $\tilde{\omega}_C$:

$$\tilde{\omega} = \overset{\vee}{\tilde{\omega}}_C \oplus \tilde{\phi}_C.1 \qquad 1.5.10,$$

where $\overset{\vee}{\tilde{\omega}}$ is $L(Spin(n))$-valued, and $\tilde{\phi}_C$ is $\mathbb{R}$-valued.

Equations 1.5.9, 1.5.10 enable a splitting of curvature, covariant derivatives, etc., into a "spin"/"rotation" part and a "conformal" part. The following example will illustrate the above ideas:

<u>Example (1.5)1</u>: (The two-sphere; S^2). To demonstrate the ideas just discussed, and to introduce the very useful spin and conformally weighted functions, we give the example of the standard two-sphere.

There is also the concept of spin and boost weighted functions (related to an embedding of S^2 into a Lorentzian four-manifold, or, more generally to a $\mathbb{C}^*$ reduction of the bundle of spacetime oriented Lorentzian-orthonormal frames) which we discuss in section 2.3. See also Geroch et al. [G 7], Penrose and Rindler [P 11], Curtis and Lerner [C 19], Dray [D 12].

Let can be the standard (round) Riemannian metric on S^2, and denote by Can $= C_{can}$ the standard conformal structure on S^2. The standard conformal two sphere turns out to play an important rôle in general relativity. Indeed, we show below that $\mathrm{Conf}(S^2, \mathrm{Can})$ is isomorphic to the restricted Lorentz group. We have already discussed spin structures on n-spheres (see 1.2.4), and for the case $n = 2$, we have the bundle of oriented can-orthonormal frames:

$$S^1 \hookrightarrow SO(3) \longrightarrow S^2 \qquad 1.5.11,$$

and the unique can-spin structure:

$$S^1 \hookrightarrow S^3 \longrightarrow S^2, \quad S^3 \xrightarrow[\eta]{\Lambda} SO(3) \qquad 1.5.12,$$

where $S^1 \hookrightarrow S^3 \longrightarrow S^2$ is a Hopf fibration and $\eta \equiv \Lambda$ is the usual double covering.

We also have a unique prolongation $(\widetilde{\widetilde{CO}}(S^2,\mathrm{Can}),\eta)$ of the bundle $CO(S^2,\mathrm{Can})$ to the group $\widetilde{\widetilde{CO}}(2) \cong \mathrm{Spin}(2) \times \mathbb{R}^+ \cong \mathbb{C}^*$. The total space of the prolongation is $\widetilde{\widetilde{CO}}(S^2,\mathrm{Can}) = \mathbb{C}^2 - \{\underline{0}\} \cong S^3 \times \mathbb{R}^+$, and the projection is the standard projection: $\mathbb{C}^2 - \{\underline{0}\} \xrightarrow{\pi} \mathbb{C}\mathbb{P}^1 \cong S^2$. Thus the unique (up to equivalence of prolongations) spin conformal structure of (S^2,Can) may be written:

$$\mathbb{C}^* \hookrightarrow \mathbb{C}^2 - \{\underline{0}\} \xrightarrow{\pi} S^2 \qquad 1.5.13.$$

Given any representation of $\mathbb{C}^*$ on a vector space, we may form the associated vector bundle as above. In particular, we may consider representations of $\mathbb{C}^* \equiv S^1 \times \mathbb{R}^+$ on $\mathbb{C}$. Let $\overset{\Delta}{\rho}_{s,w} \in \mathrm{Hom}(\mathbb{C}^*, GL(1,\mathbb{C}))$ be given by:

$$\overset{\Delta}{\rho}_{s,w}(re^{i\theta}); \quad z \mapsto r^{-2w}\, e^{2is\theta} z \qquad 1.5.14,$$

for all $re^{i\theta} \in \mathbb{C}^*$, $z \in \mathbb{C}$, where $w \in \mathbb{C}$ and $2s \in \mathbb{Z}$. The splitting $\mathbb{C}^* \cong S^1 \times \mathbb{R}^+$ corresponds to the usual splitting of $\widetilde{CO}(n) \equiv \mathrm{Spin}(n) \times \mathbb{R}^+$, and we refer to w as the <u>conformal weight</u> of $\overset{\Delta}{\rho}_{s,w}$, and to s as the <u>spin weight</u> of $\overset{\Delta}{\rho}_{s,w}$. We have changed our usual definition of conformal weight by a factor of -2 in order to conform with standard conventions in the literature of general relativity (for example, Held et al. [H 7]).

Let $E(s,w) = (\mathbb{C}^2 - \{\underline{0}\}) \times_{\mathbb{C}^*} \mathbb{C}$ be the complex line bundle over S^2 associated with the representation $\overset{\Delta}{\rho}_{s,w}$ of $\mathbb{C}^*$ on $\mathbb{C}$.

<u>Definition (1.5)3</u>: A <u>function of spin weight s and conformal weight w</u> is a section of $E(s,w)$.

It can be shown that the Chern class (which completely characterizes line bundles) of $E(s,w)$ is $-2s \in H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$, so that, although the bundles $E(s,w)$ arise from different representations of $\mathbb{C}^*$, they are not all topologically distinct ($E(s,w)$ is topologically bundle isomorphic to $E(s,w')$, for all s,w,w'). We also note the natural identifications (reflecting the various representations), $\overline{E(s,w)} = E(-s,w)$ and $E(s,w) \otimes_{\mathbb{C}} E(s',w') = E(s+s',w+w')$.

Spin and conformally weighted functions on S^2 have been used in geometry (see, for example, Parker and Rosenberg [P 3] for an application of the natural conformal Laplacian acting on such sections), but for us their relationship with structures in general relativity will be most important. We will investigate some of these relationships in section 1.9 and in Chapters Two and Three, and as preparation for this we now interpret spin and conformally weighted functions in a more suitable way. We show, in particular, how the group $SL(2,\mathbb{C})$ relates to conformal structures on S^2, and since $SL(2,\mathbb{C})$ is the "spin group" of four dimensional Lorentzian geometry (see section 1.7), this provides a basis for the use of spin and conformally weighted functions in general relativity. Similar, but less geometrical, approaches to the link between Lorentzian geometry in four dimensions on the one hand and the two sphere on the other have been made by Penrose [P 6], Newman and Penrose [N 2], Held et al. [H 7], Lind et al. [L 7] and Hansen et al. [H 3].

The standard conformal structure Can on S^2 stems in a natural way from a group action when S^2 is regarded as being one-dimensional complex projective space $\mathbb{CP}^1$, and we demonstrate this fact below (see 1.5.48). We construct the required actions as follows:

Suppose first that we are given an action

$$G \times Y \longrightarrow Y \qquad 1.5.15,$$

of a group G on a set Y. Suppose also that H is a normal subgroup of G, and then the induced action of G on Y passes to a quotient action

$$G/H \times Y/H \to Y/H \ ; \quad (gH, [y]_H) \to [gy]_H \qquad 1.5.16,$$

for all $gH \in G/H$ $[y]_H \in Y/H$. Let us now use 1.5.16 in the case where we take $G = GL(2,\mathbb{C})$, $H = \mathbb{C}^*$ $(\equiv \mathbb{C}^* \mathbb{1}_2)$ and $Y = \mathbb{C}^2 - \{\underline{0}\}$ with action given by

$$GL(2,\mathbb{C}) \times (\mathbb{C}^2 - \{\underline{0}\}) \to \mathbb{C}^2 - \{\underline{0}\} \ ; \quad (A,\underline{z}) \mapsto A\underline{z} \qquad 1.5.17,$$

for all $A \in GL(2,\mathbb{C})$, $\underline{z} \in \mathbb{C}^2 - \{\underline{0}\}$. The quotient group G/H is, in this case, $SL(2,\mathbb{C})$ (we have the short exact sequence of groups

$$1 \longrightarrow \mathbb{C}^* \xhookrightarrow{\iota} GL(2,\mathbb{C}) \xrightarrow{\alpha} SL(2,\mathbb{C}) \longrightarrow 1 \qquad 1.5.18,$$

where ι is inclusion and

$$\alpha(A) = (\det A)^{-\frac{1}{2}} A \qquad 1.5.19,$$

for all $A \in GL(2,\mathbb{C})$, where we take the principal value of the square root). The induced quotient action on $\mathbb{C}^2 - \{\underline{0}\}/\mathbb{C}^* = \mathbb{CP}^1 \cong S^2$ is given by $\phi \in \mathrm{Hom}(SL(2,\mathbb{C}), \mathrm{Diff}(S^2))$ where

$$\phi_A([\underline{z}]) = [A\underline{z}], \qquad 1.5.20,$$

for all $A \in SL(2,\mathbb{C})$ and $[\underline{z}] \in S^2$ (Here $[\underline{z}] \equiv \pi(\underline{z}) \in S^2$ denotes the $\mathbb{C}^*$-orbit of $\underline{z} \in \mathbb{C}^2 - \{\underline{0}\}$).

It is straightforward to show that $\mathrm{Ker}\,\phi \cong \mathbb{Z}_2$, and so $\mathrm{Im}\,\phi \equiv \{\phi_A : A \in SL(2,\mathbb{C})\} \leqslant \mathrm{Diff}(S^2)$ is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$. However, we demonstrate below in section 1.7 that there exists a double covering by $SL(2,\mathbb{C})$ of the restricted Lorentz group $SO^{\uparrow}(1,3)$, and hence $\mathrm{Im}\phi \cong SL(2,\mathbb{C})/\mathbb{Z}_2 \cong SO^{\uparrow}(1,3)$. We now show that $\mathrm{Conf}(S^2,\mathrm{Can}) \cong \mathrm{Im}\phi$:

Firstly we exhibit more explicitly the $SL(2,\mathbb{C})$-action given by ϕ. We introduce a family of coordinate charts on S^2 in the following way: for each $A \in SL(2,\mathbb{C})$, let

$$U_A = \{[\underline{z}] \in S^2 : (A\underline{z})^1 \neq 0\} \qquad 1.5.21,$$

where, for each $\underline{w} \in \mathbb{C}^2-\{\underline{0}\}$, we write $\underline{w} = \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$ with respect to the standard basis $\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$ of $\mathbb{C}^2$. Note that $S^2 - U_A$ consists of a single point for each $A \in SL(2,\mathbb{C})$. Now define the coordinate mappings:

$$\zeta_A : U_A \longrightarrow \mathbb{C} \; ; \quad [\underline{z}] \longmapsto (A\underline{z})^0/(A\underline{z})^1 \qquad 1.5.22,$$

and also, for future reference, the real valued functions P_A, $K_{(A,B)}$ and $\lambda_{(A,B)}$ defined by

$$P_A = \tfrac{1}{2}(1 + \zeta_A \overline{\zeta_A}) \qquad 1.5.23,$$

$$K_{(A,B)} = (1 + \zeta_B \overline{\zeta_B})(|a\zeta_B + b|^2 + |c\zeta_B + d|^2)^{-1} \qquad 1.5.24,$$

$$\exp(i\lambda_{(A,B)}) = (c\zeta_B + d)(\overline{c\zeta_B} + \overline{d})^{-1} \qquad 1.5.25,$$

for all $A, B \in SL(2,\mathbb{C})$ satisfying $BAB^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$.

We shall see shortly that $K_{(A,B)}$ may be interpreted as the local conformal factor associated with a transformation from ζ_B- to ζ_A-coordinates (regarded as an active local diffeomorphism) whilst $\lambda_{(A,B)}$ is the local angle of rotation between the corresponding ζ_A^{-1} (constant) and ζ_B^{-1} (constant) curves in $U_A \cap U_B$.

$\zeta \equiv \zeta_{\mathbb{1}_2}$ corresponds to the usual coordinate on the complement of the north pole in the Riemann sphere. For each $A \in SL(2,\mathbb{C})$, (U_A, ζ_A) is a chart of S^2 with ζ_A an (analytic) diffeomorphism of U_A onto $\mathbb{C}$ with

$$\zeta_A^{-1}(w) = [A^{-1} \binom{w}{1}] \qquad 1.5.26,$$

for all $w \in \mathbb{C}$.

From equation 1.5.22, it follows that

$$\zeta_A = \frac{a\zeta_B + b}{c\zeta_B + d} \qquad 1.5.27,$$

on $U_A \cap U_B$ if $AB^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In particular

$$\zeta_A = \frac{a\zeta + b}{c\zeta + d} \qquad 1.5.28,$$

so that

$$d\zeta_A = (c\zeta + d)^{-2} d\zeta \qquad 1.5.29,$$

for all $A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$. Another useful formula, again straightforward to verify, is

$$\zeta_A \circ \phi_B = \zeta_{AB} \quad \text{on} \quad U_{AB} \cap \phi_B^{-1}(U_A) \qquad 1.5.30,$$

and hence

$$P_A \circ \phi_B = P_{AB} \quad \text{on} \quad U_{AB} \cap \phi_B^{-1}(U_A) \qquad 1.5.31,$$

so that the ζ-coordinate representation of the $SL(2,\mathbb{C})$-action is given by

$$\zeta \circ \phi_A \circ \zeta^{-1} = \zeta_A \circ \zeta^{-1}; \quad w \longmapsto \frac{aw + b}{cw + d} \qquad 1.5.32,$$

for all $w \in (\zeta \circ \phi_A^{-1})(U)$ and $A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$.

Equation 1.5.32 (and corresponding equations for other ζ_B-coordinate systems) provides the most general holomorphic transformation of $\mathbb{CP}^1$, i.e. the most general conformal and orientation preserving transformation of (S^2,Can). Note that A and $-A(A \in SL(2\mathbb{C}))$ both give rise to the same conformal transformation of S^2, reflecting the fact demonstrated above that $\mathrm{Im}\phi \cong SL(2,\mathbb{C})/\mathbb{Z}_2$. Since any element of $\mathrm{Im}\phi$ induces a conformal transformation on S^2 as in equation 1.5.32, and conversely since any conformal transformation arises in this way, we have shown that

$$\mathrm{Conf}(S^2,\mathrm{Can}) \cong \mathrm{Im}\phi \cong SO^{\uparrow}(1,3) \qquad 1.5.33,$$

since this bijection is, by inspection, a homomorphism of Lie groups. We give an alternative demonstration of the isomorphism $\mathrm{Conf}(S^2,\mathrm{Can}) \cong SO^{\uparrow}(1,3)$ below in section 1.9.

We now calculate explicitly the effect of the $SL(2,\mathbb{C})$-action on the standard conformal structure Can, in particular on the round representative can. In this complex geometric setting, the metric can is identified with the Fubini-Study metric on $\mathbb{CP}^1$ and so

$$\mathrm{can}|U = P^{-2}\, d\zeta\, d\bar{\zeta} \qquad 1.5.34,$$

where $P \equiv P_{\mathbb{1}_2} = \frac{1}{2}(1 + \zeta\bar{\zeta}): U \longrightarrow \mathbb{R}$. In what follows, we write can rather than $\mathrm{can}|V$, where V is any open dense subset of S^2. Now we have $\phi_A^*(\mathrm{can}) = \phi_A^*(P^{-2}\, d\zeta d\bar{\zeta})$ $= (P \circ \phi_A)^{-2} d(\zeta \circ \phi_A) d(\bar{\zeta} \circ \phi_A) = P_A^{-2} d\zeta_A\, d\bar{\zeta}_A$ (by equations 1.5.30

and 1.5.31) $= K_A^2$ can, where

$$K_A \equiv K_{(A,\mathbb{1}_2)} = (1+\zeta\bar{\zeta})(|a\zeta+b|^2+|c\zeta+d|^2)^{-1} \qquad 1.5.35,$$

for all $A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$, where we have used equation 1.5.29.
To summarize,

$$\phi_A^*(\mathrm{can}) = K_A^2 \,\mathrm{can} \qquad 1.5.36,$$

showing directly the conformal action of $SO^\uparrow(1,3)$ on (S^2,Can).

The (U_A, ζ_A) charts may be used to locally trivialize the spin conformal bundle $\mathbb{C}^* \hookrightarrow \mathbb{C}^2-\{\underline{0}\} \twoheadrightarrow S^2$, and hence show that definition (1.5)3 coincides with the usual definition of spin and conformally weighted function (see the references cited above). We also refer the reader to Curtis and Lerner [C 15].

Corresponding to each $A \in SL(2,\mathbb{C})$, we define a local cross section e_A: $U_A \to \widetilde{\widetilde{CO}}(S^2,\mathrm{Can}) \cong \mathbb{C}^2-\{\underline{0}\}$ by

$$e_A([\underline{z}]) = [\sqrt{2}\,(A\underline{z})^1 P_A^{\frac{1}{2}}([\underline{z}])]^{-1}\underline{z} \qquad 1.5.37,$$

for all $[\underline{z}] \in U_A$. Then, for any $A,B \in SL(2,\mathbb{C})$, we have

$$e_A([\underline{z}]) = ((A\underline{z})^1 P_A^{\frac{1}{2}}([\underline{z}]))^{-1}((B\underline{z})^1 P_B^{\frac{1}{2}}([\underline{z}]))e_B([\underline{z}]) \qquad 1.5.38,$$

for all $[\underline{z}] \in U_A \cap U_B$. We may now define (au Steenrod) the principal $\mathbb{C}^*$-bundle $\widetilde{\widetilde{CO}}(S^2,\mathrm{Can})$ by the transition functions $g_{(A,B)}$: $U_A \cap U_B \longrightarrow \mathbb{C}^*$, given by:

$$g_{(A,B)}([\underline{z}]) = ((B\underline{z})^1 P_B^{\frac{1}{2}}([\underline{z}]))^{-1}((A\underline{z})^1 P_A^{\frac{1}{2}}([\underline{z}])) \qquad 1.5.39,$$

for all $[\underline{z}] \in U_A \cap U_B$. We obtain the corresponding transition functions for any associated vector bundle by applying the appropriate representation, so for the complex line bundle $E(s,w)$, the transition functions are $g^{E(s,w)}_{(A,B)}: U_A \cap U_B \longrightarrow GL(\mathbb{C}) \cong \mathbb{C}^*$, given by

$$g^{E(s,w)}_{(A,B)} = r^{-2w}_{(A,B)} \exp(2is\theta_{(A,B)}) \qquad 1.5.40,$$

where $g_{(A,B)} = r_{(A,B)} \exp(i\theta_{(A,B)})$ on $U_A \cap U_B$.

Using equations 1.5.24, 1.5.25, 1.5.27, 1.5.39 and 1.5.40, we obtain

$$g^{E(s,w)}_{(A,B)} = K^{w}_{(A,B)} \exp(is\,\lambda_{(A,B)}) \qquad 1.5.41,$$

with $K_{(A,B)}$, $\lambda_{(A,B)}$ defined as in equations 1.5.24, 1.5.25 respectively. Now recall that the corresponding local transformation law for a section η of $E(s,w)$ (i.e. for a spin and conformally weighted function on S^2) is given, in terms of local representatives $\{\eta_A\}$, by $\eta_A = g^{E(s,w)}_{(A,B)} \eta_B$, so we have, on $U_A \cap U_B$,

$$\eta_A = K^{w}_{(A,B)} \exp(is\,\lambda_{(A,B)})\,\eta_B \qquad 1.5.42.$$

Conversely, any family $\{\eta_A\}$ of local complex-valued functions on S^2 satisfying the transformation law given by equation 1.5.42 defines a section η of $E(w,s)$. Note that equation 1.5.42 is precisely the usual definition of a spin and conformally weighted function given by Newman and Penrose [N 2], Held et al. [H 7], so we have shown that definition (1.5)3 is consistent with the usual one. From a geometrical viewpoint the definition in terms of $\Gamma(E(s,w))$ is more illuminating, whilst equation 1.5.42 is often more useful for calculational purposes.

In order to make use of spin and conformally weighted functions, we must introduce a connection and hence covariant derivatives. There is, in fact, a natural connection $\tilde{\omega}$ in $\widetilde{CO}(S^2,\mathrm{Can})$ which is just the lift of the standard Weyl connection in $CO(S^2,\mathrm{Can})$ as in equation 1.5.8, but it is more convenient to introduce $\tilde{\omega}$ directly as a connection in the principal $\mathbb{C}^*$-bundle $\widetilde{\widetilde{CO}}(S^2,\mathrm{Can}) = \mathbb{C}^2-\{\underline{0}\} \xrightarrow{\pi} S^2$. In fact, starting from the "bare" fibration π, we may construct not only the spin conformal connection $\tilde{\omega}$, but also rederive Can as the Fubini-Study conformal structure arising in a natural way as the projection of the canonical Hermitian structure on the total space:

Let h be the Hermitian metric on $\mathbb{C}^2-\{\underline{0}\}$ defined by

$$h(\underline{z})((\underline{z},\underline{u}),\ (\underline{z},\underline{v})) \quad = \quad \langle\underline{u},\underline{v}\rangle \quad \equiv \quad u^0\bar{v}^{0'} + u^1\bar{v}^{1'} \qquad 1.5.43,$$

for all $(\underline{z},\underline{u})$, $(\underline{z},\underline{v}) \in T_{\underline{z}}(\mathbb{C}^2-\{\underline{0}\}) \;=\; \{(\underline{z},\underline{w})\colon \underline{w} \in \mathbb{C}^2\}$, $\underline{z} \in \mathbb{C}^2-\{\underline{0}\}$. Note that $\mathbb{C}^*$ acts homothetically on $(\mathbb{C}-\{\underline{0}\},h)$:

$$R_\lambda^* h \quad = \quad |\lambda|^2 \quad h \qquad 1.5.44,$$

for all $\lambda \in \mathbb{C}^*$, where R is the right action of $\mathbb{C}^*$ on $\mathbb{C}^2-\{\underline{0}\}$.

Let $V = \operatorname{Ker} D\pi$ be the vertical distribution arising from π, so that

$$V_{\underline{z}} \quad = \quad DR^{\underline{z}}(1).\mathbb{C} \quad = \quad \{(\underline{z},\ w\underline{z})\colon w \in \mathbb{C}\} \qquad 1.5.45,$$

where, for each $\underline{z} \in \mathbb{C}^2-\{\underline{0}\}$, $R^{\underline{z}}\colon \mathbb{C}^* \to \mathbb{C}^2-\{\underline{0}\}$; $\lambda \longmapsto \lambda\underline{z}$ so that $DR^{\underline{z}}(1)$ is an isomorphism of $L(\mathbb{C}^*) \cong \mathbb{C}$ onto $V_{\underline{z}} \leq T_{\underline{z}}(\mathbb{C}^2-\{\underline{0}\})$. Now let H be the h-orthogonal complement of V, so that H is a differentiable distribution on $\mathbb{C}^2-\{\underline{0}\}$ with

$$H_{\underline{z}} = \{(\underline{z},\underline{u}) \in T_{\underline{z}}(\mathbb{C}^2-\{\underline{0}\}): \langle\underline{z},\underline{u}\rangle = 0\} \qquad 1.5.46,$$

and $T_{\underline{z}}(\mathbb{C}^2-\{\underline{0}\}) = H_{\underline{z}} \oplus V_{\underline{z}}$, for each $\underline{z} \in \mathbb{C}^2-\{\underline{0}\}$. The distribution H is equivariant under the $\mathbb{C}^*$-action, i.e. $H_{\lambda\underline{z}} = DR_\lambda(\underline{z}).H_{\underline{z}}$, and hence we may define a connection in π by requiring that H is the corresponding horizontal distribution. Using 1.5.45 and 1.5.46 it follows that the connection form $\tilde{\omega}$ is given by

$$\tilde{\omega}(\underline{z}).(\underline{z},\underline{u}) = |\underline{z}|^{-1}\langle\underline{z},\underline{u}\rangle \qquad 1.5.47,$$

for all $(\underline{z},\underline{u}) \in T_{\underline{z}}(\mathbb{C}^2-\{\underline{0}\})$, $\underline{z} \in \mathbb{C}^2-\{\underline{0}\}$. We have called this connection $\tilde{\omega}$ since it may be shown that it coincides with the lift of the Weyl connection in $CO(S^2,\mathrm{Can})$ mentioned above. We shall return to $\tilde{\omega}$ below, but first let us give a description of how Can falls out of the fibration π.

Each horizontal subspace $H_{\underline{z}}$ carries a Hermitian inner product induced by the Hermitian structure h in $T(\mathbb{C}^2-\{\underline{0}\})$, and we define a unique conformal structure on S^2 by requiring that $D\pi(\underline{z})|H_{\underline{z}}: H_{\underline{z}} \to T_{[\underline{z}]}S^2$ be a homothetic isomorphism for each $\underline{z} \in \mathbb{C}^2-\{\underline{0}\}$. i.e. we define the cos(angle) of u, v, $\in T_{[\underline{z}]}S^2$ to be:

$$\|\underline{u}_{\underline{z}}\|^{-1}\|\underline{v}_{\underline{z}}\|^{-1}\, h(\underline{z})((\underline{z},\underline{u}_{\underline{z}}),\ (\underline{z},\underline{v}_{\underline{z}})) \qquad 1.5.48,$$

where $\underline{u}_{\underline{z}}$, $\underline{v}_{\underline{z}}$ are the unique horizontal vectors in $H_{\underline{z}}$ projecting onto u,v respectively. The conformal structure given by 1.5.48 is well defined because $\mathbb{C}^*$ acts on $(\mathbb{C}^2-\{\underline{0}\},h)$ homothetically (see equation 1.5.44).

The above construction is a straightforward generalization of the construction of the Fubini-Study metric from the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n$ (see, for example, Kobayashi and Nomizu [K 8]) - here we just product with $\mathbb{R}^+$ to get the fibration $\mathbb{C}^* \hookrightarrow \mathbb{C}^{n+1} - \{\underline{0}\} \longrightarrow \mathbb{CP}^n$ (n = 1 in our case), and hence the Fubini-Study conformal structure, rather than a specific representative metric. Thus, the conformal structure defined by 1.5.48 is precisely Can and contains the standard round metric can (which is, of course, just the Fubini-Study metric on $\mathbb{CP}^1$ given by equation 1.5.34). We have thus demonstrated that the standard conformal structure on S^2 may be reconstructed from the spin conformal bundle $\tilde{\tilde{CO}}(S^2,\mathrm{Can}) = \mathbb{C}^2 - \{\underline{0}\} \xrightarrow{\pi} S^2$ in a natural way.

We now return to the connection $\tilde{\omega}$ which also arises in a natural way from the fibration π, but may equivalently be regarded as coming from a conformal connection on S^2. This connection $\tilde{\omega}$ induces covariant derivatives on sections of the associated vector bundles $E(s,w)$ in the usual way, so that we may write down differential equations involving spin and conformally weighted functions on S^2. Note that $\mathbb{C}^*$ is abelian, and so the curvature in $E(s,w)$ may be regarded as a global $L(\mathbb{C}^*) \cong \mathbb{C}$ - valued 2-form on S^2. In fact it may be shown (see, for example, Dray [D 12]) that the curvature in $E(s,w)$ is just is vol(can) corresponding to the fact that the Chern number of $E(s,w)$ is $\frac{i}{2\pi}\int_{S^2}(\text{curvature}) = \frac{i}{2\pi}\int_{S^2} is\,\mathrm{vol(can)} = -2s$.

Dray [D 12] also gives local expressions for the connection 1-forms and these may be used to derive an explicit formula for the covariant derivatives acting on spin and conformally weighted functions. These covariant derivatives are essentially the $\eth$ (eth)-operators introduced by Newman and Penrose [N 2], and we now give the usual local expressions for $\eth$ using the notation given above:

Recall that we have introduced a family of charts $\{(U_A, \zeta_A)\}$ on S^2 parameterized by $SL(2,\mathbb{C})$ (see 1.5.21, 1.5.22), and also a family of functions $\{P_A\}$. It is straightforward to show that the functions P_A defined in 1.5.23 give rise to a global section P of the complex line bundle $E(0,1)$. The section P is just the Hermitian norm arising from $<,>$ (see equation 1.5.43) under the usual correspondence between sections of associated bundles on the one hand and equivariant mappings on the total space of a principal bundle on the other. For each $A \in SL(2,\mathbb{C})$, let us define the quantity

$$m_A = \sqrt{2}\, P_A \frac{\partial}{\partial \zeta_A} \qquad 1.5.49,$$

which can be regarded as either an element of $\Gamma(T^{\mathbb{C}}S^2 \otimes E(1,-1)|U_A)$, or equivalently as a complex vector field on $U_A \subseteq S^2$. In fact, m_A is a null vector field since it has zero norm in the inner product in $T^{\mathbb{C}}S^2$ obtained by extending can.

Now let $\nabla^{(s,w)}: \Gamma(E(s,w)) \longrightarrow \Omega^1(E(s,w))$ be the covariant derivative operator in $E(s,w)$ induced by the connection $\tilde{\omega}$. The eth operator on U_A is just the directional covariant derivative in the m_A-direction. Namely, let $\eth_A^{(s,w)}: \Gamma(E(s,w)|U_A) \longrightarrow \Gamma(E(s+1,w-1)|U_A)$ be defined by

$$\eth_A^{(s,w)} = \iota_{m_A} \circ \nabla^{(s,w)} \qquad 1.5.50,$$

where ι_X denotes interior multiplication by a vector field X.

To be more explicit, we restrict our attention to the open set $U \equiv U_{\Pi_2}$ and use the vector field $m \equiv m_{\Pi_2} \equiv \sqrt{2}\, P \frac{\partial}{\partial \zeta}$. The corresponding eth $\eth^{(s,w)} \equiv \eth^{(s,w)}_{\Pi_2}$ then reduces to the usual definition:

$$\eth^{(s,w)}\, \eta \;=\; 2\, P^{w-s} \frac{\partial}{\partial \zeta} (P^{s-w} \eta) \qquad 1.5.51,$$

for all spin and conformally weighted functions η on U. Note that $\eth \equiv \eth^{(s,w)}$ lowers the conformal weight by one as well as increasing the spin weight by one.

The operators $\eth^{(s,w)}_A$ defined in equation 1.5.50 give rise to a globally defined operator $\eth\colon \Gamma(E(s,w)) \longrightarrow \Gamma(E(s+1,w-1))$, since $\eth_A \eta_A$ and $\eth_B \eta_B$ are related according to equation 1.5.42 (with s replaced by s+1, w replaced by w-1). Regarding $\eta \in \Gamma(E(s,w))$ as an equivariant map from $\mathbb{C}^2 - \{\underline{0}\}$ into $\mathbb{C}$, we have an explicit formula:

$$\eth\eta \;=\; 2P^{-1}\left(\overline{z}^{1'} \frac{\partial \eta}{\partial z^1} - \overline{z}^{o'} \frac{\partial \eta}{\partial z^o} \right) \qquad 1.5.52.$$

According to equations 1.5.17 - 1.5.36, a conformal transformation of S^2 arises from a $GL(2,\mathbb{C})$ transformation of $\mathbb{C}^2 - \{\underline{0}\}$. Under this $GL(2,\mathbb{C})$-action, P will not be invariant, and so $\eth$ will not be conformally invariant in general. For special choices of (s,w) however, we do have conformal invariance. In particular, for $w - s + 1 \in \mathbb{Z}^+$, the operator $(\eth^{(s,w)})^{w-s+1}\colon \Gamma(E(s,w)) \to \Gamma(E(w+1,s-1))$ is conformally invariant (see Eastwood and Tod [E 3]).

The relationship between S^2 and $SO^{\uparrow}(1,3)$ given by 1.5.33 leads to the use of the $\eth$-operators in general relativity, and we shall take up this below. Further geometrical properties of $\eth$ have been discussed by various authors:

Further insights into $\eth$ may be gleaned by regarding S^2 as $\mathbb{CP}^1$, and associating $\eth$ with the ∂-(or $\bar{\partial}$-) operator of complex analysis. We refer the reader to Eastwood and Tod [E 3] for a treatment along these lines. The Laplacian $\bar{\eth}\eth$ acting on sections of $E(s,w)$ may also be introduced. This leads to the notion of spin and conformally weighted spherical harmonics (i.e. eigensections of $\bar{\eth}\eth$). See Dray [D 11], [D 12], Goldberg et al. [G 13], Kuwabara [K 17]. Penrose and Rindler [P 11].

Note that there exist various slightly different conventions regarding $\eth$. The differences arise both because of overall multiplications by normalizing factors, and also because of the possibility of associating $\eth$ with $\bar{\partial}$ rather than with ∂ in the complex geometric interpretation. Our conventions are essentially those of Eastwood and Tod [E 3], and the reader may refer to pp. 307-8 of Penrose and Rindler [P 11] for details of the relationship between the different conventions in use.

The importance to us of the various structures related to (S^2,Can) comes about because $S^2 \cong \mathbb{C}^2-\{\underline{0}\}/\mathbb{C}^*$ is the typical fibre of the bundle of future null directions of a (conformal) Lorentzian 4-manifold (the total space of this bundle being a Lorentzian version of the Penrose twistor space) and because the conformal group of (S^2,Can) is isomorphic to the restricted Lorentz group. These links will be taken up in section 1.9 and in Chapters Two and Three. This concludes, for the moment, our discussion of the spin conformal structure on the 2-sphere.

In this section, we have considered notions of spin associated with conformal structures. In the next section, we continue with our theme concerning the relationship between metrics and spin structures. We may regard a conformal structure as an orbit under the action of the (abelian) group $C^+(M)$ on $\mathrm{Met}(M)$. In section 1.6, we consider the action of the group $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$, and we discuss how this action interacts with spin structures.

1.6 Diffeomorphisms and Spinors

In this section, we give a brief survey of the inter-relationship between diffeomorphisms of a manifold M and spin structures on M. We consider in particular the transformation of spinor fields, in connection with the ideas of section 1.4.

Let M be an oriented manifold of dimension n, and let $\mathrm{Diff}(M)$ denote the group of orientation preserving diffeomorphisms of M. Let $\mathrm{Diff}_o(M) \triangleleft \mathrm{Diff}(M)$ be the group consisting of all orientation preserving diffeomorphisms of M which are isotopic to the identity. Note that $\mathrm{Diff}_o(M)$ is generally a simple group. See Chapter Four for more discussion of diffeomorphism groups.

The reasons for studying the action of diffeomorphisms on spinors are manifold. Spinor fields arise as important geometrical and physical objects, and so it is important to consider group actions on the space of spinor fields. In particular it is important to investigate the rôle of symmetry groups. In geometry and general relativity, one such symmetry group is the diffeomorphism group - a universal group in many respects, so any natural action of $\mathrm{Diff}(M)$ (or some subgroup thereof) on spinor fields is of interest.

One class of subgroups of Diff(M) of particular importance is the class of one parameter subgroups. These arise as the local one parameter families of (in general, local) diffeomorphisms generated by a vector field on the manifold M. The action of one parameter subgroups on geometrical objects leads to the very useful concept of Lie derivatives. The typical situation is as follows: Let $\psi \in \Gamma(B)$ where B is some bundle over M on whose sections there exists a right action by pullback of Diff(M) (denoted by $(\psi,\phi) \mapsto \phi^*\psi$). Let X be a vector field on M generating the local one-parameter group of diffeomorphisms $\{\phi_t\}$. Then the Lie derivative of ψ with respect to X is defined by:

$$L_X\psi = \left.\frac{d}{dt}\phi_t^*\psi\right|_{t=0} \qquad 1.6.1,$$

if this exists, so that $L_X\psi \in T_\psi\Gamma(B)$.

Thus, in order to define the notion of Lie derivative of spinor fields, a starting point is the action of Diff(M) on the space of spinor fields (if such an action is meaningful). Alternative approaches to Lie derivatives of spinor fields have been investigated by Lichnerowicz [L 16] (for the case of Killing vectors with respect to the metric giving rise to the spinor fields), by Huggett and Tod [H 16] (for conformal Killing vectors), and by Kosmann [K 10] (a more general and extensive treatment).

The problem here is that the relationship of spinors to the diffeomorphism group is very different to that of other geometrical objects. For instance, given any natural vector bundle B over the manifold M, there exists a pullback action of Diff(M) on B, which induces a pullback action on sections of B. Typically, such bundles B are tensor bundles associated with some reduction of the frame bundle of M, or perhaps sub-bundles of such bundles.

For example, the bundle of metrics (of a given signature) on M is a sub-bundle of $\odot^2 T^*M$, which, in turn, is a sub-bundle of $\otimes^2 T^*M$. These bundles admit an action of $\mathrm{Diff}(M)$ which induces an action on sections, so, for instance, we have the important action of the diffeomorphism group on the space of metrics (see Chapter Four). Another example which has been analyzed in the literature is the action of $\mathrm{Diff}(M)$ on conformal structures - see Fischer and Marsden [F 5].

The action of a group on a space becomes much more important if the action leaves invariant the structures of geometrical or physical importance. The diffeomorphism group itself does play the rôle of such a symmetry group in geometry and in general relativity, particularly when we consider the space of metrics: The important geometrical maps on $\mathrm{Met}(M)$ are all equivariant with respect to pullback action of $\mathrm{Diff}(M)$ on sections of tensor bundles. For instance, $F \circ \phi^* = \phi^* \circ F$, for all $\phi \in \mathrm{Diff}(M)$, where $F = \mathrm{Riem}, \mathrm{Ric}, \mathrm{Scal}, \mathrm{Vol}$ mapping $\mathrm{Met}(M)$ into the space of sections of the bundles $\mathrm{End}(\Lambda^2 TM)$, $\odot^2 T^*M$, $M \times \mathbb{R}$, $\Lambda^n T^*M$ respectively. In this sense, it is only the $\mathrm{Diff}(M)$-orbit of a metric which is important in geometry and general relativity (since $\mathrm{Ein}(g) = \mathrm{Ric}(g) - \frac{1}{2}\mathrm{Scal}(g)g$), and hence the space of interest is the space of geometries of M, $\mathrm{Geom}(M) = \mathrm{Met}(M)/\mathrm{Diff}(M)$, sometimes called the superspace of M. See Chapter Four and references therein.

The situation with spinor fields is complicated by the structure of the spinor field configuration space. The space of all Riemannian metrics, for example, forms an open cone in the vector space $S_2(M) = \Gamma(\odot^2 T^* M)$, and the action of $\mathrm{Diff}(M)$ arises by restriction of the action on $S_2(M)$. On the other hand, the spinor fields we use in geometry and physics arise from an a priori specification of a spin structure on M and of a metric (or possibly conformal structure - see section 1.5) on M. For each metric, there is a distinct space of spinor fields, and the configuration space of all spinor fields turns out to be the (infinite dimensional) vector bundle $E(M,\alpha,\tilde{\rho})$ (or $E(M,\tilde{\rho})$) described in section 1.4. This bundle has base space $\mathrm{Met}(M)$, and the fibre over the metric g on M is just the space of spinor fields of type $(s_g,\tilde{\rho},V)$ $(s_g = \tilde{r}_g(s)$, some $s \in \alpha$, $\tilde{\rho} \in \mathrm{Hom}(\mathrm{Spin}(n), GL(V)))$. Since a diffeomorphism ϕ transforms a metric g into the metric $\phi^* g$, the space of spinor fields over g should be transformed into the space of spinor fields over $\phi^* g$.

If we assume that an action of $\mathrm{Diff}(M)$ does not change the spin structure equivalence class α (for instance, any element of $\mathrm{Diff}_o(M)$ could not change α), then this action would be by automorphisms of $E(M,\alpha,\tilde{\rho})$ which covers the action of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$. The situation is even more complicated if $H^1(M;\mathbb{Z}_2) \neq 0$, so that there exist inequivalent spin structures on M. Elements of $\mathrm{Diff}(M)$ could permute the elements of $H^1(M;\mathbb{Z}_2)$ and hence transform a spin structure into an inequivalent spin structure. This would constitute an action of $\mathrm{Diff}(M)$ on $E(M,\tilde{\rho})$.

We see therefore that the diffeomorphism group might act on spaces of spinor fields at various different levels. A particular spin structure $s \in \tilde{\Sigma}(M)$ (representing $\alpha \in \Sigma(M)$) might be fixed, and we would wish to consider the action of all those diffeomorphisms which leave s invariant on spinors and spinor fields arising from s, and perhaps a choice of $g \in Met(M)$. Another level of action would be to consider transformations under arbitrary subgroups of $Diff(M)$ - possibly even the entire group. This second level of $Diff(M)$-action would be important in any considerations of the symmetries of a theory in which changes in spin structure are necessary.

We now give a brief discussion of how those actions may be realised. The basic constructions may be found in Dabrowski and Percacci [D 1]. Let us initially consider the case of diffeomorphisms which do not change the spin structure:

Let $\phi \in Diff(M)$, where M is an oriented spin manifold. Let $\hat{\phi}$ be the automorphism of $GL^+(M)$ induced by ϕ, so that:

$$\hat{\phi}(u) = \{D\phi(\pi_M(u)).e_a\} \qquad 1.6.1,$$

for all $u = \{e_a\} \in GL^+(M)$ (see definition (6.1)27).

Let $g \in Met(M)$ and consider the transformed metric $\phi^* g$. Any oriented $\phi^* g$-orthonormal frame is mapped by $\hat{\phi}$ into an oriented g-orthonormal frame, so that $\hat{\phi} \equiv \hat{\phi}|SO(M,\phi^* g)$ is an isomorphism of principal $SO(n)$-bundles:

$$\hat{\phi} : SO(M,\phi^* g) \longrightarrow SO(M,g) \qquad 1.6.2,$$

(Cf: equation 1.5.1 for the analogous isomorphism induced by an element f of the group $C^+(M)$. See also section 4.1 for a unification of 1.5.1, 1.6.2).

Now let $s = (\widetilde{\widetilde{GL}}^+(M), \eta) \in \tilde{\Sigma}(M)$ be a spin structure on M. We assume that $\hat{\phi}$ lifts to a $\widetilde{\widetilde{GL}}^+(n,\mathbb{R})$-automorphism $\overset{\Delta}{\phi}$ of $\widetilde{\widetilde{GL}}^+(M)$ such that

$$\eta \circ \overset{\Delta}{\phi} = \hat{\phi} \circ \eta \qquad 1.6.3.$$

For example, $\hat{\phi}$ will certainly admit such a lift if $\phi \in \mathrm{Diff}_o(M)$.

Definition (1.6)1: If $\phi \in \mathrm{Diff}(M)$ is such that $\hat{\phi} \in \mathrm{Aut}(GL^+(M))$ admits s lift to a $\overset{\Delta}{\phi} \in \mathrm{Aut}(\widetilde{\widetilde{GL}}^+(M))$ such that $\eta \circ \overset{\Delta}{\phi} = \hat{\phi} \circ \eta$, where $s = (\widetilde{\widetilde{GL}}^+(M), \eta) \in \tilde{\Sigma}(M)$, then we say that s is ϕ-invariant. Let $\mathrm{Diff}(M,s) = \{\phi \in \mathrm{Diff}(M)\colon\ s \text{ is } \phi\text{-invariant}\}$.

Note that $\mathrm{Diff}_o(M) \leqslant \mathrm{Diff}(M,s)$ for any $s \in \tilde{\Sigma}(M)$, but in general there may be diffeomorphisms which are not isotopic to id_M, but which do not change s.

Now let $s_g \equiv (SO(M,g), \eta_g) = \tilde{r}_g(s) \in \tilde{\Sigma}(M,g)$ (and similarly for ϕ^*g) as in section 1.1. Then $\overset{\Delta}{\phi} \equiv \overset{\Delta}{\phi}|\widetilde{\widetilde{SO}}(M,\phi^*g)$ is an isomorphism of principal Spin(n)-bundles:

$$\overset{\Delta}{\phi} : \widetilde{\widetilde{SO}}(M,\phi^*g) \longrightarrow \widetilde{\widetilde{SO}}(M,g) \qquad 1.6.4,$$

such that

$$\eta_g \circ \overset{\Delta}{\phi} = \hat{\phi} \circ \eta_{\phi^* g} \qquad 1.6.5,$$

(Cf: equation 1.5.4).

Let us now introduce spinor fields. Let $\tilde{\rho} \in \mathrm{Hom}(\mathrm{Spin}(n), GL(V))$ be any representative of Spin(n) on the vector space V. We note that $\overset{\Delta}{\phi}$ in equation 1.6.4 induces an isomorphism $\tilde{\phi}(\tilde{\rho})$ of the vector bundle $B(\phi^*g) = \widetilde{\widetilde{SO}}(M, \phi^*g) \times_{\mathrm{Spin}(n)} V$ onto the corresponding bundle $B(g)$, defined by:

$$\tilde{\phi}(\tilde{\rho}): \quad B(\phi^* g) \longrightarrow B(g); \quad [(\tilde{u},\xi)] \longmapsto [(\overset{\Delta}{\phi}(\tilde{u}),\xi)] \qquad 1.6.6,$$

for all $[(\tilde{u},\xi)] \in B(\phi^* g)$.

A spinor field of type $(s_g,\tilde{\rho},V)$ may be regarded either as a section ψ of $B(g)$ or as an equivariant map $\Psi : \widetilde{\widetilde{SO}}(M,g) \longrightarrow V$ (such that $\Psi(\tilde{u}A) = \tilde{\rho}(A^{-1}).\Psi(\tilde{u})$, for all $\tilde{u} \in \widetilde{\widetilde{SO}}(M,g)$, $A \in Spin(n)$). The group $Diff(M,s)$ transforms the latter by:

$$(\Psi,\phi) \longmapsto \Psi \circ \overset{\Delta}{\phi} \in C_{Spin(n)}(\widetilde{\widetilde{SO}}(M,\phi^* g),V) \qquad 1.6.7,$$

for all $\phi \in Diff(M,s)$, whilst the corresponding transformation on $\psi \in \Gamma(B(g))$ is easily seen to be:

$$(\psi,\phi) \longmapsto \tilde{\phi}(\tilde{\rho})^{-1} \circ \psi \circ \phi \in \Gamma(B(\phi^* g)) \qquad 1.6.8.$$

To summarize; for fixed $s \in \tilde{\Sigma}(M)$, we have an action of $Diff(M,s)$ by automorphisms on the vector bundle $E(M,s,\tilde{\rho})$ for any representation $\tilde{\rho}$ of $Spin(n)$ on the vector space V. This action transforms a spinor field ψ in the fibre above g (i.e. $B(g)$) into an element of the fibre above $\phi^* g$ (i.e. $B(\phi^* g)$) according to equation 1.6.8, or, equivalently, equation 1.6.7. We see that this action on $E(M,s,\tilde{\rho})$ covers the action of $Diff(M,s)$ on the base $Met(M)$.

Suppose now that we are interested in diffeomorphisms which change the spin structure. One may construct (Dabrowski and Percacci [D 1]), for each $\phi \in Diff(M)$, $s, s' \in \tilde{\Sigma}(M)$, a cohomology class $\kappa(\phi;s,s') \in H^1(M;\mathbb{Z}_2)$ which is the obstruction to the lifting of $\hat{\phi}$ to an isomorphism $\overset{\Delta}{\phi}$ of principal $\widetilde{\widetilde{GL}}^+(n,\mathbb{R})$-bundles:

$$\overset{\Delta}{\phi} : \widetilde{\widetilde{GL}}^{+}_{s'}(M) \longrightarrow \widetilde{\widetilde{GL}}^{+}_{s}(M) \qquad 1.6.9,$$

such that

$$\eta_s \circ \overset{\Delta}{\phi} = \hat{\phi} \circ \eta_{s'} \qquad 1.6.10,$$

where $s = (\widetilde{\widetilde{GL}}^{+}_{s}(M), \eta_s)$ and $s' = (\widetilde{\widetilde{GL}}^{+}_{s'}(M), \eta_{s'})$ are the two spin structures on M.

In other words, given any orientation preserving diffeomorphism ϕ and spin structures s, s' on M, $\hat{\phi}$ lifts to an isomorphism $\overset{\Delta}{\phi}$ satisfying equations 1.6.9 and 1.6.10 if and only if $\kappa(\phi;s,s') = 0$. In fact, given $\phi \in \mathrm{Diff}(M)$ and $s \in \tilde{\Sigma}(M)$, there exists a unique (up to equivalence, of course) $s' \equiv s'(\phi,s) \in \tilde{\Sigma}(M)$ such that $\kappa(\phi;s,s') = 0$. s' is just the pullback by $\overset{\Delta}{\phi}$ of the prolongation $s = (\widetilde{\widetilde{GL}}^{+}_{s}(M), \eta_s)$; $\widetilde{\widetilde{GL}}^{+}_{s'}(M)$ is the pullback by ϕ of the principal $\widetilde{\widetilde{GL}}^{+}(n,\mathbb{R})$-bundle $\tilde{\pi}_s : \widetilde{\widetilde{GL}}^{+}_{s}(M) \to M$, and $\eta_{s'} = \hat{\phi}^{-1} \circ \eta \circ (\tilde{\pi}_s{}^{*}\phi)$, where $\tilde{\pi}_s{}^{*}\phi : \widetilde{\widetilde{GL}}^{+}_{s'}(M) \equiv \phi^{*}(\widetilde{\widetilde{GL}}^{+}_{s}(M)) \to \widetilde{\widetilde{GL}}^{+}_{s}(M)$ is the canonical isomorphism of principal $\widetilde{\widetilde{GL}}^{+}(n,\mathbb{R})$-bundles arising in the construction of the pullback by ϕ of $\tilde{\pi}_s : \widetilde{\widetilde{GL}}^{+}_{s}(M) \to M$. The required $s' \in \tilde{\Sigma}(M)$ is given by $s' = (\widetilde{\widetilde{GL}}^{+}_{s'}(M), \eta_{s'})$.

We may now define a map:

$$\tilde{p} : \mathrm{Diff}(M) \times \tilde{\Sigma}(M) \to \tilde{\Sigma}(M); \quad (\phi,s) \mapsto s'(\phi,s) \qquad 1.6.11,$$

for all $\phi \in \mathrm{Diff}(M)$, $s \in \tilde{\Sigma}(M)$. It can be shown that $\tilde{p}$ is a right action of Diff(M) on $\tilde{\Sigma}(M)$, i.e.:

$$\tilde{p}(\phi_1 \circ \phi_2, s) = \tilde{p}(\phi_2, \tilde{p}(\phi_1, s)) \qquad 1.6.12,$$

for all $\phi_1, \phi_2 \in \mathrm{Diff}(M)$, $s \in \tilde{\Sigma}(M)$. It can further be shown that the value of $\tilde{p}$ at (ϕ,s) depends only on the isotopy class $[\phi]$

of ϕ, and if $s_1 \sim s_2$, then $\tilde{p}(\phi,s_1) \sim \tilde{p}(\phi,s_2)$. In other words, we may project $\tilde{p}$ to a well defined action p of $\Omega(M) = \mathrm{Diff}(M)/\mathrm{Diff}_o(M)$ (the group of connected components of $\mathrm{Diff}(M)$) on the set $\Sigma(M)$ of equivalence classes of spin structures on M:

$$p: \Omega(M) \times \Sigma(M) \longrightarrow \Sigma(M); \ ([\phi],[s]) \longmapsto [\tilde{p}(\phi,s)] \qquad 1.6.13,$$

for all $[\phi] \in \Omega(M)$, $[s] \in \Sigma(M)$. It is shown in Dambrowski and Percacci [D 1] that equation 1.6.13 defines an affine action of $\Omega(M)$ on $\Sigma(M)$.

The transformation rule for spinor fields under arbitrary diffeomorphisms may now be given. Let $g \in \mathrm{Met}(M)$ and $s \in \tilde{\Sigma}(M)$, so that $s_g = \tilde{r}_g(s) = (\widetilde{\widetilde{SO}}_s(M,g),\eta_g)$ is a g-spin structure on M. Let $\Psi \in C_{\mathrm{Spin}(n)}(\widetilde{\widetilde{SO}}_s(M,g),V)$ be a spinor field of type $(s_g,\tilde{\rho},V)$. Given any $\phi \in \mathrm{Diff}(M)$, we have that $s' = \tilde{p}(\phi,s) \in \tilde{\Sigma}(M)$ is the unique spin structure on M such that $\hat{\phi}$ lifts to an isomorphism $\overset{\Delta}{\phi}: \widetilde{\widetilde{GL}}{}^+_{s'}(M) \longrightarrow \widetilde{\widetilde{GL}}{}^+_s(M)$ satisfying $\eta_s \circ \overset{\Delta}{\phi} = \hat{\phi} \circ \eta_{s'}$, (see equations 1.6.9, 1.6.10). Using s' and $g' = \phi^* g$, we construct the g'-spin structure $\tilde{r}_{g'}(s') = (\widetilde{\widetilde{SO}}_{s'}(M,g'),\eta_{g'})$, and then it is easily seen that $\overset{\Delta}{\phi}$ restricts to an isomorphism of principal Spin(n)-bundles:

$$\overset{\Delta}{\phi} \ \equiv \ \overset{\Delta}{\phi}|\widetilde{\widetilde{SO}}_{s'}(M,g'): \widetilde{\widetilde{SO}}_{s'}(M,g') \longrightarrow \widetilde{\widetilde{SO}}_s(M,g) \qquad 1.6.14.$$

The transformation of the spinor field Ψ of type $(s_g,\tilde{\rho},V)$ is now given by:

$$(\Psi,\phi) \longmapsto \Psi' \ = \ \Psi \circ \overset{\Delta}{\phi} \in C_{\mathrm{Spin}(n)}(\widetilde{\widetilde{SO}}_{s'}(M,g'),V) \qquad 1.6.15,$$

so that Ψ' is a spinor field of type $(s'_{g'},\tilde{\rho},V)$. The corresponding transformation for sections may be also written down. The action described by equation 1.6.15 effectively gives us an action by Diff(M) by automorphisms on the bundle $\tilde{E}(M,\tilde{\rho})$ given in section 1.4, and so now we have a symmetry group for the metric-spin structure-spinor field configuration space, which relates fibres over different metrics (and over different spin structures on M). Actually, there is a slight problem which we now describe:

Suppose we are given $\phi \in \mathrm{Diff}(M)$ and $s \in \tilde{\Sigma}(M)$. Then, from the above, we know that there exists a unique (up to equivalence) spin structure $s' = \tilde{p}(\phi,s)$ such that $\hat{\phi}$ lifts to an isomorphism $\overset{\Delta}{\phi}$ of $\widetilde{GL}^+_{s'}(M)$ onto $\widetilde{GL}^+_s(M)$ satisfying $\eta_s \circ \overset{\Delta}{\phi} = \hat{\phi} \circ \eta_{s'}$. However, the particular lift $\overset{\Delta}{\phi}$ is <u>not</u> unique. In fact, there are precisely two lifts of $\hat{\phi}$ satisfying equations 1.6.9, 1.6.10, and these two lifts differ by the automorphism of $\widetilde{GL}^+_s(M)$ corresponding to multiplication by the generator of $\mathbb{Z}_2 \leq \widetilde{GL}^+(n,\mathbb{R})$. In general, there does not exist a consistent lift $\overset{\Delta}{\phi}$ for all $\phi \in \mathrm{Diff}(M)$ - the composition rule for lifts of diffeomorphisms will hold only up to $\mathbb{Z}_2$, so that the induced action on spinor fields, given by equation 1.6.15, is only a projective action. In order to obtain a true group action on the space of spinor fields, we must remove the $\mathbb{Z}_2$ ambiguity in composition by lifting to a double cover. For example, consider a fixed $s \in \tilde{\Sigma}(M)$ and, as above, let Diff(M,s) denote the group of diffeomorphisms leaving the spin structure s invariant. According to the remarks just made, Diff(M,s) acts only projectively on spinor fields associated with the spin structure s. Now let:

$$\widetilde{\mathrm{Diff}}(M,s) = \{f \in \mathrm{Aut}(\widetilde{GL}^+_s(M)) : \eta_s \circ f = \hat{f}_M \circ \eta_s\} \qquad 1.6.16,$$

(where $f_M \in \mathrm{Diff}(M)$ is the diffeomorphism covered by f), so that

$$f \longmapsto f_M \qquad 1.6.17,$$

defines a double cover of Diff(M.s) by the group $\widetilde{\mathrm{Diff}}(M,s)$. We now have a true (i.e. not projective) representation of $\widetilde{\mathrm{Diff}}(M,s)$ on spinor fields associated with the spin structure s on M, given by

$$(\Psi,f) \longmapsto \Psi \circ f \qquad 1.6.18,$$

for all spinor fields Ψ, and $f \in \widetilde{\mathrm{Diff}}(M,s)$.

As an example, consider (yet again) the two sphere, S^2. As we indicated in section 1.2, S^2 admits a unique can-spin structure, given by the Hopf fibration, $S^1 \hookrightarrow S^3 \rightarrow S^2$, together with the usual double cover, $S^3 \rightarrow SO(3)$. Let us denote this spin structure by $s_{can} \in \tilde{\Sigma}(S^2,\mathrm{can})$. It turns out that $\mathrm{Diff}(S^2,s_{can}) = \mathrm{Diff}(S^2)$, i.e. the spin structure on S^2 is invariant under all orientation preserving diffeomorphisms of S^2. The inclusion of $SO(3) \cong \mathrm{Isom}(S^2,\mathrm{can})$ into $\mathrm{Diff}(S^2)$ is a homotopy equivalence (see Smale [S 30]) and hence any double cover of $\mathrm{Diff}(S^2)$ must be homotopy equivalent to $S^3 \cong \mathrm{Spin}(3)$, so that the required double cover of $\mathrm{Diff}(S^2)$ is the unique non-trivial one.

We conclude this section with a remark of a more speculative nature. It can be argued that the diffeomorphism group arises naturally as the symmetry group of general relativity, regarded as a theory of pure gravity (see, for example, Isham and Kuchar [I 11]). For instance, Diff(M) is the largest group leaving invariant the Einstein-Hilbert action:

$$S : \mathrm{Met}(M) \longrightarrow \mathbb{R} ; \quad g \longmapsto \int_M \mathrm{Scal}(g)\mathrm{vol}(g) \qquad 1.6.19,$$

and so Diff(M) is important at least as far as the gravitational action is concerned. A further motivation for the fundamental nature of spinors in gravity theory would come about if a double cover $\widetilde{\mathrm{Diff}}(M)$ arose in a natural way as symmetry group of some important object in the theory - perhaps of an action. The question as to whether or not such an action exists is a matter for further investigation.

Having now, in sections 1.1 to 1.6, described the theory of spinors in general, we turn to the question of spacetime.

1.7 Spacetime Spinors

We now consider the structures we have introduced above in the context of spacetimes in general relativity. We shall see that spinor structure gels especially well with Lorentzian structures on four-manifolds, and for this reason, spinors are very useful in general relativity. Indeed, as we shall discuss in section 1.8, spinors may be taken as the foundation of global spacetime geometry.

In this section we adapt the theory given in previous sections to the special case of a four-manifold equipped with a Lorentzian metric, i.e. a metric of signature minus two, so that the local diagonal form is (+ ---). We begin with some remarks of an algebraic nature (see Penrose and Rindler [P 11] for a different approach to spinor algebra).

Let ε be the symplectic form on $\mathbb{C}^2$ with components

$$(\varepsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad 1.7.1,$$

with respect to the standard basis of $\mathbb{C}^2$. Now consider $\mathbb{C}^2$ as a four dimensional real vector space (i.e. restrict the action of $\mathbb{C}$ (scalar multiplication) to an action of $\mathbb{R}$) equipped with the almost complex structure J obtained by multiplication by i considered as an $\mathbb{R}$- linear mapping. J extends in a $\mathbb{C}$-linear fashion to the space $\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C}$ (in this section, since we are dealing with both real and complex vector spaces, we make explicit the field over which tensor products of vector spaces are taken), and we have a direct sum decomposition:

$$\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C} = S \oplus \overline{S} \qquad 1.7.2,$$

where $S(\overline{S})$ is the $+i$ $(-i)$-eigenspace of J. We identify S with $\mathbb{C}^2$ in a $\mathbb{C}$-linear way, and $\overline{S}$ with $\mathbb{C}^2$ in a $\overline{\mathbb{C}}$-linear way. S is just the representation space for the defining representation $\rho \in \mathrm{Hom}(SL(2,\mathbb{C}), GL(2\mathbb{C}))$ of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$, given by

$$\rho_A(\underline{z}) = A\underline{z} \qquad 1.7.3,$$

for all $A \in SL(2,\mathbb{C})$, $\underline{z} \in \mathbb{C}^2$, and $\overline{S}$ is the representation space for the conjugate representation $\overline{\rho}$ given by:

$$\overline{\rho}_A(\underline{z}) = \overline{A}\,\underline{z} \qquad 1.7.4.$$

The symplectic form ε induces symplectic forms (also denoted

ε) on S and on $\overline{S}$. These forms are invariant under the $SL(2,\mathbb{C})$-actions given in equations 1.7.3, 1.7.4.

We now identify (using the standard bases) $S \otimes_{\mathbb{C}} \overline{S}$ with $M(2,\mathbb{C}) = \{\text{complex } 2\times 2 \text{ matrices}\}$, and then we have the representation $\rho \otimes_{\mathbb{C}} \overline{\rho} \in \mathrm{Hom}(SL(2,\mathbb{C}), GL(M(2,\mathbb{C})))$ given by

$$(\rho \otimes_{\mathbb{C}} \overline{\rho})_A(M) = AMA^{\dagger} \qquad 1.7.5,$$

for all $A \in SL(2,\mathbb{C})$ and $M \in M(2,\mathbb{C})$. Here $\dagger$ means Hermitian adjoint of elements of $M(2,\mathbb{C})$ (in particular, of $SL(2,\mathbb{C})$). Let $H(2)$ denote the space of 2×2 Hermitian matrices. Then $H(2)$ may be regarded as a real four dimensional subspace of $M(2,\mathbb{C}) \cong S \otimes_{\mathbb{C}} \overline{S}$. The representation $\rho \otimes_{\mathbb{C}} \overline{\rho}$ reduces to a real representation of $SL(2,\mathbb{C})$ on $H(2)$, which we denote by $\rho \otimes \overline{\rho}$.

The original symplectic form ε induces on $M(2,\mathbb{C})$ a complex inner product, which restricts to a real Lorentzian inner product $\varepsilon \otimes \overline{\varepsilon}$ on $H(2)$. $SL(2,\mathbb{C})$ acts upon $(H(2), \varepsilon \otimes \overline{\varepsilon})$ by isometries, since ε is $SL(2,\mathbb{C})$-invariant. In fact, the norm associated to $\varepsilon \otimes \overline{\varepsilon}$ is just (twice) the determinant.

We may regard $(H(2), \varepsilon \otimes \varepsilon)$ as a copy of (real) Minkowski space embedded in the space of complex 2×2 matrices, $M(2,\mathbb{C})$:

Let $\underline{\sigma} \in \mathbb{R}^4 \otimes_{\mathbb{R}} H(2)$ be defined by its components σ^a, $a = 0,1,2,3$, with respect to the standard basis of $\mathbb{R}^4$, as follows:

$$\sigma^0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma^2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$$

$$\sigma^3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad 1.7.6,$$

(i.e. the σ^a are the Pauli-matrices), and define the linear map α of $\mathbb{R}^4$ onto $H(2)$ by:

$$\alpha : \mathbb{R}^4 \longrightarrow H(2); \quad \underline{x} \longmapsto \tfrac{1}{2}(\underline{x}^\eta \otimes_{\mathbb{R}} id_{H(2)})(\underline{\sigma}) \qquad 1.7.7,$$

for all $\underline{x} \in \mathbb{R}^4$, where $\underline{x}^\eta = \eta(\underline{x},\cdot) \in (\mathbb{R}^4)^*$, and $\eta = \text{diag}(1, -1, -1, -1)$ is the Minkowski inner product on $\mathbb{R}^4$. We may abuse notation slightly, and rewrite equation 1.7.7 as:

$$\alpha(\underline{x}) = \tfrac{1}{2}\eta(\underline{x}, \underline{\sigma}) \qquad 1.7.8,$$

for all $\underline{x} \in \mathbb{R}^4$. The inverse linear map α^{-1} is given by:

$$\alpha^{-1}(M) = \text{trace}(M\underline{\sigma}) \qquad 1.7.9,$$

for all $M \in H(2)$, as is easily verified, using elementary properties of the Pauli matrices, σ^a. Another important property of α is that it is an isometry of $(\mathbb{R}^4, \eta)$ onto $(H(2), \varepsilon \otimes \overline{\varepsilon})$, i.e.

$$\det \alpha(\underline{x}) = \tfrac{1}{2}\eta(\underline{x}, \underline{x}) \qquad 1.7.10,$$

for all $\underline{x} \in \mathbb{R}^4$, and so α isometrically embeds Minkowski space in $M(2,\mathbb{C})$.

We now define $\Lambda \in \text{Hom}(SL(2,\mathbb{C}), GL(4,\mathbb{R}))$ by:

$$\Lambda_A \equiv \Lambda(A) = \alpha^{-1} \circ (\rho \otimes \overline{\rho})_A \circ \alpha \qquad 1.7.11,$$

i.e.

$$\Lambda_A(\underline{x}) = \alpha^{-1}(A\alpha(\underline{x})A^\dagger) \qquad 1.7.12,$$

for all $\underline{x} \in \mathbb{R}^4$. Since α, $(\rho \otimes \overline{\rho})_A$ are isometries (for each $A \in SL(2,\mathbb{C})$), we have that Λ_A is a Lorentz transformation for each

$A \in SL(2,\mathbb{C})$. In fact Λ is a homomorphism of $SL(2,\mathbb{C})$ onto $SO^{\uparrow}(1,3)$, the subgroup of the Lorentz group consisting of all elements which preserve the standard spacetime orientation of Minkowski space (see below for a definition of spacetime orientability and orientation) and $\mathrm{Ker}\Lambda \cong \mathbb{Z}_2$, as can be seen from equations 1.7.5 and 1.7.11. Therefore, Λ is a non-trivial double cover of $SO^{\uparrow}(1,3)$, and, identifying Spin(1,3) with $SL(2,\mathbb{C})$, Λ coincides with the map introduced in section 1.1, in the special case of signature minus two metrics in dimension four.

Having now established the particular algebraic inter-relationship between $SL(2,\mathbb{C})$ and $SO^{\uparrow}(1,3)$, we turn now to bundles over spacetime. First, let M be any oriented four-manifold, and let:

$$SL(2,\mathbb{C}) \hookrightarrow \tilde{P} \longrightarrow M \qquad 1.7.13,$$

be any principal $SL(2,\mathbb{C})$-bundle over M. Note that $\tilde{P}$ is necessarily trivializable if M is non-compact (see, for example, Isham [I 8]).

Given $\tilde{P}$, we may construct associated bundles in the usual way. In particular we have the vector bundles constructed using representations of $SL(2,\mathbb{C})$ given above.

<u>Definition (1.7)1</u>: Let $\tilde{P}$ be a principal $SL(2,\mathbb{C})$-bundle over the 4-manifold M. Let $W(\tilde{P}) = \tilde{P} \times_{SL(2,\mathbb{C})} \mathbb{R}^4$ (constructed using the representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$ given by $A \longmapsto \Lambda_A$), $H(\tilde{P}) = \tilde{P} \times_{SL(2,\mathbb{C})} H(2)$ (using $\rho \otimes \bar{\rho}$), $S(\tilde{P}) = \tilde{P} \times_{SL(2,\mathbb{C})} S$ using ρ), $\bar{S}(\tilde{P}) = \tilde{P} \times_{SL(2,\mathbb{C})} \bar{S}$ (using $\bar{\rho}$).

Since η, ε are $SL(2,\mathbb{C})$-invariant under the action used in definition (1.7)1, the vector bundles have extra structure: $W(\tilde{P})$, $H(\tilde{P})$ are (real) Lorentzian vector bundles, and $S(\tilde{B})$, $\overline{S}(\tilde{B})$ (and $S^*(\tilde{B})$, $\overline{S}^*(\tilde{B})$) are (complex) symplectic vector bundles (we use the symbol ε for the symplectic structure in these bundles).

With respect to these structures, we have an isometry of Lorentzian vector bundles:

$$\sigma_{\tilde{P}} : W(\tilde{P}) \longrightarrow H(\tilde{P}) \hookrightarrow S(\tilde{P}) \otimes_{\mathbb{C}} \overline{S}(\tilde{P}) \qquad 1.7.14,$$

given by:

$$\sigma_{\tilde{P}}([\tilde{u}, \underline{x})]) \quad = \quad [(\tilde{u},\alpha(\underline{x}))] \qquad 1.7.15,$$

for all $[(\tilde{u},\underline{x})] \in W(\tilde{P})$. $\sigma_{\tilde{P}}$ induces an isomorphism of $\Gamma(W(\tilde{P}))$ onto $\Gamma(H(\tilde{P}))$ in the usual way.

We now restrict our attention to the case of spacetimes in general relativity. We assume that the 4-manifold M is either non-compact or compact with vanishing Euler invariant, so that M admits a Lorentzian metric g. The principal $SL(2,\mathbb{C})$-bundle $\tilde{P}$ will now arise from a g-spin structure on M.

The model for spacetime used in general relativity is that of a connected Lorentzian 4-manifold (M,g) which is spacetime orientable (and spacetime oriented). Spacetime orientable means that the bundle of g-orthonormal frames $O(M,g)$ has precisely four components (corresponding, in some unspecified way, to the four components of the (unrestricted) Lorentz group $O(1,3)$). We may choose one component θ of $O(M,g)$, i.e. we may give (M,g) a particular spacetime orientation, and call this component

θ the bundle of (spacetime) oriented g-orthonormal frames $SO(M,g)$. $SO(M,g)$ is then a principal $SO^{\uparrow}(1,3)$-bundle over M:

$$SO^{\uparrow}(1,3) \hookrightarrow SO(M,g) \xrightarrow{\pi_M} M \qquad 1.7.16.$$

Note that if (M,g) is not spacetime orientable, we may always construct a Lorentzian covering manifold which is spacetime orientable, and which is equivalent, so far as general relativity is concerned, to the original Lorentzian manifold (M,g) (see Geroch [G 3]).

We often assume, in addition, that M is non-compact. A physical reason for this is that it is easy to show that compact Lorentzian four-manifolds (M,g) admit the existence of closed timelike curves, and so most notions of causality would forbid such spacetimes. (See Beem and Ehrlich [B 5]). A mathematical reason for assuming M non-compact is that any principal $SL(2,\mathbb{C})$-bundle is necessarily trivializable over a non-compact four-manifold, as we have mentioned above, and so any g-spin structure on M would have simpler structure than in the general case. Recall (1.2.3) that a non-compact spacetime is spin if and only if M is parallelizable.

In any case, suppose we are given a spacetime (M,g) (the orientation θ is not usually mentioned) which is spin, i.e. $w_2(TM) \in H^2(M:\mathbb{Z}_2)$ vanishes. Let $s_g = (\widetilde{\widetilde{SO}}(M,g),\eta_g) \in \tilde{\Sigma}(M,g)$ be a g-spin structure on M, so that $\widetilde{\widetilde{SO}}(M,g)$ is a principal $SL(2,\mathbb{C})$-bundle over M, and $\eta_g\colon \widetilde{\widetilde{SO}}(M,g) \to SO(Mg)$ is a homomorphism of principal bundles, such that $\eta_g(\tilde{u}A) = \eta_g(\tilde{u})\Lambda(A)$, for all $\tilde{u} \in \widetilde{\widetilde{SO}}(M,g)$, $A \in SL(2,\mathbb{C})$.

Given $\widetilde{\widetilde{SO}}(m,g)$, we may define, as above, the associated vector bundles of particular interest (see definition (1.7)1). We change notation slightly, and write $W(\widetilde{\widetilde{SO}}(M,g)) = W(s_g)$, etc. Since s_g is a spin structure, we now have the additional vector bundle isomorphisms, which we defined in section 1.3. For example, using equation 1.3.3, we have the vector bundle isomorphism $\hat{\eta}_g$ of $W(s_g)$ onto TM:

$$\hat{\eta}_g : W(s_g) \longrightarrow TM; \quad [(\tilde{u},\underline{x})] \longmapsto [(\eta_g(\tilde{u}),\underline{x})] \qquad 1.7.17,$$

for all $[(u,\underline{x})] \in W(s_g)$. In fact, it is easily seen that $\hat{\eta}_g$ is an isometry of (real) Lorentzian vector bundles (by virtue of the fact that η_g is a homomorphism of principal bundles and also because of the action of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$ which we are using). We may combine equations 1.7.14 and 1.7.17 to give another isometry $\sigma(s_g)$ of Lorentzian vector bundles:

Definition (1.7)2: Let s_g be a g-spin structure on the space-time (M,g). Then the Infeld-Van der Waerden isomorphism (for s_g) $\sigma(s_g)$ is defined by $\sigma(s_g) = \sigma_{\widetilde{\widetilde{SO}}(M,g)} \circ \hat{\eta}_g^{-1}$, so that $\sigma(s_g)$ is an isometry of Lorentzian vector bundles:

$$\sigma(s_g): \quad TM \longrightarrow H(s_g) \hookrightarrow S(s_g) \otimes_{\mathbb{C}} \overline{S}(s_g) \qquad 1.7.18.$$

Note that a g-spin structure equivalent to s_g would give rise to an equivalent isomorphism. We obtain an isomorphism of the spaces of sections $\mathrm{Vect}(M) \equiv \Gamma(TM) \longleftrightarrow \Gamma(H(s_g))$ in the usual way. There is also the obvious extension of $\sigma(s_g)$ to an isomorphism of tensor products of TM and $H(s_g)$ (and thence to tensor fields). Any tensor equation, in, say, general relativity, may be translated, using $\sigma(s_g)$, into an equivalent equation involving sections of

$S(s_g) \otimes_{\mathbb{C}} \overline{S}(s_g)$ (and tensor products thereof). As in section 1.3, we call sections of $H(s_g)$, $S(s_g) \otimes_{\mathbb{C}} \overline{S}(s_g)$ etc., spinor fields and equation 1.7.18 means that we may translate any tensor equation into a spinor equation. The latter is often much easier to deal with, so the Infeld-Van der Waerden isomorphism $\sigma(s_g)$ is of much use in general relativity (see Penrose and Rindler [P 11], [P 12] and references therein).

Note that by restricting our attention to the representations ρ, $\overline{\rho}$ (and tensor products thereof) we ensure that all the spinors we use are based on two component or Weyl spinors. These are mathematically simpler than, say, Dirac spinors, and the (complex) two dimensionality of the basic representation space S ensures the validity of many useful identities and results: for example (using the Penrose [P 11] abstract index notation):

$$\lambda_{AB} = \lambda_{(AB)} + \tfrac{1}{2}\varepsilon_{AB}\,\lambda_C{}^C \qquad 1.7.19,$$

for any spinor (spinor field) λ in $S^*(s_g) \otimes_{\mathbb{C}} S^*(s_g)$ $(\Gamma(S^*(s_g) \otimes_{\mathbb{C}} S^*(s_g)))$. Here $A, B, C \in \{0,1\}$ to conform with the standard conventions. Equation 1.7.19 means that we may restrict our attention to completely symmetrized tensor product representations (see the remark below on representations of $SL(2,\mathbb{C})$). Dirac spinors arise from the representation $\rho \oplus \overline{\rho}^*$ of $SL(2,\mathbb{C})$ on $\mathbb{C}^4$, and are very important in particle physics. Note, however, that the effect of the spacetime orientation is to reduce $\rho \oplus \overline{\rho}^*$ to its two constituent irreducible "Weyl" representations (see, for example, Wald [W 13]).

We now make some concluding remarks concerning the results of this section: (i) We may extend the isometry $\sigma(s_g)$ by $\mathbb{C}$-linearity to give an isomorphism of complex Lorentzian vector bundles (which we denote by the same letter $\sigma(s_g)$):

$$\sigma(s_g) : T^{\mathbb{C}}M \longrightarrow S(s_g) \otimes_{\mathbb{C}} \overline{S}(s_g) \qquad 1.7.20,$$

where $T^{\mathbb{C}}M$ is the complexified tangent bundle of M, so that $T_x^{\mathbb{C}}M = T_xM \otimes_{\mathbb{R}} \mathbb{C}$, and carries the metric g extended by $\mathbb{C}$-linearity. (ii) Via $\sigma(s_g)$ (or its complexification, as in (i)), we have (using abstract indices):

$$g_{ab} = \varepsilon_{AB}\, \varepsilon_{A'B'} \qquad 1.7.21.$$

Note that it is customary to assign to spinors in $\overline{S}(s_g)$, $\overline{S}^*(s_g)$ (and tensor products thereof) primed abstract indices $A', B' \ldots \in \{0',1'\}$.

(iii) For completeness, we discuss the irreducible representations of the group $SL(2,\mathbb{C})$. We have already mentioned the Weyl representations ρ, $\overline{\rho}$, ρ^*, $\overline{\rho}^*$ and the Dirac representation $\rho \oplus \overline{\rho}^*$ (which is, of course, reducible). The irreducible representations of $SL(2,\mathbb{C})$ are, in fact, parameterized by the set $(\frac{1}{2}\mathbb{Z})^2$, where $\frac{1}{2}\mathbb{Z} = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, and we denote a general irreducible complex representation by $D^\circ(\mu,\nu)$. The representations we have so far discussed include $\rho = D^\circ(\frac{1}{2},0)$, $\overline{\rho}^* = D^\circ(0,\frac{1}{2})$. In general, $D^\circ(\mu,\nu)$ is the representation $(\odot^{2\mu}D^\circ(\frac{1}{2},0)) \otimes (\odot^{2\nu}D^\circ(0,\frac{1}{2}))$ defined such that its representation space is $S(\mu,\nu)$ - the space of spinors symmetric in the first 2μ (unprimed, contravariant) slots and also symmetric in the last 2ν (primed, covariant) slots. The dimension of $D^\circ(\mu,\nu)$ is then seen to be $(2\mu+1)(2\nu+1)$. The <u>spin</u> of the representation $D^{o}(\mu,\nu)$ is defined to be $\nu - \mu \in \frac{1}{2}\mathbb{Z}$. See Penrose and Rindler [P 11] for more details concerning the representation theory of $SL(2,\mathbb{C})$.

This section has been very much concerned with the implementation of spinor theory in a spacetime setting, noting in particular the way in which two component spinors (i.e. the representation $\rho \in \mathrm{Hom}(SL(2,\mathbb{C}), GL(2,\mathbb{C}))$) and four dimensional Lorentzian structure fit together in a natural way. Once implemented, the spinor theory greatly facilitates (via the Infeld-Van der Waerden isomorphism) many calculations in general relativity: Every tensor equation may be translated into a, usually simpler, spinor equation. The spinor theory, based on $S(s_g)$, $\overline{S}(s_g)$ etc. is, a priori, much richer than the tensor theory which is based (via $\sigma(s_g)$) on $S(s_g)\otimes\overline{S}(s_g)$ etc.; i.e. we can write down many operations and equations involving spinors which don't have an obvious tensor analogue, e.g. operations involving one spinor abstract index A rather than a pair AA',

as would occur in a (translated) tensor equation. In fact, as Penrose and Rindler [P 11] show, any equation or operation involving Weyl spinors may be translated back to an equation or operation involving tensors, but with a possible $\mathbb{Z}_2$-ambiguity (the correspondence being given formally by $\pm\sqrt{\sigma(s_g)}$). The beauty of the spinor formalism, therefore, is not so much the extra structure available (although there does exist some extra structure, especially in terms of complex geometry), but rather the simplicity of spinor equations compared to the corresponding tensor equations. Indeed certain important operations, of geometrical and, more importantly for us, physical significance, are suggested by the spinor formalism. The corresponding operations, when written in terms of tensors, are often very complicated and uninspiring. Examples of such operations are those involving dualization (with respect to the Hodge *-operator associated with $g \in \mathrm{Met}(M)$), trace reversals (with respect to g) and spinor symmetrization. See Ruse [R 4] for an early exposition of the simplicity of Dirac spinors as compared to tensors, and Penrose and Rindler [R 11] for the corresponding situation for Weyl spinors. We shall introduce operations with Weyl spinors when we need to make use of them (see Chapter Three).

The discussion of this section has been very much in the spirit of category 1. of section 1.0. To make tentative steps towards category 2., we now turn to the idea of spinors as a basis for global spacetime geometry. The next section will show how spinors are, at least, equivalent to Lorentzian metrics as a foundation for spacetime geometry. Further indications of the fundamental nature of spinors in general relativity will emerge in Chapter Three.

1.8 Spacetime from Spin

In this section, we remark on the rôle of spinors in general relativity, thereby expanding the discussion of section 1.0 (see also Chapters Two and Three). The viewpoint so far taken, and indeed the one which we will adopt generally, is that the basic model of spacetime is a Lorentzian spin manifold (M,g) (connected, spacetime oriented). Of course, we shall need to impose additional requirements on (M,g), such as various geometrical and topological assumptions reflecting physical properties like causality, and we will also add extra structure such as matter fields satisfying both physically reasonable local energy conditions and appropriate field equations. For the moment, however, our basic object will be the global geometry defined by (M,g,θ,s_g) (we add the spacetime orientation θ to the description of spacetime, because it will play a slightly more important rôle in this section), where we have assumed that a particular g-spin structure $s_g = (\tilde{\mathcal{S}\mathcal{O}}(M,g),\eta_g)$ has been chosen (perhaps picked out for or by physical requirements) on the spin manifold M, with Lorentzian metric g.

The g-spin structure gives rise to the vector bundles $W(s_g)$, $S(s_g)$, etc. and also the Infeld-Van der Waerden isomorphism $\sigma(s_g)$: $TM \rightarrow H(s_g) \hookrightarrow S(s_g) \otimes_{\mathbb{C}} \overline{S}(s_g)$, described in section 1.7. As we remarked above, $\sigma(s_g)$ is, in fact, an isometry of Lorentzian vector bundles: the metric in TM being just the metric g on M, and the metric in $H(s_g)$ being the one induced from the symplectic form ε, as in the first part of section 1.7.

Note that ε defines a unique symplectic conformal class $[\varepsilon]$ of symplectic forms on $\mathbb{C}^2$, where different representatives are non-zero complex multiples of one another. Because of the two dimensionality of $\mathbb{C}^2$, any symplectic form on $\mathbb{C}^2$ is contained in $[\varepsilon]$. Suppose, instead of picking a particular representative $\varepsilon \in [\varepsilon]$, we are just given the conformal class $[\varepsilon]$. Then the vector bundle $H(s_g)$ will be equipped with only a Lorentzian conformal structure $[\varepsilon \otimes \overline{\varepsilon}]$ in its fibres, and $\sigma(s_g)$ will map this conformal structure into the conformal structure in TM arising from the conformal class C_g defined by the metric g on M.

Recall that the symbol ε is also used to denote the symplectic structures in the vector bundles $S(s_g)$ etc. Suppose a particular complex rescaling of the original symplectic form ε (see equation 1.7.1) leads to the symplectic structure $f\varepsilon$ in $S(s_g)$, where $f \in C(M,\mathbb{C}^*)$. The image of $f\varepsilon$ under the Infeld-Van der Waerden isomorphism $\sigma(s_g)$ is just $|f|^2 g$, a particular representative of the conformal class C_g. Note that if we restrict our conformal rescalings to have modulus unity, so that $f \in C(M,S^1)$, then the image of $f\varepsilon$ under $\sigma(s_g)$ is still g.

In section 1.0, we remarked that spinors are not only useful, as we indicated in section 1.7, but essential to general relativity. The second possibility has led various workers to speculate that the whole of general relativity theory, perhaps even the spacetime manifold M itself, should be derived from spinor data (see, for example, Penrose [P 6]). We now give a brief outline of how the global spacetime geometry (M,g,θ,s_g) may be derived from a basic spinor structure on the manifold M, rather than a derivation from an a priori choice of Lorentzian metric $g \in \mathrm{Met}(M)$:

Our starting point will be a manifold M (we make no attempt to derive the spacetime manifold or to replace it with an alternative structure) equipped with the following data: A rank two complex vector bundle:

$$\mathbb{C}^2 \hookrightarrow S_o \longrightarrow M \qquad 1.8.1,$$

in which there is a (complex) conformal symplectic structure $[\varepsilon_o]$, so that each representative $\varepsilon_o \in \Gamma(S_o^* \otimes_{\mathbb{C}} S_o^*)$ gives rise to a symplectic form on each fibre of S_o, with $\varepsilon_o \sim \varepsilon_o'$ if and only if there exists $f \in C(M,\mathbb{C}^*)$ such that

$$\varepsilon_o' = f\varepsilon_o \qquad 1.8.2.$$

Given S_o, we may construct the conjugate rank two vector bundle $\overline{S}_o$ (by first constructing the principal $GL(2,\mathbb{C})$-bundle to which S_o is associated, and considering the conjugate representation), and we consider the map:

$$x \otimes_{\mathbb{C}} \overline{y} \longmapsto y \otimes_{\mathbb{C}} \overline{x} \qquad 1.8.3,$$

where $x, y \in S_o$. We extend 1.8.3 by complex bilinearity in fibres to give an isomorphism of complex vector bundles:

$$h : S_o \otimes_{\mathbb{C}} \overline{S}_o \longrightarrow S_o \otimes_{\mathbb{C}} \overline{S}_o \qquad 1.8.4,$$

and let W_o be the fixed point set of h. Then W_o is a rank four real vector bundle over M.

So far we have just assumed $(S_o, [\varepsilon_o])$ as given. We now choose two more pieces of data: (i) A particular representative symplectic form $\varepsilon_o \in [\varepsilon_o]$, and (ii) a (real) vector bundle isomorphism:

$$\sigma_o : TM \longrightarrow W_o \qquad 1.8.5.$$

(Assuming that such an isomorphism exists, i.e. that TM and W_o are members of the same vector bundle isomorphism class, is tantamount to requiring M to be spin.)

From the data $(M,S_o,\varepsilon_o,\sigma_o)$ we may now derive the geometry of spacetime (see also Plymen and Westbury [P 15]).

Let $\tilde{P}_o$ be the bundle of symplectic frames to which (S_o,ε_o) is associated. Then $\tilde{P}_o$ is a principal $SL(2,\mathbb{C})$-bundle over M:

$$SL(2,\mathbb{C}) \hookrightarrow \tilde{P}_o \longrightarrow M \qquad 1.8.6,$$

and given $\tilde{P}_o$, we may perform the vector bundle constructions we gave at the beginning of section 1.7 (see definition (1.7)1). In particular we have $W(\tilde{P}_o) \equiv \tilde{P}_o \times_{SL(2,\mathbb{C})} \mathbb{R}^4$ which is isometric to $H(\tilde{P}_o) \equiv \tilde{P}_o \times_{SL(2,\mathbb{C})} H(2)$ (equation 1.7.14) with respect to the natural Lorentzian structures.

It may easily be shown that there exists a natural isomorphism: $W_o \longrightarrow W(\tilde{P}_o)$, and hence, given σ_o, we have $W(\tilde{P}_o) \overset{\sim}{=} TM$ (qua vector bundles). We now <u>define</u> g to be the unique Lorentzian metric in TM (i.e. on M) such that the isomorphism $W(\tilde{P}_o) \overset{\sim}{=} TM$ is an isometry of Lorentzian vector bundles.

The symplectic form ε_o also plays the rôle of a volume element in S_o, and hence defines an orientation. This orientation induces an orientation in W_o, and hence in $W(\tilde{P}_o)$ which is compatible with the Lorentzian metric in $W(\tilde{P}_o)$ (i.e. a "spacetime" orientation). We now induce a spacetime orientation θ on (M,g) via the isometry $W(\tilde{P}_o) \overset{\sim}{=} TM$.

To summarize the above: Our basic data is $(M,S_o,\varepsilon_o,\sigma_o)$, and from this we derive (M,g,θ). To complete the description of global spacetime geometry as described above we require a g-spin structure s_g on M. This is easily constructed from $(M,S_o,\varepsilon_o,\sigma_o)$ in a unique way: we just take the principal $SL(2,\mathbb{C})$-bundle to be $\tilde{P}_o$, and the bundle homomorphism η : $\tilde{P}_o \longrightarrow SO(M,g)$ $(\equiv \theta)$ is constructed using the isomorphism σ_o and a reverse procedure to the one which lead to equation 1.7.18. This leads to a unique g-spin structure $s_g \equiv (\tilde{P}_o,\eta)$ on M.

Thus we may derive an entire description of global geometry (M,g,θ,s_g) starting from the spinor data $(M,S_o,\varepsilon_o,\sigma_o)$. The converse construction is also well defined: Given (M,g,θ,s_g), we let $S_o = S(s_g)$, and ε_o is just the symplectic structure in $S(s_g)$ constructed in section 1.7. W_o is taken to be $W(s_g)$ and $\sigma_o = \sigma(s_g)$, the Infeld-Van der Waerden isomorphism associated with the spin structure s_g. From (M,g,θ,s_g), we construct $(M,S_o,\varepsilon_o,\sigma_o)$, and the two constructions we have just described are mutually inverse. In other words, the two foundations for global spacetime geometry are equivalent. We shall usually work from and with (M,g,θ,s_g), and translate to spinor formalism as required, but there is a case for starting from the spinor foundation $(M,S_o,\varepsilon_o,\sigma_o)$, especially if this can be derived from a more basic spinor structure.

Note that, from $(M,S_o,[\varepsilon_o])$, we made two independent choices - that of $\varepsilon_o \in [\varepsilon_o]$, and also the vector bundle isomorphism σ_o. Making different choices of ε_o and σ_o leads to a different spacetime geometry, as has been demonstrated in the literature:

Choosing a symplectic form $f\varepsilon_o$ $(f \in C(M,\mathbb{C}^*))$ leads to the metric $|f|^2 g$ on M; in other words to a conformally related Lorentzian manifold. Plymen and Westbury [P15] have shown, in the case $f \in C(M,S^1)$ (i.e. when the same metric g is obtained), that ε_o, $f\varepsilon_o$ determine equivalent g-spin structures if and only if f admits a global square root. In fact, Plymen and Westbury's argument is mainly homotopy theory based, and may thus be generalized to the case where $f \in C(M,\mathbb{C}^*)$ in the sense that ε_o, $f\varepsilon_o$ will determine equivalent spin structures on M (corresponding, via $\tilde{r}$, to the respective g -, $|f|^2 g$-spin structures) if and only if f admits a global square root.

A different vector bundle isomorphism σ_o corresponds to a different principal bundle homomorphism η in the g-spin structure, and the effect this choice has on spin connections and spinor field Lagrangians has been discussed by Isham [I 8].

In the next section we introduce a natural S^2-bundle over any spacetime, which will turn out to have both geometrical and physical significance, and will be used in later work.

1.9 The Projective Null Bundle

The aim of this section is to discuss a natural S^2-bundle over a spacetime (M,g), and thereby bring together ideas of earlier sections concerning spinors, conformal structures, the 2-sphere and four dimensional Lorentzian geometry. The bundle we introduce arises in at least two possible ways from structures on a 4-manifold M. One way is from a Lorentzian conformal structure in which case we have the bundle of future null directions, and another way is from a g-spin structure, where g is a Lorentzian metric, in which case we have a projective spin bundle. The Infeld-Van der Waerden isomorphism of section 1.7 gives a natural isomorphism between the two bundles, and we refer to both S^2 bundles as the projective null bundle over a spacetime (M,g). We might also discuss this bundle as arising from a Lorentzian spin conformal structure (as in section 1.5), but here we make an explicit choice of representative metric.

The projective null bundle is obviously a very natural object from both geometric and physical viewpoints. Indeed, the fibres of the projective null bundle are just the anti-celestial spheres at the points of spacetime. We regard this bundle as a natural six-dimensional arena on which to consider physically interesting spacetime equations. The case of lifting up Yang-Mills (in particular Maxwell) theory on Minkowski space to $\mathbb{R}^4 \times S^2$ is being investigated by Newman and coworkers (see [K3], [K12], [K17], but our discussion describes the situation for a general spacetime, and so geometrizes earlier work relating Lorentzian geometry and the 2-sphere (see, for example, [H7], [H3], [L7]).

The section is organized as follows: We construct the bundle of future null directions for a Lorentzian conformal manifold (M,C) and also the projective spin bundle for a Lorentzian spin manifold (M,g,s_g). In the case where $C = C_g$, we use the Infeld-Van der Waerden isomorphism to relate the two constructions. We also indicate the geometric inter-relations between the various bundles over spacetime M on the one hand and over the typical fibre S^2 of the projective null bundle on the other.

The basic idea behind the use of the projective null bundle as a space on which to consider spacetime theories is that we have available the spin and conformally weighted functions on each 2-sphere fibre, together with the associated $\eth$-operators described in example (1.5)1. Spinor equations on M may then be lifted to the total space of the projective null bundle and the fields on M satisfying those equations may be represented as sections of appropriate complex line bundles over the total space. Complicated partial differential equations on M often turn out to be much simpler, and their geometric and physical significance illuminated, when considered in this way. For example, a Maxwell field on Minkowski space may be lifted to a complex valued function satisfying a couple of linear equations on $\mathbb{R}^4 \times S^2$. These equations lead to a deeper understanding of the structure of Maxwell theory - in particular the duality properties of the electromagnetic field (see Kent et al. [K 3] for more details). Another example is the case of spinor differential equations on embedded submanifolds of spacetime, and we may consider lifting up such equations to the pullback of the projective null bundle in order to elucidate their structure.

We commence our discussion with some generalities on vector bundles. First let $\pi_E: E \to M$ be any real vector bundle of rank $r \geq 1$ over the manifold M. Let $\overset{*}{E} = E - \{0_E\}$ where 0_E is the zero section of E, and consider the natural free right action of $\mathbb{R}^+$ on $\overset{*}{E}$ by (positive) dilatations: $(v,t) \mapsto tv$, $\forall (v,t) \in \overset{*}{E} \times \mathbb{R}^+$ (This action is generated by the Liouville vector field $A_E \in \mathrm{Vect}(\overset{*}{E})$ given by $A_E(v) = v$ modulo the usual identification of $(\mathrm{Ker}\, D\pi_E)_v$ with $\pi_E^{-1}(\pi_E(v))$). We now have a principal $\mathbb{R}^+$-bundle $\mathbb{R}^+ \hookrightarrow \overset{*}{E} \to SE$, where $SE \equiv \overset{*}{E}/\mathbb{R}^+$ is the total space of the so called <u>sphere bundle</u> associated to E given by $S^{r-1} \hookrightarrow SE \to M$, with obvious projection onto M.

In the case when π_E is a rank r complex vector bundle over M, the required bundles will be $\overset{*}{\mathbb{C}} \hookrightarrow \overset{*}{E} \to \mathbb{P}E$ and $\mathbb{CP}^{r-1} \hookrightarrow \mathbb{P}E \to M$, where $\mathbb{P}E \equiv \overset{*}{E}/\overset{*}{\mathbb{C}}$. We call the $\mathbb{CP}^{r-1}$-bundle the <u>projective</u> bundle of E.

Now let M be a manifold of dimension n with cotangent bundle $\tau_M: T^*M \to M$. We use T^*M, rather than the tangent bundle TM, as a starting point for our constructions for various reasons. One reason is convention (see, for instance, Penrose and Rindler [P 11], where the covariant, rather than contravariant, projective spin bundle is considered), and a second reason is that there exists a naturally much richer structure on T^*M. For example we have the canonical 1-form $\eta_M \in \Omega^1(T^*M)$ given by $\alpha \mapsto \alpha \circ D\tau_M(\alpha)$, $\forall \alpha \in T^*M$, and also the associated canonical symplectic form $\omega_M = -d\eta_M$.

We have the principal $\mathbb{R}^+$-bundle, $\mathbb{R}^+ \hookrightarrow {}^*T^*M \to ST^*M$, and also the S^{n-1}-bundle, $S^{n-1} \hookrightarrow ST^*M \to M$, as described above for a general vector bundle. In fact, the $\mathbb{R}^+$-bundle over ST^*M is trivial since it admits a global section - for instance, given any positive definite metric g on M, we may define the section μ_g: $ST^*M \to {}^*T^*M$; $[\alpha] \mapsto \|\alpha\|_g^{-1}\alpha$, for all $[\alpha] \in ST^*M$. This section just identifies ST^*M as the unit cosphere bundle of (M,g). Using μ_g (or indeed any other section) we may pull back the canonical 1-form on ${}^*T^*M$ (which is just the restriction of η_M, and which we also denote by η_M) to a contact form $\mu_g^*\eta_M \in \Omega^1(ST^*M)$ on ST^*M. Note that a metric of Lorentzian signature will not give rise to a global section of $\mathbb{R}^+ \hookrightarrow {}^*T^*M \to ST^*M$ in the same way as a positive definite metric. Indeed, for a Lorentzian metric g, μ_g, as defined above, is singular precisely on the subspace of all null covector equivalence classes. However, we shall see below that it is possible to use a Lorentzian metric g to realise, at least pointwise on M, the null sub-bundle of ST^*M as a sub-bundle (rather than as a quotient) of ${}^*T^*M$.

To proceed towards our definition of the projective null bundle, we assume M to have additional structure. First we assume that M admits a Lorentzian metric (so that M must be either non-compact or compact but with vanishing Euler invariant). Let $\mathrm{Con}(M)$ denote the space of all Lorentzian conformal structures on M:

Definition (1.9)1: Let $C \in \mathrm{Con}(M)$. We say $\alpha \in T^*M$ is C-null if $\|\alpha\|_g = 0$ for some (and hence every) $g \in C$. Let $N(M,C) \equiv \{\alpha \in T^*M\colon \alpha \text{ is C-null}\}$ denote the bundle of all C-null covectors over M. Let $SN(M,C) = \{[\alpha] \in ST^*M\colon \alpha \text{ is C-null}\}$ (note that this definition makes sense).

In general, assuming spacetime orientability of (M,C) (the definition of spacetime orientability obviously extends to n-dimensions), $SN(M,C)$ will be a sub-bundle of $S^{n-1} \hookrightarrow ST^*M \rightarrow M$ with typical fibre the disjoint union of two copies of S^{n-2}. We obtain a bundle $S^{n-2} \hookrightarrow SN^+(M,C) \rightarrow M$ by choosing a spacetime orientation for (M,C). Given a particular Lorentzian metric $g \in \mathrm{Met}(M)$, we write $N(M,g) \equiv N(M,C_g)$ etc., where, as usual C_g is the conformal class containing g.

Now let (M,g) be a spacetime so that M is a four-dimensional connected manifold, and (M,g) is spacetime oriented. We shoose a particular orientation so that we have a well defined, consistent notion of future pointing vectors at each point in M. We have the following fibrations:

$$S^3 \hookrightarrow ST^*M \longrightarrow M \qquad 1.9.1,$$

$$S^2 \hookrightarrow SN^+(M,g) \longrightarrow M \qquad 1.9.2,$$

corresponding to those for the general situation discussed above. We also have the principal bundle $\mathbb{R}^+ \hookrightarrow {}^*T^*M \rightarrow ST^*M$, but we shall be interested mainly in the corresponding construction for the bundle of future null directions:

$$\mathbb{R}^+ \hookrightarrow {}^*N^+(M,g) \longrightarrow SN^+(M,g) \qquad 1.9.3,$$

where $\overset{*}{N}^{+}(M,g)$ is the bundle of future pointing (non-zero) null covectors over M. Note that we adopt the convention of referring to a covector as being future pointing if the corresponding vector in TM is future pointing (using the usual identification of TM with $T^{*}M$ via g). The fibration 1.9.2 may thus be regarded as the bundle of future null directions of (M,g), so that the fibres are just the anti-celestial spheres at points of spacetime. We may obtain a more concrete picture of the future null directions at a particular point in spacetime in the usual way (see Held et al. [H 7], Penrose and Rindler [P 11]):

Let $x \in M$ and $u \in \pi^{-1}(x)$, where $\pi \equiv \pi_M$: $SO(M,g) \longrightarrow M$ is the principal $SO^{\uparrow}(1,3)$-bundle of oriented g-orthonormal frames over M (1.1.2). We write $u = \{u_0, u_1, u_2, u_3\}$ with $u_0 \in T_xM$ a timelike future pointing unit vector. Define the hyperplane $\Pi_u \subseteq T_x^{*}M$ by $\Pi_u = \{\alpha \in T_x^{*}M\colon \; g(x)(\alpha^{\#}, u_0) = 1\}$, so that $S_u^2 \equiv \Pi_u \cap \overset{*}{N}_x^{+}(M,g)$, topologically a 2-sphere, is a <u>cut</u> of the space of future null directions at x. Note that $[\alpha] \cap \Pi_u$ consists of a unique element, say α_u, for each $[\alpha] \in SN_x^{+}(M,g)$, so we may define:

$$c_u : SN_x^{+}(M,g) \longrightarrow \overset{*}{N}_x^{+}(M,g); \quad [\alpha] \longmapsto \alpha_u \qquad 1.9.4,$$

for all $[\alpha] \in SN_x^{+}(M,g)$. The image of $SN_x^{+}(M,g)$ under c_u is precisely the cut S_u^2 and so c_u realizes the space of future null directions at x as a concrete 2-sphere sitting inside $T_x^{*}M$. The map c_u is, to some extent, analogous to the section μ_g defined above for the case of a positive definite metric g, but note that in order to define c_u, we have chosen a frame u at a particular

$x \in M$. In any case, c_u is a diffeomorphism of the space of future null directions at x onto S^2_u. Choosing a different frame $ua \in \pi^{-1}(x)$, $a \in SO^{\uparrow}(1,3)$, leads to the cut S^2_{ua} which is a supertranslation (see appendix 6.3) of the cut S^2_u, and the map $c_{ua} \circ c_u^{-1} \in \mathrm{Diff}(S^2)$ is the conformal transformation of (S^2, Can) corresponding to $a \in SO^{\uparrow}(1,3)$ (see 1.5.33). This construction thus gives an alternative realization of the Lorentz group as the conformal group of the 2-sphere (Cf. Example (1.5)1).

If M is parallelizable, then there exists a global section of $SO(M,g)$ and we perform the above construction at each point of M to obtain an S^2-subbundle of $\overset{*}{T}{}^{*}M$ which is bundle isomorphic to $SN^+(M,g)$ (equivalently, we obtain a section of the fibration 1.9.3). In general, however, no such section exists, and we only have a local "unit sphere" (local sections, of course, always exist), and even if $SO(M,g)$ is trivializable, the isomorphism between the unit sphere bundle and $SN^+(M,g)$ obtained depends on the choice of trivialization. Since we wish to avoid any such choices, we prefer to regard $SN^+(M,g)$ as a quotient, rather than as a sub-bundle, of $N^+(M,g)$.

The six-dimensional total space of the projective null bundle provides a natural arena on which to consider physically interesting fields lifted from spacetime. Before discussing such lifts, let us first discuss another construction of this space, this time using spinors:

Assume now that M is spin and let $s_g = (\widetilde{\widetilde{SO}}(M,g), \eta_g) \in \tilde{\Sigma}(M,g)$ be a g-spin structure on M, so that $\widetilde{\widetilde{SO}}(M,g)$ is a principal $SL(2,\mathbb{C})$-bundle over M. Obviously we do not need to assume the existence of such a g-spin structure in order to construct

$SN^+(M,g)$, but this construction gives rise to a useful representation of the space. A natural isomorphism between the two constructions arises from the Infeld-Van der Waerden isomorphism $\sigma(s_g)$ (see definition (1.7)2).

Using the notation defined in section 1.7, let $S^*(s_g)$ denote the rank two complex vector bundle over M associated to $\widetilde{SO}(M,g)$ by the representation ρ^* of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$. Elements of $S^*(s_g)$ are unprimed spinors λ_A (in abstract index notation). We then have the bundles:

$$\mathbb{C}^* \hookrightarrow {}^*S^*(s_g) \longrightarrow \mathbb{P}S^*(s_g) \qquad 1.9.5,$$

$$\mathbb{CP}^1 \hookrightarrow \mathbb{P}S^*(s_g) \longrightarrow M \qquad 1.9.6,$$

as above. Note that 1.9.6 may be regarded as the bundle associated to $\widetilde{SO}(M,g)$ via the action ϕ of $SL(2,\mathbb{C})$ on $\mathbb{CP}^1 \cong S^2$ given by equation 1.5.20. We now show that $\mathbb{P}S^*(s_g)$ is bundle isomorphic to the S^2-bundle $SN^+(M,g)$: Recall (1.7.18) the Infeld-Van der Waerden isomorphism (for s_g), $\sigma(s_g)$: $TM \rightarrow H(s_g) \hookrightarrow S(s_g) \otimes \bar{S}(s_g)$. Now define $\sigma^*(s_g)$: $T^*M \rightarrow H^*(s_g) \hookrightarrow S^*(s_g) \otimes \bar{S}^*(s_g)$ in the obvious manner using the identification of T^*M with TM (via g) and the identification of $S^*(s_g)$ with $S(s_g)$ (via the natural symplectic structure ε in $S(s_g)$). The map $\sigma^*(s_g)$ is an isomorphism of vector bundles which, when restricted to the space of all null covectors, projects down to a well defined map on $SN^+(M,g)$: We define

$$\nu(s_g)\colon\ SN^+(M,g) \longrightarrow \mathbb{P}S^*(s_g); \quad [\alpha] \longmapsto [\lambda_\alpha] \qquad 1.9.7,$$

for all $[\alpha] \in SN^+(M,g)$, where λ_α is any element of $S^*(s_g)$

satisfying $(\sigma^*(s_g))(\alpha) = \lambda_\alpha \otimes \overline{\lambda}_\alpha$ (Note that any null covector α corresponds, under the Infeld-Van der Waerden isomorphism, to a decomposable element of $S^*(s_g) \otimes \overline{S}^*(s_g)$ of the form $\lambda_\alpha \otimes \overline{\lambda}_\alpha$, where λ_α is unique up to multiplication by an element of S^1). Replacing the representative α of the $\mathbb{R}^+$-orbit $[\alpha]$ by any other representative $t\alpha$, $t \in \mathbb{R}^+$, changes any λ_α by a scaling $(t^{\frac{1}{2}})$ and hence does not change $[\lambda_\alpha]$. This demonstrates that equation 1.9.7 defines a map from $SN^+(M,g)$ into $\mathbb{P}S^*(s_g)$. The map $\nu(s_g)$ is easily seen to be a bijection and, moreover, an isomorphism of S^2-bundles. Hence, given the g-spin structure s_g, we may identify the bundle of future null directions $SN^+(M,g)$ with the projective spin bundle $\mathbb{P}S^*(s_g)$:

<u>Definition (1.9)2</u>: Let (M,g) be a spacetime which is spin, and let s_g be a g-spin structure on M. The <u>projective null bundle</u> of (M,g) is defined to be $SN^+(M,g) \cong \mathbb{P}S^*(s_g)$.

The $SN^+(M,g)$ description is physically more tangible, whilst the $\mathbb{P}S^*(s_g)$ description is more useful from a geometric viewpoint. A more concrete realization of the projective null bundle may be obtained by using the maps c_u, $\nu(s_g)$ given by 1.9.4, 1.9.7 respectively. We regard the bundle $\mathbb{P}S^*(s_g) \longrightarrow M$ (1.9.5) as the S^2-bundle associated to $\widetilde{\widetilde{SO}}(M,g)$ by the action given in 1.5.20. We then have, for each spin frame $\tilde{u} \in \widetilde{\widetilde{SO}}(M,g)$ at x, the diffeomorphism $\kappa_{\tilde{u}}: S^2 \longrightarrow \mathbb{P}S^*_x(s_g)$; $[\underline{z}] \longmapsto [(\tilde{u},[\underline{z}])]$, for each $[\underline{z}] \in S^2$. Let $\nu_x = \nu(s_g)|SN^+_x(M,g)$ (see 1.9.7) and define the diffeomorphism:

$$\ell_{\tilde{u}} = c_u \circ \nu_x^{-1} \circ \kappa_{\tilde{u}} : S^2 \longrightarrow S^2_u \subseteq {}^*N^+_x(M,g) \qquad 1.9.8,$$

where $u = \eta_g(\tilde{u}) \in SO(m,g)$. For each spin frame $\tilde{u}$ at x, we thus have a field of null vectors defined on S^2; as $[\underline{z}]$ varies over S^2, $\ell_{\tilde{u}}([\underline{z}])$ sweeps out the entire space of (normalized) future null directions at x. Choosing a different frame $\tilde{u}A$ at x, $A \in SL(2,\mathbb{C})$, gives rise to $\ell_{\tilde{u}A} = K_A \ell_{\tilde{u}}$, where K_A is the conformal factor associated with the conformal action of $SL(2,\mathbb{C})$ (or rather $SO^{\uparrow}(1,3)$) on S^2 (see equation 1.5.36). An explicit diffeomorphism of S^2 onto S^2_u may also be constructed without using spinors by projecting the cut S^2_u into the 3-space orthogonal to u^b_o and reducing $SO^{\uparrow}(1,3)$ to $SO(3)$ which then acts on S^2 (Held et al. [H 7]), but the construction follows more directly from $\widetilde{\widetilde{SO}}(M,g)$, once a g-spin structure has been chosen.

Before returning to our discussion of the projective null bundle, we remark on the analogous construction for a Riemannian 4-manifold (M,g), which is that of the Penrose twistor space (see Atiyah et al. [A 30]). This is the six-dimensional space obtained either as the unit sphere bundle of the bundle of anti-self-dual 2-forms $\Lambda^-(M,g)$ or, again, as a projective spin bundle. The Penrose twistor space admits a natural almost complex structure which is integrable if and only if (M,g) is half-conformally flat, and this construction yields a very important link between self-duality and algebraic geometry. There is also a twistor construction for Lorentzian 4-manifolds (Wells [W 14], Woodhouse [W 16]) but this involves a more indirect correspondence with the manifold M - rather than a fibration over M itself, one has a real 5-manifold fibred only over each spacelike hypersurface in M. The projective null bundle approach is to exploit the fact that we have a fibration over spacetime itself, and a comparison with the twistor methods would be a subject for further study.

The structure of the projective null bundle will now be considered in more detail. We use the $\mathbb{P}S^*(s_g)$ description since we may regard $\mathbb{P}S^*(s_g)$ as the S^2-bundle associated to $\widetilde{SO}(M,g)$ via the action ϕ given by equation 1.5.20. Having given the theory of the complex line bundles $E(s,w)$ over S^2 in section 1.5, we will now apply this fibrewise to $S^2 \hookrightarrow \mathbb{P}S^*(s_g) \rightarrow M$ to obtain a copy of $E(s,w)$ above each point of spacetime.

We have the following diagram of fibrations:

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r, hook] & \mathbb{C}^2-\{\underline{0}\} \arrow[r] \arrow[d, hook] & S^2 \arrow[d, hook] & SL(2,\mathbb{C}) \arrow[d, hook] \\
\mathbb{C}^* \arrow[r, hook] & {}^*S^*(s_g) \arrow[r] \arrow[dr] & \mathbb{P}S^*(s_g) \arrow[d] & \widetilde{SO}(M,g) \arrow[dl] \\
 & & M &
\end{tikzcd} \quad 1.9.9,$$

where $\mathbb{C}^* \hookrightarrow \mathbb{C}^2-\{\underline{0}\} \rightarrow S$ is the unique (up to equivalence of prolongations) spin conformal structure of (S^2,Can) of 1.5.13, $\mathbb{C}^* \hookrightarrow {}^*S^*(s_g) \rightarrow \mathbb{P}S^*(s_g)$ is the principal $\mathbb{C}^*$-fibre bundle of 1.9.5, and $SL(2,\mathbb{C}) \hookrightarrow \widetilde{SO}(M,g) \rightarrow M$ is the principal $SL(2,\mathbb{C})$-bundle given by the g-spin structure s_g. The diagram 1.9.9 indicates that the spin conformal structure of S^2 is attached fibrewise to the projective null bundle $S^2 \hookrightarrow \mathbb{P}S^*(s_g) \longrightarrow M$ (1.9.6).

Now recall the complex line bundle $\mathbb{C} \hookrightarrow E(s,w) \rightarrow S^2$ defined in section 1.5 for each $w \in \mathbb{C}$, $2s \in \mathbb{Z}$. We attach such a line bundle to each point $x \in M$ in a natural way:

<u>Definition (1.9)3</u>: Let (M,g) be a spacetime which is spin, and let s_g be a g-spin structure. Let $w \in \mathbb{C}$, $2s \in \mathbb{Z}$ and define the

complex line bundle $\mathbb{C} \hookrightarrow E(s,w;s_g) \rightarrow \mathbb{P}S^*(s_g)$, where $E(s,w;s_g) = {}^{*}S^{*}(s_g) \times_{\mathbb{C}^*} \mathbb{C}$ is associated to the principal $\mathbb{C}^*$-bundle ${}^{*}S^{*}(s_g)$ by the representation $\overset{\Delta}{\rho}_{s,w}$ defined in equation 1.5.14.

Consider the diagram associated to diagram 1.9.9:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook] & E(s,w) \arrow[r] \arrow[d, hook] & S^2 \arrow[d, hook] & V \arrow[d, hook] \\
\mathbb{C} \arrow[r, hook] & E(s,w;s_g) \arrow[r] \arrow[dr] & \mathbb{P}S^*(s_g) \arrow[d] & \widetilde{SO}(M,g)\times_{SL(2,\mathbb{C})}V \arrow[dl] \\
& & M &
\end{tikzcd} \tag{1.9.10}$$

where $\widetilde{SO}(M,g) \times_{SL(2,\mathbb{C})} V$ is the vector bundle associated to the spin bundle $SL(2,\mathbb{C}) \hookrightarrow \widetilde{SO}(M,g) \rightarrow M$ via some given representation of $SL(2,\mathbb{C})$ on the vector space V. For example, this associated vector bundle could be a bundle of spinors obtained from the irreducible representation $D^{\circ}(\mu,\nu)$, which we defined in section 1.7. Sections of $\widetilde{SO}(M,g) \times_{SL(2,\mathbb{C})} V$ will be physical fields on spacetime, and we wish to associate to each such field on M a section of the complex line bundle $E(s,w;s_g)$ over $\mathbb{P}S^*(s_g)$ (for some s,w), i.e. we wish to lift fields on spacetime to sections of line bundles on the total space of the projective null bundle. The particular $E(s,w;s_g)$ which arises will obviously depend on the representation under which the spacetime field transforms. Before discussing the geometry underlying this realization of fields on spacetime as sections of line bundles over the projective null bundle, we discuss briefly the relationship between representations of $SL(2,\mathbb{C})$ on the one hand and spin and conformally weighted functions on the two sphere on the other. For more details concerning the representation theory involved, see Goldberg et al. [G 13], Held et al. [H 7], and Lind et al. [L 7].

Recall the irreducible representation $D^o(\mu,\nu)$ $(2\mu, 2\nu \in \mathbb{Z})$ of $SL(2,\mathbb{C})$ discussed at the end of section 1.7. This representation is defined on the vector space $S(\mu,\nu) = (\odot^{2\mu} S) \otimes (\odot^{2\nu} \bar{S})$ of all tensors on $S(\cong \mathbb{C}^2)$ which are totally symmetric in both their 2μ unprimed slots and in their 2ν primed slots. The dimension of the representation is $(2\mu+1)(2\nu+1)$. The existence of the symplectic form ε means that we only need consider covariant tensors Φ (= $\Phi_{A_1 \ldots A_{2\mu} A'_1 \ldots A'_{2\nu}}$ in abstract index notation or with respect to the standard basis of $\mathbb{C}^2$). Let us write

$$\Phi(\underline{z}) = \Phi_{A_1 \ldots A_{2\mu} A'_1 \ldots A'_{2\nu}} z^{A_1} \ldots z^{A_{2\mu}} \bar{z}^{A'_1} \ldots \bar{z}^{A'_{2\nu}} \qquad 1.9.11,$$

for the image in $\mathbb{C}$ of an element $\underline{z}$ of $\mathbb{C}^2$ under the tensor Φ. Then the representation $D^o(\mu,\nu)$ is given by:

$$D^o \equiv D^o(\mu,\nu)\colon SL(2,\mathbb{C}) \times S(\mu,\nu) \longrightarrow S(\mu,\nu); \quad (A,\Phi) \longmapsto D^o_A \Phi,$$

$$\text{where} \quad (D^o_A\Phi)(\underline{z}) = \Phi(A^T \underline{z}) \qquad 1.9.12,$$

for all $\underline{z} \in \mathbb{C}^2$, $\Phi \in S(\mu,\nu)$, $A \in SL(2,\mathbb{C})$.

We now consider another irreducible representation of $SL(2,\mathbb{C})$, this time on a space of spin and conformally weighted functions on S^2. Let $2s \in \mathbb{Z}$, $w \in \mathbb{C}$, and consider $\Gamma(E(s,w))$ as the space $C_{\mathbb{C}^*}(\mathbb{C}^2 - \{\underline{0}\}, \mathbb{C})$ of equivariant maps of $\mathbb{C}^2-\{\underline{0}\}$ into $\mathbb{C}$, so that $\eta \in \Gamma(E(s,w))$ implies $\eta(\lambda\underline{z}) = \overset{\Delta}{\rho}_{s,w}(\lambda^{-1})\eta(\underline{z})$, for all $\underline{z} \in \mathbb{C}^2-\{\underline{0}\}$, $\lambda \in \mathbb{C}^*$. We define a representation $\Delta(s,w)$ of $SL(2,\mathbb{C})$ on $\Gamma(E(s,w))$

by:

$$\Delta \equiv \Delta(s,w):\ SL(2,\mathbb{C}) \times \Gamma(E(s,w)) \longrightarrow \Gamma(E(s,w));(A,\eta) \mapsto \Delta_A\eta,$$

$$\text{where} \quad (\Delta_A\eta)(\underline{z}) = \eta(A^T\underline{z}) \qquad 1.9.13,$$

for all $\underline{z} \in \mathbb{C}^2-\{\underline{0}\}$, $\eta \in \Gamma(E(s,w))$, $A \in SL(2,\mathbb{C})$.

Δ is not an irreducible representation, but an irreducible representation Δ^o is obtained by considering an invariant subspace. We now assume that $2w \in \mathbb{Z}$ with $w \geq |s|$. Let $\Gamma^o(E(s,w))$ $= \text{span}\ \{{}_sY_{\ell m}\colon |s| \leq \ell \leq w\}$, where $\{{}_sY_{\ell m}\colon m \leq |\ell|,\ \ell \geq |s|\}$ are the spin s-spherical harmonics, so that $\text{span}\{{}_sY_{\ell m}\colon m \leq |\ell|\}$ is the eigenspace of the operator $\bar{\eth}\eth$ corresponding to the eigenvalue $(s-\ell)(s+\ell+1)$. Then $\Gamma^o(E(s,w))$ is a subspace of $\Gamma(E(s,w))$ with dimension $(w-s+1)(w+s+1)$ which is invariant under the $SL(2,\mathbb{C})$-action defined by Δ. Using the behaviour of the spin s-spherical harmonics under $\eth$, it may be shown that $\Delta^o \equiv \Delta|SL(2,\mathbb{C}) \times \Gamma^o(E(s,w))$ is an irreducible representation of $SL(2,\mathbb{C})$. We now show that $D^o(\mu,\nu)$ and $\Delta^o(s,w)$ are equivalent if $2\mu = w-s$ and $2\nu = w+s$, by defining an isomorphism θ which intertwines the two actions D^o, Δ^o of $SL(2,\mathbb{C})$:

We define the linear map $\theta\colon\ S(\mu,\nu) \to \Gamma^o(E(s,w))$ by:

$$\theta(\Phi) = \Phi|\mathbb{C}^2-\{\underline{0}\} \qquad 1.9.14,$$

for all $\Phi \in S(\mu,\nu)$, where we are regarding any $\Phi \in S(\mu,\nu)$ as a map on $\mathbb{C}^2$ as in 1.9.11. Suppose $A \in SL(2,\mathbb{C})$, $\Phi \in S(\mu,\nu)$ and $\underline{z} \in \mathbb{C}^2-\{\underline{0}\}$, then we have $((\Delta^o_A\circ\theta)(\Phi))(\underline{z}) = (\Delta^o_A(\theta(\Phi)))(\underline{z})$ $= \theta(\Phi)(A^T\underline{z}) = \Phi(A^T\underline{z}) = (D^o_A\Phi)(z) = (\theta(D^o_A\Phi))(\underline{z}) = ((\theta\circ D^o_A)(\Phi))(\underline{z})$, i.e. $\Delta^o_A \circ \theta = \theta \circ D^o_A$ so that the linear map θ intertwines the representations D^o, Δ^o. By inspection, θ is injective, and hence, since $\dim_{\mathbb{C}} S(\mu,\nu) = (2\mu+1)(2\nu+1) = (w-s+1)(w+s+1)$ $= \dim_{\mathbb{C}}\Gamma^o(E(s,w))$, θ is an isomorphism of vector spaces. We have demonstrated that the two representations are equivalent, and from now on we identify $D^o(\mu,\nu)$ with $\Delta^o(\nu-\mu, \mu+\nu)$ unless the identifying map θ is explicitly required.

Let us now return to spacetime:

<u>Definition (9.1)4</u>: Let (M,g) be a spacetime which is spin and let s_g be a g-spin structure on M. Let 2μ, $2\nu \in \mathbb{Z}$ and define $D^o(\mu,\nu;s_g)$ to be the vector bundle associated to $\widetilde{SO}(M,g)$ via the irreducible representation $D^o(\mu,\nu)$ of $SL(2,\mathbb{C})$ on $S(\mu,\nu) \cong \Gamma^o(E(\nu-\mu, \mu+\nu))$.

In general, a field on spacetime will transform under a representation of the spin group $SL(2,\mathbb{C})$ and also under an additional group G (e.g. the structure group of a Yang-Mills theory). Such a field is a section of $(\bigoplus_{\mu,\nu} D(\mu,\nu;s_g)) \otimes F$, where F is some G-vector bundle over M, but for simplicity we focus our attention on a particular finite dimensional irreducible representation of $SL(2,\mathbb{C})$, and hence on $D^o(\mu,\nu;s_g)$ for some choice of μ,ν.

We now define a linear isomorphism of $\Gamma(D^o(\mu,\nu;s_g))$ onto $\Gamma(E(\nu-\mu, \mu+\nu;s_g))$, so that to each field on spacetime we may associate a section of the complex line bundle $E(\nu-\mu, \mu+\nu;s_g)$ over the projective null bundle. We regard a section of $D^o(\mu,\nu;s_g)$ as an equivariant map $\Psi: \widetilde{SO}(M,g) \to S(\mu,\nu)$ $\cong \Gamma^o(E(\nu-\mu, \mu+\nu))$, and a section of $E(\nu-\mu, \mu+\nu:s_g)$ as an equivariant map $H: {}^{**}S(s_g) \to \mathbb{C}$, where, in turn, ${}^{**}S(s_g)$ is considered as the $\mathbb{C}^2-\{\underline{0}\}$-bundle associated to $\widetilde{SO}(M,g)$ via the restriction of the representation ρ^* of $SL(2,\mathbb{C})$.

<u>Definition (1.9)5</u>: Define the <u>null lift</u> $L: \Gamma(D^o(\mu,\nu;s_g)) \to \Gamma(E(\nu-\mu,\ \mu+\nu;s_g))$; $\Psi \mapsto \Psi^L$, where $\Psi^L([(\tilde{u},\ \underline{z})]) = \Psi(\tilde{u})(\underline{z})$, for all $[(\tilde{u},\ \underline{z})] \in {}^*S^*(s_g)$.

We show that definition (1.9)5 makes sense: Let $\Psi \in \Gamma(D^o(\mu,\nu;s_g))$, $\underline{z} \in \mathbb{C}^2-\{0\}$, $\tilde{u} \in \tilde{\mathcal{SO}}(M,g)$. Suppose $A \in SL(2,\mathbb{C})$, then $\Psi(\tilde{u}A)((\rho^*)^{-1}_{A^{-1}}\underline{z}) = (D^o_{A^{-1}}(\Psi(\tilde{u})))(A^T\underline{z}) = \Psi(\tilde{u})((A^{-1})^T A^T\underline{z}) = \Psi(\tilde{u})(\underline{z})$, so that Ψ^L is well defined. Now let $\lambda \in \mathbb{C}^*$, then $\Psi^L([(\tilde{u},\underline{z})]\lambda) = \Psi(\tilde{u})(\lambda\underline{z}) = \overset{\Delta}{\rho}_{\nu-\mu,\mu+\nu}(\lambda^{-1})\Psi(\tilde{u})(\underline{z}) = \overset{\Delta}{\rho}_{\nu-\mu,\mu+\nu}(\lambda^{-1})\Psi^L([(\tilde{u},\underline{z})])$, so that Ψ^L is equivariant and is indeed a section of $E(\nu-\mu,\ \mu+\nu;\ s_g)$.

Since the null lift L is, by inspection, linear and bijective, we have a one-to-one correspondence between fields on spacetime transforming under $D^o(\mu,\nu)$ on the one hand and spin and conformally weighted functions on the projective null bundle on the other. The philosophy which may be adopted is to use L to lift up equations satisfied by physical fields on spacetime to equations on the projective null bundle. The lifted equations are often simpler (see Kent et al. [K 3] for a discussion of the lifted Maxwell equations when $M = \mathbb{R}^4$), and also more natural, especially when spinors and null structures are involved, as in Chapter Three below. Hansen et al. [H 3] have demonstrated, using the concrete realization S^2_u ($u \in \pi^{-1}(x)$) of the space of future null directions at $x \in M$, how properties of spacetime fields evaluated at x may have a simple expression in terms of the spin and conformally weighted function on S^2_u obtained by null lifting the spacetime field at z. For example, the norm of a vector at x turns out to be just the product of the two critical values of the corresponding function on S^2_u.

This concludes our present discussion of the projective null bundle. In this section we have demonstrated how several important ideas from earlier sections interact and we have set up a geometrical framework which will form the basis for future work. These further investigations will appear elsewhere.

2.0 Introduction - Why Embed?

This chapter concerns itself with various aspects of the theory of embeddings. The reasons for including such a chapter are three-fold:

Firstly, embeddings have found many interesting applications in the theory of general relativity (and in other areas of gravity theory), both at the finite dimensional and infinite dimensional levels. Each of the following three sections contains a review of some of these applications; finite dimensional aspects in section 2.1, infinite dimensional aspects in section 2.2, and spinorial aspects in section 2.3.

Secondly, the theory of embeddings makes contact with and interrelates several parts of this thesis. In particular, Chapters One, Three, Four and sections 6.2, 6.3. The spinor ideas of sections 1.7, 1.8 and 1.9 come together with embeddings in the very useful GHP formalism in general relativity, and we use this formalism in Chapter Three (see also the conformal aspects in sections 1.5 and 6.2). In the theory of asymptotically flat spacetimes (see section 6.3), null infinity is an embedded submanifold of the compactified spacetime and it is this submanifold which provides a framework for the study of gravitational radiation. Embeddings also interact with the two levels of everywhere invariance discussed in Chapter Four: On the one hand, the natural flavour of everywhere invariance is present when we study spaces of embeddings (see section 2.2), and on the other hand, many of the families of metrics considered in Chapter Four are embedded submanifolds of Met(M).

The third reason for discussing embeddings is that we make explicit use of (null) embeddings in Chapter Three in our treatment of spinor propagation equations and quasi-local momentum in general relativity. Several parts of this chapter (and of Chapter One) will be utilized in Chapter Three.

The main purpose of this chapter is to review, and much of the material is standard. However, we hope that we have clarified certain interrelationships between the theory of embeddings on the one hand and aspects of general relativity on the other. We also remark that several of the suggestions made in section 2.2 are novel and deserve further study.

As usual, we make no attempt to discuss analytical details. For a thorough treatment of the infinite dimensional manifolds involved (especially in section 2.2), we refer the reader to Binz and Fischer [B 12] and to Hamilton [H 2]. All concepts are appropriately smooth.

2.1 Embeddings

The object of this section is to give a brief survey of the basic ideas relating to the theory of embeddings. We first give the differential topological framework and then introduce related differential geometrical concepts. Since much of the discussion in this thesis is conducted within the language of principal fibre bundles, we express the ideas of the latter part of this section in this language also. In this section, we also review certain applications of embeddings to topics in general relativity.

Let M, N be smooth manifolds, possibly infinite dimensional. Typically, M and N will be each modelled on some nice topological vector space. Recall that if E is a topological vector space, and $F \leqslant E$ a closed subspace of E, then we say that F splits if there exists a closed subspace G of E such that $E = F \oplus G$ (topological direct sum). For example, if E is a Hilbert space, then any closed subspace F splits; we just take $G = F^{\perp}$. In general, however, a closed subspace of even a Banach space need not possess a closed complement. If E is finite dimensional, then every subspace of E splits.

Definition (2.1)1: A smooth map $f: M \to N$ is said to be an immersion if, for all $x \in M$, the map $Df(x)$ is injective and $Df(x).T_xM$ splits (as a closed subspace of $T_{f(x)}N$). The smooth map $f: M \to N$ is said to be an embedding if f is an immersion and, in addition, f is a homeomorphism of M onto $f(M)$ ($f(M)$ with the topology inherited from N). If f is an embedding, we write $f: M \hookrightarrow N$.

The importance of embeddings lies in their relation to submanifolds; a subset A of a manifold N is a (closed) submanifold of N if and only if A is the image of a (closed) embedding. We may also use an embedding to pullback covariant tensor fields in the usual way:- Suppose $f: M \hookrightarrow N$, and $\omega \in \Gamma(\otimes^k T^*N)$. Then $f^*\omega \in \Gamma(\otimes^k T^*M)$ is defined by:

$$(f^*\omega)(x)(v_1,\ldots,v_k) = \omega(f(x))(Df(x).v_1,\ldots, Df(x).v_k) \quad 2.1.1,$$

for all $v_1,\ldots,v_k \in T_xM$ and $x \in M$. An important case is when ω is a (weak) Riemannian or symplectic structure on N.

For more details concerning the above definitions, see Lang [L2].

For the rest of this section, we assume that all embeddings are between finite dimensional manifolds. Note that the infinite dimensional case arises in Chapter Four where we consider embeddings into the space of metrics.

The manifolds we use in geometry and in physics are usually regarded as abstract objects, and not as being embedded in a manifold of higher dimension. Nevertheless, it is occasionally useful to consider such embeddings, since the degrees of freedom inherent in the additional dimensions may be of help in gaining extra geometrical insight. In general relativity, this insight may lead to new links between geometry and physics. For this reason, and others, we present a brief survey of this concretization of manifolds:

From a "bare" differential topological viewpoint, the most important result is the Whitney embedding theorem (see Hirsch [H 18]). For $n \geqslant 1$, given any (paracompact, Hausdorff) n-manifold M, there exists an embedding $f: M \hookrightarrow \mathbb{R}^{2n}$ (and an immersion $h: M \looparrowright \mathbb{R}^{2n-1}$, if $n \geqslant 2$). The Whitney theorem demonstrates that we may regard any finite dimensional (paracompact, Hausdorff) manifold as a submanifold of Euclidean space of twice the dimension. Of course, in spacific instances, we may be able to find an embedding in $\mathbb{R}^m$, where $m < 2n$.

We now consider the geometrical aspects of embedding. From now on in this section, the term Riemannian will refer to a metric of any (non-degenerate) signature.

Definition (2.1)2: Given Riemannian manifolds (M,g), (N,k), an embedding $f: M \hookrightarrow N$ is said to be isometric if $f^{*}k = g$.

We are often given f and (N,k) and we define the metric g

on M by $g = f^*k$, so that f is an isometric embedding. If k is definite, then g is necessarily definite, but if k is indefinite (e.g. Lorentzian), then there are various possibilities for g; g could be indefinite, degenerate (null) or definite, depending on f.

Suppose now we are given (M,g) and wish to isometrically embed (M,g) in some Riemannian manifold (N,k). We may ask what obstructions, if any, prevent the existence of such an embedding. (N,k) is often taken to be $(\mathbb{R}^{p+q}, \mathrm{can}(p,q))$, where $\mathrm{can}(p,q)$ has components $\mathrm{diag}(1,\ldots, 1, -1,\ldots,-1)$ (p positive eigenvalues and q negative eigenvalues; signature $= p - q$) with respect to the standard coordinates on $\mathbb{R}^{p+q}$, but other embedding spaces (N,k) may also be considered; for example, Riemannian manifolds which are of constant curvature, conformally flat or Ricci flat. We restrict our attention to the Euclidean case where the obstruction to embedding in $(\mathbb{R}^{p+q}, \mathrm{can}(p,q))$ depends only on p,q.

Three important (overlapping) theorems on isometric embedding in Euclidean space are the following:

First, we have the celebrated theorem of Nash:

Theorem (2.1)3 (Nash [N 6]): Let M be a smooth n-manifold and g a $C^k (k \geq 3)$ positive definite metric (i.e. signature = n). Then there exists a C^k isometric embedding of (M,g) in $(\mathbb{R}^p, \mathrm{can}(p,0))$, where $p = \frac{1}{2}n(3n+11)$ (M compact) or $p = \frac{1}{2}n(3n^2 + 14n + 11)$ (M non-compact).

A generalization, and improvement, of Nash's theorem is the following:

Theorem (2.1)4 (Clarke [C 10]): Let M be a smooth n-manifold

and g a C^k $(k \geq 3)$ metric of signature s. Then there exists a C^k isometric embedding of (M,g) in $(\mathbb{R}^{p+q}, \mathrm{can}(p,q))$, where $p = \frac{1}{2}(n + s + 2)$ and $q = \frac{1}{2}n(3n + 11)$ (M compact),
$q = \dfrac{1}{6}(2n^3 + 15n^2 + 37n + 6)$ (M non-compact).

Note that for non-compact definite metrics, Clarke's result improves on that of Nash (Nash's result may obviously be rewritten for negative definite metrics). If we take (M,g) to be a spacetime, so that $n = 4$ and $s = -2$, then we see that Clarke's result implies that (M,g) may be isometrically embedded in $(\mathbb{R}^{48}, \mathrm{can}(2,46))$ (M compact) or $(\mathbb{R}^{89}, \mathrm{can}(2,87))$ (M non-compact). Only two timelike directions are necessary for embedding any spacetime and this result is the best possible; Clarke [C 10] demonstrates that there exist spacetimes that cannot be isometrically embedded in $(\mathbb{R}^{1+q}, \mathrm{can}(1,q))$. However, if the spacetime is globally hyperbolic, then it can be isometrically embedded in $(\mathbb{R}^{1+q}, \mathrm{can}(1,q))$ (with q as above).

The dimensions 48,89 given above are not the best possible for spacetimes if we require the metric to be smooth, as the following result of Greene shows:

Theorem (2.1)5 (Greene [G 17]): Let M be a smooth n-manifold and g a smooth metric (of any (non-degenerate) signature). Then there exists a smooth isometric embedding of (M,g) in $(\mathbb{R}^{2p}, \mathrm{can}(p,p))$, where $p = \frac{1}{2}n(n + 5)$ (M compact) or $p = 2(2n + 1)(n + 3)$ (M non-compact).

For example, any smooth spacetime may be isometrically embedded in $(\mathbb{R}^{36}, \mathrm{can}(18,18))$ (compact case) or $(\mathbb{R}^{252}, \mathrm{can}(126,126))$. The latter result is not as good as that of Clarke and, indeed, the Greene theorem only gives an improvement in the non-compact smooth case for $n \geq 20$.

The theorems quoted above enable us to define certain arithmetic invariants associated with a Riemannian manifold. One such invariant is the embedding class. Let $\mathrm{Met}(M)$ denote the space of smooth metrics of signature s ($|s| \leq n$) on the smooth n-manifold M (analogous definitions for the C^k and the analytic cases may also be given). For $g \in \mathrm{Met}(M)$, theorem (2.1)4 gives restrictions on the possible values of the signature of a Euclidean space in which (M,g) may be isometrically embedded. For each $r \in \mathbb{Z}$, prescribed as the signature of a Euclidean embedding space for (M,g) within the limits of theorem (2.1)4, we define the <u>embedding class</u> $\nu(g,r)$ by $\nu(g,r) = \min\{u \in \mathbb{Z} :$ there exists a smooth isometric embedding of (M,g) in $(\mathbb{R}^{n+u}, \mathrm{can}(\frac{1}{2}(n+u+r), \frac{1}{2}(n+u-r)))\}$. In other words, for given (g,r), the embedding class $\nu(g,r)$ is the smallest integer u such that (M,g) may be regarded as a Riemannian submanifold of Euclidean space of dimension $(\dim M + u)$ and signature r. Theorem (2.1)4 gives an upper bound for $\nu(g,r)$. More generally embedding classes may be defined if we consider other classes of embedding spaces; for example, we could ask for the smallest dimension of a Ricci flat Riemannian manifold of given signature in which (M,g) may be smoothly isometrically embedded.

The embedding class is an invariant which may be used in a classification programme for Riemannian manifolds of a particular type; for example, solutions of the Einstein equations in general relativity. Goenner [G 11] gives examples and applications to general relativity of the <u>local embedding class</u> - this is defined as above except that only <u>local isometric embeddings</u> are considered (a local isometric embedding need only be defined on an open set of the domain). For example, necessary conditions for spacetimes to be of a given local embedding class may be written down as a relation involving the curvature tensor or as requirement that the spacetime admit a certain class of geodesic congruences. In the latter case, the congruences may be interpreted as the world lines of particular kinds of matter or radiation, thus giving a more direct link between embeddings and physics.

Another important invariant of any Riemannian manifold is its isometry group. In particular, the isometry group is an important ingredient in the classification programme in general relativity (Cf. section 4.5). The interaction between isometry groups on the one hand and embedding classes on the other is therefore important to understand. It has been known for a long time that the isometry group does not determine the embedding class (see Goenner for examples), but amongst the exact solutions of Einstein's equations known, a large isometry group is accompanied by a low (local) embedding class. On the other hand, there exist spacetimes of class one with trivial isometry group. In fact, it is the orbit structure of the isometry group action, not just the isomorphism class of the isometry group, which interacts with the embedding class.

As well as considering the possibility of embedding a spacetime into a higher dimensional Riemannian manifold, the embedded <u>submanifolds</u> of a spacetime play an important rôle in general relativity. One dimensional submanifolds are curves in spacetime and particularly important are null and timelike curves which are possible world lines of radiation and matter. Among two dimensional submanifolds, spacelike ones have probably received the most attention in general relativity. For example, an important concept in singularity theory is that of a closed trapped surface which is a compact spacelike two dimensional submanifold with a certain extrinsic curvature condition (see Hawking and Ellis [H 5]). Another area in which two dimensional submanifolds arise is in the study of quasi-local kinematical quantities (see Chapter Three); to obtain, say, the gravitational energy intercepted by some region in a spacelike hypersurface, we integrate over the compact spacelike surface which is the boundary of the hypersurface region. Indeed, we may regard the energy calculated in this way as being surrounded or "linked" by the 2-surface. Three dimensional submanifolds considered in general relativity are null (see Chapter Three), spacelike (as Cauchy surfaces and in the 3+1 initial value problem - see Hawking and Ellis [H 5]) or timelike (in cosmology).

It is fair to say that in four dimensional geometry, in particular in general relativity, all possible codimensions for submanifolds play an important rôle; codimension zero corresponds to spacetime, codimension one to hypersurfaces, codimension two to (2-) surfaces (on which curvature has perhaps its most essential manifestation), codimension three to curves, and codimension four to discrete collections of events.

There are also some connections between submanifolds of geometric significance on the one hand, and embedding class on the other. For example, any product spacetime has (local) embedding class less than six, and any spacetime with a non-null totally geodesic hypersurface has embedding class not greater than five.

We refer the reader to Goenner [G 11] for more examples of these connections.

We now move away from embedding class and mention alternative possible reasons for studying embeddings in general relativity:

The most important global aspects of general relativity are those related to causal structure, and causal properties of a spacetime may be related to the possibility of certain kinds of embedding. For example, if a spacetime (M,g) admits an isometric embedding in $(\mathbb{R}^n, \mathrm{can}(1,n-1))$, then since the embedding space contains no closed non-spacelike curves, neither does (M,g). i.e. such a spacetime is necessarily causal. Moreover, it can be shown (see Clarke [C 10]) that a spacetime embeddible in $(\mathbb{R}^n, \mathrm{can}(1,n-1))$ is actually stably causal (i.e. there exists a C^0-neighbourhood of g in $\mathrm{Met}(M)$ whose elements are all causal, so that the spacetime metric g remains causal under small continuous perturbations). Conversely, any stably causal spacetime is conformeomorphic (see section 6.2) to a spacetime which does admit an isometric embedding in $(\mathbb{R}^n, \mathrm{can}(1,n-1))$. We have already stated above that any globally hyperbolic spacetime can be isometrically embedded in some $(\mathbb{R}^n, \mathrm{can}(1,n-1))$, and since a spacetime is globally hyperbolic if and only if it admits a Cauchy surface (see [H 5], pp. 211-212), we see another connection between submanifolds and embeddibility.

Another area in which an investigation of embeddings may shed light on global aspects of spacetimes is in the definition of boundary points (or singularities). There exist various possible ways of attaching a boundary to a spacetime (see Hawking and Ellis [H 5]), but another means of doing this is by minimally embedding the maximal extension (still incomplete) of the spacetime in the Euclidean space of dimension equal to the embedding class (for some signature). The boundary would be obtained by taking the closure of M regarded as a submanifold of Euclidean space. Other embedding spaces could also be used. We refer the reader to Goenner [G 11] for more discussion on this and other possible applications of embeddings to general relativity. Note that, as Goenner emphasizes, it is important to realize that a mere cataloguing of spacetimes according to their embedding properties is not sufficient; in order that the embedding ideas mentioned above be useful, it is necessary to investigate further those concepts that enable physical questions to be answered.

We now give a description of the differential geometric aspects of embeddings. For a thorough account, see Kobayashi and Nomizu [K 8]. For ease of exposition, we deal only with the positive definite case. Analogous results hold for spacelike or timelike embeddings into a Lorentzian manifold. We deal with null embeddings in spacetime in Chapter Three.

Let us first consider an embedding $f: M \hookrightarrow N$. We have the tangent bundle $\tau_N: TN \to N$, and hence the pullback bundle (see definition (6.1)9) $f^*\tau_N: f^*(TN) \to M$. The tangent bundle $\tau_M: TM \to M$ may be regarded as a subbundle of $f^*(TN)$ via the vector bundle monomorphism, $v \mapsto (\tau_M(v), Df(\tau_M(v)).v) \in (f^*\tau_N)^{-1}(\tau_M(v)) \subseteq f^*(TN)$, for all $v \in TM$. We may therefore take the quotient of $f^*(TN)$ by (the image of) TM. This is a vector bundle over M whose fibre over $x \in M$ is (naturally isomorphic to) $T_{f(x)}N/Df(x).T_xM$ (since $(f^*\tau_N)^{-1}(x) \equiv \{x\} \times T_{f(x)}N$). Let us denote this vector bundle by $\nu_f: N(f) \to M$. Obviously, rank $(\nu_f) = \dim N - \dim M$.

Definition (2.1)6: The vector bundle $\nu_f\colon N(f) \to M$ is called the normal bundle of f.

A canonical example is the following; let $d\colon \mathrm{diag}(M) \hookrightarrow M\times M$ be the inclusion of the diagonal $\mathrm{diag}(M) \equiv \{(x,x)\colon x \in M\}$. Then the normal bundle ν_d is isomorphic to the tangent bundle τ_M of M.

We now assume that M,N are equipped with metrics g, k respectively, and that the embedding f is isometric, i.e. $g = f^*k$. We may now realize the quotient ν_f as a subbundle of $f^*\tau_N$:- For $x \in M$, let $T_x^\perp M$ denote the k-orthogonal complement of $Df(x).T_xM$ in $T_{f(x)}N$, so that $T_{f(x)}N = (Df(x).T_xM) \oplus T_x^\perp M$. Then the normal bundle $\nu_f\colon N(f) \to M$ is isomorphic to the vector subbundle of $f^*\tau_N$ obtained by taking as fibre over $x \in M$ the subspace $T_x^\perp M$. Although this geometric realization of the normal bundle as a subbundle of $f^*\tau_N$ depends on $k \in \mathrm{Met}(N)$, we use the same notation $\nu_f\colon N(f) \to M$ to denote this isomorph, so that $\nu_f^{-1}(x) = T_x^\perp M$ and $f^*(TN) = TM \oplus N(f)$ (Here, we use the monomorphism $\tau_N^* f$ (see (6.1)9) to identify $(f^*\tau_N)^{-1}(x) = \{x\} \times T_{f(x)}N$ with $T_{f(x)}N$ and we are regarding TM as a subbundle of $f^*(TN)$). $T_x^\perp M$ is called the normal space to M at x.

Let $\dim M = m$ and $\dim N = n$. We have the principal bundles $O(m) \hookrightarrow O(M,g) \xrightarrow{\pi_M} M$, $O(n) \to O(N,k) \xrightarrow{\pi_N} N$ and $O(n) \hookrightarrow f^*O(N,k) \xrightarrow{f^*\pi_N} M$. Let $O(f) = \{(x,u) \in f^*O(N,k)\colon$ $u = \{e_1,\ldots,e_m, e_{m+1},\ldots, e_n\}$ with $e_1,\ldots,e_m \in Df(x).T_xM$, $e_{m+1},\ldots e_n \in T_x^\perp M\}$. Then, regarding $O(m) \cong \begin{pmatrix} O(m) & 0 \\ 0 & 1\!\!1_p \end{pmatrix} \leqslant O(n)$, $O(p) \cong \begin{pmatrix} 1\!\!1_m & 0 \\ 0 & O(p) \end{pmatrix} \leqslant O(n)$ $(p = n - m)$, the group $O(m) \times O(p)$ acts freely on the right on $O(f)$ in an obvious manner. We then have the principal $O(m) \times O(p)$-bundle, $\pi_f \colon O(f) \longrightarrow M$.

<u>Definition (2.1)7</u>: The principal $O(m) \times O(p)$-bundle π_f is called the <u>bundle of f-adapted frames</u>.

Obviously, $O(f)$ is a principal subbundle of $f^*O(N,k)$ (via inclusion of the total space and the natural inclusion of $O(m) \times O(p)$ as a subgroup of $O(n)$). The natural epimorphism $O(m) \times O(p) \longrightarrow O(m)$ induces a principal bundle epimorphism, $O(f) \longrightarrow O(M,g)$ given by $(x,u) \equiv (x,\{e_1,\ldots,e_n\}) \longmapsto \{Df(x)^{-1}e_1,\ldots, Df(x)^{-1}e_m\}$, for all $(x,u) \in O(f)$. Therefore, $O(M,g)$ is naturally isomorphic to $O(f)/_{O(p)}$. Similarly, there exists a natural isomorphism of $O(N(f))$ onto $O(f)/_{O(m)}$, where $O(N(f)) \subseteq GL(N(f))$ (see definition (6.1)24) is the bundle of k-orthonormal frames of the vector bundle $\nu_f\colon N(f) \longrightarrow M$. Let $\pi_f^{\perp} : O(N(f)) \longrightarrow M$ denote projection.

<u>Definition (2.1)8</u>: The bundle $O(p) \hookrightarrow O(N(f)) \xrightarrow{\pi^{\perp}_f} M$ is called the <u>normal frame bundle of f</u>.

Clearly, the vector bundle associated to $O(N(f))$ via the standard action of $O(p)$ on $\mathbb{R}^p$ is just the normal bundle ν_f of f.

The bundles discussed above may be summarized in the following diagram:

$$\begin{tikzcd}
 & & O(m)\times O(p) \arrow[rrr, hook] \arrow[dl] \arrow[dr] \arrow[dd, hook] & & & O(m+p) \arrow[r, leftrightarrow] \arrow[dd, hook] & O(m+p) \arrow[dd, hook] \\
 & O(p) \arrow[dr, hook] & & O(m) \arrow[dl, hook'] & & & \\
 & & O(f) \arrow[rrr, hook] \arrow[dl] \arrow[dr] \arrow[dd, "\pi_f"] & & & f^*O(N,k) \arrow[r, "\pi_N^* f"] \arrow[dd, "f^*\pi_N"] & O(N,k) \arrow[dd, "\pi_N"] \\
O(p) \arrow[r, hook] & O(N(f)) \arrow[dr, "\pi_f^{\perp}"] & & O(M,g) \arrow[dl, "\pi_M"] & O(m) \arrow[l, hook'] & & \\
 & & M \arrow[rrr, leftrightarrow] & & & M \arrow[r, "f"] & N
\end{tikzcd}$$

2.1.2.

The group $O(m) \times O(p)$ acts on $O(m+p)$ on the left in the following way: $((b,c),a) \mapsto \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} a$, for all $(b,c) \in O(m) \times O(p)$, $a \in O(m+p)$. We can therefore form the associated bundle $O(f) \times_{O(m)\times O(p)} O(m+p)$ over M. The following will be used below:

<u>Proposition (2.1)9</u>: $O(f) \times_{O(m)\times O(p)} O(m+p)$ is naturally isomorphic to $f^*O(N,k)$.

<u>Proof</u>: Define $\mu: O(f) \times_{O(m)\times O(p)} O(m+p) \to f^*O(N,k)$ by $\mu([((x,u),a)]) = (x,ua)$, for all $[((x,u),a)] \in O(f) \times_{O(m)\times O(p)} O(m+p)$.

We first show that μ is well defined. Suppose $[((x,u),a)] = [((x',u'),a')]$. Then there exists $(b,c) \in O(m) \times O(p)$ such that $(x,u)(b,c) = (x',u')$ and $(b,c)^{-1}a = a'$. This implies that $x' = x$, $u' = \{e_1 b, \ldots, e_m b, e_{m+1} c, \ldots, e_{m+p} c\}$ (where $\{e_1, \ldots, e_{m+p}\} = u$) and $a' = \begin{pmatrix} b^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix} a$. Hence, $(x',u'a') = (x,ua)$, so that μ

is well defined.

Given $(x,u) \in f^*O(N,k)$, choose any $u' \in \pi_f^{-1}(x) \subseteq (f^*\pi_N)^{-1}(x)$, and let $a \in O(m+p)$ such that $u'a = u$. Then $\mu([((x,u'),a)]) = (x,u)$, so that μ is surjective.

Suppose $\mu([((x,u),a)]) = \mu([((x',u'),a')])$. Then $x = x'$ and $ua = u'a'$. Let $b = a'a^{-1} \in O(m) \times O(p)$, so that $u = u'b$ and $a = b^{-1}a'$. Hence, $[((x,u),a)] = [((x,u')b, b^{-1}a')] = [((x',u'),a')]$, so that μ is injective.

By inspection, μ is fibre preserving, so that μ is a bundle isomorphism □

We now consider differential forms on the various principal bundles introduced above. First we need to consider the principal bundle epimorphisms $\alpha\colon O(f) \to O(M,g)$; $(x,u) \longmapsto \{Df(x)^{-1}e_1,\ldots, Df(x)^{-1}e_m\}$, $\beta\colon O(f) \to O(N(f))$; $(x,u) \mapsto \{e_{m+1},\ldots, e_{m+p}\}$, for all $(x,u) \equiv (x,\{e_1,\ldots,e_m, e_{m+1},\ldots,e_{m+p}\}) \in O(f)$, referred to above. We regard $\mathbb{R}^n \equiv \mathbb{R}^{m+p} = \mathbb{R}^m \oplus \mathbb{R}^p$, where $\mathbb{R}^m$ is the subspace of $\mathbb{R}^n$ spanned by $\{\varepsilon_1,\ldots, \varepsilon_m\}$, and $\mathbb{R}^p$ is the subspace spanned by $\{\varepsilon_{m+1},\ldots, \varepsilon_{m+p}\}$. Here $\{\varepsilon_1,\ldots, \varepsilon_n\}$ is the natural basis of $\mathbb{R}^n$. Then, regarded as subgroups of $O(m+p)$, the groups $O(m)$, $O(p)$ induce the identity transformation on the subspaces $\mathbb{R}^p$, $\mathbb{R}^m$ respectively. Let $\iota\colon O(f) \hookrightarrow f^*O(N,k)$ denote inclusion.

It is straightforward to see that:

$$Df(x) \circ \kappa_{\alpha(x,u)} = \kappa_u|\mathbb{R}^m \qquad 2.1.3,$$

$$\kappa_{\beta(x,u)} = \kappa_u|\mathbb{R}^p \qquad 2.1.4,$$

for all $(x,u) \in O(f) \xhookrightarrow{\iota} f^*O(N,k) \xrightarrow{\pi_N^* f} O(N,k)$ (see 6.1.1. for the definition of κ).

Let $\theta_M \in \Omega^1(O(M,g), \mathbb{R}^m)$, $\theta_N \in \Omega^1(O(N,k), \mathbb{R}^{m+p})$ denote the restriction to the orthonormal frame bundles of the canonical 1-forms of M,N respectively (see definition (6.1)26).

<u>Proposition (2.1)10</u>: $((\pi_N^* f)\circ\iota)^*\theta_N = \alpha^*\theta_M$ $(\in \Omega^1(O(f),\mathbb{R}^m))$.

<u>Proof</u>: Let $(x,u) \in O(f)$. Then $(((\pi_N^* f)\circ\iota)^*\theta_N)(x,u)$

$$= \theta_N((\pi_N^* f)(x,u))\circ D((\pi_N^* f)\circ\iota)(x,u) = \theta_N(u)\circ D(\pi_N^* f)(x,u)$$

$$= \kappa_u^{-1}\circ D\pi_N(u)\circ D(\pi_N^* f)(x,u) = \kappa_u^{-1}\circ D(\pi_N\circ(\pi_N^* f))(x,u)$$

$$= \kappa_u^{-1}\circ D(f\circ(f^*\pi_N))(x,u) = \kappa_u^{-1}\circ Df(x)\circ D(f^*\pi_N)(x,u)$$

$$= (\kappa_u^{-1}\Big| Df(x).T_xM)\circ Df(x)\circ D(\pi_M\circ\alpha)(x,u)$$

$$= (\kappa_u\Big|\mathbb{R}^m)^{-1}\circ Df(x)\circ D\pi_M(\alpha(x,u))\circ D\alpha(x,u)$$

$$= \kappa_{\alpha(x,u)}^{-1}\circ D\pi_M(\alpha(x,u))\circ D\alpha(x,u) = \theta_M(\alpha(x,u))\circ D\alpha(x,u)$$

$= (\alpha^*\theta_M)(x,u)$. Hence, $((\pi_N^* f)\circ\iota)^*\theta_N = \alpha^*\theta_M$ □

Let $\omega_g \equiv LC(g) \in \mathrm{Conn}(O(M,g))$, $\omega_k \equiv LC(k) \in \mathrm{Conn}(O(M,k))$ denote the Levi-Cività connection 1-forms of the metrics g,k respectively (see (6.1.4). The map $\pi_N^* f\colon f^*O(N,k) \to O(N,k)$ (together with $id_{O(m+p)}$) is a homomorphism of principal bundles, and $(\pi_N^* f)^*\omega_k$ is the connection in $f^*O(N,k)$ induced from ω_k (see definition (6.1)22). To obtain a connection in $O(f)$, we must restrict to a particular subspace of the Lie algebra of $O(m+p)$:-

Let $L(m,p)$ denote the orthogonal complement to $LO(m)\oplus LO(p)$ in $LO(m+p)$ with respect to the Cartan-Killing form (given by $(\xi,\eta)\mapsto -\operatorname{trace}(\operatorname{ad}(\xi)\circ\operatorname{ad}(\eta))$, for all $\xi,\eta \in LO(m+p)$). Then $L(m,p) = \{\begin{pmatrix} 0 & A \\ -A^T & 0\end{pmatrix} : A \in M(m\times p)\}$. Let $\omega_f \in \Omega^1(O(f), L(O(m)\times O(p)))$ denote the $LO(m)\oplus LO(p)$-component of $((\pi_N^* f)\circ\iota)^*\omega_k$ with respect to the decomposition $LO(m+p) = LO(m) \oplus LO(p) \oplus L(m,p)$.

Proposition (2.1)11: $\omega_f \in \mathrm{Conn}(O(f))$.

Proof: Note that $Ad|O(m) \times O(p)$ leaves $L(m,p)$ invariant, so by proposition 6.4 on p. 83 of [K 7], the $LO(m) \oplus LO(p)$-component of $((\pi_N^* f) \circ \iota)^* \omega_k$ is a connection in $O(m) \times O(p) \hookrightarrow O(f) \xrightarrow{\pi_f} M$. i.e. $\omega_f \in \mathrm{Conn}(O(f))$ □

We have the principal bundle epimorphism $\alpha: O(f) \to O(M,g)$ with corresponding group epimorphism $pr_1 \equiv \alpha'': O(m) \times O(p) \to O(m)$. Hence, $D\alpha''(1): LO(m) \oplus LO(p) \to LO(m)$ is projection onto the $LO(m)$ factor. The image of $\omega_f \in \mathrm{Conn}(O(f))$ under α (see definition (6.1)21) is a connection ω in $O(M,g)$ such that $\alpha^* \omega = D\alpha''(1) \circ \omega_f = (\omega_f)_{LO(m)}$.

Proposition (2.1)12: The image ω of ω_f under $\alpha: O(f) \to O(M,g)$ is ω_g, the Levi-Cività connection of g.

Proof: The connection ω is in $O(M,g)$. Hence, if we show that ω has zero torsion, then ω must be the Levi-Cività connection of g (by the fundamental theorem of Riemannian geometry - see section 6.1).

The torsion form Θ^ω of ω is given by the Cartan structure equation, $\Theta^\omega = d\theta_M + \omega(\theta_M)$. Since α is surjective, it suffices to show that $\alpha^* \Theta^\omega = 0$. We have $\alpha^* \Theta^\omega = d(\alpha^* \theta_M) + (\alpha^* \omega)(\alpha^* \theta_M)$ $= d(\gamma^* \theta_N) + (\omega_f)_{LO(m)}(\gamma^* \theta_N)$, where we have used proposition (2.1)10, putting $\gamma = (\pi_N^* f) \circ \iota$.

Now note that $0 = d\theta_N + \omega_k(\theta_N)$, since ω_k has zero torsion. Hence, $0 = d(\gamma^* \theta_N) + (\gamma^* \omega_k)(\gamma^* \theta_N) = d(\gamma^* \theta_N) + (\gamma^* \omega_k)_{LO(m) \oplus LO(p)}(\gamma^* \theta_N)$ $+ (\gamma^* \omega_k)_{L(m,p)}(\gamma^* \theta_N) = d(\gamma^* \theta_N) + \omega_f(\gamma^* \theta_N) + (\gamma^* \omega_k)_{L(m,p)}(\gamma^* \theta_N)$, which takes its values in $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^p$. We now project this equation onto the $\mathbb{R}^m$ factor to obtain $d(\gamma^* \theta_N) + (\omega_f)_{LO(m)}(\gamma^* \theta_N) = 0$, where we have used the fact that $\gamma^*\theta_N$ takes its values in $\mathbb{R}^m$ (see proposition (2.1)10). Hence $\alpha^*\Theta^\omega = 0$, and so $\Theta^\omega = 0$.

Therefore ω is the Levi Cività connection of g □

We also have the epimorphism β: $O(f) \longrightarrow O(N(f))$ (pr_2: $O(m) \times O(p) \longrightarrow O(p)$) and we denote by $\overset{\perp}{\omega}_f$ the image of ω_f under β. Hence, $\overset{\perp}{\omega}_f \in \mathrm{Conn}(O(N(f)))$ satisfies $\beta^*\overset{\perp}{\omega}_f = (\omega_f)_{LO(p)}$. The connection $\overset{\perp}{\omega}_f$ gives rise to parallel transport (definition (6.1)15) of $\overset{\perp}{T}_xM$ onto $\overset{\perp}{T}_yM$ along any curve c in M from x to y.

To summarize the above exposition, the connection ω_k in $O(N,k)$ gives rise to a connection ω_f in $O(f)$, which in turn gives rise to a connection ω in $O(M,g)$ (which coincides with ω_g) and to a connection $\overset{\perp}{\omega}_f$ in $O(N(f))$. We may unify the discussion by considering the principal bundle isomorphism $\alpha \times \beta$: $O(f) \longrightarrow O(M,g) \times_M O(N(k))$ (with group isomorphism just the identity of $O(m) \times O(p)$. Then, by proposition 6.3 on p. 82 of [K 7], we have the following:

<u>Proposition (2.1)13</u>: $\omega_f = \alpha^*\omega_g + \beta^*\overset{\perp}{\omega}_f$.

In the above, we have described the various connections associated with an embedding in terms of principal bundles. For calculations, we often utilize the formalism of covariant derivatives in vector bundles (see definition (6.1)17 and equations 6.1.7, 6.1.8. The covariant derivative formulae may be obtained by translating the results involving connections in principal bundles, but here we just summarize the results (see Klingenberg [K 5]):

Let $f: M \hookrightarrow N$ be an embedding as usual, and let $k \in \mathrm{Met}(N)$ and $g \equiv f^*k \in \mathrm{Met}(M)$. The metric k in TN may be pulled back to $f^*(TN)$ using the vector bundle homomorphism τ_N^*f. We then obtain metrics in the vector subbundles $Df.TM$, $N(f)$ of $f^*(TN)$ by restricting $(\tau_N^* f)^* k$. The metric in $Df.TM$ is just the push forward of the metric g in TM, and we denote the metric in the normal bundle $N(f)$ by $k^\perp$. We then have the orthogonal Whitney sum of Riemannian vector bundles $f^*(TN) = Df.TM \oplus N(f)$ and corresponding direct sum of sections; $\mathrm{Vect}_f(M) \cong \Gamma(f^*(TN)) = Df.\mathrm{Vect}(M) \oplus \Gamma(N(f))$. For $X \in \mathrm{Vect}(M)$, we define $Df.X \in Df.\mathrm{Vect}(M) \subseteq \mathrm{Vect}_f(N)$ by $(Df.X)(x) = Df(x).X(x) \in T_{f(x)}N$, for all $x \in M$.

Let the covariant derivative operators of k, g be ∇^k, ∇^g respectively. Hence, for example, $\nabla_X^g \equiv \nabla_X^{\omega g}$ $(X \in \mathrm{Vect}(M))$ acts on all spaces of tensor fields of M according to equation 6.1.9. We also denote by ∇^k the covariant derivative induced in $f^*(TN)$ by $\tau_N^* f$ from ∇^k (see definition (6.1)23). Then, for each $X \in \mathrm{Vect}(M)$ and $V \in \mathrm{Vect}_f(N) \cong \Gamma(f^*(TN))$, the covariant derivative of V along f in the direction X is denoted $\nabla_X^k V \in \mathrm{Vect}_f(N)$. In particular, we have $\nabla_X^k\, Df.Y \in \mathrm{Vect}_f(N)$, for each $X,Y \in \mathrm{Vect}(M)$.

The first formula concerning covariant derivatives describes the manner in which Df, ∇^g and ∇^k interact:

$$Df.\nabla_X^g Y \;=\; h.\nabla_X^k\, Df.Y \qquad 2.1.5,$$

for all $X,Y \in \mathrm{Vect}(M)$. Here, $h \in \Gamma(f^*(T^*N) \otimes Df.TM) \cong \Gamma((f^*(TN))^* \otimes TM)$ is the orthogonal projection onto $Df.TM$, so that $h(x)$ is a linear map of $T_{f(x)}N$ onto $Df(x).T_xM \cong T_xM$, for each $x \in M$.

Let us now consider the decomposition of the covariant derivative $\nabla_X^k : \mathrm{Vect}_f(N) \longrightarrow \mathrm{Vect}_f(N)$ $(X \in \mathrm{Vect}(M))$ with respect to the orthogonal splitting, $\mathrm{Vect}_f(M) = Df.\mathrm{Vect}(M) \oplus \Gamma(N(f))$. First, we consider the action on a tangential vector field. We have

$$\nabla^k_X \, Df.Y \;=\; Df.\nabla^g_X \, Y + K^f(X,Y) \qquad 2.1.6,$$

for all $Y \in Vect(M)$ (i.e. for all $Df.Y \in Df.Vect(M)$). Here, $K^f \in \Gamma((\otimes^2 T^* M) \otimes N(f))$ is called the second fundamental form of f. Equation 2.1.6 is known as the Gauss formula.

For normal vector fields, we have the Weingarten formula:

$$\nabla^k_X N \;=\; - Df.A_N X + \overset{\perp}{\nabla}_X N \qquad 2.1.7,$$

for all $N \in \Gamma(N(f))$ (= {normal vector fields}). Here, $A_N X \in Vect(M)$ defines an element of $\Gamma(N(f)^* \otimes T^* M \otimes TM)$, and satisfies;

$$g(A_N X,Y) \;=\; \overset{\perp}{k}(N,K^f(X,Y)) \qquad 2.1.8,$$

for all $X,Y \in Vect(M)$, $N \in \Gamma(N(f))$. An important consequence of 2.1.8 is that A_N is (pointwise) self-adjoint with respect to g, i.e.:

$$g(A_N X,Y) \;=\; g(X,A_N Y) \qquad 2.1.9,$$

for all $X,Y \in Vect(M)$, $N \in \Gamma(N(f))$.

The normal compontnt of 2.1.7 is $\overset{\perp}{\nabla}_X N$, where $\overset{\perp}{\nabla}$ is a covariant derivative in the vector bundle $N(f)$. In fact $\overset{\perp}{\nabla}$ is just the covariant derivative induced by the connection $\overset{\perp}{\nabla}_f \in Conn(O(N(f)))$, and is a metric connection, since $\overset{\perp}{\nabla}\overset{\perp}{k} = 0$.

We now relate the second fundamental form K^f back to differential forms in principal bundles. Let us define $\tilde{K}^f \in \Gamma((\otimes^2 T^* O(f)) \otimes (O(f) \times \mathbb{R}^p))$ by:

$$\tilde{K}^f(x,u).(v.w) \;=\; ((\omega_k(u).v).(\theta_N(u).w))_{\mathbb{R}^p} \qquad 2.1.10,$$

for all $v, w \in T_{(x,u)}O(f)$, $(x,u) \in O(f)$. Here the subscript $\mathbb{R}^p$ denotes the $\mathbb{R}^p$-component in $\mathbb{R}^m \oplus \mathbb{R}^p$, and we regard $LO(m+p)$ as a subalgebra of $gl(\mathbb{R}^{m+p})$. It may be shown (see proposition 3.5 on p. 21 of [K 8]) that:

$$K^f(x)(D\pi_f(x,u).v,\ D\pi_f(x,u).w) = \kappa_u(\tilde{K}^f(x,u).(v,w)) \quad 2.1.11,$$

for all $v,w \in T_{(x,u)}O(f)$, $(x,u) \in O(f)$, and this is the equation relating the second fundamental form to differential forms in the principal $O(m)\times O(p)$-bundle $\pi_f\colon O(f) \to M$.

To conclude this section we make several remarks which will be useful when we apply the theory of embeddings below. Firstly, the above discussion goes through in the case when M and N are oriented (with the embedding f respecting these orientations). In this case we replace $O(M,g)$, $O(N,k)$ by the oriented frame bundles $SO(M,g)$, $SO(N,k)$ and we obtain the principal $SO(m)\times SO(p)$-bundle $\pi_f\colon SO(f) \to M$ of f-adapted oriented frames. We may also consider the pullback by f of a k-spin structure $(\tilde{SO}(N,k),\eta_k)$ if N is spin (Cf. section 2.3).

The codimension one embeddings are especially interesting since embedded hypersurfaces often arise in geometry and in general relativity. Suppose $f\colon (M,g) \hookrightarrow (N,k)$ is an isometric embedding, where $\dim N = m + 1 = \dim M + 1$. For simplicity, we assume that M,N are oriented with f-compatible orientations. Then there exists a unique normal vector field $N \in \Gamma(N(f))$ specified by (i) $k^{\perp}(N,N) = 1$; and (ii) for each frame $u \equiv \{e_1,\ldots, e_m\} \in SO(M,g)$, the frame $\{Df(\pi_M(u)).e_1,\ldots, Df(\pi_M(u)).e_m,\ N(x)\} \in \pi_N^{-1}(f(\pi_M(u))) \subseteq SO(N,k)$.

<u>Definition (2.1)14</u>: The normal vector field N specified by properties (i), (ii) is called the <u>unit normal</u> of the (orientation

compatible) embedding f.

For a spacelike embedding of an oriented 3-manifold into a spacetime oriented Lorentzian 4-manifold, the requirement that the embedding be compatible with the orientations means that the (timelike) unit normal N is necessarily future directed.

Let $k_f = (\tau_N^* f)^* k$ denote the metric in $f^*(TN)$ induced from the metric k in TN. Then, since $\nabla^k k = 0$, we have $\nabla^k k_f = 0$ (where, in the second equality, ∇^k denotes the covariant derivative in $f^*(TN)$). Let N be the unit normal of f, so that $k_f(N,N) \equiv \overset{\perp}{k}(N,N) = 1$. Hence, for each $X \in \mathrm{Vect}(M)$, $k_f(\nabla_X^k N,N) = 0$. We now use the Weingarten formula 2.1.7 to obtain $\overset{\perp}{k}(\overset{\perp}{\nabla}_X N,N) = 0$. But since $\overset{\perp}{\nabla}_X N \in \Gamma(N(f))$ is given by $\overset{\perp}{\nabla} N = hN$ for some $h \in C(M)$, we must have that $\overset{\perp}{\nabla}_X N = 0$, and hence $\nabla_X^k N = -\, Df.A_N X$, for all $X \in \mathrm{Vect}(M)$.

Definition (2.1)15: Let $f\colon (M,g) \hookrightarrow (N,k)$ be a codimension one orientation compatible isometric embedding of oriented Riemannian manifolds with unit normal N. The extrinsic curvature of f is the tensor field $K_N^f \in S_2(M) \equiv \Gamma(\Theta^2 T^* M)$ defined by $K_N^f(X,Y) = \overset{\perp}{k}(K^f(X,Y),N)$, for all $X,Y \in \mathrm{Vect}(M)$.

Using the Gauss formula (2.1.6), we obtain $K_N^f(X,Y) = k_f(N, \nabla_X^k\, Df.Y)$, and hence

$$K_N^f(X,Y) \;=\; -\, k_f(\nabla_X^k N,\, Df.Y) \qquad 2.1.11,$$

for all $X,Y \in \mathrm{Vect}(M)$. If the embedding f is an inclusion, then we may write 2.1.11 in a more convenient manner as $K_N^f = -\,\nabla^k N^b$ (where $N^b = k(N,\cdot)$) $= -\frac{1}{2} L_N k$.

We return to hypersurfaces (codimension one embeddings) in Chapter Three. In particular, in section 3.3, we consider a one parameter family of spacelike codimension one embeddings associated with a given spacelike codimension two embedding. The "null limit" of this family is used in the Ludvigsen-Vickers definition of quasi-local momentum in general relativity.

In this section, we have discussed the geometric properties and applications to general relativity of particular embeddings f of one manifold into another. In the next section we consider the structure of the space of all such embeddings.

2.2 The Manifold of Embeddings

Having considered the various geometric notions associated with a particular embedding $f: M \hookrightarrow N$ in section 2.1, we now consider the space of all such embeddings. This space admits a natural manifold structure and is related to other infinite dimensional manifolds arising from M and N. A consideration of the manifold of embeddings leads to a natural framework for bringing together important geometric ideas and it also provides a useful tool in applications to physics. We review some important uses of the manifold of embeddings in general relativity at the end of this section.

Many of the constructions considered in this section are standard (see, for example, Hamilton [H 2], but, inspired by the "everywhere invariance" outlook of Chapter Four (see, in particular, section 4.1), we also describe some additional natural features of the manifold of embeddings. Possible uses for such constructions in general relativity are also suggested.

As usual, we assume that all structures are appropriately smooth. An especially useful category in which to work is the

Nash-Moser category of "tame" Fréchet manifolds and maps, for then we have the Nash-Moser inverse function theorem at our disposal. For all analytical details concerning the differentiable structure of the concepts considered in this section, we refer the reader to Binz and Fischer [B 12] and to Hamilton [H 2].

In this section, M,N denote connected smooth manifolds without boundary and with $\dim M \leq \dim N < \infty$. For certain constructions, we also require M and N to be oriented. For technical reasons, we assume that M is compact.

Let $C(M,N)$ denote the manifold of smooth maps from M into N. The tangent space to $C(M,N)$ at the point f is given by $T_f C(M,N) = \{X \in C(M,TN): \tau_N \circ X = f\} \equiv \mathrm{Vect}_f(N) \equiv \Gamma(f^* TN)$.

The diffeomorphism group $\mathrm{Diff}(M)$ is open in $C(M,M)$ (with respect to the compact-open topology) and is thus a manifold with tangent bundle $T\mathrm{Diff}(M) = \{X \in C(M,TM): \tau_M \circ X \in \mathrm{Diff}(M)\}$ and tangent space $T_\phi \mathrm{Diff}(M) = \mathrm{Vect}_\phi(M)$, for each $\phi \in \mathrm{Diff}(M)$. The group multiplication (composition) and inversion are both smooth and $\mathrm{Diff}(M)$ may be regarded as a Lie group (see section 4.4 for more details regarding $\mathrm{Diff}(M)$).

Similarly, if $\mathrm{Emb}(M,N)$ denotes the space of all embeddings of M into N, then $\mathrm{Emb}(M,N)$ is an open submanifold of $C(M,N)$. The tangent bundle is given by $T\mathrm{Emb}(M,N) = \{X \in C(M,TN): \tau_N \circ X \in \mathrm{Emb}(M,N)\}$ and the tangent space is $T_f \mathrm{Emb}(M,N) = \mathrm{Vect}_f(N)$, for each $f \in \mathrm{Emb}(M,N)$.

We have the composition map $\mathrm{comp}: \mathrm{Emb}(M,N) \times \mathrm{Diff}(M) \longrightarrow \mathrm{Emb}(M,N)$ given by

$$\mathrm{comp}(f, \phi) = f \circ \phi \qquad 2.2.1,$$

for all $(f,\phi) \in \mathrm{Emb}(M,N) \times \mathrm{Diff}(M)$. This map is smooth with derivative given by:

$$(D\mathrm{comp}(f,\phi).(X,Y))(x) = Df(\phi(x)).Y(x) + X(\phi(x)) \qquad 2.2.2,$$

for all $x \in M$, $(X,Y) \in T_{(f,\phi)}(\mathrm{Emb}(M,N) \times \mathrm{Diff}(M))$ $\cong \mathrm{Vect}_f(N) \oplus \mathrm{Vect}_\phi(M)$ and $(f,\phi) \in \mathrm{Emb}(M,N) \times \mathrm{Diff}(M)$ (see Irwin [I 5]).

Equation 2.2.1 defines a right action of $\mathrm{Diff}(M)$ on $\mathrm{Emb}(M,N)$ and we write $R_\phi = \mathrm{comp}(\cdot,\phi)$, for all $\phi \in \mathrm{Diff}(M)$. We denote the other partial map by $\omega_f\colon \mathrm{Diff}(M) \to \mathrm{Emb}(M,N)$, for each $f \in \mathrm{Emb}(M,N)$. Since each embedding is injective, the right action of $\mathrm{Diff}(M)$ on $\mathrm{Emb}(M,N)$ is free, and, for each $f \in \mathrm{Emb}(M,N)$, ω_f is a diffeomorphism of $\mathrm{Diff}(M)$ onto the orbit of f. Since $f,f' \in \mathrm{Emb}(M,N)$ are $\mathrm{Diff}(M)$- equivalent if and only if $f(M) = f'(M)$, we see that the orbit of f consists precisely of all diffeomorphisms of M onto $f(M)$ (a closed submanifold of N). In order to discuss the structure of the orbit space, we first consider the space of all submanifolds of N:

Let $\mathrm{Sub}(N)$ denote the space of all (compact) smooth submanifolds of N. Then $\mathrm{Sub}(N)$ is a manifold and, for each $S \in \mathrm{Sub}(N)$, $T_S\mathrm{Sub}(N) = \Gamma(N(\iota_S))$, where $\iota_S\colon S \hookrightarrow N$ is inclusion, and $\nu_S \equiv \nu_{\iota_S}\colon N(\iota_S) \to S$ is the normal bundle of S (definition (2.1)6). Note that all the submanifolds S in a given connected component of $\mathrm{Sub}(N)$ are necessarily diffeomorphic, but, on the other hand, diffeomorphic submanifolds could lie in different components. In general, there may be many components.

Now let $\mathrm{Sub}_M(N) = \{S \in \mathrm{Sub}(N)\colon\ S$ is diffeomorphic to $M\}$.

$Sub_M(N)$ is an open submanifold of $Sub(N)$, so that $T_S Sub_M(N) = \Gamma(N(\iota_S))$, for all $S \in Sub_M(N)$. We define a projection $\gamma \equiv \gamma^N_M$ by:

$$\gamma\colon Emb(M,N) \to Sub_M(N);\quad f \mapsto f(M) \qquad 2.2.3,$$

for all $f \in Emb(M,N)$. We then have the principal $Diff(M)$-bundle

$$Diff(M) \hookrightarrow Emb(M,N) \xrightarrow{\gamma} Sub_M(N) \qquad 2.2.4,$$

where $Diff(M)$ acts on $Emb(M,N)$ by equation 2.2.1.

We may calculate the vertical distribution of γ by considering the fundamental vector fields (see appendix 6.1). Let $X \in LDiff(M) \equiv Vect(M)$ (see section 4.4) and let $X_E \equiv X_{Emb(M,N)}$ be the fundamental vector field corresponding to X. Then $X_E(f) = D\omega_f(id_M).X$, for all $f \in Emb(M,N)$. From equation 2.2.2, we see that $(D\omega_f(\phi).Y)(x) = Df(\phi(x)).Y(x)$, for all $x \in M$ and $Y \in T_\phi Diff(M) = Vect_\phi(M)$, so that $(X_E(f))(x) = Df(x).X(x) = (Df.X)(x)$, for all $x \in M$. Hence, $X_E(f) = Df.X \in Df.Vect(M) \subseteq Vect_f(N) \equiv T_f Emb(M,N)$.

Let us denote the vertical subspace at $f \in Emb(M,N)$ by V_f, so that $V_f = T_f\gamma^{-1}(\gamma(f))$ (the tangent space at f of the fibre $\gamma^{-1}(\gamma(f))$ through f). From the general theory of principal fibre bundles (see section 6.1), the map $X \mapsto X_E(f)$ is a linear isomorphism of $LDiff(M) = Vect(M)$ onto V_f, for each $f \in Emb(M,N)$. Noting that the map $X \mapsto Df.X$ is a linear isomorphism of $Vect(M)$ onto $Df.Vect(M) \subseteq Vect_f(N)$, we have demonstrated the following:

<u>Proposition (2.2)1</u>: The vertical distribution $V(\gamma) \equiv Ker\, D\gamma$ of the principal $Diff(M)$-bundle γ (2.2.4) has total space $V(\gamma) = \bigcup_f V_f$, where $V_f = Df.Vect(M)$, for all $f \in Emb(M,N)$.

Now let $k \in \mathrm{Met}(N)$. As shown in section 2.1, the normal bundle $\nu_f\colon N(f) \longrightarrow M$ may be realized as a subbundle of $f^*(TN)$ using the metric k. In fact, we have the k-orthogonal Whitney sum, $f^*(TN) = Df.TM \oplus N(f)$, and this induces a splitting of the corresponding spaces of sections:

$$\mathrm{Vect}_f(N) \;=\; Df.\mathrm{Vect}(M) \oplus \Gamma(N(f)) \qquad 2.2.5,$$

where we have, as usual, identified $\Gamma(f^*(TN))$ with $\mathrm{Vect}_f(N)$.

We may rewrite equation 2.2.5 in terms of the bundle $\gamma\colon \mathrm{Emb}(M,N) \longrightarrow \mathrm{Sub}_M(N)$:- Define $H^k(\gamma) = \bigcup_f H^k_f$, where $H^k_f = \Gamma(N(f))$, for each $f \in \mathrm{Emb}(M,N)$ (Recall that, regarded as a subbundle of $f^*(TN)$, $N(f)$ depends on k). Then $H^k(\gamma)$ is the total space of a vector bundle over $\mathrm{Sub}_M(N)$ and we have $T_f\mathrm{Emb}(M,N) = V_f \oplus H^k_f$, for each $f \in \mathrm{Emb}(M,N)$. Thus, for each $k \in \mathrm{Met}(M)$, we have the following decomposition of the tangent bundle of $\mathrm{Emb}(M,N)$:

$$T\mathrm{Emb}(M,N) \;=\; V(\gamma) \oplus H^k(\gamma) \qquad 2.2.6.$$

<u>Proposition (2.2)2</u>: For each $k \in \mathrm{Met}(N)$, the distribution $H^k(\gamma)$ defines a connection in the principal Diff(M)-bundle $\gamma\colon \mathrm{Emb}(M,N) \longrightarrow \mathrm{Sub}_M(N)$.

<u>Proof</u>: Let $k \in \mathrm{Met}(N)$. We already know that $H^k(\gamma)$ is complementary to the vertical distribution $V(\gamma)$, so that we must now show that $H^k(\gamma)$ is equivariant under the action of $\mathrm{Diff}(M)$ on the total space $\mathrm{Emb}(M,N)$; i.e. we must demonstrate that $H^k_{R_\phi(f)} = DR_\phi(f).H^k_f$, for all $f \in \mathrm{Emb}(M,N)$, $\phi \in \mathrm{Diff}(M)$.

From equation 2.2.2, we see that $DR_\phi(f).X = X\circ\phi$, for all $X \in \mathrm{Vect}_f(N)$, so in particular $DR_\phi(f).N = N\circ\phi$, for all $N \in \Gamma(N(f)) \subseteq \mathrm{Vect}_f(N)$. Hence $DR_\phi(f).H^k_f = \{N\circ\phi\colon N \in \Gamma(N(f))\}$.

Let U^k denote the k-orthogonal complement of any subspace U of $T_{f(x)}N$, $x \in M$. We have $\nu_f^{-1}(x) = (Df(x).T_xM)^k$, so that $\nu_f^{-1}(\phi(x)) = (Df(\phi(x)).T_{\phi(x)}M)^k = (D(f\circ\phi)(x).T_xM)^k$ (since $D\phi(x).T_xM = T_{\phi(x)}M) = \nu_{f\circ\phi}^{-1}(x)$.

Let $N' \in H^k_{R_\phi(f)} = \Gamma(N(f\circ\phi))$. Then $N'(x) \in \nu_{f\circ\phi}^{-1}(x) = \nu_f^{-1}(\phi(x))$, and so $(N'\circ\phi^{-1})(x') \in \nu_f^{-1}(x')$, for all $x' \in M$. We conclude that $N'\circ\phi^{-1} \in \Gamma(N(f))$. Hence, $N' \in DR_\phi(f).H^k_f$, and so $H^k_{R_\phi(f)} \subseteq DR_\phi(f).H^k_f$.

Conversely, suppose $N\circ\phi \in DR_\phi(f).H^k_f$. Then $(N\circ\phi)(x) = N(\phi(x)) \in \nu_f^{-1}(\phi(x)) = \nu_{f\circ\phi}^{-1}(x)$, for all $x \in M$, so that $N\circ\phi \in \Gamma(N(f\circ\phi)) = H^k_{R_\phi(f)}$. Hence, $DR_\phi(f).H^k_f \subseteq H^k_{R_\phi(f)}$.

Therefore $H^k_{R_\phi(f)} = DR_\phi(f).H^k_f$, for all $f \in Emb(M,N)$ and $\phi \in Diff(M)$. Hence, for each $k \in Met(N)$, $H^k(\gamma)$ is the horizontal distribution of a connection in γ □

A consequence of proposition (2.2)2 is that there exists a natural map η: $Met(N) \longrightarrow Conn(Emb(M,N))$. We now give the corresponding connection 1-forms:

Proposition (2.2)3: For $k \in Met(N)$, the connection 1-form $\eta(k) \in \Omega^1(Emb(M,N), LDiff(M))$ is given by $(\eta(k)(f).X)^b = k_f(X, Df.(\cdot))$, for all $X \in T_fEmb(M,N)$, $f \in Emb(M,N)$. Here $k_f = (\tau_N^* f)^* k$ is the metric in the vector bundle $f^*(TN)$ induced from the metric k in TN by the embedding f, and, for each $Y \in Vect(M)$, $Y^b \in \Omega^1(M)$ is the 1-form corresponding to Y via $f^*k \in Met(M)$.

Proof: Let $f \in Emb(M,N)$ and $X \in T_fEmb(M,N) = Vect_f(N)$. Then there exists unique $ver(X) \in V_f = Df.Vect(M)$ and $hor(X) \in H^k_f = \Gamma(N(f))$ such that $X = ver(X) + hor(X)$. Let $X_M \in Vect(M) = LDiff(M)$ be the unique vector field such that $Df.X_M = ver(X)$ (so that $(X_M)_E(f) = ver(X)$ - see proposition (2.2)1). Then

$\eta(k)(f).X = X_M$ (see section 6.1).

We have $\langle X_M^b, Y\rangle = (f^*k)(X_M, Y) = k_f(Df.X_M, Df.Y) = k_f(X, Df.Y)$ since $k_f = k\circ f$ and $hor(X)$ is k_f-orthogonal to $Df.Y$), for all $Y \in Vect(M)$.

Hence, $(\eta(k)(f).X)^b = k_f(X, Df.(\cdot))$, for all $X \in T_f Emb(M,N)$ and $f \in Emb(M,N)$ as required □

We now investigate the behaviour of the map $\eta\colon Met(N) \longrightarrow Conn(Emb(M,N))$ under the action of natural groups. First consider the right action of $Diff(M)$ on $Emb(M,N)$ given by 2.2.1. Since, for each $k \in Met(N)$, $\eta(k)$ is a connection in the principal bundle $Diff(M) \hookrightarrow Emb(M,N) \xrightarrow{\gamma} Sub_M(N)$, we have $R_\phi^*(\eta(k)) = Ad_{\phi^{-1}} \circ \eta(k)$, for all $\phi \in Diff(M)$ (see definition (6.1)11). By equation 4.4.9, we have $Ad_\phi = \phi_*$ (acting on $LDiff(M) = Vect(M)$), and so the behaviour of η under $Diff(M)$ may be written:

$$R_\phi^*(\eta(k)) = {\phi^{-1}}_* \circ \eta(k) \qquad 2.2.7,$$

for all $k \in Met(N)$ and $\phi \in Diff(M)$. Equation 2.2.7 may also be obtained by direct computation by differentiating the right action.

Now consider $Diff(N)$. This group acts on $Emb(M,N)$ by left composition $L\colon Diff(N) \times Emb(M,N) \longrightarrow Emb(M,N)$ defined by:

$$L_\psi(f) = \psi \circ f \qquad 2.2.8,$$

for all $f \in Emb(M,N)$, $\psi \in Diff(N)$. For each $\psi \in Diff(N)$, the map L_ψ is a partial map of the composition map and is smooth (Irwin [I 5]) with derivative given by (Cf. 2.2.2):

$$(DL_\psi(f).X)(x) = D\psi(f(x)).X(x) \qquad 2.2.9,$$

for all $x \in M$, $X \in T_f\mathrm{Emb}(M,N) = \mathrm{Vect}_f(N)$ and $f \in \mathrm{Emb}(M,N)$.

We have $L_\psi \circ R_\phi = R_\phi \circ L_\psi$, for all $\phi \in \mathrm{Diff}(M)$ and $\psi \in \mathrm{Diff}(N)$, and therefore L defines a homomorphism of $\mathrm{Diff}(N)$ into $\mathrm{Aut}(\mathrm{Emb}(M,N))$. Via this homomorphism, we have an action of $\mathrm{Diff}(N)$ on the space $\mathrm{Conn}(\mathrm{Emb}(M,N))$ given by $(\omega,\psi) \mapsto L_\psi^*\omega$, for all $(\omega,\psi) \in \mathrm{Conn}(\mathrm{Emb}(M,N)) \times \mathrm{Diff}(N)$.

Proposition (2.2)4: The map $\eta\colon \mathrm{Met}(N) \longrightarrow \mathrm{Conn}(\mathrm{Emb}(M,N))$ is equivariant with respect to the actions of $\mathrm{Diff}(N)$ on $\mathrm{Met}(N)$ and on $\mathrm{Conn}(\mathrm{Emb}(M,N))$ respectively; i.e. $L_\psi^* \circ \eta = \eta \circ \psi^*$, for all $\psi \in \mathrm{Diff}(N)$.

Proof: Let $k \in \mathrm{Met}(N)$, $\psi \in \mathrm{Diff}(N)$, $f \in \mathrm{Emb}(M,N)$, $X \in T_f\mathrm{Emb}(M,N) = \mathrm{Vect}_f(N)$, $Y \in \mathrm{Vect}(M)$, $x \in M$. Let $X_1 = (L_\psi^* \circ \eta)(k)(f).X$ and $X_2 = (\eta \circ \psi^*)(k)(f).X$.

By proposition (2.2)3, $X_1 = \eta(k)(\psi\circ f).DL_\psi(f).X$ satisfies $((\psi\circ f)^*k)(X_1,Y) = k_{\psi\circ f}(DL_\psi(f).X,\ D(\psi\circ f).Y)$, and $X_2 = \eta(\psi^*k)(f).X$ satisfies $(f^*(\psi^*k))(X_2,Y) = (\psi^*k)_f(X,Df.Y)$. Let $f_i = ((\psi\circ f)^*k)(X_i,Y) \in C(M)$ for $i \in \{1,2\}$.

Then, $f_1(x) = k((\psi\circ f)(x))((DL_\psi(f).X)(x),\ D(\psi\circ f)(x).Y(x))$ $= k(\psi(f(x)))(D\psi(f(x)).X(x),\ D\psi(f(x)).Df(x).Y(x))$ (using 2.2.9) $= (\psi^*k)(f(x))(X(x),\ (Df.Y)(x)) = f_2(x)$. Hence, $f_1 = f_2$.

Assuming that $(\psi\circ f)^*k$ is a metric of non-degenerate signature, we now have $X_1 = X_2$, and hence $L_\psi^*\circ\eta = \eta\circ\psi^*$, for all $\psi \in \mathrm{Diff}(N)$ □

In addition to the natural map $\eta\colon \mathrm{Met}(N) \longrightarrow \mathrm{Conn}(\mathrm{Emb}(M,N))$, we also have the natural map $\zeta\colon \mathrm{Met}(N) \longrightarrow \mathrm{Met}(\mathrm{Emb}(M,N))$ defined, for each $k \in \mathrm{Met}(N)$, $f \in \mathrm{Emb}(M,N)$, $X,Y \in T_f\mathrm{Emb}(M,N)$ by:

$$\zeta(k)(f)(X,Y) = \int_M k_f(X,Y)\,\mathrm{vol}(f^*k) \qquad 2.2.10,$$

where, as above, k_f denotes the metric in $f^*(TN)$ induced by f from k.

Proposition (2.2)5: The metric $\zeta(k)$ is right invariant for each $k \in \mathrm{Met}(N)$.

Proof: Let $k \in \mathrm{Met}(N)$, $\phi \in \mathrm{Diff}(M)$, $f \in \mathrm{Emb}(M,N)$ and $X,Y \in T_f\mathrm{Emb}(M,N)$.

Then $(R_\phi^* \circ \zeta)(k)(f)(X,Y) = \zeta(k)(f\circ\phi)(DR_\phi(f).X, DR_\phi(f).Y)$

$$= \int_M k_{f\circ\phi}(X\circ\phi, Y\circ\phi)\mathrm{vol}(\phi^* f^* k) = \int_M \phi^*(k_f(X,Y)\mathrm{vol}(f^* k))$$

$$= \int_M k_f(X,Y)\mathrm{vol}(f^* k) = \zeta(k)(f)(X,Y).$$ Hence $R_\phi^* \circ \zeta = \zeta$, for all $\phi \in \mathrm{Diff}(M)$ □

(Note that Diff(M) refers to orientation preserving diffeomorphisms in the above proposition.)

Proposition (2.2)6: The map ζ: $\mathrm{Met}(N) \to \mathrm{Met}(\mathrm{Emb}(M,N))$ is equivariant with respect to the actions of Diff(N) on Met(N) and on Met(Emb(M,N)) respectively; i.e. $L_\psi^* \circ \zeta = \zeta \circ \psi^*$, for all $\psi \in \mathrm{Diff}(N)$.

Proof: Let $k \in \mathrm{Met}(N)$, $\psi \in \mathrm{Diff}(N)$, $f \in \mathrm{Emb}(M,N)$ and $X,Y \in T_f\mathrm{Emb}(M,N)$.

Then $(L_\psi^* \circ \zeta)(k)(f)(X,Y) = \zeta(k)(\psi\circ f)(DL_\psi(f).X,\ DL_\psi(f).Y)$

$$= \int_M k_{\psi\circ f}(DL_\psi(f).X,\ DL_\psi(f).Y)\mathrm{vol}((\psi\circ f)^* k).$$ Let

$h = k_{\psi\circ f}(DL_\psi(f).X,\ DL_\psi(f).Y) \in C(M)$. Then, for $x \in M$,

$h(x) = k(\psi(f(x)))(D\psi(f(x)).X(x),\ D\psi(f(x)).Y(x))$ (by (2.2.9)

$= (\psi^* k)(f(x))(X(x),Y(x)) = ((\psi^* k)_f(X,Y))(x)$. Hence,

$$(L_\psi^* \circ \zeta)(k)(f)(X,Y) = \int_M (\psi^* k)_f(X,Y)\mathrm{vol}(f^*\psi^* k)$$

$= \zeta(\psi^* k)(f)(X,Y) = (\zeta\circ \psi^*)(k)(f)(X,Y)$.

Therefore, $L_\psi^* \circ \zeta = \zeta \circ \psi^*$, for all $\psi \in \mathrm{Diff}(N)$ □

The above ideas concerning natural metrics may be unified if we consider the left action S of the group $G(M,N) \equiv \mathrm{Diff}(M)\times\mathrm{Diff}(N)$

on the manifold $Q(M,N) \equiv Emb(M,N) \times Met(N)$ given by:

$$S_{(\phi,\psi)}(f,k) = (\psi \circ f \circ \phi^{-1}, \psi_* k) \qquad 2.2.11,$$

for all $(f,k) \in Q(M,N)$ and $(\phi,\psi) \in G(M,N)$. Note that the projection of the action S onto $Emb(M,N)$ is given by $R_{\phi^{-1}} \circ L_\psi$, for all $(\phi,\psi) \in G(M,N)$, and that the projection onto $Met(N)$ is the (lower star) action of $Diff(N)$ discussed at great length in Chapter Four.

The manifold $Q(M,N)$ admits a natural $G(M,N)$-invariant (weak) Riemannian metric as we now demonstrate. Define $K \in Met(Q(M,N))$ by:

$$K(f,k)((X_1,h_1),(X_2,h_2)) = \zeta(k)(f)(X_1,X_2) + G_o(k)(h_1,h_2) \qquad 2.2.12,$$

for all (X_1,h_1), $(X_2,h_2) \in T_{(f,k)}Q(M,N) \cong Vect_f(N) \oplus S_2(N)$ and $(f,k) \in Q(M,N)$. In 2.2.12, ζ is defined by equation 2.2.10 and $G_o \in Met(Met(N))$ is defined by equation 4.1.2. Note that here we require that N be compact.

<u>Proposition (2.2)7</u>: $G(M,N)$ acts by isometries on the Riemannian manifold $(Q(M,N),K)$.

<u>Proof</u>: Let $(\phi,\psi) \in G(M,N)$, $(f,k) \in Q(M,N)$ and $(X_1,h_1),(X_2,h_2) \in T_{(f,k)}Q(M,N)$.

Then, $(S^*_{(\phi,\psi)}K)(f,k)((X_1,h_1),(X_2,h_2))$

$$\begin{aligned}
&= K((R_{\phi^{-1}} \circ L_\psi)(f),\psi_*k)((D(R_{\phi^{-1}} \circ L_\psi)(f).X_1,\ \psi_*h_1), \\
&\qquad (D(R_{\phi^{-1}} \circ L_\psi)(f).X_2,\psi_*h_2)) \\
&= \zeta(\psi_*k)((R_{\phi^{-1}} \circ L_\psi)(f))(D(R_{\phi^{-1}} \circ L_\psi)(f).X_1,\ D(R_{\phi^{-1}} \circ L_\psi)(f).X_2) \\
&\qquad + G_o(\psi_*k)(\psi_*h_1,\ \psi_*h_2) \\
&= \big(R_{\phi^{-1}} \circ L_\psi)^*(L^*_{\psi^{-1}}(\zeta(k)))(f)(X_1,X_2) + G_o(k)(h_1,h_2)
\end{aligned}$$

(using proposition (2.2)6 together with the invariance of G_o under Diff(N) as demonstrated in section 4.1)

$= (R^*_{\phi^{-1}} \circ L^*_\psi \circ L^*_{\psi^{-1}})(\zeta(k))(f)(X_1,X_2) + G_o(k)(h_1,h_2)$ (since $R_{\phi^{-1}} \circ L_\psi = L_\psi \circ R_{\phi^{-1}}$)

$= \zeta(k)(f)(X_1,X_2) + G_o(k)(h_1,h_2)$ (using proposition (2.2)5)

$= K(f,k)((X_1,h_i),(X_2,h_2))$.

Thus, $S^*_{(\phi,\psi)}K = K$, for all $(\phi,\psi) \in G(M,N)$ □

The manifold Q(M,N) has applications in general relativity and higher dimensional gravitational physics; we take N to be the spacetime arena and M a diffeomorph of an extended spacetime object or "membrane". For example, for a particle theory, we take M to be an interval and for a string theory, we take M to be a two dimensional surface. The manifold Q(M,N), or, more usually, some open submanifold of Q(M,N), is the configuration space for the theory. For instance, if Met(N) denotes the space of Lorentzian metrics on N, then we may restrict to the open submanifold $Q_T(M,N) = \{(f,k) \in Q(M,N): f(M)$ is a timelike submanifold of $(N,k)\}$.

In connection with physical applications, there exists a natural smooth function A on Q(M,N) known in the literature as the membrane action (see [H 15]). A is defined by:

$$A(f,k) = \int_M \mathrm{vol}(f^*k) \qquad 2.2.13,$$

for all $(f,k) \in Q(M,N)$. Note that, for $(\phi,\psi) \in G(M,N)$, $(f,k) \in Q(M,N)$, we have $(A \circ S_{(\phi,\psi)})(f,k) = A(\psi \circ f \circ \phi^{-1}, \psi_* k)$ $= \int_M \mathrm{vol}((\phi_* \circ f^* \circ \psi^* \circ \psi_*)k) = \int_M \mathrm{vol}(f^*k) = A(f,k)$. Hence A is invariant under the action of G(M,N), so that A projects to a function on the space $(\mathrm{Sub}_M(N)/\mathrm{Diff}(N)) \times \mathrm{Geom}(N)$, where $\mathrm{Geom}(N) \equiv \mathrm{Met}(N)/\mathrm{Diff}(N)$ denotes the space of geometries on the manifold N (see section 4.1).

Let $T^*Q(M,N)$ be the L^2-cotangent bundle of $Q(M,N)$. Then $T^*Q(M,N)$ is equipped with the canonical (weak) symplectic form ω, and the symplectic manifold $(T^*Q(M,N),\omega)$ is the phase space corresponding to the configuration space $Q(M,N)$. The action S of $G(M,N)$ on $Q(M,N)$ (see 2.2.11) lifts to a symplectomorphic action on $(T^*Q(M,N),\omega)$ in the usual way.

The (weak) Riemannian metric K defined by equation 2.2.12 gives rise to the smooth map (actually a homomorphism of vector bundles) $b_k\colon\ TQ(M,N) \longrightarrow T^*Q(M,N);\ W \mapsto K(W,\cdot)$. Hence, we have the symplectic form $\omega_K = b_K^*\omega$ on $TQ(M,N)$. Using proposition (2.2)7, we see that the action S lifts to a symplectomorphic action of $G(M,N)$ on $(TQ(M,N),\omega_K)$ (see Abraham and Marsden [A 2] for the general theory of symplectic actions).

We remark that it would be interesting to investigate the spray of the Riemannian manifold $(Q(M,N),K)$ and to study the geodesic flow. We might also consider more general motion in the presence of natural potentials such as $A \in C(Q(M,N))$ (see 2.2.13) and functions on $Q(M,N)$ constructed from curvature quantities.

Before reviewing further applications of the theory of the manifold of embeddings to general relativity, we briefly return to the principal $\mathrm{Diff}(M)$-bundle $\gamma\colon \mathrm{Emb}(M,N) \longrightarrow \mathrm{Sub}_M(N)$ (2.2.3, 2.2.4). Given a manifold F together with $\rho \in \mathrm{Hom}(\mathrm{Diff}(M), \mathrm{Diff}(F))$, we may construct the associated bundle $E_\rho = \mathrm{Emb}(M,N)\times_\rho F$ (see definition (6.1)2).

Examples of such associated bundles include the following:-
(i) Take $F = \mathrm{Diff}(M)$ and $\rho = \mathrm{conj}$; $\phi \longmapsto \mathrm{conj}_\phi$; $\psi \longmapsto \phi \circ \psi \circ \phi^{-1}$. Then, $E_\rho = \mathrm{Conj}(\mathrm{Emb}(M,N))$, the conjugation bundle. The space of sections of $\mathrm{Conj}(\mathrm{Emb}(M,N))$ is isomorphic to $\mathrm{Gau}(\mathrm{Emb}(M,N))$, the group of gauge transformations of $\gamma\colon \mathrm{Emb}(M,N) \longrightarrow \mathrm{Sub}_M(N)$;
(ii) Take $F = \mathrm{Vect}(M)$ and $\rho = \mathrm{Ad}$; $\phi \longmapsto \phi_* \in GL(\mathrm{Vect}(M))$ (see equation 4.4.9). Then, $E_\rho = \mathrm{Ad}(\mathrm{Emb}(M,N))$, the Lie algebra bundle, and the space of sections of this bundle may be regarded as the Lie algebra of $\mathrm{Gau}(\mathrm{Emb}(M,N))$. For more details of the bundle constructions in (i), (ii), see section 6.1; (iii) Take $F = O(M) = \{(g,u) \in \mathrm{Met}(M) \times GL(M)\colon u \in O(M,g)\}$, the total space of the principal $O(m)$-bundle of M (see 1.4.1, 4.1.16), and ρ; $\phi \longmapsto \phi_* \times \hat{\phi}$; $(g,u) \longmapsto ((\phi^{-1})^* g, \hat{\phi}(u))$ (see 1.6.1 and proposition (4.1)11). E_ρ is then a fibre bundle over $\mathrm{Sub}_M(N)$ whose fibre over the submanifold S of N may be regarded as the total space of the principal fibration $\mathrm{Diff}(S) \hookrightarrow O(S) \xrightarrow{\pi_O} \mathrm{Geom}_O(S)$ (see equation 4.1.32). The manifold $\mathrm{Geom}_O(S)$ is a resolution of the singularities of the space of geometries on the submanifold S (see section 4.1 for a discussion of the resolution of singularities in the space of geometries); (iv) Take $F = C(M)$ and $\rho = $ (lower star); $\phi \longmapsto \phi_* = (\phi^{-1})^*$; $h \longmapsto h \circ \phi^{-1}$. Then E_ρ is a vector bundle over $\mathrm{Sub}_M(N)$ containing all smooth functions on all submanifolds (of diffeomorphism type M) of N:

Let $C\mathrm{Sub}_M(N) = \{h\colon h$ is a smooth function on some submanifold S of N, with S diffeomorphic to $M\}$ and define the projection $p\colon C\,\mathrm{Sub}_M(N) \longrightarrow \mathrm{Sub}_M(N)$; $h \longmapsto S$ if and only if $h \in C(S)$. Then p is a vector bundle over $\mathrm{Sub}_M(N)$ with fibres $p^{-1}(S) = C(S)$, for all $S \in \mathrm{Sub}_M(N)$. We now define a vector bundle isomorphism Ψ of $E_\rho \equiv \mathrm{Emb}(M,N)\times_{\mathrm{Diff}(M)} C(M)$ onto $C\,\mathrm{Sub}_M(N)$:- For $[(f,h)] \in E_\rho$, $\Psi([(f,h)]) \in p^{-1}(f(M))$ is given by:

$$\Psi([(f,h)])(f(x)) = h(x) \qquad 2.2.14,$$

for all $f(x) \in f(M)$. The map Ψ is easily seen to be a well defined isomorphism of vector bundles over $\mathrm{Sub}_M(N)$. Using Ψ together with the usual identification of sections with equivariant maps (see the remarks following definition (6.1)6), we have $\Gamma(\mathrm{Emb}(M,N)\times_{\mathrm{Diff}(M)} C(M)) \cong \Gamma(C\mathrm{Sub}_M(N)) \cong C_{\mathrm{Diff}(M)}(\mathrm{Emb}(M,N),C(M))$. Each function j on N now gives rise to a section of $\mathrm{Emb}(M,N)\times_{\mathrm{Diff}(M)} C(M)$; the value of j at $S \in \mathrm{Sub}_M(N)$ is just $j|S \in p^{-1}(S) \subseteq C\mathrm{Sub}_M(N) \cong \mathrm{Emb}(M,N)\times_{\mathrm{Diff}(M)} C(M)$.

We now consider some important applications of the space of embeddings to the theory of general relativity:

A major use for the space of embeddings arises in the 3+1 approach to general relativity. In this approach, the space $\mathrm{Emb}_S(M,N)$ of all spacelike embeddings of the (oriented) 3-manifold M into the (spacetime oriented) Lorentzian 4-manifold (N,k) is considered. $\mathrm{Emb}_S(M,N)$ is an open submanifold of $\mathrm{Emb}(M,N)$ and is the total space of a principal Diff(M)-bundle over the manifold $\mathrm{Sub}_M^S(N)$ of spacelike "slices" of type M in N.

The evolution of a given initial spacelike slice $S_o = f_o(M)$ is represented by a smooth curve $I \subseteq \mathbb{R} \to \mathrm{Emb}_S(M,N)$; $t \mapsto f_t$. For each $t \in I$, the velocity vector is $\dot{f}_t \in T_{f_t}\mathrm{Emb}_S(M,N)$ $= Df_t.\mathrm{Vect}(M) \oplus \Gamma(N(f_t))$ (using the fact that $\mathrm{Emb}_S(M,N)$ is open in $\mathrm{Emb}(M,N)$ together with equation 2.2.6). Since each f_t is a codimension one (orientation compatible) spacelike embedding, we have the future directed unit timelike normal vector field

$N_t \in \Gamma(N(f_t))$. Any element of $\Gamma(N(f_t))$ may be written as $L_t N_t$, for some $L_t \in C(M)$. Thus, for each $t \in I$, we may write:

$$\dot{f}_t = Df_t . X_t + L_t N_t \qquad 2.2.15,$$

where $X_t \in \mathrm{Vect}(M)$ and $L_t \in C(M)$. The time dependent sections X_t, L_t are called the <u>shift vector field</u>, <u>lapse function</u> respectively (see Arnowitt et al. [A 8]).

If $L_t \in C^+(M)$, for all $t \in I$, then the map $F: I \times M \longrightarrow N$; $(t,x) \mapsto f_t(x)$ is a diffeomorphism of $I \times M$ onto a tubular neighbourhood of S_o in N if $I = (-\varepsilon, \varepsilon)$ is sufficiently small. In this case, the curve $t \mapsto f_t$ is called a <u>slicing</u> of (N,k). Each slicing gives rise to a foliation of N (at least in a neighbourhood of S_o) into spacelike hypersurfaces, each diffeomorphic to M.

For fixed $x \in M$, the map $t \mapsto L_t(x)$ gives the <u>proper time</u> elapsed in moving from $f_o(x) \in S_o$ to $f_t(x) \in S_t$. Similarly, the map $t \mapsto X_t(x)$ gives the local change of spatial frames after a time t has elapsed.

For each $t \in I$, let $g_t = f_t^* k$ denote the induced (negative definite) metric on M, and let $k_t = K_{N_t}^{f_t}$ (see definition (2.1)15) denote the corresponding extrinsic curvature. The imposition of the vacuum Einstein equations $\mathrm{Ric}(k) = 0$ on (N,k) leads to a set of twelve first order evolution equations and four non-linear constraint equations for (g_t, k_t). Conversely, if $t \mapsto f_t$ is a slicing of (N,k) satisfying the evolution and constraint equations, then k is necessarily Ricci flat (see [A 8]).

Fischer and Marsden [F 6], in a very elegant piece of work, formulate the evolution equations as a Hamiltonian flow on the phase space $(T^*\mathrm{Met}(M), \omega)$. Here, ω is the canonical (weak) symplectic form on the L^2-cotangent bundle of $\mathrm{Met}(M)$ (see section 4.1). The constraints are maintained by the evolution equations for any lapse and shift, and, generically, the constraint set is a smooth submanifold of $T^*\mathrm{Met}(M)$. The Marsden-Weinstein reduction technique (see [M 4]) may be applied to give a dynamical representation of the space of true gravitational degrees of freedom.

The Fischer and Marsden approach to the initial value problem is based on $\mathrm{Met}(M)$ or, after the diffeomorphism group has been factored out, on <u>superspace</u> $\mathrm{Geom}(M)$. The problem with superspace is that it is not a manifold and possesses singularities (see section 4.1). An alternative approach by Kuchař and coworkers (see, for example, [K 14]) utilizes the manifold $\mathrm{Sub}_M^S(N)$ more directly. Kuchar refers to $\mathrm{Sub}_M^S(N)$ as <u>hyperspace</u>. The connection between the two approaches may be realized by considering the natural map $\mathrm{Emb}_S(M,N) \to \mathrm{Met}(M)$; $f \mapsto f^*k$, which projects to a map $\mathrm{Sub}_M^S(N) \to \mathrm{Geom}(M)$.

A third method for dealing with the 3+1 splitting of Einstein's equations is due to Binz [B 11] and this method also avoids the problem of dealing with a non-manifold. Binz formulates the evolution equations on the manifold $\mathrm{Emb}(M,\mathbb{R}^p)$. Here, p is a sufficiently large integer so that (M,g) may be isometrically embedded in $(\mathbb{R}^p, \mathrm{can}(p,0))$ (this can always be done by the theorem of Nash, (2.1)3). The submersion $m: \mathrm{Emb}(M,\mathbb{R}^p) \longrightarrow \mathrm{Met}(M)$: $f \mapsto f^* \mathrm{can}(p,0)$ induces a projection: $\mathrm{Sub}_M(\mathbb{R}^p) \longrightarrow \mathrm{Geom}(M)$ which is, in fact, a resolution of the singularities of $\mathrm{Geom}(M)$ (see definition (4.1)18).

In fact Binz generalizes the discussion to the manifold $\mathrm{Imm}(M,\mathbb{R}^p)$ of all immersions of M in $\mathbb{R}^p$, with corresponding

$m: \mathrm{Imm}(M,\mathbb{R}^p) \longrightarrow \mathrm{Met}(M)$. The important ingredient in the Binz approach is the fact that the differential of any $j \in \mathrm{Imm}(M,\mathbb{R}^p)$ may be expressed by $Dj(x) = a(x)\circ Di(x)\circ F(x)$, for all $x \in M$, where i is some fixed immersion of M in $\mathbb{R}^p$, $F \in \mathrm{End}(TM)$ is self-adjoint with respect to $m(i)$, and $a \in C(M,O(p))$ is an "integrating factor" which converts the $\mathbb{R}^p$-valued 1-form $Di\circ F$ into a differential by left composition.

Having fixed the initial immersion $i \in \mathrm{Imm}(M,\mathbb{R}^p)$, Binz considers the manifold $\{(a,F)$: there exists $j \in \mathrm{Imm}(M,\mathbb{R}^p)$ with $Dj = a\circ Di\circ F\}$. A Lagrangian is defined on the tangent bundle of this manifold together with constraint equations. The extremals of the Lagrangian satisfying the constraint equations then project down to solutions of the Einstein evolution equation on $\mathrm{Met}(M)$. Note that the lapse is one and the shift zero in the Binz formalism.

Another area of general relativity in which a space of embeddings arises is the theory of <u>cone space</u>. There are various (equivalent) ways of defining cone space A. The simplest is $A = \Gamma(\pi)$, where $\pi: S^2\times\mathbb{R} \longrightarrow S^2$ is the trivial affine bundle discussed in section 6.3. A is thus an affine space modelled on the vector space $C(S^2)$. Alternatively, we may regard cone space as the manifold of smooth cuts of the future null infinity $\mathscr{I}^+$ of an asymptotically flat spacetime (N,k). This space of cuts may be naturally identified with a submanifold of $\mathrm{Emb}(S^2,\mathscr{I}^+)$ and also with the space of outgoing null embeddings of $S^2\times\mathbb{R}$ into a neighbourhood of $\mathscr{I}^+$ in the compactified spacetime.

Since asymptotic moments of the gravitational field are obtained by integrating certain expressions around a given cut of $\mathscr{I}^+$, these moments should be regarded as tensor or spinor fields on cone space. For example, the Bondi 4-momentum described in section 3.2 may be regarded as a vector field on cone space. For more examples and for a detailed description of cone space, we refer the reader to Bramson [B 21]. Note that Bramson adopts a philosophy in which cone space replaces spacetime as the physical arena in many problems. This is because, by formulating physical laws in cone space rather than in (curved) spacetime, certain features of special relativistic theory persist. Cone space is also more intimately connected with that which an asymptotic observer experiences.

For a study of quasi-local, rather than asymptotic moments, there is no analogous theory of cone space. One possibility would be to consider the manifold $\mathrm{Emb}_S(S^2,N)$ of spacelike embeddings of S^2 in spacetime (N,k). If we wish to allow the possibility of varying the spacetime metric, then we could utilize $Q_S(S^2,N) = \{(f,k) \in Q(S^2,N) : f(S^2)$ is a spacelike submanifold of $(N,k)\}$ (here, of course, $Q(S^2,N) = \mathrm{Emb}(S^2,N) \times \mathrm{Met}(N)$, where $\mathrm{Met}(N)$ denotes the space of Lorentzian metrics on the 4-manifold N). The analogue of the BMS group would be some subgroup of the isometry group of $(Q_S(S^2,N),K_S)$, where K_S is the metric on $Q_S(S^2,N)$ induced from K (defined in equation 2.2.12).

In this section, we have discussed various infinite dimensional aspects of the theory of embeddings with particular reference to applications in general relativity. We conclude this chapter with a section concerned with the spinorial aspects of embeddings. Applications of the interaction between spinors and embeddings will appear in Chapter Three.

2.3 Spinors and Embeddings

In this section, we give an indication of how the spin structures introduced in Chapter One behave under embeddings as discussed in section 2.1. In particular, we develop certain tools which will be utilized in Chapter Three of this thesis, and so this section is especially concerned with embeddings in a spacetime.

The section is organized as follows:- First, we give a brief discussion of the way spin structures interact with a general isometric embedding $f: (M,g) \hookrightarrow (N,k)$. We then specialize to the case of spacelike embeddings in a spacetime. To conform with the notation of Chapter One, we denote a typical spacetime by (M,g), so that M is a connected, orientable, smooth 4-manifold and g is a Lorentzian metric (signature = -2) on M such that (M,g) is spacetime orientable.

Both codimension one and codimension two spacelike embeddings are important in general relativity. Moreover, the interplay between two, three and four dimensional structures is clearly manifested when spinor structures in general relativity are considered. For this reason, in this section, we describe both codimension one (hypersurface) spinors and codimension two (2-surface) spinors. The former are utilized in our treatment of spinor propagation equations as used in the definition of quasi-local moments (see sections 3.3 and 3.4) and the latter are the basis of the extremely useful GHP formalism (see Geroch et al. [G 7]) and also of the Penrose quasi-local programme (see [P 8]).

Note that we restrict our attention to spacelike embeddings in this section. In section 3.3, we use a null limit of spacelike embeddings to obtain a useful spinor propagation equation related to a codimension one <u>null</u> embedding.

Let us first give a brief description of the general situation. Suppose M,N are oriented manifolds of dimensions $m,n = m+p$ respectively. Let $f \in \mathrm{Emb}(M,N)$ and $k \in \mathrm{Met}(N)$ (the space of positive definite metrics on N; the indefinite case follows in a similar fashion - see below for examples). We define $g = f^*k$ so that f is an isometric embedding of codimension p. As in section 2.1, we have the following principal bundles:-

$$SO(m) \hookrightarrow SO(M,g) \xrightarrow{\pi_M} M; \quad SO(n) \hookrightarrow SO(N,k) \xrightarrow{\pi_N} N;$$

$$SO(p) \hookrightarrow SO(N(f)) \xrightarrow{\pi_f^\perp} M; \quad SO(m)\times SO(p) \hookrightarrow SO(f) \xrightarrow{\pi_f} M.$$

Here, $SO(N(f))$ is the bundle of oriented k-orthonormal frames of the normal bundle $\nu_f \colon N(f) \to M$. As above we regard $SO(m)$ and $SO(p)$ (and hence $SO(m)\times SO(p)$) as subgroups of $SO(n) \equiv SO(m+p)$.

In order to discuss spin structures, it is necessary to consider the spin groups corresponding to the various special orthogonal groups:- Let $\Lambda\colon \mathrm{Spin}(q) \to SO(q)$ denote the unique, non-trivial double covering for any $q \geq 2$. We also use the notation Λ for the induced double covering: $\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p) \to SO(m)\times SO(p)$: $[(B,C)] \longmapsto (\Lambda(B),\Lambda(C))$, for all $[(B,C)] \in \mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p) \equiv \mathrm{Spin}(m)\times_{\mathbb{Z}_2} \mathrm{Spin}(p)$. The inclusion of $SO(m)\times SO(p)$ in $SO(m+p)$ induces an inclusion of $\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p)$ in $\mathrm{Spin}(m+p)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p) & \longrightarrow & \mathrm{Spin}(m+p) \\ \downarrow \Lambda & & \downarrow \Lambda \\ SO(m) \times SO(p) & \longrightarrow & SO(m+p) \end{array} \qquad 2.3.1.$$

We now make <u>two</u> further assumptions. Firstly, we assume that M is spin, i.e. $w_2(TM) = 0$, and we choose a g-spin structure $s_g = (\widetilde{SO}(M,g),\eta_g) \in \tilde{\Sigma}(M,g)$. Secondly, we assume that $w_2(N(f)) = 0$, and we choose a Λ-prolongation $s^{\perp}_{(f,k)} = (\widetilde{\widetilde{SO}}(N(f)), \eta^{\perp}_{(f,k)})$ of $\widetilde{\widetilde{SO}}(N(f))$ to Spin(p).

Let us consider the principal bundle $\mathrm{Spin}(m) \times \mathrm{Spin}(p) \hookrightarrow \widetilde{\widetilde{SO}}(M,g)\times_M \widetilde{\widetilde{SO}}(N(f)) \to M$ together with the double covering μ: $\mathrm{Spin}(m)\times\mathrm{Spin}(p) \to \mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p)$. Let $\tilde{P}$ denote the μ-extension of $\widetilde{\widetilde{SO}}(M,g)\times_M \widetilde{\widetilde{SO}}(N(f))$, so that $\tilde{P}$ is a principal $\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p)$ bundle over M (see definition (6.1)8). Now define η: $\tilde{P} \to SO(M,g)\times_M SO(N(f))$ by $\eta([((\tilde{u},\tilde{u}'),[(B,C)])]) = (\eta_g(\tilde{u}), \eta^{\perp}_{(f,k)}(\tilde{u}'))\Lambda([(B,C)])$, for all $[((\tilde{u},\tilde{u}'),[(B,C)])] \in \tilde{P} \equiv (\widetilde{\widetilde{SO}}(M,g)\times_M \widetilde{\widetilde{SO}}(N,f)))\times_\mu(\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p))$. It is straightforward to verify that η is well defined. Moreover, we have $\eta([((\tilde{u},\tilde{u}'),[(B,C)])][(B',C')]) = \eta([((\tilde{u},\tilde{u}'),[(BB',CC')])])$
$= (\eta_g(\tilde{u}), \eta^{\perp}_{(f,k)}(\tilde{u}'))\Lambda([(BB',CC')]) = (\eta_g(\tilde{u}), \eta^{\perp}_{(f,k)}(\tilde{u}'))\Lambda([(B,C)])\Lambda([(B',C')])$
$= \eta([((\tilde{u},\tilde{u}'),[(B,C)])])\Lambda([(B',C')])$, for all $[(B',C')] \in \mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p)$ and $[((\tilde{u},\tilde{u}'),[(B,C)])] \in P$. Hence, $(\tilde{P},\eta)$ is a Λ-prolongation of $SO(M,g)\times_M SO(N(f))$ to the group $\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p)$.

We now utilize the diffeomorphism $\alpha\times\beta$: $SO(f) \to SO(M,g)\times_M SO(N(f))$ (see the remark immediately preceding proposition (2.1)13) to pullback the principal $\mathbb{Z}_2$-bundle η: $\tilde{P} \to SO(M,g)\times_M SO(N(f))$ to a principal $\mathbb{Z}_2$-bundle η_f: $\widetilde{\widetilde{SO}}(f) \to SO(f)$. Here, $\widetilde{\widetilde{SO}}(f) = (\alpha\times\beta)^* \tilde{P}$ and $\eta_f = (\alpha\times\beta)^*\eta$. The pair $(\widetilde{\widetilde{SO}}(f), \eta_f)$ is now a Λ-prolongation of the principal $SO(m) \times SO(p)$-bundle π_f: $SO(f) \to M$ to the group $\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p)$. Let $\tilde{\pi}_f$: $\widetilde{\widetilde{SO}}(f) \to M$ denote the projection.

We now demonstrate a construction that enables Spin(m+p)-spinors to be defined on M. In order to do this, it is necessary to prolong the principal SO(m+p)-bundle $f^*\pi_N$: $f^*SO(N,k) \to M$ to Spin(m+p). Let us denote by $\tilde{\pi}_{(f,k)}$: $\widetilde{\widetilde{SO}}(f,k) \to M$ the extension of $\tilde{\pi}_f$: $\widetilde{\widetilde{SO}}(f) \to M$ corresponding to the inclusion homomorphism: $\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p) \to \mathrm{Spin}(m+p)$ (see definition (6.1)8). Then, $\widetilde{\widetilde{SO}}(f,k) \equiv \widetilde{\widetilde{SO}}(f)\times_{\mathrm{Spin}(m)\tilde{\times}\mathrm{Spin}(p)} \mathrm{Spin}(m+p)$ is a principal Spin(m+p)-bundle over M. Define η': $\widetilde{\widetilde{SO}}(f,k) \to SO(f)\times_{SO(m)\times SO(p)} SO(m+p)$ by

$\eta'([(t,A)]) = [(\eta_f(t), \Lambda(A))]$, for all $[(t,A)] \in \tilde{\tilde{SO}}(f,k)$. The map η' is easily seen to be well defined. Moreover, we have $\eta'([(t,A)]A') = \eta'([(t,AA')])$ (by definition of the right action of Spin(m+p) on $\tilde{\tilde{SO}}(f,k)) = [(\eta_f(t),\Lambda(AA'))] = [(\eta_f(t), \Lambda(A)\Lambda(A')))]$ $= [(\eta_f(t), \Lambda(A))]\Lambda(A') = \eta'([(t,A)])\Lambda(A')$, for all $[(t,A)] \in \tilde{\tilde{SO}}(f,k)$ and $A' \in \mathrm{Spin}(m+k)$. Hence, the pair $(\tilde{\tilde{SO}}(f,k),\eta')$ is a Λ-prolongation of the principal SO(m+p)-bundle $SO(f)\times_{SO(m)\times SO(p)}SO(m+p)$ to the group Spin(m+p). Finally, we utilize the principal bundle isomorphism $\mu\colon SO(f)\times_{SO(m)\times SO(p)}SO(m+p) \to f^*SO(N,k)$ (see proposition (2.1)9) to yield the Λ-prolongation $(\tilde{\tilde{SO}}(f,k),\eta_{(f,k)})$ of $f^*SO(N,k)$ to Spin(m+p). Here $\eta_{(f,k)} = \mu\circ\eta'$.

To summarize the above; we have proved:

<u>Proposition (2.3)1</u>: Let M,N,f,k,s_g and $s^{\perp}_{(f,k)}$ be as above. Then there exists a Λ-prolongation $(\tilde{\tilde{SO}}(f,k),\eta_{(f,k)})$ of the principal SO(m+p)-bundle $f^*\pi_N\colon f^*SO(N,k) \to M$ to the group Spin(m+p).

Note that we have <u>not</u> assumed that the target space N is spin; we require only the vanishing of the two obstructions $w_2(TM)$ and $w_2(N(f))$ - these are the obstructions to prolonging the oriented orthonormal frame bundles of the vector bundles TM and N(f) respectively. However, since M is <u>orientable</u>, $w_2(TM) = 0 = w_2(N(f))$ if and only if both M and N are spin.

<u>Definition (2.3)2</u>: The principal bundle $\mathrm{Spin}(m+p) \hookrightarrow \tilde{\tilde{SO}}(f,k) \xrightarrow{\tilde{\pi}_{(f,k)}} M$ is called <u>the spin frame bundle corresponding to f, s_g and $s^{\perp}_{(f,k)}$</u>.

Given any action $\tilde{\rho} \in \mathrm{Hom}(\mathrm{Spin}(m+p), \mathrm{Diff}(F))$, where F is some manifold, we may form the associated bundle $\tilde{\tilde{SO}}(f,k)\times_{\tilde{\rho}}F$ over M. In particular, for $F = V$, a vector space, and $\tilde{\rho} \in \mathrm{Hom}(\mathrm{Spin}(m+p), GL(V))$, we may form the vector bundle $\tilde{\tilde{SO}}(f,k)\times_{\tilde{\rho}} V$ over M. Sections of such a vector bundle are spinor fields on M which transform under the group $\mathrm{Spin}(m+p)$ rather than the group $\mathrm{Spin}(m)$ (the usual spin group for M). Note that we have the vector bundle isomorphism $\gamma\colon \tilde{\tilde{SO}}(f,k)\times_{\tilde{\rho}} V \to \tilde{\tilde{SO}}(f)\times_{\tilde{\rho}\circ\iota} V$ (where $\iota\colon \mathrm{Spin}(m)\tilde{\times}\mathrm{Spin}(p) \hookrightarrow \mathrm{Spin}(m+p)$ is inclusion) defined by:

$$\gamma([([(t,A)],\xi)]) \;=\; [(t,\rho_A(\xi))] \qquad 2.3.2,$$

for all $[([(t,A)],\xi)] \in \tilde{\tilde{SO}}(f,k)\times_{\tilde{\rho}} V \equiv (\tilde{\tilde{SO}}(f)\times_{\mathrm{Spin}(m)\tilde{\times}\mathrm{Spin}(p)}\mathrm{Spin}(m+p))\times_{\tilde{\rho}} V$. It is straightforward to check that γ is a well defined isomorphism of vector bundles.

Let us now consider connections. Let $\omega_k \in \mathrm{Conn}(SO(N,k))$ denote the Levi-Civitâ connection 1-form of the metric k and let us denote the induced connection $(\pi_N^{\;*}f)^*\omega_k$ in $f^*SO(N,k)$ by $\omega_{(f,k)}$ (see definition (6.1)22 and section 2.1). We have the Lie algebra isomorphism $\Lambda_* \equiv D\Lambda(1)\colon L\mathrm{Spin}(m+p) \longrightarrow LSO(m+p)$, and this enables us to define the induced connection $\tilde{\omega}_{(f,k)} \in \mathrm{Conn}(\tilde{\tilde{SO}}(f,k))$ by:

$$\tilde{\omega}_{(f,k)} \;=\; \Lambda_*^{-1} \circ \eta^*_{(f,k)}\,\omega_{(f,k)} \qquad 2.3.3.$$

(Cf. equations 1.3.4 and 1.3.5).

<u>Definition (2.3)3</u>: Given M,N,f,k,s_g and $s^{\perp}_{(f,k)}$ as above, we call $\tilde{\omega}_{(f,k)} \in \mathrm{Conn}(\tilde{\tilde{SO}}((f,k))$ (as defined by 2.3.3) the <u>induced spin connection</u>.

Given $\tilde{\omega}_{(f,k)}$ together with $\tilde{\rho} \in \mathrm{Hom}(\mathrm{Spin}(m+p), GL(V))$, we may define the corresponding covariant derivative acting on sections of $E_{\tilde{\rho}} \equiv \tilde{\tilde{SO}}(f,k)\times_{\tilde{\rho}} V$ as in equation 6.1.9. Let us denote this covariant derivative by $\tilde{\nabla}^{(f,k)}$, or just $\tilde{\nabla}^k$ for short. Thus, we have:

$$\tilde{\nabla}^{(f,k)}\colon\; \Gamma(E_{\tilde{\rho}}) \longrightarrow \Omega^1(E_{\tilde{\rho}}) \qquad 2.3.4.$$

An alternative method of defining a covariant derivative acting on sections of $E_{\tilde{\rho}}$ may be described as follows: By a proposition analogous to (2.1)11, we have the connection ω_f in the principal $SO(m) \times SO(p)$-bundle $SO(f)$. This connection is constructed from the Levi-Cività connection of g, $\omega_g \in \mathrm{Conn}(SO(M,g))$, together with the normal connection $\omega_f^{\perp} \in \mathrm{Conn}(SO(N(f)))$ (see proposition (2.1)13). We have the Λ-prolongation $(\widetilde{\widetilde{SO}}(f), \eta_f)$ of $SO(f)$ to $\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p)$ and so we may define the induced connection $\tilde{\omega}_f$ in $\widetilde{\widetilde{SO}}(f)$. The latter connection is given by:

$$\tilde{\omega}_f = \Lambda_*^{-1} \circ \eta_f^* \, \omega_f \qquad 2.3.5.$$

(Cf. equations 1.3.4, 1.3.5 and 2.3.3, but note that here, $\Lambda_*: L(\mathrm{Spin}(m) \tilde{\times} \mathrm{Spin}(p)) \cong L\mathrm{Spin}(m) \oplus L\mathrm{Spin}(p) \longrightarrow L(SO(m) \times SO(p)) \cong LSO(m) \oplus LSO(p))$.

<u>Definition (2.3)4</u>: Given M, N, f, k, s_g and $s^{\perp}_{(f,k)}$, as above, we call $\tilde{\omega}_f \in \mathrm{Conn}(\widetilde{\widetilde{SO}}(f))$ (as defined by equation 2.3.5) the <u>adapted spin connection</u>.

As with the induced spin connection $\tilde{\omega}_{(f,k)}$, we may use the adapted spin connection $\tilde{\omega}_f$ to define the corresponding covariant derivative acting on sections of $E_{\tilde{\rho}}$ (identified with $\widetilde{\widetilde{SO}}(f) \times_{\tilde{\rho} \circ \iota} V$ using the isomorphism γ as given by equation 2.3.2). Let us denote the covariant derivative arising from $\tilde{\omega}_f$ by $\tilde{\nabla}^{(f,g)}$, or just $\tilde{\nabla}^g$ for short. Thus, we have:

$$\tilde{\nabla}^{(f,g)} : \Gamma(E_{\tilde{\rho}}) \longrightarrow \Omega^1(E_{\tilde{\rho}}) \qquad 2.3.6,$$

as in equation 6.1.9.

We may summarize the above discussion as follows:- Given an (orientation compatible) isometric embedding $f: (M,g) \hookrightarrow (N,k)$ and also Λ-prolongations $s_g = (\widetilde{\widetilde{SO}}(M,g), \eta_g)$,

$s^{\perp}_{(f,k)} = (\widetilde{\widetilde{SO}}(N,f)), \eta^{\perp}_{(f,k)})$ of the orthonormal frame bundles $SO(M,g)$, $SO(N(f))$ respectively. we may define two further Λ-prolongations. Firstly, we have the Λ-prolongation $(\widetilde{\widetilde{SO}}(f),\eta_f)$ of the principal $SO(m)\times SO(p)$-bundle $\pi_f\colon SO(f) \longrightarrow M$ of f-adapted frames to the group $Spin(m) \tilde{\times} Spin(p)$. Secondly, we have the Λ-prolongation $(\widetilde{\widetilde{SO}}(f,k),\eta_{(f,k)})$ of the principal $SO(m+p)$-bundle $f^*\pi_N$: $f^*SO(N,k) \longrightarrow M$ to the group $Spin(m+p)$. For any representation $\tilde{\rho}$ of $Spin(m+p)$ on a vector space V, the associated vector bundles $\widetilde{\widetilde{SO}}(f)\times_{\tilde{\rho}\circ\iota} V$ and $\widetilde{\widetilde{SO}}(f,k)\times_{\rho} V$ are naturally isomorphic. We identify them and denote this vector bundle by $E_{\tilde{\rho}}$ (although, of course, $E_{\tilde{\rho}}$ depends on f,k,s_g and $s^{\perp}_{(f,k)}$). The embedding and spin structure data lead to <u>two covariant derivatives</u> acting on sections of $E_{\tilde{\rho}}$. Firstly, we have the adapted covariant derivative, $\tilde{\nabla}^g \equiv \tilde{\nabla}^{(f,g)^{\rho}}$, arising from the adapted spin connection $\tilde{\omega}_f \in Conn(\widetilde{\widetilde{SO}}(f))$, and secondly, we have the induced covariant derivative, $\tilde{\nabla}^k \equiv \tilde{\nabla}^{(f,k)}$, arising from the induced spin connection $\tilde{\omega}_{(f,k)} \in Conn(\widetilde{\widetilde{SO}}(f,k))$. The adapted covariant derivative may be regarded as a $Spin(m) \tilde{\times} Spin(p)$ -operator whereas the induced covariant derivative is a $\mathcal{S}pin(m+p)$-operator.

Note that if $\tilde{\rho} = \rho \circ \Lambda$ for some $\rho \in Hom(SO(m+p), GL(V))$, then we have an isomorphism of $\widetilde{\widetilde{SO}}(f,k)\times_{\tilde{\rho}}V$ onto $f^*SO(N,k)\times_{\rho}V$ analogous to that defined by equation 1.3.2. Similarly, we have an isomorphism of $\widetilde{\widetilde{SO}}(f)\times_{\tilde{\rho}\circ\iota} V$ onto $SO(f)\times_{\rho\circ\iota}V$ (where ι also denotes inclusion of $SO(m)\times SO(p)$ in $SO(m+p)$). Note also that $f^*SO(N,k)\times_{\rho}V$ is naturally isomorphic to $f^*(SO(N,k)\times_{\rho}V)$. Hence, the five vector bundles $\widetilde{\widetilde{SO}}(f,k)\times_{\tilde{\rho}}V$, $\widetilde{\widetilde{SO}}(f)\times_{\tilde{\rho}\circ\iota} V$, $f^*SO(N,k)\times_{\rho}V$, $f^*(SO(N,k)\times_{\rho}V)$ and $SO(f)\times_{\rho\circ\iota}V$ are mutually isomorphic for $\rho \in Hom(SO(m+p),GL(V))$ and $\tilde{\rho} = \rho\circ\Lambda$. We refer to any one of these five as E_ρ.

The connection ω_k in $SO(N,k)$ gives rise to a covariant derivative ∇^k in $SO(N,k)\times_\rho V$ and hence, by pullback, a covariant derivative in $f^*(SO(N,k)\times_\rho V) \cong E_\rho$. The same covariant derivative is given by the connection $\omega_{(f,k)}$ in $f^*SO(N,k)$ and indeed by the connection $\tilde{\omega}_{(f,k)}$ in $\tilde{\tilde{SO}}(f,k)$. Thus, the induced covariant derivative $\tilde{\nabla}^{(f,k)}$ in E_ρ coincides with the covariant derivative induced from ∇^k in $SO(N,k)\times_\rho V$ by f.

The connection ω_f in $SO(f)$ also gives rise to a covariant derivative in E_ρ, and this covariant derivative coincides with the adapted covariant derivative $\tilde{\nabla}^{(f,g)}$ obtained from the connection $\tilde{\omega}_f$ in $\tilde{\tilde{SO}}(f)$.

Thus, each of the five isomorphs of E_ρ is furnished with the two covariant derivatives $\tilde{\nabla}^{(f,k)}$ and $\tilde{\nabla}^{(f,g)}$.

In particular, suppose ρ is the defining representation of $SO(m+p)$ on $\mathbb{R}^{m+p}$. Then the vector bundle E_ρ is just $f^*(TN)$ (Cf. equation 1.3.3) $= Df.TM \oplus N(f)$ (see section 2.1). In this case, the induced covariant derivative $\tilde{\nabla}^{(f,k)}$ is just the covariant derivative ∇^k given by equations 2.1.6 and 2.1.7. The adapted covariant derivative is given by the equation:

$$\tilde{\nabla}^{(f,g)}_X V = Df.\nabla^g_X V_M + \nabla^\perp_X V^\perp \qquad 2.3.7,$$

for all $X \in Vect(M)$, $V = Df.V_M + V^\perp \in Df.Vect(M) \oplus \Gamma(N(f)) \equiv Vect_f(N) \equiv \Gamma(f^*TN)$. Here, as usual, ∇^g is the covariant derivative in TM arising from the Levi-Cività connection of g, $\omega_g \in Conn(SO(M,g))$, and $\nabla^\perp$ is the connection in $N(f)$ arising from the normal connection $\omega^\perp_f \in Conn(SO(N(f)))$. Note that equation 2.3.7 reflects the fact that $\tilde{\nabla}^{(f,g)}$ is obtained from the connection ω_f; ω_f being constructed from ω_g together with $\omega_f^{\perp}$ (see proposition (2.1)13).

Returning now to principal bundles, we remark that, in addition to the connection 1-forms $\tilde{\omega}_f$, $\tilde{\omega}_{(f,k)}$, there exist other natural forms on the bundles $\widetilde{\widetilde{SO}}(f)$, $\widetilde{\widetilde{SO}}(f,k)$. We have the maps $\alpha\circ\eta_f$: $\widetilde{\widetilde{SO}}(f) \to SO(M,g)$, $(\pi_N^* f)\circ\eta_{(f,k)}$: $\widetilde{\widetilde{SO}}(f,k) \to SO(N,k)$. Therefore, we may pullback the canonical 1-forms of M,N respectively. Let $\theta_M^f = (\alpha\circ\eta_f)^*\theta_M \in \Omega^1(\widetilde{\widetilde{SO}}(f),\mathbb{R}^m)$ and $\theta_N^f = ((\pi_N^* f)\circ\eta_{(f,k)})^*\theta_N$ $\in \Omega^1(\widetilde{\widetilde{SO}}(f,k),\mathbb{R}^{m+p})$. As in section 2.1, we denote the inclusion of $SO(f)$ in $f^*SO(N,k)$ by ι, and let $\tilde{\iota} \equiv \eta_{(f,k)}^*\iota$ denote the induced inclusion of $\widetilde{\widetilde{SO}}(f)$ in $\widetilde{\widetilde{SO}}(f,k)$, so that we have $\iota\circ\eta_f = \eta_{(f,k)}\circ\tilde{\iota}$.

<u>Proposition (2.3)5</u>: If $\tilde{\iota}$: $\widetilde{\widetilde{SO}}(f) \to \widetilde{\widetilde{SO}}(f,k)$ is the principal bundle monomorphism arising from the isometric embedding f, then $\tilde{\iota}^*\theta_N^f = \theta_M^f \in \Omega^1(\widetilde{\widetilde{SO}}(f),\mathbb{R}^m)$.

<u>Proof</u>: $\tilde{\iota}^*\theta_N^f = \tilde{\iota}^*((\pi_N^* f)\circ\eta_{(f,k)})^*\theta_N = ((\pi_N^* f)\circ\eta_{(f,k)}\circ\tilde{\iota})^*\theta_N$

$= ((\pi_N^* f)\circ\iota\circ\eta_f)^*\theta_N = \eta_f^*((\pi_N^* f)\circ\iota)^*\theta_N = \eta_f^*\alpha^*\theta_M$ (by proposition (2.1)10)

$= (\alpha\circ\eta_f)^*\theta_M = \theta_M^f$ $\square$

Another form, this time symmetric and related to the second fundamental form, may also be defined; we have $\tilde{K}^f \equiv \eta_f^*\tilde{K}^f$ $\in \Gamma((\Theta^2 T^*\widetilde{\widetilde{SO}}(f)) \otimes (\widetilde{\widetilde{SO}}(f)\times\mathbb{R}^p))$ (see 2.1.10, 2.1.11).

For convenience, we summarize the various bundles and maps used above in the following diagram:

$$\begin{tikzcd}
& \widetilde{SO}(f) \arrow[rr, "\tilde{\iota}"] \arrow[d, "\eta_f"] & & \widetilde{SO}(f,k) \arrow[d, "\eta_{(f,k)}"] & \\
& SO(f) \arrow[rr, "\iota"] \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] \arrow[dd, "\pi_f"] & & f^{*}SO(N,k) \arrow[r, "\pi_N^{*}f"] \arrow[dd, "f^{*}\pi_N"] & SO(N,k) \arrow[dd, "\pi_N"] \\
SO(N(f)) \arrow[dr, "\pi_f^{\perp}"'] & & SO(M,g) \arrow[dl, "\pi_M"] & & \\
& M \arrow[rr, leftrightarrow, "id_M"'] & & M \arrow[r, "f"'] & N
\end{tikzcd}$$

2.3.8.

The absence of spin structure on the target space N is reflected in the vacuous nature of the top right hand corner of diagram 2.3.8. However, the only spinor fields we will need in applications are defined on the domain M, so that the prolongations $(\widetilde{SO}(f), \eta_f)$ and $\widetilde{SO}(f,k), \eta_{(f,k)})$ are all that we require.

Let us now consider the general formalism develpped above in the context of certain useful special cases in general relativity theory. As indicated in the introductory remarks of this section, we will denote a typical spacetime by (M,g).

Our first special case is that of a spacelike embedding of codimension one. Let H be an oriented 3-manifold and (M,g) a (spacetime oriented) spacetime. Let $f: (H,h) \hookrightarrow (M,g)$ be an orientation compatible spacelike isometric embedding, so that $h \equiv f^{*}g$ has signature equal to -3. To conform with the notation used in the general relativity literature, we denote the unit normal of f by t, so that t is a future directed unit normal vector field such that $u = \{e_1, e_2, e_3\} \in \pi_H^{-1}(x) \subseteq SO(H,h)$ implies $\{t(x), Df(x).e_1, Df(x).e_2, Df(x).e_3\} \in \pi_M^{-1}(f(x)) \subseteq SO(M,g)$ (see definition (2.1)14).

Let $k \equiv K_t^f \in S_2(H)$ denote the extrinsic curvature of f (definition (2.1)15). Then, we may write (see 2.1.12) $k(X,Y) = -g(\nabla_X^g t, Df.Y)$, for all $X,Y \in Vect(H)$, where $g \equiv g_f$ denotes the induced metric in the vector bundle $f^*(TM) = Df.TM \oplus N(f)$.

Note that the normal bundle $\nu_f\colon N(f) \to H$ is trivializable with total space $N(f) = \{rt(x)\colon r \in \mathbb{R},\ x \in H\}$. Similarly, the bundle $SO(N(f))$ of oriented $g^{\perp}$-orthonormal frames is trivializable with structure group $SO(1) = \{1\}$. Hence, the bundle $SO(f)$ of f-adapted oriented orthonormal frames is isomorphic to the bundle $SO(H,h)$ of oriented h-orthonormal frames. We shall henceforth identify $SO(f)$ and $SO(H,h)$ and denote both by $SO(H,h)$. The oriented version of diagram 2.1.2 now collapses to the following:

$$\begin{tikzcd}
SO(3) \arrow[r, hook] \arrow[d, hook] & SO^{\uparrow}(1,3) \arrow[r] \arrow[d, hook] & SO^{\uparrow}(1,3) \arrow[d, hook] \\
SO(H,h) \arrow[r, "\iota"] \arrow[d, "\pi_H"] & f^* SO(M,g) \arrow[r, "\pi_M^* f"] \arrow[d, "f^*\pi_M"] & SO(M,g) \arrow[d, "\pi_M"] \\
H \arrow[r, leftrightarrow] & H \arrow[r, hook, "f"] & M
\end{tikzcd}$$

2.3.9.

Note that, in the case of a codimension one spacelike embedding in spacetime, proposition (2.1)9 reduces to $SO(H,h)\times_{SO(3)}SO^{\uparrow}(1,3) \cong f^{*}SO(M,g)$, and proposition (2.1)10 reduces to $((\pi_N^{*}f)\circ\iota)^{*}\theta_M = \theta_H \in \Omega^1(SO(H,h),\mathbb{R}^3)$.

The connections available are $\omega_h \in \mathrm{Conn}(SO(H,h))$ and $\omega_g \in \mathrm{Conn}(SO(M,g))$. By propositions (2.1)11 and (2.1)12, the $LSO(3)$-component of $((\pi_N^{*}f)\circ\iota)^{*}\omega_g$ with respect to the decomposition $LSO^{\uparrow}(1,3) = LSO(3) \oplus L(1,3)$ (note the slight change of notation) is precisely ω_h, the Levi-Civitá connection of h. The normal connection $\omega_f^{\perp}$ is trivial, and hence the unit normal vector field is parallel along all curves in H - this corresponds to the fact, already noted (see the remarks immediately preceding definition (2.1)15), that $\nabla^{\perp}t = 0$. The relationship between the covariant derivatives $\nabla^h \equiv \nabla^{\omega_h}$ and $\nabla^g \equiv \nabla^{\omega_g}$ may be obtained from equations 2.1.5, 2.1.6 and 2.1.7. We shall give more explicit formulae for those covariant derivatives below.

Let us now consider the question of spin structures. Since the normal bundle of f is trivial, the only assumption that we need to make is that H is spin, i.e. $w_2(TH) = 0$. For example (see 1.2.2), if H is compact, then the obstruction to the existence of a spin structure on H vanishes. In any case, suppose H is spin and choose an h-spin structure $s_h = (\widetilde{\widetilde{SO}}(H,h),\eta_h) \in \tilde{\Sigma}(H,h)$ (up to equivalence of h-spin structures, this corresponds to choosing an element of $H^1(H;\mathbb{Z}_2)$ - see 1.1.5). Thus $\widetilde{\widetilde{SO}}(H,h)$ is a principal bundle over H with structure group $Spin(3) \cong SU(2) \cong S^3$. The Λ-prolongation $(\widetilde{\widetilde{SO}}(f),\eta_f)$ is identified with $(\widetilde{\widetilde{SO}}(H,h),\eta_h)$ since we have identified $SO(f)$ with $SO(H,h)$, and $\mathbb{Z}_2 \tilde{\times} SU(2) \cong SU(2)$.

In order to utilize $SL(2,\mathbb{C})$-spinors on H, it is necessary to consider the Λ-prolongation $(\widetilde{\widetilde{SO}}(f,g),\eta_{(f,g)})$ of $f^*SO(M,g)$ $\cong SO(H,h)\times_{SO(3)}SO^{\uparrow}(1,3)$ to the group $SL(2,\mathbb{C})$. Here, $\widetilde{\widetilde{SO}}(f,g)$ $=\widetilde{\widetilde{SO}}(H,h)\times_{SU(2)}SL(2,\mathbb{C})$ is the spin frame bundle corresponding to f and s_h (see definition (2.3)2).

We may now wheel out our favourite representations of the group $SL(2,\mathbb{C})$ and construct bundles over H associated with the principal bundle $SL(2,\mathbb{C}) \hookrightarrow \widetilde{\widetilde{SO}}(f,g) \to H$. Using the notation of section 1.7, we have, in particular, the representations ρ, $\bar{\rho}$, ρ^*, $\bar{\rho}^*$ $\in$ $\mathrm{Hom}(SL(2,\mathbb{C}),GL(2,\mathbb{C}))$ with corresponding associated vector bundles S_f, $\bar{S}_f$, S_f^*, $\bar{S}_f^*$, where $S_f=\widetilde{\widetilde{SO}}(f,g)\times_{\rho}\mathbb{C}^2$, etc. Since the symplectic form ε (see equation 1.7.1) is invariant under $SL(2,\mathbb{C})$, the vector bundles S_f, etc. each have the structure of a (complex) symplectic vector bundle over the 3-manifold H.

<u>Definition (2.3)6</u>: Given H,M,f and s_h as above, the symplectic vector bundle $\mathbb{C}^2 \hookrightarrow S_f \to H$ is called the <u>bundle of contravariant unprimed Weyl spinors over H</u>.

Similarly, $\bar{S}_f$ is called the bundle of contravariant primed Weyl spinors over H, etc.

Given any $\rho \in \mathrm{Hom}(SL(2,\mathbb{C}),GL(V))$, we may form the associated bundle $\widetilde{\widetilde{SO}}(H,h)\times_{\tilde{\rho}\circ\iota} V$, where $\iota\colon SU(2)\hookrightarrow SL(2,\mathbb{C})$ is inclusion. As in 2.3.2, we have an isomorphism of $\widetilde{\widetilde{SO}}(f,g)\times_{\tilde{\rho}}V$ onto $\widetilde{\widetilde{SO}}(H,h)\times_{\tilde{\rho}\circ\iota} V$, and we shall identify these two vector bundles, denoting both by $E_{\tilde{\rho}}$. For example, we identify $\widetilde{\widetilde{SO}}(H,h)\times_{\rho\circ\iota}\mathbb{C}^2$ with S_f, so that $\widetilde{\widetilde{SO}}(H,h)\times_{\rho\circ\iota}\mathbb{C}^2$ carries a natural symplectic structure ε.

Another way of constructing vector bundles is to use representations of $SO^{\uparrow}(1,3)$. In particular, when composed with the covering $\Lambda\colon SL(2,\mathbb{C}) \to SO^{\uparrow}(1,3)$, the defining representation of $SO^{\uparrow}(1,3)$ on $\mathbb{R}^4$ yields a representation of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$. Let

$W_f \equiv \widetilde{\widetilde{SO}}(f,g) \times_{SL(2,\mathbb{C})} \mathbb{R}^4$ denote the corresponding associated vector bundle over H (Cf. definition (1.7)1). Since the Minkowski inner product $\eta \equiv \mathrm{can}(1,3)$ on $\mathbb{R}^4$ is invariant under $SL(2,\mathbb{C})$, the vector bundle W_f is equipped with a fibre metric, also denoted by η. In fact, by the remarks above (arising from equation 1.3.3), the Lorentzian vector bundles (W_f,η) and (f^*TM,g) (where $g \equiv g_f$) are isometric (Cf. equation 1.7.17).

Now consider the representation $\rho \otimes \bar{\rho}$ of $SL(2,\mathbb{C})$ on $H(2)$, the vector space of 2×2 Hermitian matrices. We denote by H_f the corresponding vector bundle over H. The Lorentzian inner product $\varepsilon \otimes \bar{\varepsilon}$ on $H(2)$ is $SL(2,\mathbb{C})$-invariant and therefore furnishes H_f with a Lorentzian fibre metric, also denoted by $\varepsilon \otimes \bar{\varepsilon}$. Now define $\sigma_f\colon W_f \longrightarrow H_f$ by

$$\sigma_f([(\tilde{u},\underline{x})]) = [(\tilde{u},\alpha(\underline{x}))] \qquad 2.3.10,$$

for all $[(u,\underline{x})] \in W_f$. Here, α is the isometry of $(\mathbb{R}^4,\eta)$ onto $(H(2), \varepsilon \otimes \bar{\varepsilon})$ defined by equation 1.7.7. The map σ_f is an isometry of Lorentzian vector bundles, as is easily demonstrated (Cf. equations 1.7.14, 1.7.15).

We now compose the isometry of (f^*TM,g) onto (W_f,η) with that of (W_f,η) onto $(H_f, \varepsilon \otimes \bar{\varepsilon})$ to give an isometry of (f^*TM,g) onto $(H_f,\varepsilon \otimes \bar{\varepsilon})$. We denote this isometry by $\sigma(f)$.

Definition (2.3)7: Let H,M,f and s_h be as above. The isometry $\sigma(f)\colon f^*TM \longrightarrow H_f$ of Lorentzian vector bundles over H is called the Infeld-Van der Waerden isomorphism corresponding to f and s_h (Cf. definition (1.7)2).

Note that H_f is naturally embedded (qua vector bundle) in $S_f \otimes_{\mathbb{C}} \bar{S}_f$, so that the isometry $\sigma(f)$ embeds f^*TM in $S_f \otimes_{\mathbb{C}} \bar{S}_f$

(Cf. equation 1.7.18). The isometry $\sigma(f)$ extends to tensor products and, as for a spacetime, tensor field equations on H may be translated into equivalent equations involving $SL(2,\mathbb{C})$-spinor fields. As usual, we may restrict our attention to completely symmetric spinor fields.

In calculations, the Infeld-Van der Waerden isomorphism $\sigma(f)$ is not usually explicitly mentioned, and we have the usual abstract index identifications; for example, the metric induced in $f^{*}TM$ from g by f is written $g_{ab}(\equiv (g_f)_{ab}) = \varepsilon_{AB}\,\varepsilon_{A'B'}$ (Cf. equation 1.7.21). For further remarks concerning the use of 2-component Weyl spinors in general relativity theory, we refer the reader to the comments following definition (1.7)2 in section 1.7.

We now make a few remarks concerning the interaction between the groups $SL(2,\mathbb{C})$ and $SU(2)$. The inclusion of $SU(2)$ in $SL(2,\mathbb{C})$ which we are using is induced by the inclusion $a \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ of $SO(3)$ in $SO^{\uparrow}(1,3)$. The latter inclusion may be regarded as arising from the choice of a timelike direction in $(\mathbb{R}^4,\eta)$ and corresponds on the manifold level to the (timelike) normal bundle $N(f)$. In other words, we may regard the embedding f as a means of reducing the $SO^{\uparrow}(1,3)$ symmetry to an $SO(3)$ symmetry, and hence the $SL(2,\mathbb{C})$ symmetry to an $SU(2)$ symmetry.

Let $\underline{t} \in \mathbb{R}^4$ denote the unit future directed timelike vector in $(\mathbb{R}^4,\eta)$ corresponding to the inclusion $SO(3) \hookrightarrow SO^{\uparrow}(1,3)$. We define a Hermitian inner product on $\mathbb{C}^2$ by $G = \sqrt{2}\,\alpha(\underline{t}) \in H(2)$ (the $\sqrt{2}$ factor ensures that $\det G = 1$; Cf. equation 1.7.10). The inner product G is invariant under the action of $SU(2)$ on $\mathbb{C}^2$ (but not under the action of $SL(2,\mathbb{C})$), and so the vector bundle $\mathbb{C}^2 \hookrightarrow S_f \equiv \widetilde{SO}(f,g)\times_{\rho}\mathbb{C}^2 \longrightarrow H$ carries a Hermitian fibre metric, which we shall also denote by G.

Upon activating the Infeld-Van der Waerden isomorphism $\sigma(f)$, we may write $G_{AA'} = \sqrt{2}\, t_{AA'}$, where $t_{AA'} \equiv t^{BB'}\varepsilon_{BA}\varepsilon_{B'A'}$ is the 1-form corresponding to $t \in \Gamma(N(f))$.

Thus, the vector bundle S_f is equipped with the Hermitian fibre metric G in addition to the symplectic structure ε. Noting that $S_f \overset{\sim}{=} \widetilde{\widetilde{SO}}(H,h)\times_{\rho\circ\iota}\mathbb{C}^2$, we regard the principal $SU(2)$-bundle $\widetilde{\widetilde{SO}}(H,h)$ as the Hermitian frame bundle of (S_f,G) just as the principal $SL(2,\mathbb{C})$-bundle $\widetilde{\widetilde{SO}}(f,g)$ may be regarded as the symplectic frame bundle of (S_f,ε).

The Hermitian metric G is positive definite and possesses the inverse $G^{AA'} = \sqrt{2}\, t^{AA'}$ which satisfies $G^{AA'}\, G_{BA'} = \varepsilon_B{}^A$ (using the fact that $t^{AA'}t_{BA'} = \frac{1}{2}\varepsilon_B{}^A$). Hence, we have an isomorphism of vector bundles: $\overline{S}_f \longrightarrow S_f^*$; $\lambda^{A'} \longmapsto \lambda^{A'}G_{AA'}$, and by hermiticity, $\overline{S}_f^* \longrightarrow S_f$; $\mu_{A'} \longmapsto G^{AA'}\mu_{A'}$. The existence of G enables us to work entirely in terms of unprimed spinors - we just "convert" all primed indices to unprimed ones using the isomorphisms induced by G. For more details concerning the use of the unprimed $SU(2)$ spinors, we refer the reader to Sen [S 13]. We shall allow ourselves the freedom of using $SL(2,\mathbb{C})$ spinors.

We now consider the spin connections arising from our embedding and spin structure data. The Levi-Cività connection $\omega_g \in \mathrm{Conn}(SO(M,g))$ induces the connection $\omega_{(f,g)} \in \mathrm{Conn}(f^*SO(M,g))$ and the latter connection gives rise to the induced spin connection $\tilde{\omega}_{(f,g)}$ in $\widetilde{\widetilde{SO}}(f,g)$ (see definition (2.3)3). The connection $\tilde{\omega}_{(f,g)}$ then leads to the induced covariant derivative $\tilde{\nabla}^{(f,g)}$ acting on sections of $E_{\tilde{\rho}} \equiv \widetilde{\widetilde{SO}}(f,g)\times_{\tilde{\rho}}V \overset{\sim}{=} \widetilde{\widetilde{SO}}(H,h)\times_{\tilde{\rho}\circ\iota} V$, where $\tilde{\rho} \in \mathrm{Hom}(SL(2,\mathbb{C}),$ $GL(V))$.

We also have the connection $\omega_h \in \mathrm{Conn}(SO(H,h))$ and this induces the adapted spin connection $\tilde{\omega}_f \equiv \tilde{\omega}_h$ in $\widetilde{\widetilde{SO}}(H,h)$ (see definition (2.3)4). We then have the adapted covariant derivative $\tilde{\nabla}^{(f,h)}$ acting on sections of $E_{\tilde{\rho}}$.

As usual, we have two covariant derivative operators acting on the sections of any associated vector bundle $E_{\tilde{\rho}}$. Therefore, in constructing spinor differential equations on the 3-manifold H, we may utilize either the $SL(2,\mathbb{C})$ derivative $\tilde{\nabla}^{(f,g)}$ or, alternatively, the $SU(2)$ derivative $\tilde{\nabla}^{(f,h)}$.

The most important associated vector bundle for us is $S_f \equiv \widetilde{\widetilde{SO}}(f,g)\times_\rho \mathbb{C}^2 = \widetilde{\widetilde{SO}}(H,h)\times_{\rho\circ\iota}\mathbb{C}^2$ equipped with the symplectic structure ε together with the Hermitian fibre metric G. We also utilize S_f^*, $\overline{S}_f$, $\overline{S}_f^*$ and tensor products thereof; each of these vector bundles is also equipped with a symplectic structure and a Hermitian structure induced from ε and G respectively.

The two covariant derivatices $\tilde{\nabla}^{(f,g)}$ and $\tilde{\nabla}^{(f,h)}$ act on $\Gamma(S_f)$. The symplectic form ε is parallel with respect to the induced covariant derivative $\tilde{\nabla}^{(f,g)}$, but not with respect to the adapted covariant derivative $\tilde{\nabla}^{(f,h)}$. On the other hand, the Hermitian fibre metric G is parallel with respect to the adapted covariant derivative, but not with respect to the induced covariant derivative. These results follow from the fact that $\tilde{\nabla}^{(f,g)}$ is obtained from a connection in the bundle $\widetilde{\widetilde{SO}}(f,g)$ of symplectic frames of (S_f,ε), whilst $\tilde{\nabla}^{(f,h)}$ is obtained from a connection in the bundle $\widetilde{\widetilde{SO}}(H,h)$ of Hermitian G-orthonormal frames.

We now present explicit formulae for the covariant derivatives which will be utilized in Chapter Three. We use abstract indices corresponding to the vector bundle f^*TM, its complexification, its dual and tensor products thereof. We refer to tensors as <u>spatial</u> if they arise from tensor products of the subbundle Df.TH of f^*M. Spatial tensors are characterized by the property that their contracted tensor products with t^a and t_a all vanish. For convenience, we will suppress any explicit mention of the embedding f in our abstract index formulae.

The metric in f^*TM is given by g_{ab} and the induced metric on H is $h_{ab} = g_{ab} - t_a t_b$. All indices are raised and lowered with the metric g_{ab} (and its inverse), although for spatial tensors, raising and lowering with h_{ab} (and $h^{ab} \equiv g^{ab} - t^a t^b$) also gives the same result. The orthogonal projection onto Df.TH is given by $h_a^b = \delta_a^b - t_a t^b$. Thus, h_a^b projects out the spatial part of any tensor arising from f^*TM. For example, the abstract index version of the extrinsic curvature is $k_{ab} = - h_a^c \, h_b^d \, \nabla_c^g \, t_f$ (see equation 2.1.12).

In our formulae, we express all covariant derivatives in terms of the covariant derivative $\tilde{\nabla}^{(f,g)}$. In the special case of vector bundles constructed from $f^*(TM)$, this derivative reduces to the derivative $\nabla^g \equiv (\tau_M^* f)^* \nabla^g$ as we remarked above (see the discussion immediately preceding equation 2.3.7).

For example, on suppression of explicit mention of f, equation 2.1.5 gives us $\nabla_X^h Y = h.\nabla_X^g Y$, for all $X,Y \in \text{Vect}(H)$. Thus, we have $X^c \nabla_c^h Y^b = h_d^b X^c \nabla_c^g Y^d$, so that $h_a^c X^a \nabla_c^h Y^b = h_d^b h_a^c X^a \nabla_c^g Y^d$ for all $X \in \text{Vect}(H)$. Therefore, $h_a^c \nabla_c^h Y^b = h_a^c h_d^b \nabla_c^g Y^d$, and hence $\nabla_a^h Y^b = h_a^c h_d^b \nabla_c^g Y^d$, for all $Y \in \text{Vect}(H)$ (using the fact that ∇^h is spatial). The extension to a general spatial tensor field $T^{b\ldots c}{}_{d\ldots e}$ is given by:

$$\nabla^{h}_{a} T^{b\ldots c}{}_{d\ldots e} = h^{m}_{a} h^{b}_{n} \ldots h^{c}_{p} h^{q}_{d} \ldots h^{r}_{e} \nabla^{g}_{m} T^{n\ldots p}{}_{q\ldots r} \qquad 2.3.11.$$

Equation 2.3.11 gives the spatial component of the adapted covariant derivative $\tilde{\nabla}^{(f,h)}$ (see equation 2.3.7). The normal component is given by $\nabla^{\perp}(at) = da \otimes t$, for any $at \in \Gamma(N(f))$.

We may deal with the induced covariant derivative $\tilde{\nabla}^{(f,g)}$ in a similar manner:- Let $V \in \Gamma(f^{*}TM)$ and $X \in \mathrm{Vect}(H)$. Then, $\tilde{\nabla}^{(f,g)}_{X} V^{b} = \nabla^{g}_{X} V^{b} = X^{c} \nabla^{g}_{c} V^{b} = h^{c}_{a} X^{a} \nabla^{g}_{c} V^{b}$, so that $\tilde{\nabla}^{(f,g)}_{a} V^{b} = h^{c}_{a} \nabla^{g}_{c} V^{b}$, and hence $\tilde{\nabla}^{(f,g)}_{a} = h^{b}_{a} \nabla^{g}_{b}$. For convenience, let us write $D^{f} = \tilde{\nabla}^{(f,g)}$, so that $D^{f}_{a} = h^{b}_{a} \nabla^{g}_{b}$ acting on $\Gamma(f^{*}TM)$ and, by extension qua derivation, on tensor fields. The induced covariant derivative D^{f} also acts on $\Gamma(f^{*}T^{\mathbb{C}}M)$ (as a real operator) and hence, on using the complexification of the Infeld-Van der Waerden isomorphism $\sigma(f)$: $f^{*}T^{\mathbb{C}}M \longrightarrow S_{f} \otimes_{\mathbb{C}} \overline{S}_{f}$ (cf. equation 1.7.20) together with the fact that $\sigma(f)$ is $\tilde{\nabla}^{(f,g)}$-parallel, we may write $D^{f}_{AA'} = h^{BB'}_{AA'} \nabla^{g}_{BB'}$ acting on sections of S_{f}, $\overline{S}_{f}$, S^{*}_{f}, $\overline{S}^{*}_{f}$ and tensor products thereof.

The induced covariant derivative D^{f} was used by Witten [W 9] in his famous proof of the positive energy theorem (in Witten's paper, D^{f} acts on Dirac spinor fields, i.e. on sections of $S_{f} \oplus \overline{S}^{*}_{f}$) and by Sen [S 12] who regarded the kernel of the map: $\Gamma(S_{f}) \longrightarrow \Gamma(\overline{S}^{*}_{f})$; $\lambda^{A} \longmapsto D^{f}_{AA'}\lambda^{A}$ as the space of neutrino zero modes corresponding to the "initial" hypersurface $f(H)$.

Definition (2.3)8: Let H, M, f, g and s_{h} be as above. The induced covariant derivative $D^{f} \equiv \tilde{\nabla}^{(f,g)}$ acting on $\Gamma(S_{f})$ (and on conjugate-dual-tensor products) is called the Sen-Witten operator.

The equation

$$D^{f}_{AA'} \lambda^{A} = 0 \qquad 2.3.12,$$

is called the Sen-Witten equation.

The existence of solutions to this equation satisfying the spatially asymptotically constant boundary condition $\lambda - \lambda_{(0)} = O(r^{-1})$ for some constant spinor $\lambda_{(0)}$, has been established by Choquet-Bruhat and Christodoulou [C 16] and (in the Dirac case) by Parker and Taubes [P 4]. See also Reula [R 3].

Note that the Sen-Witten equation (2.3.12) is defined for a spacelike embedding f using the non-degenerate projection h onto Df.TH. In the case of a null embedding, such a projection does not exist and so we cannot define a corresponding Sen-Witten operator in such a simple fashion. However, it is useful to be able to define a null version of equation 2.3.12, for example in relation to the propagation of spinor fields on null hypersurfaces as is needed in a consideration of the quasi-local version of the Bondi-Sachs 4-momentum, and in Chapter Three we obtain such a null Sen-Witten equation by taking a certain null limit of the space-like equation.

Note that the extrinsic curvature may be expressed using the Sen-Witten operator; we have $k_{ab} = -h_a^c h_b^d \nabla_c^g t_d = -h_a^c \nabla_c^g t_b + h_a^c t_b t^d \nabla_c^g t_d = -D_a^f t_b$ (since $t_d t^d = 1$). Using this formula, it is straightforward to derive the following expression for the commutator of the Sen-Witten operator:

$$D^f_{[a} D^f_{b]} = 2\delta^d_{[a} S^c_{b]} \nabla^g_{[c} \nabla^g_{d]} - t_{[a} k_{b]}{}^d D^f_d \qquad 2.3.13,$$

where $S_{ab} = t_{AB'}\, t_{BA'} \equiv t_a t_b - \tfrac{1}{2} g_{ab}$.

Before leaving codimension one spacelike embeddings and associated spin concepts, we make a remark concerning an important recent use of hypersurface spinors. This is the work of Ashtekar [A15] on a spinor reformulation of the Hamiltonian formalism in general relativity. Ashtekar considers a 3-manifold H equipped with complex Riemannian metric q. H is assumed to be spin and a q-spin structure is chosen. Since the complexification of the group Spin(3) $\equiv$ SU(2) is SL(2,$\mathbb{C}$), the spinors under consideration are SL(2,$\mathbb{C}$)-spinors, although it is important to note that here, the group SL(2,$\mathbb{C}$) does not originate as the four dimensional (real) Lorentzian spin group. The dynamical variables are deemed to be the Infeld-Van der Waerden isomorphism σ corresponding to the q-spin structure chosen, together with an SL(2,$\mathbb{C}$) connection D (not the connection arising from q) which is the conjugate of σ. The traditional dynamical variables (q,k) are regarded as derived from the variables (σ,D), and a major advantage of utilizing the new variables is that the Einstein constraints (see section 2.2) are at worst quadratic rather than non-polynomial as is the case when the variables (q,k) are used. A further advantage is the fact that every constrained initial data set (σ,D) for (complex) general relativity provides an initial data set for SL(2,$\mathbb{C}$)-Yang-Mills theory. Conversely, any initial data set for Yang-Mills theory that satisfies certain additional algebraic constraints yields an initial data set for general relativity. Thus, techniques from Yang-Mills and general relativity theory may be interchanged.

The spacetime interpretation of the new 3+1 variables is also very satisfying. Indeed, suppose (M,g) is a solution of Einstein's equations obtained from a constrained initial data set (σ,D). Then

D is just the Sen-Witten operator arising from the embedding $(H,q) \hookrightarrow (M,g)$ and the corresponding connection form is a potential for the anti-self-dual part of the Weyl tensor field. The use of the new variables should also shed light on the relationship between $SL(2,\mathbb{C})$-spinors on the one hand and gravitational energy on the other (see Chapter Three for further discussion on spinors and kinematical quantities in general relativity).

We now return to spinors and embeddings. Having discussed co-dimension one spacelike embeddings in spacetime, we turn now to co-dimension two spacelike embeddings. As mentioned above, spacelike 2-surfaces have found various applications in general relativity theory; as closed trapped surfaces, they are an important tool in the analysis of spacetime singularities, and they also form the basic framework for defining quasi-local kinematical quantities. In particular, the codimension two spinor theory is an essential component of both the Ludvigsen-Vickers and of the Penrose approach to quasi-local momentum. A careful study of spinor structures on 2-surfaces also sheds light on the geometry of the GHP formalism, a formalism which has found applications in many areas of the theory of general relativity.

Let S be an oriented 2-manifold and (M,g) a (spacetime oriented) spacetime. Let $f\colon (S,h) \hookrightarrow (M,g)$ be an orientation compatible spacelike isometric embedding, so that $h \equiv f^*g$ has signature equal to -2.

We have the following principal bundles:- $S^1 \hookrightarrow SO(S,h) \xrightarrow{\pi_S} S$; $SO^{\uparrow}(1,3) \hookrightarrow SO(M,g) \xrightarrow{\pi_M} M$; $\mathbb{R}^+ \hookrightarrow SO(N(f)) \xrightarrow{\pi_f^{\perp}} S$; $\mathbb{C}^* \hookrightarrow SO(f) \xrightarrow{\pi_f} S$. Here, we have used the isomorphisms $SO(2) \cong S^1$, $SO^{\uparrow}(1,1) \cong \mathbb{R}^+$ (using the isomorphism

$\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \longmapsto e^{\phi}$) and $S^1 \times \mathbb{R}^+ \cong \mathbb{C}^*$.

We now demonstrate that the normal bundle of such a codimension two spacelike embedding is necessarily trivializable (Cf. the co-dimension one case described above, and also Newman [N 5]):

Proposition (2.3)9: Suppose S, M, f, g are as just described. Then the normal bundle $\nu_f : N(f) \to S$ is trivializable.

Proof: First note that, since $f^*TM = Df.TS \oplus N(f)$, the fact that both S and M are orientable implies that the vector bundle $N(f)$ is orientable.

Now we use the fact that (M,g) is spacetime oriented, so that there exists a globally defined timelike vector field on M. We can choose this vector field t such that t is orthogonal to $f(S)$, i.e. such that $t \circ f \in \Gamma(N(f))$.

The section $t \circ f$ is necessarily nowhere vanishing (since t is timelike) and hence we may define the line bundle $N_t = \{r(t \circ f)(x) : r \in \mathbb{R},\ x \in S\}$ over S. N_t is a line subbundle of $N(f)$, and we denote by $N_t^{\perp}$ the $g^{\perp}$-orthogonal complement of N_t in $N(f)$. Hence, $N(f) = N_t \oplus N_t^{\perp}$.

We now utilize the fact that a line bundle is orientable if and only if it is trivializable. The trivial line bundle N_t is thus orientable and hence, since $N(f)$ is orientable, we must have that $N_t^{\perp}$ is orientable and thus trivializable.

Now, $N(f)$ is the Whitney sum of trivializable vector bundles and therefore $N(f)$ is itself trivializable □

In applications, we always choose a global trivialization of $N(f)$, and it is convenient to use a pair of null normal fields to achieve this trivialization:-

First note that, for each $x \in S$, the spacetime orientation of (M,g) together with the orientation of S uniquely defines an ordered pair (L_x^{out}, L_x^{in}) of future pointing null directions in $T_{f(x)}M$. The spaces L_x^{out} and L_x^{in} are respectively the outward and inward pointing future null directions contained in the timelike subspace $T_x^{\perp}S$ of $T_{f(x)}M$.

Definition (2.3)10: A trivialization $\{l,n\}$ of $N(f)$ is said to be null if (i) $l(x) \in L_x^{out}$ and $n(x) \in L_x^{in}$, for all $x \in S$; and (ii) $g^{\perp}(l,n) = 1$.

Note that the action of $SO^{\uparrow}(1,1) \cong \mathbb{R}^+$ on $SO(N(f))$ gives rise to the free transitive action of Gau $SO(N(f)) \cong C^+(S)$ on the space of null trivializations given by $(r, \{l,n\}) \mapsto \{rl, r^{-1}n\}$, for all $r \in C^+(S)$ and for all null trivializations $\{l,n\}$.

Given a null trivialization $\{l,n\}$, the induced metric h on S may be written

$$h = g - l^b \otimes n^b - n^b \otimes l^b \qquad 2.3.14,$$

where $b: N(f) \longrightarrow f^*T^*M$ is the lowering map induced by $g^{\perp}$. In equation 2.3.14, we have suppressed mention of f. The orthogonal projection of f^*TM onto $Df.TS$ is given by

$$h_a^b = \delta_a^b - l_a n^b - n_a l^b \qquad 2.3.15,$$

using abstract index notation, and the normal projection onto $N(f)$ is given by:

$$(g^{\perp})_a^b = l_a n^b + n_a l^b \qquad 2.3.16.$$

A 2-surface tensor is one which arises from any tensor product of $Df.TS$ and its dual. 2-surface tensors may be characterized by the property that their contracted tensor products with $(g^{\perp})_a^b$ all vanish.

For more details concerning this "2+2" orthogonal decomposition, we refer the reader to d'Inverno and Stachel [I 4] and to Smallwood [S 21]. These papers give a reformulation of the initial value problem within a 2+2 framework.

Returning now to the bundles arising from the embedding f, we have the following version of diagram 2.1.2:

$$\begin{tikzcd}
 & & \mathbb{C}^* \arrow[dl] \arrow[dr] \arrow[dd, hook] \arrow[rrr, hook] & & & SO^{\uparrow}(1,3) \arrow[dd] \arrow[rr] & & SO^{\uparrow}(1,3) \arrow[dd, hook] \\
 & \mathbb{R}^+ \arrow[dr, hook] & & S^1 \arrow[dl, hook] & & & & \\
\mathbb{R}^+ \arrow[dr, hook] & & SO(f) \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] \arrow[dd, "\pi_f"] \arrow[rrr, hook, "\iota"] & & S^1 \arrow[dl, hook] & f^*SO(M,g) \arrow[dd, "f^*\pi_M"] \arrow[rr, "\pi_M^*f"] & & SO(M,g) \arrow[dd, "\pi_M"] \\
 & SO(N(f)) \arrow[dr, "\pi_S^{\perp}"'] & & SO(S,h) \arrow[dl, "\pi_S"] & & & & \\
 & & S \arrow[rrr, leftrightarrow] & & & S \arrow[rr, hook, "f"] & & M
\end{tikzcd}$$

2.3.17,

and $SO(f)\times_{\mathbb{C}^*} SO^{\uparrow}(1,3) \cong f^*SO(M,g)$, by proposition (2.1)9.

The theory of connections follows from the general ideas developed in section 2.1. For instance, the $\mathbb{R}^2$- component of $((\pi_N^* f)\circ\iota)^*\omega_g$ with respect to the decomposition $LSO^{\uparrow}(1,3) = \mathbb{R}^2 \oplus L(2,2)$ defines a connection ω_f in $SO(f)$. Proposition (2.1)13 gives us that $\omega_f = \alpha^*\omega_h + \beta^*\omega_f^{\perp}$, where $\omega_h \in \text{Conn } SO(S,h)$ is the Levi-Cività connection of h and $\omega_f^{\perp} \in \text{Conn } SO(N(f))$ is the normal connection. For covariant derivatives we utilize equations 2.1.5, 2.1.6 and 2.1.7, and we have a direct analogue of equation 2.3.11 for 2 -surface tensors; simply replace h_a^b in 2.3.11 by the projection given by equation 2.3.15.

We now consider spin structures. The spin groups corresponding to the various special orthogonal groups are as follows:- We have $\mathrm{Spin}(2) \cong S^1$ with double cover $\Lambda: S^1 \to S^1$; $e^{i\theta} \mapsto e^{2i\theta}$, and $\mathrm{Spin}(1,1) \cong \mathbb{R}^*$ with double cover $\Lambda: \mathbb{R}^* \longrightarrow \mathbb{R}^+$; $t \mapsto t^2$. Then, $\mathrm{Spin}(1,1) \tilde{\times} \mathrm{Spin}(2) \equiv \mathbb{R}^* \times_{\mathbb{Z}_2} S^1$ is isomorphic to $\mathbb{C}^*$ via the isomorphism; $[(t,e^{i\theta})] \longmapsto te^{i\theta}$. The double cover $\Lambda: \mathbb{C}^* \cong \mathrm{Spin}(1,1) \tilde{\times} \mathrm{Spin}(2) \to \mathbb{C}^* \cong SO^{\uparrow}(1,1) \times SO(2)$ is now seen to be just the squaring map; $z \longmapsto z^2$, for all $z \in \mathbb{C}^*$.

The bundle of oriented normal frames $\mathbb{R}^+ \hookrightarrow SO(N(f)) \to S$ is trivializable, therefore there is no obstruction to a Λ-prolongation. The question as to whether or not the bundle $S^1 \hookrightarrow SO(S,h) \to S$ admits a Λ-prolongation depends of course on the topology of S. For simplicity, suppose that S is compact, connected and without boundary, so that S is characterized by its genus g (recall that all our manifolds are orientable). The second Stiefel-Whitney class vanishes in this case, so that there is no obstruction to prolonging $SO(S,h)$. In fact, $H^1(S,\mathbb{Z}_2) = \mathbb{Z}_2^{2g}$, therefore there exist 4^g inequivalent h-spin structures on S (see 1.2.2). Choosing one of these equivalence classes and taking a representative $s_h = (\widetilde{\widetilde{SO}}(S,h),\eta_h) \in \tilde{\Sigma}(S,h)$, we have the principal circle bundle $\widetilde{\widetilde{SO}}(S,h)$ over S. We also have the trivializable principal $\mathbb{R}^*$-bundle $\widetilde{\widetilde{SO}}(N(f))$.

On performing the constructions given above in this section, we obtain the bundle of adapted spin frames $\mathrm{Spin}(1,1) \tilde{\times} \mathrm{Spin}(2) \cong \mathbb{C}^* \hookrightarrow \widetilde{\widetilde{SO}}(f) \to S$ together with the spin frame bundle $SL(2,\mathbb{C}) \hookrightarrow \widetilde{\widetilde{SO}}(f,g) \to S$, where $\widetilde{\widetilde{SO}}(f,g) = \widetilde{\widetilde{SO}}(f) \times_{\mathbb{C}^*} SL(2,\mathbb{C})$. As with the codimension one case, we may construct the vector bundles

S_f, etc., over S (see definition (2.3)6). The vector bundle S_f may be thought of as either $\widetilde{\widetilde{SO}}(f,g)\times_\rho \mathbb{C}^2$ or as $\widetilde{\widetilde{SO}}(f)\times_{\rho\circ\iota}\mathbb{C}^2$, where $\iota\colon \mathbb{C}^* \hookrightarrow SL(2,\mathbb{C})$ is inclusion. As before, the vector bundles S_f, etc., carry a symplectic structure ε. We also have an Infeld-Van der Waerden isomorphism corresponding to f and s_h (see definition (2.3)7). Thus, we possess the full power of the 2-component spinor formalism on the 2-manifold S. In order to do calculus, we introduce the spin connections $\tilde{\omega}_{(f,g)} \in \mathrm{Conn}(\widetilde{\widetilde{SO}}(f,g))$ and $\tilde{\omega}_f \in \mathrm{Conn}(\widetilde{\widetilde{SO}}(f))$ as above.

To complete this section, let us consider the relationship between codimension two embeddings and related spinor structures on the one hand and certain constructions arising in other parts of this thesis and in the literature on the other. In particular, we recall section 1.5 in which we discussed spinors and conformal structure.

Given the isometric embedding $f\colon (S,h) \hookrightarrow (M,g)$ together with the h-spin structure $s_h = (\widetilde{\widetilde{SO}}(S,h),\eta_h)$, we may construct the principal $\mathbb{C}^*$-bundle $\widetilde{\widetilde{SO}}(f)$ and also the connection $\tilde{\omega}_f$. Let us now consider vector bundles associated to $\widetilde{\widetilde{SO}}(f)$ and their corresponding covariant derivatives arising from $\tilde{\omega}_f \in \mathrm{Conn}(\widetilde{\widetilde{SO}}(f))$:-

For each $(s,w) \in \frac{1}{2}\mathbb{Z} \times \mathbb{C}$, we have the representation $\overset{\Delta}{\rho}_{s,w}$ of $\mathbb{C}^*$ on $\mathbb{C}$ (defined by equation 1.5.14). We define the complex line bundle $E(f;s,w)$ by

$$E(f;s,w) = \widetilde{\widetilde{SO}}(f)\times_{\overset{\Delta}{\rho}_{s,w}} \mathbb{C} \qquad 2.3.18,$$

and, by analogy with definition (1.5)3, we have the following:

<u>Definition (2.3)11</u>: Given S,M,f,g,s_h and (s,w) as above, we define a <u>function of spin weight s and boost weight w</u> to be a

section of $E(f;s,w)$.

The reason for the term boost weight is that

$SO^{\uparrow}(1,1) \equiv \left\{ \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} : \phi \in \mathbb{R} \right\}$ is the group of boosts of

$(\mathbb{R}^2, \mathrm{can}(1,1))$. Definition (2.3)11 may be regarded as a geometrization of the definition of a spin- and boost- weighted scalar given in Geroch et al. [G 7] in the special case of the GHP formalism on an embedded 2-manifold. We also refer the reader to Ehlers [E 7] who discusses a modified GHP formalism in which the basic principal $\mathbb{C}^*$-bundle is a reduction of the entire bundle $SO^{\uparrow}(1,3) \hookrightarrow SO(M,g) \longrightarrow M$ of oriented g-orthonormal frames over spacetime. We do <u>not</u> assume that such a reduction of $SO^{\uparrow}(1,3)$ to $\mathbb{C}^*$ exists - all our constructions arise from the embedding $f: S \hookrightarrow M$.

The connection $\tilde{\omega}_f$ in $\widetilde{SO}(f)$ now gives rise to a covariant derivative acting on spin- and boost- weighted functions. For given (s,w), we denote this derivative by $\tilde{\nabla}^{(f;s,w)}$, so that

$$\tilde{\nabla}^{(f;s,w)}: \Gamma(E(f;s,w)) \longrightarrow \Omega^1(E(f;s,w)) \qquad 2.3.19.$$

We may deal with spin- and boost-weighted spinor fields over S by taking tensor products. For example, we have the vector bundle $\mathbb{C}^2 \hookrightarrow S_f \to S$, and we may construct $E(f;s,w)\otimes_{\mathbb{C}} S_f$. Sections of the latter bundle are contravariant unprimed Weyl spinor fields of spin weight s and boost weight w. Regarding $E(f;s,w)$ as a rank two real vector bundle over S, we may also form such tensor products as $E(f;s,w)\otimes TS$ and $E(f;s,w) \otimes N(f)$, thereby obtaining spin- and boost-weighted vector fields. The covariant derivatives in $E(f;s,w)$ and in the spinor bundles combine to give a covariant derivative acting on weighted spinor fields. Given a null trivialization $\{l,n\}$ together with a local section of $SO(S,h)$, we may take the corresponding four components of the covariant derivative. The resulting four operators are the GHP operators $Þ$, $Þ'$, $ð_{GHP}$ and $ð'_{GHP}$.

Let us now consider the special case $S = S^2$. For any metric h on S^2 arising from an embedding $f: S^2 \hookrightarrow (M,g)$, there exists a unique h-spin structure (because $H^1(S^2;\mathbb{Z}_2) = 0$). We therefore have the principal $\mathbb{C}^*$-bundle $\widetilde{\widetilde{SO}}(f)$ over S^2 (and $\widetilde{\widetilde{SO}}(f)$ depends only on f and the spacetime metric g).

Now recall that there exists a canonical principal $\mathbb{C}^*$-bundle over S^2 (see section 1.5). This is given by $\mathbb{C}^* \hookrightarrow \mathbb{C}^2-\{\underline{0}\} \xrightarrow{\pi} S^2$ (1.5.13) and corresponds to the unique spin conformal structure $(\widetilde{\widetilde{CO}}(S^2,\mathrm{Can}),\eta)$ of (S^2,Can). Here $\eta: \widetilde{\widetilde{CO}}(S^2,\mathrm{Can}) \cong \mathbb{C}^2-\{\underline{0}\} \cong S^3 \times \mathbb{R}^+ \longrightarrow CO(S^2,\mathrm{Can}) \cong SO(3) \times \mathbb{R}^+$ is given by $\eta = \Lambda \times \mathrm{id}_{\mathbb{R}^+}$, where $\Lambda: S^3 \cong \mathrm{Spin}(3) \rightarrow SO(3)$ is the double covering (Cf. 1.2.4).

The principal $\mathbb{C}^*$-bundle π leads to the notion of functions of spin weight s and conformal weight w (definition (1.5)3) and also to the eth operator $ð_o$ (see equations 1.5.51 and 1.5.52, but note that here we use the notation $ð_o$ rather than $ð$).

Thus, given an isometric (codimension two) spacelike embedding $f: (S^2,h) \hookrightarrow (M,g)$, we have <u>two</u> principal $\mathbb{C}^*$-bundles over S^2, namely $\widetilde{\widetilde{SO}}(f)$ and $\widetilde{\widetilde{CO}}(S^2,\mathrm{Can})$, and two corresponding eth operators, namely $ð_{GHP}$ (acting on spin- and boost-weighted functions) and $ð_o$ (acting on spin- and conformally-weighted functions). In fact, a third eth operator is also used in the literature (see, for example, Newman and Tod [N 4]), and we denote this third eth by $ð_{NP}$. The third eth may be either regarded as the GHP eth acting on functions of boost weight zero, or as arising in the same manner as $ð_o$ in section 1.5, but with a general metric h on S^2 rather than can (of course any such h may be obtained from can by a conformorphism by the uniformization theorem for Riemann surfaces).

In the usual general relativity notation, the relationship between the three eths may be written as follows:-

$$\eth_{GHP}\eta = \eth_{NP}\eta - \tau b(\eta)\eta \qquad 2.3.20,$$

$$\eth_{NP}\eta = v\eth_o\eta + (\eth_o v)s(\eta)\eta \qquad 2.3.21,$$

where $b(\eta)$, $s(\eta)$ are respectively the boost- and spin-weights of the (local) section η, and v is such that $h = 4v(1+\zeta\bar{\zeta})^{-2}d\zeta d\bar{\zeta}$ in local isothermal coordinates (Cf. 1.5.23, 1.5.34). Equations 2.3.20 and 2.3.21 are verified in a straightforward manner using the GHP formalism together with equation 1.5.51.

The two principal bundles $\widetilde{\widetilde{SO}}(f)$ and $\widetilde{\widetilde{CO}}(S^2,\mathrm{Can})$ over S^2 both possess the same structure group, namely $\mathbb{C}^*$, but this group arises in two different ways. In $\widetilde{\widetilde{SO}}(f)$, $\mathbb{C}^*$ is an isomorph of $\mathrm{Spin}(1,1) \tilde{\times} \mathrm{Spin}(2)$, whereas in $\widetilde{\widetilde{CO}}(S^2,\mathrm{Can})$, $\mathbb{C}^*$ is an isomorph of $\widetilde{\widetilde{CO}}(2)$. The $\mathbb{R}^+$-factor in $\mathbb{C}^*$ is responsible for boost-weight in $\widetilde{\widetilde{SO}}(f)$, and for conformal-weight in $\widetilde{\widetilde{CO}}(S^2,\mathrm{Can})$.

The notions of boost- and conformal-weight may be united in two ways. Firstly, we may consider the standard embedding f_o of S^2 in Minkowski spacetime $(\mathbb{R}^4, \mathrm{can}(1,3))$; S^2 is embedded as a round sphere of radius one in $\mathbb{R}^3$ and this embedding is composed with the spacelike embedding of $\mathbb{R}^3$ as any hyperplane in $(\mathbb{R}^4, \mathrm{can}(1,3))$. Then, $h = \mathrm{can} \in \mathrm{Met}(S^2)$ and $\widetilde{\widetilde{SO}}(f_o) = \widetilde{\widetilde{CO}}(S^2,\mathrm{Can})$, so that, in this case, there is no distinction between boost- and conformal-weight.

On the other hand, if we wish to maintain both weights, then a second approach is possible; we consider conformal rescalings of the spacetime metric g. Let $CO^{\uparrow}(1,3) \hookrightarrow CO(M,C_g) \to M$ denote the conformal frame bundle of spacetime (M,g). Then $f^{*}CO(M,C_g)$ is a principal $CO^{\uparrow}(1,3)$-bundle over S, where f is an isometric embedding of (S,h) (S any oriented 2-manifold) in (M,g). We may therefore consider the real line bundle $\mathbb{R}^f_w = f^{*}CO(M,C_g) \times_{\hat{w}} \mathbb{R}$ over S (Cf. definition (6.2)4). Sections of $\mathbb{R}^f_w$ are called functions of conformal weight w on S. By tensoring $\mathbb{R}^f_w$ with $E(f;s,w')$ (and with other vector bundles over S) we may obtain quantities with spin-weight s, boost-weight w' and conformal-weight w. This approach is adopted in Penrose and Rindler [P 11] (see pp. 352-362), but we hope that our remarks have clarified the geometry of the situation. We also refer the reader to the formulae 6.2.12 - 6.2.44 which give the conformal transformation properties of important spin- and boost-weighted quantities.

We note also the work of Ludwig [L 13]. Ludwig considers complex Lorentz transformations and complex conformal rescalings within the framework of the group $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$. Ludwig also calculates the generalized transformation properties of useful geometrical quantities, albeit in a very algebraic fashion. An interesting avenue for further study would involve a consideration of the Ludwig ideas within the context of embeddings and principal bundles. This work will be left for future investigation.

We hope that the remarks of this section have demonstrated the natural geometric manner in which embeddings and spinor structures interact with one another. Other ideas, such as conformal structure, also come into play when we consider the specific case of embeddings in a spacetime. We have already indicated certain applications of the ideas of this section to general relativity theory. In Chapter Three, in particular in sections 3.3 and 3.4, we consider another application; this time to spinor field null propagation and the definition of quasi-local momentum.

3.0 Introduction

In this chapter, we continue our discussion of spinors and embeddings in general relativity. The emphasis is on spinor propagation equations on null hypersurfaces and their application to the definition of quasi-local momenta in spacetimes. The chapter highlights the natural interaction between null embeddings, $SL(2,\mathbb{C})$-spinors, 2-surfaces and kinematical quantities.

The chapter is organized as follows:- Having discussed spacelike embeddings in section 2.3, we first introduce null embeddings in spacetime. Null embeddings present problems when we attempt to push through the ideas developed for spacelike, and other non-degenerate, embeddings. Nevertheless, null embeddings are important in gravity theory (the reasons for this importance will emerge below), and therefore we outline the approaches to circumventing these difficulties.

In section 3.2, we describe how concepts arising from spacelike embeddings may be transferred to the null context by taking a limit of spacelike embeddings. We refer to this as a null limit. Our approach is straightforward and it is based upon the normal bundle - a trivial line bundle for a codimension one, orientation compatible, spacelike embedding. Our main application of the null limit technique is to derive a natural spinor field propagation equation on a null hypersurface - this is a limit of the Sen-Witten equation described in section 2.3. The propagation equation has been used in general relativity theory in the definition of quasi-local momentum due to Ludvigsen and Vickers [L 10] and also in the derivation of a fundamental inequality between mass and charge (see [L 11]). In section 3.4, we describe how our propagation equation is used in these applications.

In order to set the scene for the quasi-local momentum discussion of section 3.4, in section 3.3 we give a brief overview of gravitational momentum in general. We discuss the historical, physical and geometrical development and we emphasize the importance of kinematical quantities in general relativity. In recent years, it has become apparent that spin structure seems to underly mass-momentum-angular momentum in general relativity theory. The full significance of the interaction between spinors and momentum is not yet clear, but our discussion covers the current state of understanding.

The important novel idea in this chapter is that of the null limit as a means of obtaining null versions of spacelike concepts. We will indicate further applications of this technique in sections 3.2 and 3.4.

3.1 Null Embeddings

In this section, we give a brief discussion of the theory of null embeddings in a Lorentzian manifold. We are particularly interested in the case of codimension one null embeddings in a spacetime, since null hypersurfaces play an important rôle in the theory of general relativity. For example, null hypersurfaces appear as the smooth parts of achronal boundaries such as event horizons and Cauchy horizons, and they are also an essential ingredient in the theory of black holes and of singularity theory in general (see Hawking and Ellis [H 5]). Null hypersurfaces also arise as the future and past null infinities of asymptotically flat spacetimes (see section 6.3) and as characteristic surfaces in the theory of radiation (see Friedrich and Stewart [F 12]). Our use for null embeddings is within the context of quasi-local momentum in general relativity and we discuss this particular application in section 3.4.

We now summarize the main facts concerning the structure of null embeddings. For details, we refer the reader fo Kupeli [K 16] and references therein.

Let (M,g) be a connected, spacetime oriented Lorentzian n-manifold, so that the signature of g is equal to $2 - n$. Let $f: H \hookrightarrow M$ be an embedding of the m-manifold H $(m < n)$ in M and let $h = f^*g \in S_2(H)$.

Definition (3.1)1: The embedding $f: (H,h) \hookrightarrow (M,g)$ is said to be null if h is degenerate.

Let $f: (H,h) \hookrightarrow (M,g)$ be a null embedding and, as usual - see section 2.1, let $\nu_f: N(f) \to H$ be the normal bundle of f, g. Hence, for each $x \in H$, $\nu_f^{-1}(x) \equiv T_x^\perp H = \{v \in T_{f(x)}M: g(f(x))(v,w) = 0,$ for all $w \in Df(x).T_xH\}$, and $T_x^\perp H$ is a $(n-m)$-dimensional subspace of $T_{f(x)}M$. Note, however, that $Df(x).T_xH \oplus T_x^\perp H \neq T_{f(x)}M$, since the inner product induced on $Df(x).T_xH$ is degenerate.

For each $x \in H$, we let $L_x = Df(x).T_xH \cap T_x^\perp H$, so that L_x is a one-dimensional degenerate subspace of both $Df(x).T_xH$ and of $T_x^\perp H$. The orthogonal complement of L_x in $T_{f(x)}M$ is given by $L_x^\perp = Df(x).T_xH + T_x^\perp H$. Let $L(f) = \bigcup_{x \in M} L_x$ and define the projection $\lambda_f: L(f) \to H$ in the obvious manner. Then $L(f)$ possesses the structure of a line bundle over H. In fact, $L(f)$ is the only subbundle of both $Df.TH$ and $N(f)$ with one-dimensional null fibres.

Definition (3.1)2: The line bundle $\lambda_f\colon L(f) \to H$ is called the null bundle of the null embedding f.

Proposition (3.1)3: The null bundle $L(f)$ is trivializable.

Proof: Let t be a globally defined future directed timelike vector field on (M,g) and define $l \in \Gamma(L(f))$ by $x \mapsto l(x)$, where $l(x)$ is the unique element of $\lambda_f^{-1}(x)$ satisfying $g(f(x))(l(x),t(x)) = 1$. Then, l is a smooth (future directed) nowhere vanishing section of $L(f)$ which provides a trivialization for $L(f)$ □

Note that, since $L(f)$ is a line bundle, any future directed section of $L(f)$ is given by $l' = rl$ for some $r \in C^+(H)$. Moreover, since $L(f)$ is the unique null subbundle of $Df.TH$, the section $(Df)^{-1}l$ is a future directed null vector field on H which is unique up to an element of $C^+(H)$.

We have the following corollary to proposition (3.1)3:

Corollary (3.1)4: Let $f\colon (H,h) \hookrightarrow (M,g)$ be a null embedding of codimension one or two. Then H is necessarily orientable.

Proof: Let t be any unit future directed timelike vector field on (M,g) and l a future directed section of $L(f)$ such that $g_f(l,t) = 1$.

Suppose first that $\operatorname{codim}(f) = 1$, then, since t is nowhere tangent to $f(H)$, $f^*(\iota_t \operatorname{vol}(g))$ is a nowhere vanishing $(n-1)$-form on H. Hence, H is orientable.

Now suppose that $\operatorname{codim}(f) = 2$ and let $l_f \in \Gamma(TM|f(H))$ be defined by $l_f(f(x)) = l(x)$, for all $f(x) \in f(H)$. Then we have that $s = t|f(H) - l_f$ is a spacelike section of $TM|f(H)$ which is nowhere tangent to $f(H)$. Hence, $f^*(\iota_s \iota_{t|f(H)} (\operatorname{vol}(g)|f(H)))$ is a nowhere vanishing $(n-2)$-form on H, so, again, H is orientable □

Note that if H is given the orientations referred to in the proof of corollary (3.1)4, then f is an orientation compatible embedding.

Definition (3.1)5: Let $f: (H,h) \hookrightarrow (M,g)$ be a null embedding and l a future directed, nowhere vanishing section of $L(f)$, so that $(Df)^{-1}l$ is a future directed null vector field on H. The image of an inextendable integral curve σ of $(Df)^{-1}l$ is called a (null) generator of f.

The idea of a null generator is very useful in the analysis of null embeddings, in particular, the causal structure of null embeddings. We do not discuss such matters here, but refer the reader to Kupeli [K 16]. However, for completeness, we state the following theorem:

Theorem (3.1)6 (Kupeli [K 16]): Let H be simply connected and $f: (H,h) \hookrightarrow (M,g)$ a null embedding. Suppose S is a closed, connected, spacelike hypersurface of (H,h) with the property that every generator of f intersects S. Then there exists a diffeomorphism $\psi: H \to S \times \mathbb{R}$ such that, for each $x \in S$, $\psi^{-1}(\{x\} \times \mathbb{R})$ is a generator of f. In particular, since H is diffeomorphic to $S \times \mathbb{R}$, S is simply connected.

The major difficulty in the study of null embeddings arises because of the non-existence of a well defined projection of f^*TM onto $Df.TH$. Thus, we cannot construct expressions such as those contained in equations such as 2.1.5 - 2.1.7 and 2.3.11 which utilize the projection h. In particular, there is no well defined induced connection on H. However, there do exist certain useful Riemannian vector bundles over H:-

Let E denote any one of the vector bundles $Df.TH$, $N(f)$,

$L^{\perp}(f) \cong Df.TH \dotplus N(f)$ over H. Hence, $L(f) \subseteq E \subseteq f^{*}TM$. Define $\bar{E} = E/L(f)$ with Riemannian fibre metric $\bar{g}$ given by $\bar{g}(x)(\bar{v},\bar{w}) = g(x)(v,w)$, where $v,w \in \pi_E^{-1}(x)$ with $p(v) = \bar{v}$, $p(w) = \bar{w}$ (where $p: E \to \bar{E}$ is the quotient map), for all $\bar{v}, \bar{w} \in \pi_{\bar{E}}^{-1}(x)$ and $x \in H$. It is straightforward to verify that the metric $\bar{g}$ is a well defined Riemannian fibre metric in the vector bundle $\bar{E}$ over M. Note that $\overline{L^{\perp}(f)} = \overline{Df.TH} \oplus \overline{N(f)}$ is an orthogonal Whitney sum of Riemannian vector bundles over H.

Definition (3.1)7: The Riemannian vector bundle $\rho_f: R(f) \equiv \overline{Df.TH} \to H$ is called the canonical Riemannian vector bundle associated with the null embedding f.

Note that $R(f)$ is a rank $(m-1)$ vector bundle over H. This bundle may be used in a study of null embeddings utilizing the techniques of Riemannian geometry. In particular, a null second fundamental form $K^f \in \Gamma((\Theta^2 R(f)^{*}) \otimes L(f))$ together with a covariant derivative operator (which differentiates sections of $R(f)$ along the direction of $L(f)$) may be defined. Kupeli [K16] uses these concepts to investigate the deviation of null congruences in an embedded null submanifold and also to analyze the structure of totally geodesic null submanifolds.

Since we shall be especially interested in codimension one null embeddings, it is worth remarking on particular features of this case. Suppose $f: (H,h) \hookrightarrow (M,g)$ is a codimension one null embedding. Then the vector bundles $Df.TH$, $N(f)$ have ranks equal to $(n-1), 1$ respectively, and $L(f) = N(f)$. Hence, $R(f) \equiv Df.TH/N(f)$ has rank $(n-2)$, $\overline{N(f)} \cong H$ and $\overline{L^{\perp}(f)} = R(f)$.

In particular, if $n = 4$, then $R(f)$ is a rank two Riemannian vector bundle over the 3-manifold H. In this spacetime case, theorem (3.1)6 gives a relationship between embedded null hypersurfaces of spacetime on the one hand and embedded spacelike 2-surfaces on the other. We return to the interaction between embedded null hypersurfaces and embedded spacelike 2-surfaces in section 3.2.

To complete this section, we draw the reader's attention to the paper of Crampin [C 11]. In this paper, Crampin discusses degenerate metrics on manifolds using the appropriate frame bundles. In particular, a vector valued function on the orthonormal frame bundle is defined, and the vanishing of this function implies that the manifold admits a torsion free linear connection whose parallel translation leaves the degenerate metric invariant. This principal bundle approach to connections associated with degenerate metrics complements the vector bundle approach of Kupeli [K 16]. We do not provide an analysis of the relationship between the two approaches here, but such an analysis would shed further light on the structure of null embeddings.

We turn now to an alternative technique for dealing with null embeddings, namely the null limit.

3.2 The Null Limit

In section 3.1, we outlined the reasons why an analysis of null embeddings is more difficult than of the non-degenerate case. In this section, we present a technique which enables certain concepts associated with a spacelike embedding to be also associated with a particular class of null embeddings. This is done by taking a null limit of spacelike concepts. We shall apply this technique to obtain a spinor differential equation on a null hypersurface in section 3.4, this particular equation being an important part of the Ludvigsen-Vickers definition of quasi-local momentum in general relativity theory. We also suggest other possible applications for the null limit technique.

The class of null embeddings for which our technique is appropriate is the class constructed from null geodesic congruences orthogonal to spacelike 2-surfaces. Note that not every null embedding arises in this way (see Kupeli [K 16]) but this class will be sufficient for our purposes. Indeed, in the application to quasi-local momentum we take a spacelike 2-surface as the starting point.

In what follows, (M,g) is a connected, (spacetime oriented) spacetime and S is a connected, oriented 2-manifold. As in section 2.2, we denote by $\mathrm{Emb}_S(S,M)$ the manifold of spacelike embeddings of S in (M,g).

Let $f \in \mathrm{Emb}_S(S,M)$. Then we have $f^*TM = Df.TS \oplus N(f)$. In addition, we have the further decomposition $N(f) = L^{out}(f) \oplus L^{in}(f)$, since $N(f)$ is trivializable (see proposition (2.3)9). Here, $L^{out}(f)$ and $L^{in}(f)$ are the naturally defined null line bundles over S defined by f, so that $L^{out}(f) = \bigcup_{x \in S} \{t\, l(x) : t \in \mathbb{R}\}$ and $L^{in}(f) = \{t\, n(x) : t \in \mathbb{R}\}$, where $\{l,n\}$ is any null trivialization of $N(f)$ (see definition (2.3)10).

Let $\exp_g$ denote the exponential map of g. Then there exists an open neighbourhood V of the zero section in TM such that $\exp_g : V \longrightarrow M$ is defined. Let $V^\perp = (\tau_M^* f)^{-1}(V) \cap N(f)$, so that $V^\perp$ is an open neighbourhood of the zero section in $N(f)$, and define $\exp^\perp = (\tau_M^* f)^* \exp_g \big|_{V^\perp}$. The map $\exp^\perp$ is called the <u>normal exponential map</u> (of f and g). Now let $U = V^\perp \cap L^{out}(f)$, then U is an open neighbourhood of the zero section in $L^{out}(f)$, such that $\exp^{\perp}|U$ is an embedding of U into M. Let $\tilde{H} = \exp^{\perp}(U)$. Then $\tilde{H}$ is an embedded null hypersurface of (M,g).

The null hypersurface $\tilde{H}$ has the special property that there exists a geodesic null vector field on $\tilde{H}$; this is obtained by taking the tangent vectors of the null geodesics with initial data $l(x)$, $x \in S$, where $\{l,n\}$ is some trivialization of $N(f)$. A null hypersurface with the property that it admits a geodesic null vector field is called a geodesic null hypersurface. Kupeli [K 16] demonstrates that if a null hypersurface $\tilde{H}$ in a spacetime (M,g) is causally separated by a spacelike 2-surface $\tilde{S}$, then $\tilde{H}$ is geodesic. Here, $\tilde{S}$ causally separates $\tilde{H}$ means that there exists a diffeomorphism $\tilde{\psi}: \tilde{H} \to \tilde{S} \times \mathbb{R}$ such that, for each $x \in \tilde{S}$, $\tilde{\psi}^{-1}(\{x\} \times \mathbb{R})$ is a null generator of $\iota_{\tilde{H}}: \tilde{H} \to M$ (Cf. theorem (3.1)6)

Thus, the null hypersurface $\tilde{H}$ constructed from our original $f \in \mathrm{Emb}_S(S,M)$ is a geodesic hypersurface. Let $\{l,n\}$ be any null trivialization of $N(f)$. Then $\{l,n\}$ is unique up to an element of $C^+(S)$. Denote by $\tilde{l}$ the extension of l to a future directed null geodesic vector field on the null hypersurface $\tilde{H}$ and denote by $\{\phi_u: u \in I \subseteq \mathbb{R}\}$ the local 1-parameter group of (local) diffeomorphisms generated by $\tilde{l} \in \Gamma(N(\iota_{\tilde{H}})) \subseteq D\iota_{\tilde{H}}.\mathrm{Vect}(\tilde{H}) \equiv \mathrm{Vect}(\tilde{H})$. Since $f(S)$ is an embedded submanifold of $\tilde{H}$, we may define the embedding $f_u = \phi_u \circ f: S \hookrightarrow \tilde{H}$, for each $u \in I$. The map $u \mapsto f_u$ is thus a curve of spacelike embeddings of S in (M,g). We also have the diffeomorphism $F: S \times I \to \tilde{H}; (x,u) \mapsto f_u(x) \equiv \phi_u(f(x))$, and F is then an element of $\mathrm{Emb}_N(S \times I, M)$, where $\mathrm{Emb}_N(S \times I, M)$ denotes the space of null embeddings of $S \times I$ in (M,g). $\mathrm{Emb}_N(S \times I, M)$ may be regarded as a boundary of $\mathrm{Emb}_S(S \times I, M)$ in $\mathrm{Emb}(S \times I, M)$.

Note that $\tilde{H} = \bigcup_{u\in I} f_u(S)$ and let $E_f = \bigcup_{u\in I} Df_u.TS$. We regard E_f as a subbundle of $T\tilde{H}$ by defining the projection $\tau_f\colon E_f \to \tilde{H}$; $Df_u(x).v \mapsto f_u(x)$. Hence, E_f is also a subbundle of $\iota_{\tilde{H}}^*(TM)$, and we may define the g-orthogonal complement $E_f^\perp$ of E_f in $\iota_{\tilde{H}}^*(TM)$. Thus, E_f and $E_f^\perp$ are rank two vector bundles over $\tilde{H}$ with spacelike and timelike fibres respectively.

The vector bundle $E_f \xrightarrow{\tau_f} \tilde{H}$ may be related to the vector bundle $\rho_F\colon R(F) \to S\times I$. Here, $F\colon S\times I \to M$; $(x,u) \mapsto f_u(x)$ and $R(F)$ is defined by (3.1)7. The image of F is $\tilde{H} \subseteq M$, and, for all $(x,u) \in S\times I$ and $(v,t) \in T_{(x,u)}(S\times I) \cong T_xS \oplus \mathbb{R}$, we have

$$DF(x,u).(v,t) = Df_u(x).v + \frac{df_u}{du}(x).t = Df_u(x).v + t\tilde{l}(f_u(x))$$
$$\in T_{f_u(x)}\tilde{H} \subseteq T_{f_u(x)}M.$$

Here, we have used the fact that $f_u = \phi_u \circ f$, where $\{\phi_u\}$ is the local 1-parameter group of diffeomorphisms generated by $\tilde{l} \in \mathrm{Vect}(\tilde{H})$. We also have that $\nu_F^{-1}(x,u) = \{t\tilde{l}(f_u(x))\colon t\in\mathbb{R}\}$, so that the fibre of the vector bundle $R(F) \equiv DF.T(S\times I)/N(F)$ at $(x,u) \in S\times I$ is naturally isomorphic to $\{Df_u(x).v\colon v \in T_xS\}$ $= Df_u(x).T_xS$. Hence, $R(F) = \bigcup_{(x,u)} Df_u(x).T_xS = \bigcup_u Df_u.TS$, and $\rho_F\colon R(F) \to S\times I$; $Df_u(x).v \mapsto (x,u)$.

We have $\tau_f\colon E_f \to \tilde{H}$ and $F\colon S\times I \to \tilde{H}$, so we may construct the pullback bundle $F^*\tau_f\colon F^*E_f \to S\times I$. In fact, F^*E_f $= \{((x,u),Df_u(x).v)\colon (x,u) \in S\times I \text{ and } v \in T_xS\}$ with $(F^*\tau_f)((x,u),Df_u(x).v) = (x,u) = \rho_F(Df_u(x).v)$. Hence, the map $\Psi\colon F^*E_f \to R(F)$; $((x,u),Df_u(x).v) \mapsto Df_u(x).v$ is an isomorphism of vector bundles over $S\times I$. We summarize the above in the following:

<u>Proposition (3.2)1</u>: Let $f \in \mathrm{Emb}_S(S,M)$ and $\{l,n\}$ a null trivialization of $N(f)$. Let $F \in \mathrm{Emb}_N(S\times I,M)$ be the codimension one

geodesic null embedding constructed from $(f, \{l,n\})$ as above and let $\tilde{H} \equiv F(S \times I)$ denote the corresponding null hypersurface in (M,g). Let $\rho_F: R(F) \to S \times I$ and $\tau_f: E_f \to \tilde{H}$ be the rank two vector bundles defined above. Then $R(F)$ is naturally isomorphic to $F^* E_f$.

We make the following remarks:- (i) $T\tilde{H} = E_f \oplus N(\iota_{\tilde{H}})$; (ii) The Riemannian structure $\bar{g}$ in the vector bundle $R(F)$ may be transferred to $F^* E_f$ using Ψ and, since $F: S \times I \to \tilde{H}$ is a diffeomorphism, we also obtain a Riemannian structure in the vector bundle $\tau_f: E_f \to \tilde{H}$.

Since E_f is a subbundle of $T\tilde{H}$, then $N(\iota_{\tilde{H}})$ is a subbundle of $E_f^{\perp}$. Hence, $\tilde{l} \in \Gamma(N(\iota_{\tilde{H}}))$ is a section of $E_f^{\perp}$. We may define another null section $\tilde{n}$ of $E_f^{\perp}$ by requiring that $(g \circ \iota_{\tilde{H}})(\tilde{l},\tilde{n}) = 1$. By definition, $\tilde{n} \notin \Gamma(N(\iota_{\tilde{H}}))$ (since $\tilde{l} \in \mathrm{Vect}(\tilde{H})$), and therefore $\{\tilde{l},\tilde{n}\}$ provides a trivialization of the vector bundle $E_f^{\perp}$ (we may regard $E_f^{\perp}$ as being obtained by Lie transporting over $\tilde{H}$ the $\{l,n\}$-trivialized vector bundle $N(f)$ using the null vector field $\tilde{l}$).

We summarize the various Whitney sum decompositions as follows:- $N(f) = L^{out}(f) \oplus L^{in}(f)$; $T\tilde{H} = E_f \oplus N(\iota_{\tilde{H}})$; $\iota_{\tilde{H}}^*(TM) = E_f \oplus E_f^{\perp}$; and $E_f^{\perp} = N(\iota_{\tilde{H}}) \oplus N'(\iota_{\tilde{H}})$, where $N'(\iota_{\tilde{H}}) = \{t\tilde{n}(f_u(x)):\ t \in \mathbb{R}, (x,u) \in S \times I\}$ with projection $\nu'_{\tilde{H}}: N'(\iota_{\tilde{H}}) \to \tilde{H}$ given by $t\tilde{n}(f_u(x)) \to f_u(x)$. Thus, $N(\iota_{\tilde{H}})$ and $N'(\iota_{\tilde{H}})$ are obtained by Lie transporting over $\tilde{H}$, $L^{out}(f)$ and $L^{in}(f)$ respectively, using $\tilde{l} \in \Gamma(N(\iota_{\tilde{H}})) \subseteq \mathrm{Vect}(\tilde{H})$. Obviously, $\tilde{n}$ is the extension of n to a future directed null section of $E_f^{\perp}$ with $(g \circ \iota_{\tilde{H}})(\tilde{l},\tilde{n}) = 1$.

The initial data $(f, \{l,n\})$ has yielded in a unique way the null embedding $F: S \times I \hookrightarrow M$ together with the null vector fields $\{\tilde{l},\tilde{n}\}$. By construction, $\tilde{H} = F(S \times I)$ is foliated by spacelike

2-surfaces. We remark that Kupeli [K 16] gives necessary and sufficient conditions for the foliation of a null hypersurface in a spacetime by spacelike 2-surfaces. These conditions are stated in terms of the self-adjointness of the E_f-component of the covariant derivative of $\tilde{n}$.

We now consider the problem of defining useful concepts on the null hypersurface $\tilde{H}$ (constructed from $(f,\{l,n\})$) analogous to concepts which may be defined on a spacelike hypersurface. The concepts we have in mind are those which, a priori, require a well defined projection h; for example, the Sen-Witten operator (see definition (2.3)8). We perform the null construction by taking a limit of the analogous spacelike constructions. In order to do this, it is first necessary to define a 1-parameter family of spacelike embeddings whose limit is the given null embedding F in some sense:-

Let $\delta > 0$ and $s \in (0,\delta)$. Define $L_s = \bigcup_{x \in S} \{t(s^{-1}l(x) + s\,n(x)):t \in \mathbb{R}\}$ and $\lambda_s\colon L_s \to S$; $t(s^{-1}l(x) + sn(x)) \mapsto x$. Then L_s is a line subbundle of $N(f)$. Moreover, the fibres of L_s are timelike, since $\|t(s^{-1}l + sn)\|^2 = 2t^2 > 0$ for $t \neq 0$. Let $L_s^{\perp}$ denote the g-orthogonal complement of L_s in $f^*(TM)$, so that $L_s^{\perp}$ is a rank three spacelike vector subbundle of $f^*(TM)$ and define a spacelike line subbundle W_s of $L_s^{\perp}$ by $W_s = L_s^{\perp} \cap N(f) = \bigcup_{x\in S} \{t(l(x) - s^2n(x)):t \in \mathbb{R}\}$ with obvious projection onto S.

Now consider the pullback of the exponential map of g, namely $(\tau_M^*f)^*\exp_g\colon (\tau_M^*f)^{-1}(V) \to M$, where V is the open neighbourhood of the zero section in TM referred to above. Let $U_s = W_s \cap (\tau_M^*f)^{-1}(V)$, so that U_s is an open neighbourhood of the zero section in the line bundle W_s over S. Now we define $\tilde{H}'_s = ((\tau_M^* f)^* \exp_g)(U_s)$. Then $\tilde{H}'_s$ is an embedded spacelike hypersurface of (M,g) which contains $f(S)$ as a codimension one embedded submanifold.

Recall that we have the future directed null sections $\tilde{l}$, $\tilde{n}$ of $\iota_{\tilde{H}}^*(TM)$. We may extend $\tilde{l}$, $\tilde{n}$ to future directed null sections, also denoted $\tilde{l}$, $\tilde{n}$, of TM defined on an open neighbourhood of $\tilde{H}$ in M such that $g(\tilde{l},\tilde{n}) = 1$. In particular, the extended vector fields $\tilde{l}$, $\tilde{n}$ are defined on a neighbourhood of $f(S)$ in M, and hence on a neighbourhood $\tilde{H}_s$ of $f(S)$ in $\tilde{H}'_s$. The 3-manifold $\tilde{H}_s$ is also an embedded spacelike hypersurface of (M,g) which contains $f(S)$ as a codimension one embedded submanifold.

We define the unit tangent vector field $\tilde{k}$ on $\tilde{H}_s$ by $\tilde{k} = \frac{1}{\sqrt{2}}(s^{-1}\tilde{l} - s\,\tilde{n})\circ\iota_s$, where $\iota_s \equiv \iota_{\tilde{H}_s}: \tilde{H}_s \hookrightarrow M$ is inclusion, and denote by $\{\psi_q : q \in J \subseteq \mathbb{R}\}$ the local 1-parameter group of (local) diffeomorphisms generated by $\tilde{k} \in D\iota_s.\mathrm{Vect}(\tilde{H}_s) \equiv \mathrm{Vect}(\tilde{H}_s)$. Since $f(S)$ is an embedded submanifold of $\tilde{H}_s$, we may define the embedding $f_q = \psi_q \circ f: S \hookrightarrow \tilde{H}_s$, for each $q \in J$. We then have the diffeomorphism $F_s: S\times J \longrightarrow \tilde{H}_s$; $(x,q) \longmapsto f_q(x) \equiv \psi_q(f(x))$ and, by definition, $F_s \in \mathrm{Emb}_S(S\times J, M)$.

Since I and J are open with non-empty intersection, we may, without loss of generality, take $J = I$. We then have $F \in \mathrm{Emb}_N(S\times I, M)$ and also the curve $\mathcal{F}: (0,\delta) \longrightarrow \mathrm{Emb}_S(S\times I, M)$. For each $s \in (0,\delta)$, the unit normal of F_s is given by $t_s = \frac{1}{\sqrt{2}}(s^{-1}\tilde{l} + s\tilde{n})\circ F_s$. Note that the component of the normal of F_s in the $\tilde{n}$-direction vanishes as $s \longrightarrow 0$, so that the normal is entirely in the $\tilde{l}$-direction in the limit. For this reason, we regard $\{F_s : s \in (0,\delta)\}$ as a 1-parameter family of codimension one spacelike embeddings with the codimension one null embedding F as null limit.

The idea of null limit provides a technique for obtaining null versions of spacelike concepts. For example, suppose, for $s \in (0,\delta)$, Q_s is some expression arising from the spacelike embedding F_s, and constructed using the (non-degenerate) projection h_s. A corresponding quantity for the null embedding F does not exist a priori since there is no well defined projection. However, we may obtain an analogous null version of the Q_s by taking the limit of Q_s as $s \rightarrow 0$. We say that Q is the null limit of $\{Q_s\}$.

In section 3.4, we use the null limit technique to obtain a version of the Sen-Witten equation (2.3.12) which may be utilized on null, rather than spacelike, hypersurfaces. Other possible applications include the characteristic initial value problem (where the limit of the spacelike constraints is taken), null canonical quantization and also the study of the interaction between asymptotic structure at spacelike infinity on the one hand and at null infinity on the other.

Since our main use of the null limit technique is to show how the null limit of the Sen-Witten equation is an essential ingredient in the Ludvigsen-Vickers definition of quasi-local momentum, we first give a brief overview of momentum in general relativity.

3.3 Gravitational Momentum

In this section, we present an overview of momentum in general relativity. We outline the importance of gravitational momentum, the problems encountered when attempting to define it and also the various approaches to solving these problems. In the next section, we continue our discussion with a description of quasi-local momentum in general relativity, emphasizing the importance of spinor and null embedding ideas.

Let us first consider the general framework of kinematical quantities or moments and the reasons why this is important in the study of a physical system. The central principle is Noether's theorem which implies the existence of conservation laws for a Hamiltonian system with symmetry. It is very useful to be able to characterize a system using a small number of conserved parameters, for example momentum, angular momentum and other "charges" arising from Yang-Mills theories.

The starting point is a phase space with symmetry. Geometrically, we have a symplectic manifold (P, ω) on which a Lie group G acts by symplectomorphisms. A moment for the G-action is then a smooth map $j: P \longrightarrow (LG)^*$ such that $d(\hat{j}(\xi)) = \iota_{\xi_P}\omega$, for all $\xi \in LG$. Here, $\hat{j} : LG \longrightarrow C(P)$ is the map dual to j and $\xi_P \in Vect(P)$ is the infinitesimal generator corresponding to ξ, for all $\xi \in LG$. In other words, each infinitesimal generator ξ_P has $\hat{j}(\xi)$ as a Hamiltonian function. Note that a moment, if it exists, is defined up to an element of $(LG)^*$.

If the Hamiltonian $H \in C(P)$ is invariant under the action of G, then any moment is a constant of the motion for the Hamiltonian flow. This is a version of Noether's theorem and demonstrates the sense in which symmetries of a Hamiltonian system lead to conserved quantities. In physics, the phase space usually consists of an infinite dimensional manifold of fields, which is equipped with a weak symplectic form. Moments arising from a symmetry may often be interpreted as fluxes of physical quantities. For more details concerning Hamiltonian systems with symmetry, we refer the reader to Abraham and Marsden [A 2].

It is useful to compare the situation for a special relativistic field theory on the one hand with general relativity on the other:- In special relativity we have Minkowski spacetime $(\mathbb{R}^4, \eta)$ as an arena in which physics takes place. We also have a group action, namely that of the Poincaré group $O(1,3) \ltimes \mathbb{R}^4$ which acts by isometries on Minkowski spacetime. For simplicity, let us consider the case of massless scalar field theory. The phase space may be taken to be the space P of η-harmonic functions of compact support equipped with the weak (constant) symplectic form ω given by:

$$\omega(f_1, f_2) = \int_H (f_1 {*df_2} - f_2 {*df_1}) \qquad 3.3.1,$$

for all $f_1, f_2 \in P$. Here, $*$ is the Hodge dual arising from η and $\tilde{H}$ is any Cauchy surface in $(\mathbb{R}^4, \eta)$. Since f_1 and f_2 are η-harmonic, the symplectic form ω is independent of the choice of H.

The action of the Poincaré group on Minkowski spacetime induces a symplectomorphic action on (P, ω) in the obvious manner. The corresponding moment may be given in terms of the <u>stress-energy-momentum tensor</u> for the massless scalar field theory:- For $f \in P$, let $T_f \equiv df \otimes df - \frac{1}{2}\|df\|^2 \eta \in S_2(\mathbb{R}^4)$ denote the corresponding stress-energy-momentum tensor. Consider $\xi \in L$ Poincaré (regarded as a subalgebra of $\mathrm{Vect}(\mathbb{R}^4)$) and define $J \equiv J_f(\xi) = T_f(\xi, \cdot) \in \Omega^1(\mathbb{R}^4)$. Then, since T_f is divergence free and ξ is a Killing vector field of $(\mathbb{R}^4, \eta)$, we have $\delta J = 0$. Thus the 1-form J may be regarded as the <u>conserved current</u> describing the particular momentum or angular momentum component associated with ξ. In fact, it may be shown (see Abraham and Marsden [A ℒ]) that the moment for the action of the Poincaré group on (P,ω) is given (up to a constant) by:

$$\langle j(f),\xi\rangle = \int_H *J_f(\xi) \qquad 3.3.2,$$

for all $\xi \in L$ Poincaré and $f \in P$. Thus the moment gives the total net "charge" intercepted by the spacelike hypersurface H. Alternatively, we may interpret 3.3.2 as giving the flux through H of the momentum or angular momentum component associated with the particular Lie algebra element ξ chosen.

We may generalize the above discussion to other Poincaré invariant field theories on Minkowski spacetime. For example, we could take Maxwell theory (or, more generally, a Yang-Mills theory) or a fluids theory. In each case, we have the stress-energy-momentum tensor T_Φ for each field Φ in the phase space and the moments are given by an equation similar to 3.3.2.

It is very useful to rewrite equation 3.3.2 as one involving a 2-surface integral rather than a hypersurface integral. Suppose $S = \partial H'$, where H' is some region of H. Suppose also that there exists $F \equiv F_\Phi(\xi) \in \Omega^2(\mathbb{R}^4)$ such that $\delta F = J$ $(\equiv J_\Phi(\xi))$. Then the integral in equation 3.3.2 may be rewritten as $\int_S *F$ (using Stokes' theorem). This 2-surface integral may be regarded as giving the total charge surrounded, or <u>linked</u>, by the spacelike 2-surface S. we write:

$$Q_\Phi(\xi;S) = \int_S *F_\Phi(\xi) \qquad 3.3.3.$$

Equation 3.3.3 is regarded as a basic ingredient in the calculation of conserved kinematical quantities. Conservation of the momentum or angular momentum component corresponding to the Lie algebra element ξ shows up in two ways:- Firstly, the value of $Q \equiv Q_\Phi(\xi;S)$ is independent of the spacelike hypersurface chosen to span S, i.e. the total charge intercepted by an "earlier" such hypersurface is equal to that intercepted by a "later" one. Secondly, suppose S is deformed continuously outside the support of T_Φ. Then the value of Q does not change.

Ideally, we would like to be able to write down an expression analogous to 3.3.3 for kinematical quantities in general relativity: Minkowski spacetime $(\mathbb{R}^4,\eta)$ is replaced by a general vacuum (or possibly non-vacuum) spacetime (M,g) and instead of calculating moments of the special relativistic field Φ, we wish to calculate gravitational moments. If an equation like 3.3.3 could be written down for gravity, we would still require that the charge integral be independent of any particular choice of hypersurface spanning the 2-surface S. However, we cannot hope for a conservation law in the second stronger sense; the gravitational field in empty space carries energy (and other kinematical quantities) and therefore even an empty space continuous deformation of the 2-surface S will change the value of the charge integral. We consider this problem of non-localizability of gravitational energy shortly.

Let us now consider the factors inherent in the theory of general relativity which cause problems when an attempt is made to write down an expression such as 3.3.3.

A basic problem is the choice of phase space and symmetry group. For special relativistic field theories, we have Minkowski spacetime $(\mathbb{R}^4,\eta)$ which plays the rôle of an arena in which physics takes place. The phase space is a space of fields on this arena and the Poincaré group is a group of symmetries of the phase space. The arena and group are "universal" in that they do not depend on the particular solution of the field theory under consideration. What arena do we use for general relativity? Spacetime (M,g) itself is no good because, even after fixing the background manifold M, the metric g is a particular solution of the theory and so certainly cannot be regarded as providing a universal geometry like the Minkowski metric η does in special relativistic theories. Even if we do dispense with the idea of universal arena and focus attention on a particular solution (M,g), we still have no analogue of the Poincaré group – a generic metric g possesses trivial isometry group (see section 4.1). Thus, we have no Killing vector fields ξ to plug into equation 3.3.3.

The fact that there exists no universal group analogous to the Poincaré group means that it is not entirely clear which kinematical quantities make sense in general relativity. In special relativity, we know that the elements of L Poincaré give rise to specific kinematical quantities. For example, a translation generator gives rise to a component of 4-momentum and, once a Lorentz subalgebra has been picked out by choosing an origin, a rotation generator gives rise to a component of angular momentum. Without a group with such a physical interpretation as the Poincaré group in special relativity (or the Galileo group in Newtonian theory), it is difficult to give <u>mathematical</u> reasons for the existence of given moments such as energy, momentum and angular momentum. One way out of this problem is to restrict the class of spacetimes by imposing <u>physically</u> reasonable requirements. For instance we could consider isolated systems only - this leads to a consideration of asymptotically flat spacetimes for which well defined and physically meaningful universal asymptotic symmetry groups do exist. We return to this point below.

In addition to the problem caused by the absence of a universal phase space and symmetry group, there also exists another factor inherent to gravitational theory which prevents the existence of a straightforward analogue of equation 3.3.3. Note that the construction of J_Φ (and hence F_Φ) involves the local tensor field T_Φ - the stress-energy-momentum tensor of the field Φ. No such local moment density exists in gravitational theory due to the non-localizability of gravitational momentum:-

From a foundational viewpoint we have the Principle of Equivalence, a cornerstone of general relativity and of other theories of gravity. The Principle of Equivalence postulates the existence of local inertial frames and hence the non-existence of local gravitational fields. The absence of local gravitational fields implies that gravitational momentum cannot be localized. Thus, the localization of gravitational momentum is forbidden by the Principle of Equivalence.

In the model of gravitation provided by the theory of general relativity, we may give an alternative reason for the non-existence of a local momentum density:- On dimensional grounds, any such density should depend only on the (1-jet of the metric $g)^2$. However, on a Lorentzian manifold (M,g), there does not exist any non-zero field constructed solely from the 1-jet of the metric. Thus, we do not expect to find a local momentum density in general relativity theory.

On the other hand, gravitational momentum certainly exists:- In the Newtonian limit of general relativity theory, we know that the empty space gravitational field carries energy. For example, the gravitational potential energy contributes (negatively) to the total mass-energy of a system; the total mass-energy of two gravitating bodies instantaneously at rest with respect to one another is less than the sum of the mass-energies of the individual bodies by an amount equal to the gravitational potential energy of interaction.

Another manifestation of the empty space gravitational non-local contribution to momentum is given by the existence of gravitational waves - assuming that we believe the observational evidence; for example, that which seems to imply the speeding up of the binary pulsar (see Taylor [T 8]).

The conclusion that we draw from the above discussion is that if charge integrals analogous to the one given by equation 3.3.3 exist in the theory of general relativity, then the integrand cannot be constructed from a local moment density as is the case in special relativistic field theory. There exists no local quantity which describes the gravitational contribution to the total momentum.

We now consider the various approaches that have been made to solving the problem of momentum in general relativity theory. It is clear that any good approach should address both the question of phase space and symmetry group and also the question of non-localizability. We should be guided by clear physical and geometric principles.

Historically, the first approaches to the problem were neither physical nor geometric; in the early days, gravitational momentum was described by means of a "pseudo-tensor" t_{ab}. This was a coordinate-dependent quantity such that the coordinate-divergence of $T_{ab} + t_{ab}$ vanished. This led to an integral conservation law for gravitational and matter field momentum. The non-localizability of gravitational momentum manifested itself in the fact that, at any given spacetime event, the quantity measuring gravitational momentum density could be reduced to zero by a suitable choice of local coordinates. The only physically meaningful quantity was the total momentum, obtained by integrating the non-tensorial objects out at infinity. Even when taken to infinity, coordinate dependence remained, so from a geometric point of view, there was a basic obstacle to giving a suitable interpretation of these expressions. For more details concerning these early pseudo-tensorial attempts, see Einstein [E 9], Tolman [T 7], Landau and Lifshitz [L 1] and Møller [M 12].

In the late 1950's, new, more geometrical alternatives were suggested. These were due to Bel and Robinson (see Bel [B 6]) and to Komar (see [K 9]), and both of these approaches have had an impact on later work in the area, particularly in quasi-local definitions (see section 3.4).

The Bel-Robinson approach was an attempt to obtain an energy-density-like quantity for gravity based on an analogy with Maxwell electrodynamics. The basic idea was to consider the Bel-Robinson tensor field $BR(g) \in S_4(M)$ associated with a spacetime (M,g). Using abstract indices, $BR(g)$ is given by

$$BR_{abcd} = C_{amcn}C_{b\ d}^{\ m\ n} + {}^*C_{amcn}{}^*C_{b\ d}^{\ m\ n} \qquad 3.3.4,$$

where C_{abcd} is the Weyl tensor of g and ${}^*C_{abcd} \equiv \frac{1}{2}\varepsilon_{abmn}C^{mn}{}_{cd}$ is the dual of the Weyl tensor. For vacuum spacetimes, the Bel-Robinson tensor is totally symmetric, trace-free, and divergence-free. Thus, the Bel-Robinson tensor possesses similar properties to the stress-energy-momentum tensor T_F of a Maxwell field.

A further analogy with electrodynamics is obtained if a unit timelike vector field t is chosen. Then the Weyl tensor is uniquely determined by its electric and magnetic components:

$$E_{ab} = C_{ambn}t^m t^n \qquad 3.3.5,$$

$$B_{ab} = {}^*C_{ambn}t^m t^n \qquad 3.3.6.$$

E and B are symmetric, trace-free, and spatial with respect to t. If we evaluate the Bel-Robinson tensor on t, we obtain:

$$BR(g)(t,t,t,t) = \| E \|^2 + \| B \|^2 \qquad 3.3.7,$$

where the pointwise norm is the one induced by g. Thus, the timelike component of $BR(g)$ is non-negative and equal to zero if and only if (M,g) is conformally flat. Equation 3.3.7 should be compared with the analogous equation $T_F(t,t) = \| E \|^2 + \| B \|^2$ for Maxwell theory. Here, E and B are respectively the electric and magnetic components of the electromagnetic 2-form F with respect to the unit timelike vector field t. The fact that there is such a strong resemblance between the Bel-Robinson tensor and the Maxwell stress-energy-momentum tensor leads to the possibility of regarding $BR(g)$ as some kind of "stress-energy-momentum tensor for the spacetime (M,g)". However, the Bel-Robinson tensor does not have the correct dimensions; in geometrized units, momentum density has dimensions $(\text{length})^{-2}$, whereas $BR(g)$ has dimensions $(\text{length})^{-4}$. Hence, there can be no direct interpretation of $BR(g)$ as a gravitational stress-energy-momentum tensor.

It turns out, however, that the Bel-Robinson tensor is important in certain aspects of quasi-local energy in general relativity (see section 3.4). In particular, BR(g) makes an appearance in the "small 2-surface" limit of both the Ludvigsen-Vickers quasi-local energy (see Bergqvist and Ludvigsen [B 9]) and also in the Hawking quasi-local energy (see Horowitz and Schmidt [H 13]). In both cases, in the absence of matter, the leading-order (r^5) component of the quasi-local energy is determined by the Bel-Robinson tensor. These results for the Ludvigsen-Vickers and Hawking quasi-local energies should be contrasted with the result for the Penrose quasi-local energy; in this case, the energy vanishes at the fifth order in a vacuum spacetime (see Kelly et al. [K 2]).

Thus, in the Ludvigsen-Vickers and Hawking definitions of quasi-local gravitational energy, the Bel-Robinson tensor may be viewed as a measure of the gravitational energy per unit $(\text{length})^5$. It is not an energy density, but it is the dominant contribution to these quasi-local energy integrals in the absence of matter.

The second major contribution to gravitational kinematics in the late fifties came from Komar [K 9]. Komar constructed covariant conservation laws in general relativity in certain special cases and his work may be regarded as a prototype quasi-local approach. The linkage framework of Geroch-Tamburino-Winicour (see Tamburino and Winicour [T 1], Winicour [W 7] and Geroch and Winicour [G 6]) also owes much to the Komar approach.

We now present a brief description of the Komar kinematical quantities. Let (M,g) be a spacetime and S an embedded spacelike 2-surface in (M,g). Our starting point is the diffeomorphism group Diff(M) regarded as a symmetry group in the sense that we shall write down a charge integral such as 3.3.3 which defines an element of the dual of LDiff(M) = Vect(M). It turns out, however, that for physically meaningful quantities to be defined, it is necessary either to restrict to Killing vector fields for g (of which there may be no non-trivial ones, of course) or to consider asymptotically embedded 2-surfaces S.

Define the linear map $J: \mathrm{Vect}(M) \to \Omega^1(M)$ by

$$J(\xi) \quad = \quad \delta d\xi^b \qquad 3.3.8,$$

for all $\xi \in \mathrm{Vect}(M)$. Now let H be any spacelike hypersurface in (M,g) with $\partial H = S$, and define the <u>Komar integral</u> $Q(\cdot;S) \in \mathrm{Vect}(M)^*$ by:

$$Q(\xi;S) \quad = \quad \int_H *J(\xi) \qquad 3.3.9,$$

for all $\xi \in \mathrm{Vect}(M)$. Note that $Q(\cdot;S)$ depends only on S and not on the choice of spanning hypersurface:- Suppose H_1, H_2 are spacelike hypersurfaces spanning the given spacelike 2-surface S. Then $\int_{H_1} *J(\xi) - \int_{H_2} *J(\xi) = \int_V d*J(\xi)$ (where V is the four-dimensional region in M with $\partial V = H_1 \cup (-H_2)) = \int_V d*\delta d\xi^b = 0$, for all $\xi \in \mathrm{Vect}(M)$. Hence, $Q(\cdot;S)$ depends only on S and is thus conserved.

We may write $Q(\cdot;S)$ as an explicit 2-surface integral over S as follows:- Choose any spanning hypersurface H and let $\xi \in \mathrm{Vect}(M)$. Then, $Q(\xi;S) = \int_H *\delta d\xi^b = \int_H d*d\xi^b = \int_S *d\xi^b$. Hence, the Komar integral may be written in the charge integral form as:

$$Q(\xi;S) \quad = \quad \int_S *d\xi^b \qquad 3.3.10,$$

for all $\xi \in \mathrm{Vect}(M)$ (the expression given by Komar in equation (3.7) of [K 9] differs from 3.3.10 by a factor of $(16\pi G)^{-1}$, where G is the gravitational constant).

In order to extract physically meaningful information from the Komar integral, we must assume that g is non-generic, i.e. g admits a non-trivial Killing vector field ξ. In addition, we assume that S lies entirely inside the matter-free region of spacetime. It then follows that $Q(\xi;S)$ is independent of S:- Suppose S_1 and S_2 are embedded spacelike 2-surfaces in the matter-free region and suppose H is an embedded 3-manifold with $\partial H = S_1 \cup (-S_2)$. Then, $Q(\xi;S_2) - Q(\xi;S_1) = \int_{\partial H} {*d\xi^b} = \int_H {*\delta d\xi^b} = 0$, since $\nabla^2\xi^b + g(\xi, R(\cdot,\cdot)(\cdot)) = 0$ $(R \equiv \mathrm{Riem}(g))$ and $\mathrm{Ric}(g) = 0$. Hence, $Q(\xi;S)$ is independent of S.

The Komar integral now leads to satisfactory moments. For example, suppose (M,g) is stationary with timelike Killing vector field ξ generating time translations. Then $Q(\xi;S)$ is ($16\pi G$ times) the mass of the spacetime (provided that S lies outside the matter region). If (M,g) is axisymmetric with spacelike Killing vector field ξ generating rotations, then $Q(\xi;S)$ is ($16\pi G$ times) the component of angular momentum associated with ξ.

It is important to note that the Komar integral is performed over a 2-surface which lies entirely in the matter-free region. If $\mathrm{Ric}(g)|S$ is not identically zero, then the Komar integral exhibits certain anomalous behaviour as was noted by Tod [T 3], who showed that the Komar integral does not give the correct answer for the mass of the Reissner-Nordstrom spacetime (by "correct answer", we mean the one which agrees with linearized theory - the agreement with linearized theory is a very important constraint for definitions of kinematical quantities in general relativity theory).

The Komar integral cannot be used in a generic spacetime in a meaningful way. However, for an asymptotically flat spacetime, the existence of the BMS group at null infinity (see appendix 6.3) means that the Komar integral at null infinity is physically appropriate and it reduces to the Bondi-Sachs momentum (see Tamburino and Winicour [T 1]). We now discuss isolated systems and asymptotic kinematics in more detail:-

The above remarks concerning the problems encountered when attempting to define gravitational moments have made it clear that the absence of a universal arena and symmetry group means that we should restrict our attention to a restricted class of spacetimes in order to circumvent some of the problems. One such restricted class which has strong physical motivation is that consisting of isolated systems.

Many theories in physics admit a class of solutions which may be regarded as representing isolated systems. For example, in Newtonian gravitational theory, an isolated system is one whose mass density possesses spatially compact support and also has asymptotically vanishing gravitational potential. Although such isolated systems are not expected to represent the Universe in every detail, they are very useful in that they are a good approximation to certain subsystems of the Universe encountered in the physical world; for example, in Newtonian theory, our solar system. Indeed, it is only through a useful notion of isolated system that we acquire the ability to describe various subsystems of the Universe - in particular, to characterize these subsystems using parameters such as momentum, angular momentum and charge.

In general relativity theory, examples of isolated systems include stars and black holes, and a study of the solutions of the Einstein equations representing such systems has made a very important contribution to our understanding of the structure of the theory. It turns out that a very useful definition of isolated system is essentially that of asymptotic flatness. This idea encapsulates the notion that the spacetime metric "approaches some flat metric far from the sources of the gravitational field". The current definition of asymptotic flatness is both geometrical and physically reasonable and much of the important work in general relativity done in the 1960's fits neatly into this framework.

There are three distinct regimes in which asymptotic flatness may be considered. These correspond to passage from the isolated sources to infinity in spacelike directions, in null directions and in timelike directions. The timelike case has little interest for us in a discussion of gravitational momenta, although it may play a rôle in cosmology. The spacelike and null cases are both very important in the definition of moments in general relativity, although for us, the null case is more important. In fact, it is fair to say that, from a physical point of view, the null asymptotic structure of an isolated system is more important than is the spacelike asymptotic structure. We summarize the important properties of (null) asymptotically flat spacetimes in appendix 6.3.

From both a mathematical and physical viewpoint, the null and spacelike regimes have certain important features in common. For example, in both cases, the "boundary at infinity" may be detached from the original physical spacetime and the boundary becomes a universal asymptotic arena, i.e. it doesn't depend on the particular asymptotically flat spacetime. Various spacetime fields induce corresponding fields on the boundary and these fields are classified as either universal or physical. The former, together with the boundary space itself, provide the universal asymptotic structure. The physical fields provide asymptotic information about the physical spacetime. This splitting up of fields into two classes solves a basic problem always present in the theory of general relativity, namely the fact that the spacetime metric is both a geometrical and a physical object. Extracting physical information without a non-dynamical background is difficult, but the asymptotic splitting up of the fields provides a means of getting hold of physically meaningful information which was hitherto unobtainable.

Having obtained the universal arena (i.e. the boundary at infinity equipped with the geometrical fields), it is then possible to define a universal symmetry group, namely the group of diffeomorphisms of the boundary which leave invariant the universal fields. The existence of the asymptotic symmetry group leads to the possibility of defining moments. These moments may be interpreted physically in terms of the knowledge they give us concerning the original physical spacetime. The physical interpretation of the moments is arrived at by either evaluating the moments for particular spacetimes for which we can be "sure" of the meaning of the moments (e.g. for stationary spacetimes) or by comparing the asymptotic symmetry group with the Poincaré group. It turns out that the asymptotic symmetry group is similar in structure to the

Poincaré group, but with a less recognizable translation subgroup. Since we know which physically interesting moments correspond to the various Poincaré Lie algebra elements, we may use the resemblance of the asymptotic symmetry group to the Poincaré group to give physical meaning to at least some of the asymptotic moments.

In addition to the similarities between null and spacelike asymptotics, there exist important differences. In the null case, the asymptotic boundary is a null submanifold of the compactified spacetime whereas in the spacelike case, the boundary is timelike. From a physical viewpoint, information may travel from the physical spacetime to reach null infinity, but not spatial infinity. Hence, null infinity may be used to study the dynamics of the isolated system, whereas at spatial infinity, there is no dynamical information. This state of affairs is reflected in the behaviour of the masses defined in the two cases; the ADM mass (see Arnowitt et al. [A 6], [A 7], [A 8]) defined at spatial infinity is a fixed number representing the total mass of the spacetime, but the Bondi mass (see Bondi et al. [B 15]) is a dynamical quantity representing the mass at a particular retarded time. Indeed, in the null case, we have a formula expressing the retarded time rate of change of the Bondi mass in terms of the mass lost by radiation to null infinity.

The asymptotic boundary at spatial infinity, being a timelike submanifold, is equipped with a non-degenerate metric and the symmetry group is just the Lorentz group. On the other hand, the metric in the null case is degenerate and the asymptotic symmetry group is the infinite dimensional BMS group (see definition (6.3)3). Thus, in the spacelike case, the translation subgroup has been reduced from four dimensions (for the Poincaré group) to zero, whilst in the null case, the dimension has increased from four to infinity. The infinite dimensional translation group does play a physical rôle at null infinity, whereas translations play no rôle at spatial infinity.

Our main interest is in spacetimes asymptotically flat at null infinity since these are physically more interesting and they also provide a class of spacetimes for which the Ludvigsen-Vickers quasi-local momentum may be defined (see section 3.4). For a thorough discussion of spatial infinity, we refer the reader to Geroch [G 5] and to Shaw [S 15]. Note that the relationship between asymptotic structure at null infinity on the one hand and spatial infinity on the other may be investigated within the unified framework of Ashtekar and Hansen [A 17]. For example, given an isolated system which is asymptotically empty and flat at both null and spatial infinity and which also satisfies a boundedness condition on the Bondi news tensor, it may be shown that the difference between the ADM momentum and the Bondi momentum associated with a given retarded time is equal to the momentum carried away by the gravitational radiation emitted between the infinite past and the given retarded instant (see Ashtekar and Magnon-Ashtekar [A 19]).

Let us now discuss asymptotic null momentum in more detail. The original definition in the early 1960's (see Bondi et al [B 15]) was not within the framework of null infinity, but, since it is natural to consider null momentum within the context of asymptotically flat spacetimes, we shall do so here; we indicate how the universal arena and symmetry group lead directly to a physically useful concept of asymptotic momentum. For details, we refer the reader to Ashtekar [A 12] and Ashtekar and Streubel [A 23].

The basic idea is to utilize the universal kinematical arena which exists for any asymptotically flat spacetime (M,g). This arena consists of the manifold $\mathcal{J}^+ \equiv S^2 \times \mathbb{R}$ equipped with the strong conformal geometry S (see definition (6.3)2). The automorphism group of this structure is the Bondi-Metzner-Sachs group, $BMS \equiv SO^{\uparrow}(1,3) \ltimes C(S^2)$ which acts on $\mathcal{J}^+$ according to equation 6.3.2.

The strong conformal geometry is the first order structure on $\mathcal{J}^+$. In addition to this structure, we also have second order structure:- Suppose $S = q \otimes n \otimes n$ and consider the collection Γ' of torsion-free connections ω on $\mathcal{J}^+$ such that q and n are ω-parallel. Since q is degenerate, each such connection is defined only up to an element of $C(\mathcal{J}^+)$, and we define Γ to be the space $\Gamma'/C(\mathcal{J}^+)$. Each element of Γ possesses two independent components per point of $\mathcal{J}^+$, and these represent the two radiative degrees of freedom of the gravitational field.

The space Γ has a natural affine structure and is equipped with a weak symplectic form Ω. The BMS group acts symplectomorphically on the phase space (Γ,Ω) and, for each $\xi \in LBMS$, the moment function $\hat{j}(\xi) \in C(\Gamma)$ is precisely the flux through $\mathcal{J}^+$ of the conserved quantity associated with ξ. For ξ a translation generator, the flux obtained is precisely the flux of the Bondi 4-momentum.

In order to obtain a charge integral over a cut of $\mathcal{J}^+$, it is necessary to integrate the flux. Ashtekar and Streubel [A23] demonstrate that this integration may be performed for generators of supermomentum. If a restriction to the four dimensional translation subalgebra LT is made, then the Bondi 4-momentum is obtained.

In order to give a formula for the Bondi 4-momentum, it is convenient to re-introduce the physical spacetime and its compactification. Let (M,g) be an asymptotically flat spacetime and $(\hat{M},\hat{g},f,\phi)$ a maximal regular asymptote where, for simplicity, we assume that the conformal embedding ϕ is inclusion of (M,g) in $(\hat{M},\hat{g})$. Let S be a cut of future null infinity $\mathscr{I}^+$, so that S is a section of $\mathscr{I}^+$ diffeomorphic to S^2. Let $\iota_S : S \hookrightarrow \hat{M}$ be the inclusion of S in $\hat{M}$, so that ι_S is a codimension two spacelike embedding of S in $(\hat{M},\hat{g})$. Let $\{\hat{l},\hat{n}\}$ be the unique null trivialization (with respect to $\hat{g}$) of $N(\iota_S)$ (see definition (2.3)10) such that $\hat{n}$ is the restriction to $S \subseteq \mathscr{I}^+$ of the null normal to $\mathscr{I}^+$, namely $(-df)^{\hat{\#}}$. Exponentiating $\hat{l}$ gives an outgoing null hypersurface $\tilde{\hat{H}}$ of $(\hat{M},\hat{g})$ in a neighbourhood of $\mathscr{I}^+$ for which the images of the integral curves of the extension of $\hat{l}$, also called $\hat{l}$, are generators. Let $\hat{\sigma}$ denote the <u>shear</u> of the null geodesic generators of the null hypersurface $\tilde{\hat{H}}$. Then, with respect to a local $\hat{g}$-null tetrad $\{\hat{l},\hat{n},\hat{m},\hat{\bar{m}}\}$ (obtained by extending $\hat{l}$ and $\hat{n}$ to a neighbourhood of $\mathscr{I}^+$ in $\hat{M}$ such that $\hat{g}(\hat{l},\hat{n}) = 1$, and then adjoining the null extensions of m and $\bar{m}$, where m is a local null section of $T^{\mathbb{C}}S$, such that $\hat{g}(\hat{m},\hat{\bar{m}}) = -1$, we have:

$$\hat{\sigma} = \hat{m}^a \hat{m}^b \hat{\nabla}_a \hat{l}_b \qquad 3.3.11.$$

Note that the corresponding g-null tetrad is $\{l,n,m,\bar{m}\}$, where $l = f^2\hat{l}$, $n = \hat{n}$ and $m = f\hat{m}$. The formula analogous to 3.3.11 using the physical tetrad gives $\sigma = f^2\hat{\sigma}$.

The restriction of $\hat{\sigma}$ to $S \subseteq \mathscr{I}^+$ is a measure of the trace-free part of the extrinsic curvature of the embedding $S \hookrightarrow \mathscr{I}^+$. In fact, $|\hat{\sigma}|$ gives the magnitude of this curvature and $\arg\hat{\sigma}$ gives the directions of maximum extrinsic curvature relative to the null tetrad (see Penrose [P 7]).

The other ingredient in the Bondi 4-momentum is conformal curvature information. In fact, we need only the following component of the Weyl tensor:

$$\hat{\Psi}_2 = \hat{C}_{abcd}\,\hat{l}^a\,\hat{m}^b\,\hat{\bar{m}}^c\,\hat{n}^d \equiv f^{-4}\Psi_2 \qquad 3.3.12.$$

Given the cut S, it is now possible to define the Bondi 4-momentum as an element of the dual of the four dimensional subalgebra LT of $LBMS$ consisting of generators of translations. Using the isomorphism $BMS \cong SO^{\uparrow}(1,3) \ltimes C(S^2)$ together with the action 6.3.2, we may regard LT as the four dimensional subalgebra of $\mathrm{Vect}(\mathscr{I}^+)$ consisting of vector fields $\xi = \alpha n$, where $n \equiv (-df)^{\hat{\#}}$ is the null normal and $\alpha \in C(S^2)$ is a linear combination of spherical harmonics with $l = 0$ and 1.

It is convenient to introduce standard Bondi-type coordinates in a neighbourhood of S in $\hat{M}$ and compatible with the tetrad (see Penrose and Rindler [P]). Then the <u>news function</u> $N \equiv -\frac{1}{2}\hat{R}_{ab}\hat{\bar{m}}^a\,\hat{\bar{m}}^b$ $(\hat{R} \equiv \mathrm{Ric}(\hat{g}))$ is given by:

$$\overline{N} = -\frac{\partial}{\partial u}\hat{\sigma} \qquad 3.3.13,$$

and satisfies:

$$\frac{\partial}{\partial u}N = \hat{\Psi}_4 \qquad 3.3.14,$$

where $\hat{\Psi}_4$ is given by:

$$\hat{\Psi}_4 = \hat{C}_{abcd}\,\hat{\bar{m}}^a\,\hat{n}^b\,\hat{\bar{m}}^c\,\hat{n}^d \equiv f^{-2}\,\Psi_4 \qquad 3.3.15.$$

Note that $\hat{\Psi}_4$ is the part of the Weyl tensor corresponding to <u>gravitational radiation</u>, so that the presence of gravitational radiation is an obstruction to the constancy of the shear $\hat{\sigma}$ over the family of cuts given by $u = \text{constant}$.

The Bondi 4-momentum $p \in (LT)^*$ for the cut S (corresponding to $u = 0$) is now given by:

$$p(\xi;S) = \frac{1}{4\pi}\int_S \alpha\,(\hat{\sigma}N - \hat{\Psi}_2)\,dS \qquad 3.3.16,$$

for all $\xi = \alpha n \in LT$, where dS is the volume element corresponding to the metric induced on S from $\hat{g}$. For a derivation of equation 3.3.16 from the point of view of the Geroch-Tamburino-Winicour linkages, we refer the reader to Walker [W 2].

The Bondi 4-momentum is geometrically well motivated (it arises both from the linkage framework of Geroch-Tamburino-Winicour and from the purely asymptotic phase space framework of Ashtekar and Streubel - two very different approaches) and it possesses extremely desirable properties:-

As we have mentioned above, the Bondi 4-momentum p is interpreted as the total 4-momentum of the spacetime (M,g) at the retarded time given by the cut S. A very important property of p is that it satisfies the momentum loss formula on $\mathcal{I}^+$:- Suppose S_1 and S_2 are two cuts of $\mathcal{I}^+$ with S_2 entirely to the future of S_1, and let $\xi = \alpha n \in LT$ with $\alpha > 0$ (so that ξ is a future pointing vector field on $\mathcal{I}^+$). Then

$$p(\xi;S_2) \leq p(\xi;S_1) \qquad 3.3.17,$$

if the weak energy condition (see Hawking and Ellis [H 5]) holds in a neighbourhood of $\mathcal{I}^+$. Note that there also exists a similar result for $\mathcal{I}^-$, but on $\mathcal{I}^-$, the Bondi 4-momentum is non-decreasing rather than non-increasing with time.

The momentum loss formula 3.3.17 is important because it shows that gravitational radiation emitted by an isolated system carries positive energy. However, the formula gives no information about the total energy of the system, and a long-standing conjecture until the early 1980's was that the total energy of a system satisfying a reasonable local energy condition is positive. More precisely, the Bondi 4-momentum was conjectured to correspond to a future-directed timelike or null vector for an asymptotically flat spacetime satisfying the dominant energy condition (i.e. $T_{ab}t^b$ is future-directed and timelike for all future-directed and timelike vector fields t). This conjecture was referred to as the positive energy conjecture at null infinity.

There is also a positive energy conjecture at spatial infinity for the ADM 4-momentum. For a review of the spatial conjecture, we refer the reader to Horowitz [H 11]. Note that the spatial conjecture is weaker than the null version, because the Bondi 4-momentum is a retarded time (i.e. cut)-dependent quantity, whereas the ADM 4-momentum is constant.

The positive energy conjecture at null infinity has now been verified. One method is due to Schoen and Yau [S 10] and is a modification of the variational technique which they used to prove the spatial conjecture (see [S 8], [S 9]). The other method, due to Ludvigsen and Vickers (see [L 8], [L 9], [L 10]) and also to Horowitz and Perry [see [H 12]), utilizes spinor techniques in a fundamental way. The spinorial attack on the null conjecture is analogous to that used by Witten [W 9] in his proof of the spatial conjecture, but the null version presents different problems due to the degeneracy of the null hypersurface on which the spinor fields are required to propagate. In section 3.4, we demonstrate a link between the two approaches by showing that the Ludvigsen-Vickers propagation equation is a null limit of the equation used by Witten.

Another important link between the null and spatial cases is that the 2-surface spinor integrand used in defining the Bondi 4-momentum is the same as that used in the definition of the ADM 4-momentum (see Israel and Nester [I 9]) and this means that a spinorial version of the Ashtekar-Magnon-Ashtekar result [A 19] concerning the relationship between the two 4-momenta may be obtained (see Horowitz [H 11]). A result of Ashtekar and Horowitz [A 18] also shows that neither of the two momenta can be null (i.e. the total 4-momentum must be <u>strictly</u> timelike).

The fact that total energy in the theory of general relativity is positive is a very important result and indicates a fundamental difference between general relativity theory and Newtonian gravitational theory:- In Newtonian theory, any bound system possesses negative total energy and, even if the rest mass of the matter is included in the total energy, it is still possible to have systems with negative total energy, because the Newtonian gravitational potential is unbounded from below. If it were possible for a general relativistic system to have negative energy, then it would be possible to extract an unlimited amount of energy from such a system; gravitational radiation carries away positive energy by 3.3.17, thereby reducing the (already negative) energy of the system. If there were no lower bound on the energy, then the radiation could continue to carry away energy from the system indefinitely. The validity of the positive energy conjecture at null infinity ensures that such a phenomenon cannot occur within general relativity theory; if a system is compactified in order to try to make the total energy negative due to binding energy, then a black hole is necessarily formed and this black hole possesses positive total energy.

Various further consequences, extensions and generalizations of the positive energy theorem have been explored. One such generalization takes into account the presence of black holes in the asymptotically flat spacetime (see Ludvigsen and Vickers [L 10], Gibbons et al. [G 9]) and another considers the Einstein-Maxwell case; it is possible to show that there exists an inequality implying that the total energy is bounded below by the total electromagnetic charge (see Ludvigsen and Vickers [L 11] for the null case, and Gibbons and Hull [G 8] for the spacelike case). Another extension involves a consideration of higher dimensional theories - for example, Moreschi and Sparling [M 17] formulate the positive energy theorem in Kaluza-Klein theory, thereby answering questions concerning the attractiveness of the effective gravitational interactions and also the classical stability of the theory.

For further remarks concerning the interaction between the various approaches, we refer the reader to Horowitz [H 11], Horowitz and Tod [H 14] and Shaw [S 17].

Before moving to the topic of quasi-local momentum in general relativity theory, we make a few remarks concerning angular momentum. In Newtonian theory and in special relativity theory, angular momentum arises as the moment corresponding to generators of rotations in the Galileo group $SO(3) \ltimes \mathbb{R}^3$ and in the Poincaré group $O(1,3) \ltimes \mathbb{R}^4$ respectively. In order to define a rotation subgroup of these semi-direct products, it is necessary to choose an "origin" about which the angular momentum is to be taken. Having chosen an origin, the stabilizer of this point under the entire semi-direct product group is isomorphic to a copy of the rotation group. Elements of the Lie algebra of this rotation group then give rise to corresponding components of angular momentum.

The asymptotic symmetry group of null infinity $\mathscr{I}^+$ also has the structure of a semi-direct product of a rotation group and a translation group; we have $BMS = SO^{\uparrow}(1.3) \ltimes C(S^2)$ (see 6.3.4). In order to define a Lorentz subgroup of BMS, we must first fix an origin in cone space A. This is tantamount to fixing a cut S of $\mathscr{I}^+$.

First consider Minkowski spacetime $(\mathbb{R}^4, \eta)$. Each point in $\mathbb{R}^4$ gives rise to a cut of $\mathscr{I}^+$ obtained by taking the intersection with $\mathscr{I}^+$ of the future null cone of the given point. Such a cut is called a good cut and is characterized by the shear-free condition $\hat{\sigma} = 0$. Having chosen a particular good cut S_o, we obtain a Lorentz subgroup of BMS by taking the stabilizer of S_o. Since the four-dimensional translation subgroup T is uniquely specified (independent of choice of cut), we then have a Poincaré group P corresponding to S_o. Any other Poincaré subgroup of BMS is obtained by conjugating P with some supertranslation. The Poincaré group P, being a subgroup of BMS, leaves invariant the strong conformal geometry of $\mathscr{I}^+$ (see definition (6.3)2), and, in addition, P leaves invariant the family of good cuts (note that any good cut is obtained from S_o by some translation in T).

For a general asymptotically flat spacetime (M,g), we may define a good cut to be one for which $\hat{\sigma} = 0$ (we cannot take the intersection of $\mathscr{I}^+$ with future null cones due to the presence, in general, of caustics). Unfortunately, the presence of gravitational radiation is an obstruction to the existence of a family of good cuts, as equations 3.3.13 and 3.3.14 indicate. Therefore, for a general asymptotically flat spacetime, although there exists a naturally defined translation subgroup T of BMS, there is no family of good cuts with which to define Lorentz subgroups. Even if there exists a good cut at some retarded time u_1 and also at a later time u_2, the emission of gravitational radiation during a retarded time interval contained in (u_1,u_2) generally implies that the Lorentz subgroups corresponding to u_1 and u_2 are conjugates by a non-trivial supertranslation of one another. Thus, as Penrose [P 7] remarks, "the very concept of angular momentum gets 'shifted' with time".

It seems that an attempt to define angular momentum in the same way that 4-momentum is defined is doomed to failure in general relativity theory. Indeed, the Bondi 4-momentum occurs naturally in that there is a unique canonical translation subgroup T of BMS, but what right do we have to expect a corresponding definition of angular momentum given that, in general, the good cuts necessary for extracting Lorentz subgroups of BMS, do not exist?

The angular momentum problem has been attacked in various ways over the years, but still has not been fully resolved. An obvious approach is to define angular momentum in special cases for which the above problems do not arise. For example in an axisymmetric spacetime, there exists a rotational isometry and so the Komar charge 3.3.10 may be used and there exist good physical arguments for regarding this charge as representing angular momentum (see Prior [P 18]). Another special case is that of a radiation-free spacetime. For such a spacetime, there exists a four dimensional space of cuts whose shear possesses zero electric part (see Newman and Penrose [N 2]). If the spacetime is stationary, then this four dimensional space consists of good cuts, so that a situation analogous to that for Minkowski spacetime exists. The subgroup of BMS leaving invariant the four dimensional space of cuts is precisely T and the stabilizer of any one of them gives rise to a Lorentz subgroup and thence to a notion of angular momentum - see Bramson [B 20]. Bramson also discusses the radiation of spin and shows that the spin vectors of the system, long before and long after the emission of radiation, are supertranslation invariant (see Bramson [B22]). A third possibility for investigating a special case of general relativistic angular momentum is to consider past null infinity $\mathscr{I}^-$, where a no incoming radiation condition is usually assumed to hold.

More general definitions of angular momentum arise from the Geroch-Tamburino-Winicour linkage formalism (see [G 6]), from a twistor framework (Streubel [S24]), from Yang-Mills theory (Bramson [B20]) and also from quasi-local definitions (Penrose [P 8], Ludvigsen-Vickers [L10]). The relationship between these definitions, in particular, the "anomalous factor of two problem", is explored in Dray and Streubel [D 13] and in Shaw [S17]. For an approach to angular momentum based on 4-momentum, we refer the reader to Cresswell and Zimmerman [C 13], and for a review based on physical considerations, see Winicour [W 8].

This section has given an overview of various aspects of total (or asymptotic) gravitational momentum. In the next section, we consider the problem of defining quasi-local kinematical quantities in the theory of general relativity.

3.4 Quasi-local Momentum in General Relativity

This section serves two purposes; firstly to continue our discussion of gravitational momentum, in particular to discuss quasi-local aspects of momentum, and secondly to give an application of the null limit technique described in section 3.2. The section is organized as follows:- After a brief discussion of the quasi-local philosophy, we review the various attempts to define quasi-local kinematical quantities. We then describe the Ludvigsen-Vickers definition in more detail and we demonstrate how this definition is very natural within a spinor-null embedding context. The null limit technique is then used to give a link between the Ludvigsen-Vickers framework on the one hand and the Witten proof of the positive energy conjecture on the other. We also make further suggestions concerning quasi-local momentum in general relativity theory.

In the previous section, we reviewed ideas relating, in the main, to <u>total</u> gravitational momentum, i.e. momentum defined asymptotically which represents, in the appropriate sense, the total momentum content of the spacetime. From the problems outlined at the beginning of section 3.3, we know that there exists no possibility of defining <u>local</u> gravitational momentum, but do we really need to go out all the way out to infinity in order to obtain a physically meaningful concept of momentum? Another possibility is to try to define <u>quasi-local</u> kinematical quantities; given an arbitrary embedded, spacelike, closed 2-surface S in an arbitrary spacetime (M,g), is it possible to assign to S some quantity representing the total momentum or angular momentum (gravitational plus that due to matter fields) surrounded by (or threading through) S? Such a quantity could also be interpreted as the total momentum or angular momentum intercepted by any spacelike hypersurface H with $\partial H = S$. To be useful, such a quasi-local quantity should possess physically reasonable positivity and semi-additivity properties and also should give the "correct" answer for the special cases for which we already have a notion of quasi-local kinematical quantities (e.g. special relativity, linearized limit).

Given the spacetime (M,g), we may regard a quasi-local quantity as a map $Q\colon D \times \mathrm{Sub}_2^S(M) \longrightarrow \mathbb{R}$, where D is a space of "descriptors", each descriptor being responsible for picking out a particular component of momentum, angular momentum or other kinematical quantity, and $\mathrm{Sub}_2^S(M)$ is the manifold of two-dimensional, closed, spacelike submanifolds of (M,g) (Cf. section 2.2). A restricted quasi-local quantity would arise if we specified the diffeomorphism type S_o of the closed 2-surface (for example, we might require that $S_o = S^2$); then we would have $Q\colon D \times \mathrm{Sub}_{S_o}^S(M) \longrightarrow \mathbb{R}$, where $\mathrm{Sub}_{S_o}^S(M)$ denotes the manifold of spacelike submanifolds of type S_o in (M,g). In the latter case, it may be convenient to work with embeddings rather than with submanifolds:- Thus, we would consider a map $\tilde{Q}\colon D \times \mathrm{Emb}_S(S_o,M) \longrightarrow \mathbb{R}$, where $\mathrm{Emb}_S(S_o,M)$ denotes the manifold of spacelike embeddings of S_o in (M,g). Provided that $\tilde{Q}(D,\cdot)$ is invariant under $\mathrm{Diff}(S_o)$, we may project to obtain a quasi-local quantity defined on the manifold $\mathrm{Sub}_{S_o}^S(M)$ (Cf. equation 2.2.4).

In practice, the quasi-local quantities considered are of the form $Q\colon D \times \mathrm{Sub}_2^S(M) \longrightarrow \mathbb{R}$, where:

$$Q(\alpha;S) = \int_S *F(\alpha;S) \qquad 3.4.1,$$

for all $\alpha \in D$ and $S \in \mathrm{Sub}_2^S(M)$. Here $*F(\cdot;S)\colon D \longrightarrow \Omega^2(S)$ is a 2-form valued map on the descriptor space defined for each $S \in \mathrm{Sub}_2^S(M)$. One possibility is that $*F(\cdot;S) = \iota_S^* \circ * \circ F_M$, for all $S \in \mathrm{Sub}_2^S(M)$, where $F_M\colon D \longrightarrow \Omega^2(M)$ is some given 2-form valued map, but ideally the integrand in 3.4.1 should be completely intrinsic to the 2-surface.

Various possibilities for F have been explored in the literature. The simplest special case is when the spacetime (M,g) is Minkowski spacetime $(\mathbb{R}^4,\eta)$. Then the momentum-angular momentum is due to the (non-gravitational) fields Φ and we have equation 3.3.3 giving the total charge linked by the 2-surface. In this case, we have $*F(\cdot;S) \equiv *F_\Phi(\cdot;S) = \iota_S^* \circ * \circ F$, where $F\colon L\ \text{Poincaré} \longrightarrow \Omega^2(\mathbb{R}^4)$, so that the descriptor space is the Lie algebra of the Poincaré group, or, equivalently, the space of Killing vector fields of Minkowski spacetime.

Moving on from Minkowski spacetime to a general non-generic spacetime, we may consider the Komar approach. The paper of Komar [K 9] may be regarded as a prototype for the study of quasi-local quantities in general relativity theory. Let (M,g) be a spacetime admitting a Killing vector field ξ and let S be a closed, space-like 2-surface in (M,g) lying entirely within the matter-free region of spacetime. Then the charge $Q(\xi;S)$ given by equation 3.3.10 gives the momentum component corresponding to ξ. In the Komar case, $*F(\cdot;S)\colon L\ \mathrm{Isom}(M,g) \longrightarrow \Omega^2(S);\ \xi \longmapsto \iota_S^*(*d\xi^b)$, so that the descriptor space is the space of Killing vector fields of (M,g).

The Komar charge is not a good candidate for a measure of quasi-local kinematical quantities for two obvious reasons:- The first is that we have to restrict to a subspace $\mathscr{f}\,\mathrm{Sub}_2^S(M)$ (see the remarks following equation 3.3.10 concerning the anomalous behaviour of the Komar charge for 2-surfaces within the matter support). Secondly, the Komar descriptor space is zero for a generic spacetime.

A third problem with the Komar charge is that the integrand is not completely intrinsic to the 2-surface. In fact, the integrand depends on derivatives of the vector field ξ in non-tangential directions.

As we have remarked above, a modification of the Komar charge may be utilized at the (future) null infinity of a generic asymptotically flat spacetime. Then the descriptor space is the (super) translation subalgebra of LBMS, and we get back to asymptotic or total measures of momentum (see Tamburino and Winicour [T 1]).

This development of the Komar approach leads to the linkage framework of Geroch-Tamburino-Winicour (see also Geroch and Winicour [G 6]). The linkage integrand generalizes that of Komar and reduces to $*d\xi^b$ when ξ is a Killing vector field. In general, the linkage descriptor space is LBMS transported to the 2-surface using a propagation equation on the outgoing null hypersurface from S to $\mathcal{I}^+$.

Unfortunately, the linkage approach does not lead to a useful quasi-local momentum. There are ambiguities inherent in the framework (for example in the formulation of conservation laws) and the integrand is still not intrinsic to S; it contains derivatives of ξ in the direction of the outgoing null hypersurface.

We now turn to a definition of quasi-local <u>energy</u> defined by Hawking in 1968 (see [H 4]). Let (M,g) be a spacetime and S a spacelike, embedded 2-sphere in (M,g). Let $A \equiv \int_S \mathrm{vol}(\overset{*}{\iota}_S g)$ be the area of S and denote by k the second fundamental form of S (see equation 2.1.6). Using abstract indices corresponding to the vector bundle $\overset{*}{\iota}_S(TM)$, we have $k_{ab}{}^c = - h^m_a\, h^n_b\, \nabla^g_m\, (g^\perp)^c_n$, where h is the projection onto TS (see equation 2.3.15) and $g^\perp$ is the projection onto the normal bundle $N(\iota_S)$ (see equation 2.3.16).

This formula for k is the direct analogue of equation 2.1.12 for a codimension two embedding. The $(\overset{*}{\iota}_S g)$-trace of k, denoted N, is known as the <u>normal mean curvature</u> (see Kobayashi and Nomizu [K 5]), so that $N \in \Gamma(N(\iota_S))$ is given by $N^a = k^{b\ a}_{\ b}$. Note that the $g^{\perp}$-(pointwise) norm of N is given by $\|N\|^2 = 8\rho\rho'$, where ρ and ρ' are respectively the convergences of the outgoing and ingoing null geodesics normal to S (see Horowitz and Schmidt [H 13]). The <u>Hawking quasi-local energy</u> is now given by the following integral:

$$E(S) = \tfrac{1}{4}\left(\frac{A}{\pi}\right)^{\frac{1}{2}} \left(1 + \frac{1}{16\pi}\int_S \|N\|^2\, dS\right) \qquad 3.4.2,$$

where $dS \equiv vol(\overset{*}{\iota}_S\, g)$ is the area 2-form of S. Note that the Hawking quasi-local energy may be written in the form 3.4.1 if we put:

$$*F(S) = \frac{1}{64}\,(A\pi^3)^{-\frac{1}{2}}(16\pi + A\,\|N\|^2)\,dS \qquad 3.4.3,$$

but $*F(S)$ is <u>not</u> the pullback of a 2-form on M.

The expression 3.4.2 was investigated by Eardley [E 1] and was shown to possess several desirable properties. In particular, it vanishes in the limit when the radius of S tends to zero, it coincides with the ADM and Bondi expressions when applied asymptotically in an asymptotically flat spacetime, and, under certain conditions, it increases monotonically with the radius of S. The "small sphere" behaviour of $E(S)$ was discussed by Horowitz and Schmidt [H 13] and they discovered that, in the presence of matter, the leading-order contribution (r^3) is the energy density of the matter, whilst in the vacuum case, the leading-order contribution (r^5) is proportional to the "time component" of the Bel-Robinson tensor (see equation 3.3.7). Note that the perturbation technique used by Horowitz and Schmidt is natural in the sense that it is based on the null cone at a given point in spacetime, although it is not clear that the Bel-Robinson tensor would occur in alternative perturbation schemes.

In addition to the desirable properties of the Hawking quasi-local energy, there also exist not so desirable properties. The main disadvantage with the energy is that it gives non-zero answers for certain 2-spheres in Minkowski spacetime:- The Hawking energy certainly vanishes for 2-spheres S such that $\overset{*}{\iota}_S g$ is sufficiently close to the round metric can, but for "less round" 2-spheres, the energy can be non-zero. For this reason, we should only apply the Hawking quasi-local energy to 2-spheres with induced metrics close to can, although a precise definition of this class of "sufficiently round" embeddings of S^2 in a general spacetime (M,g) is difficult to give.

Another problem with $E(S)$ arises if the "large sphere" behaviour is considered (see Shaw [S 19]). It turns out that, even to first order, the Hawking energy gives non-zero contributions to the energy of Minkowski spacetime. This problem may be avoided by using shear-free 2-spheres (in any stationary spacetime), but then unphysical contributions to the energy occur at third order.

We remark that another definition of "quasi-local" general relativistic energy was suggested in the early 1970's by Geroch (see [G 4]). The Geroch definition is given for a 2-sphere embedded in some spacelike hypersurface in a spacetime, so that <u>two</u> embeddings are specified rather than just one. Thus, the Geroch energy is not a bona fide quasi-local moment. It may be shown (see [H 13]) that the Geroch energy is just the corresponding Hawking energy modified by a term involving the extrinsic curvature of the given hypersurface. This extra term modifies the "small sphere" behaviour of the energy and the Geroch energy appears to be physically less appropriate than the Hawking energy. Indeed, the Geroch energy cannot be regarded as a hopeful candidate for a definition of quasi-local energy for the reason alluded to above; the energy depends on two embeddings.

We turn now to a much better definition of quasi-local kinematical quantities in the theory of general relativity, namely that of Penrose [P 8], [P 12]. The Penrose approach is based on twistor theory and therefore utilizes spin structure in a fundamental way. Indeed the successes of this approach, along with the essential use of spinors in the proofs of the positivity conjectures (see section 3.3) and in the Ludvigsen-Vickers definition of quasi-local moments (see below), provides solid evidence supporting the belief that spinors and gravitational moments are intimately linked (see section 1.0).

The basic Penrose construction may be described as follows:- Let (M,g) be a spacetime and S an embedded, spacelike, closed 2-surface in (M,g). We assume that M is spin and that a g-spin structure s_g has been fixed. We may then discuss the <u>twistor equation</u>:

$$\nabla^{(A}_{A'}\,\omega^{B)} = 0 \qquad 3.4.3,$$

where $\nabla_{AA'}$ is the covariant derivative induced in the bundle $S(s_g)$ and $\omega \in \Gamma(S(s_g))$ (see sections 1,3, 1.7). Equation 3.4.3 is projected onto the 2-surface S and we obtain the <u>superficial twistor equations for S</u> for $\omega \equiv \omega \circ \iota_S \in \Gamma(\iota_S^* S(s_g))$:

$$\eth' \omega^0 = \sigma' \omega^1 \qquad 3.4.4,$$

$$\eth \omega^1 = \sigma \omega^0 \qquad 3.4.5,$$

with respect to a choice of null trivialization of $N(\iota_S)$ and a local section of $T^{\mathbb{C}}S$. Here, $\eth$ and $\eth'$ are the GHP operators and σ and σ' describe the shear of the null geodesics normal to S (or, equivalently, the trace-free part of the extrinsic curvature of S in (M,g)) (see Geroch et al. [G 7], Penrose and Rindler [P 11]).

The equations 3.4.4, 3.4.5 correspond to an elliptic differential operator and, using the Atiyah-Singer index theorem, it may be demonstrated that the kernel of this operator has (complex) dimension equal to four, at least for S an embedded 2-sphere with $\overset{*}{\iota_S} g$ not too far from can. Let $\mathbb{T}(S)$ denote the kernel of the elliptic operator defined by the superficial twistor equations for S. We assume that $\dim_{\mathbb{C}} \mathbb{T}(S) = 4$, but see Jeffryes [J 3] for a discussion of the possibility of "extra" solutions to the superficial twistor equations. In any case, the vector space $\mathbb{T}(S)$ is called the <u>2-surface twistor space associated with S</u>.

Given $\mathbb{T}(S)$, the kinematical quantities are given by a preferred element $A(S)$ of $\odot^2\mathbb{T}^*(S)$. The symmetric 2-surface twistor $A(S)$ is called the <u>kinematic twistor for S</u> and is given by:

$$A(S)(\omega_1,\omega_2) = -\frac{i}{4\pi}\int_S \nu\, J(\omega_1,\omega_2) \qquad 3.4.5,$$

where the 2-form valued twistor $J \in (\odot^2\mathbb{T}^*(S)) \otimes_{\mathbb{R}} \Omega^2(S)$ is given by:

$$J(\omega_1,\omega_2) = -\tfrac{1}{2}iR_{ABCDA'B'C'D'}\varepsilon^{A'B'}\omega_1^{(A}\omega_2^{B)}dS^{CC'DD'} \qquad 3.4.7,$$

for all $\omega_1, \omega_2 \in \mathbb{T}(S)$. Here, $R \equiv \overset{*}{\iota_S}\,\mathrm{Riem}(g)$ is the restriction of the Riemann tensor field to S.

The factor ν was taken to be unity in the original Penrose quasi-local paper [P 8], but a modification with $\nu \neq 1$ was proposed by Penrose at a later stage (see [P 12]). The modification takes into account the behaviour of $\mathbb{T}(S)$ under complex conformal rescalings of the symplectic structure (Cf. section 1.8 and see Shaw [S 19] for further discussion on this matter).

Equation 3.4.6 defines a notion of quasi-local kinematical quantity as follows:- The descriptor space is taken to be $\odot^2\mathbb{T}(S)$ and, noting that $A(S)$ defines an element of $(\odot^2\mathbb{T}(S))^*$ (each element γ of $\odot^2\mathbb{T}(S)$ may be written in the form $\gamma = \sum_{i,j} \omega_i \odot \omega_j$, for some $\omega_i, \omega_j \in \mathbb{T}(S)$ and $i,j \in \{1,2\}$), we may utilize our previous notation and write

$$Q(\gamma;S) = A(S)(\gamma) \qquad 3.4.8,$$

for all $\gamma \in \odot^2\mathbb{T}(S)$. Equations 3.4.6 and 3.4.8 should be compared with equation 3.4.1.

Note that the descriptor space $\odot^2\mathbb{T}(S)$ is a ten dimensional complex vector space. If any correspondence with physical kinematical quantities in special relativity is required (as it must be!), then we might expect that the space $\odot^2\mathbb{T}(S)$ corresponds in some sense to LPoincaré. In order to make this correspondence, it is necessary to reduce the ten complex dimensions to ten real dimensions using some hermiticity property. In fact, it turns out that, although the Penrose procedure is perfectly well defined up to and including the definition of the kinematic twistor $A(S)$, in order to extract physically useful information, it is necessary to have further structures. Thus, in order to define the Penrose quasi-local energy, a pseudo-Hermitian inner product (see Penrose [P 8]) or, alternatively, a volume element (see Tod [T 3]) on $\mathbb{T}(S)$ is required. To proceed further, it is necessary to separate out angular momentum components from momentum components, and for this a notion of infinity twistor is required. Unfortunately, the general definitions of these additional structures is not yet clear.

However, in certain special cases, there exist obvious candidates for these extra structures and calculations yielding physically interesting information may be performed:-

In Tod [T 3], various examples of the Penrose quasi-local mass are given. The 2-surface twistor space $\mathbb{T}(S)$ is calculated for a number of particular embedded 2-surfaces in particular spacetimes for which there exists a clear candidate for the pseudo-Hermitian inner product. The Penrose quasi-local (rest) mass is then given by:

$$m_P^2(S) \quad = \quad - \tfrac{1}{2}\|A(S)\|^2 \qquad\qquad 3.4.9,$$

where $\|\cdot\|$ is the norm arising from the induced inner product on $\Theta^2\mathbb{T}^*(S)$. Note that equation 3.4.9 is the direct analogue of the one in standard flat space twistor theory (see Penrose and MacCallum [P10]). Indeed, the entire Penrose quasi-local approach is motivated by standard twistor theory; instead of basing the framework on Minkowski spacetime, the embedded 2-surface is used.

The Tod calculations give physically reasonable answers:- For S constrained to lie in a constant t hypersurface in the Schwarzschild spacetime, the Schwarzschild mass parameter is obtained if S links the source, and zero is obtained otherwise. For a general vacuum spacetime containing a hypersurface of time-symmetry, the Penrose quasi-local mass is invariant under continuous deformations of S within the hypersurface. If such a spacetime describes a configuration of black holes, momentarily at rest with respect to one another, then the Penrose mass depends only on the topological relationship of S with the black holes. As expected, the mass is not additive, but includes a negative contribution due to gravitational potential energy in the case when S links two or more black holes.

Other calculations performed by Tod include the case of a spherically symmetric 2-surface in the Reissner-Nordstrom and Friedmann-Robertson-Walker spacetimes. It turns out that the Penrose quasi-local mass for the Reissner-Nordstrom spacetime includes the correct negative contribution due to electrostatic field energy and the mass for the Friedmann-Robertson-Walker spacetime is equal to the product of the mass density with the volume enclosed by a surface in flat space with the same surface area as S.

The Penrose construction also works well at the null infinity of an asymptotically flat spacetime. In particular, the momentum and angular momentum components may be separated out from the kinematic twistor. The 4-momentum is precisely the Bondi 4-momentum (see equation 3.3.16) and the angular momentum defined does not suffer from the shortcomings of previously defined expressions. We refer the reader to Penrose [P 8], Dray and Streubel [D 13], Shaw [S 16], [S 19] and Dray [D 14] for a discussion of the null asymptotics of the Penrose quasi-local moments, and to Bizon [B 13] and Shaw [S 15], [S 18] for the corresponding spacelike structure.

The "small surface" perturbation approach to the Penrose quasi-local moments has been studied by Kelly et al. [K 2]. The technique employed in this paper is the same as that used by Horowitz and Schmidt [H 13] in their analysis of the Hawking quasi-local energy and by Bergqvist and Ludvigsen [B 9] in their analysis of the Ludvigsen-Vickers quasi-local momentum, namely an expansion in powers of an affine parameter along the generators of the null cone. It turns out that for sensible (i.e. non-complex) answers to be obtained, the Penrose modification $(\nu \neq 1)$ is required. For non-vacuum spacetimes, the lowest order (r^3) contribution to the mass is from the stress-energy-momentum tensor and it is positive for spacetimes satisfying the dominant energy condition. For vacuum spacetimes, the mass vanishes at the next lowest order (r^5) - this result should be compared with the analogous one for the Hawking and Ludvigsen-Vickers quasi-local masses for which the Bel-Robinson tensor gives the fifth order contribution. Note that the calculations in [K 2] are performed both for small spheres and for small surfaces of more general shape.

As a complement to the "small surface" treatment, Shaw [S 19] performs analogous calculations for "large surfaces". The idea here is to consider a cut of the future null infinity of an asumptotically flat spacetime. The cut defines a unique outgoing null hypersurface in a neighbourhood of $\mathcal{I}^+$. It is then possible to analyse the quasi-local moments associated to 2-sphere sections of this null hypersurface by labelling the sections using a suitable parameter r and then expanding all quantities in inverse powers of r. Shaw shows that the Penrose angular momentum contains no unphysical contributions, at least to third order for stationary spacetimes. In order to give further support to the use of the Penrose definition, it will be necessary to extend the work of Shaw to higher orders in the parameter r and also to the case of non-stationary spacetimes.

We now mention further work on the Penrose quasi-local definition. Tod [T 5] has shown that a static black hole satisfies the inequality $A \leqslant 16\pi m_P^2$, where A is the area and m_P the Penrose quasi-local mass. This result should be compared with that of Ludvigsen and Vickers [L 12] (here, the same inequality is proved relating the

Bondi mass to the area of a closed trapped surface).

Jeffryes has discussed the Newtonian limit of Penrose's quasi-local mass (see [J 4]) and also the relationship between the twistor space $\mathbb{T}(S)$ on the one hand and the possibility of embedding S in real and complex conformally flat spaces on the other (see [J 5]). Tod [T 6] has also considered the problem of embedding in conformally flat spaces. It turns out that a necessary and sufficient condition for an embedded 2-surface S to be embeddable in a (locally) conformally flat spacetime, with the same induced metric and second fundamental form, is that the standard twistor norm is constant on S. If the twistor norm be constant, then S is called a <u>non-contorted surface</u>; otherwise, S is called <u>contorted</u> (these terms are due to Penrose [P 12]).

All of the examples calculated by Tod [T 3] were of non-contorted surfaces, the calculation being possible due to the constancy of the norm. In [T 6], Tod considers a calculation involving a contorted surface:- The twistor space and kinematic twistor may be obtained, but the non-constancy of the norm prevent further progress. A further problem is pointed out by Woodhouse [W 16], who shows that new ideas are required even for the calculation of the Penrose quasi-local mass for <u>small</u> contorted surfaces.

Tod [T 6] concludes that, although the idea of the Penrose quasi-local mass works extremely well for non-contorted surfaces, the definition for contorted surfaces presents problems which remain to be solved.

One possible means of clarifying the situation is by using the fact that certain concepts relating to contorted surfaces may be obtained from the corresponding concepts for non-contorted surfaces via a complex conformal rescaling (see Jeffryes [J 5] and Tod [T6]). Whether or not this transformation property will help in the calculation and interpretation of the Penrose quasi-local mass for contorted surfaces remains to be seen. For other considerations relating to the use of complex conformal rescalings, see Penrose [P9] and also section 6.2.

Before leaving the Penrose approach, we remark that a similar definition of qaasi-local moments has been given for a general Yang-Mills theory. We refer the reader to Tod [T4] for details.

Let us now turn to the Ludvigsen-Vickers definition of quasi-local kinematical quantities in the theory of general relativity. We derive the definition from basic principles, but the original motivation from the proofs of the positive energy conjecture at null infinity (see section 3.3) may be found in [L 8], [L 9] and [L 10]. Further applications for the Ludvigsen-Vickers techniques may be found in [L 11] and [L 12]. Special cases of the Ludvigsen-Vickers definition are given by Bergqvist and Ludvigsen [B 9] (the "small surface" limit) and by Shaw [S 19] (the "large surface" limit). For further comments regarding the development of the Ludvigsen-Vickers framework, we refer the reader to Swift and Vickers [S25].

We now demonstrate that the Ludvigsen-Vickers ideas are extremely natural and may be derived from clear geometric principles. Our starting point is expression 3.4.1 for a general quasi-local charge integral. We will show that there exists an essentially unique integrand satisfying certain physically reasonable requirements. First we define some basic notions:-

Let (M,g) be an asymptotically flat spacetime and $(\hat{M},\hat{g},f,\phi)$ a maximal regular asymptote (see definition (6.3)1) where, for simplicity, we take ϕ to be inclusion of M in $\hat{M}$. We consider future null infinity $\mathcal{J}^+$ only, but past null infinity may also be considered. Note that (M,g) is asymptotically flat implies that (M,g) is globally hyperbolic, so that M has topology $\Sigma \times \mathbb{R}$, where Σ is a Cauchy surface for (M,g). Therefore M is non-compact and parallelizable and we deduce that M is spin (see example 1.2.3). We assume that a g-spin structure s_g has been chosen.

We now fix an embedded, spacelike, closed 2-surface S in (M,g) and denote by H the geodesic null hypersurface constructed from ι_S: $S \hookrightarrow M$ as in section 3.2. Since the inclusion of (M,g) in $(\hat{M},\hat{g})$ is conformal, $\tilde{H}$ is also a null hypersurface in $(\hat{M},\hat{g})$. Given a null trivialization $\{l,n\}$ of $N(\iota_S)$, we may follow through the programme discussed in section 3.2:- We have the embedding f_r: $S \hookrightarrow \tilde{H}$ for $r \in I \subseteq \mathbb{R}$ (we use r instead of u to conform with the notation of Ludvigsen and Vickers [L 10]), and we let $S_r = f_r(S) \subseteq \tilde{H}$, for all $r \in I$.

We now <u>assume</u> that $\tilde{H} \cap \mathcal{J}^+ = S_\infty$, where S_∞ is a cut of $\mathcal{J}^+$, and that the range of the affine parameter r may be extended so that $I = [0,\infty)$ with $S_o = S$ and $S_\infty = \lim\limits_{r\to\infty} S_r$. In other words, we assume that no caustics occur.

As in section 3.2, we have the null vector fields l,n on H such that $(g\circ\iota_H)(l,n) = 1$ (we drop all tildes on l,n and H for convenience). For each $r \in I$, as explained in section 2.3, we have the full power of the $SL(2,\mathbb{C})$-spinor formalism on S_r. In particular, using the Infeld-Van der Waerden isomorphisms arising from $\iota_{S_r}: S \hookrightarrow M$ (as r runs over I), the null trivialization $\{l,n\} \equiv \{l|S_r, n|S_r\}$ of $N(\iota_{S_r})$ may be written $\{o^A o^{A'}, \iota^A \iota^{A'}\}$ in the usual way (note that r-independence has been suppressed).

We now define a notion that will be very important in what follows. By a spinor field λ on H, we mean a section of $\iota_H^* S(s_g)$ (see section 1.7 for our spacetime spinor notation):

Definition (3.4)1: A spinor field λ on H is said to be asymptotically constant if $\lim_{r\to\infty}(o_A\lambda^A)$ and $\lim_{r\to\infty}(\iota_A\lambda^A)$ exist and, in addition,

$$\lim_{r\to\infty}(r\,\iota^A\,\bar{o}^{A'}o_B\nabla_{AA'}\,\lambda^B) = 0 \qquad 3.4.10,$$

and

$$\lim_{r\to\infty}(r\,o^A\,\bar{\iota}^{A'}o_B\,\nabla_{AA'}\,\lambda^B) = 0 \qquad 3.4.11.$$

This is the definition given by Ludvigsen and Vickers (see [L 10]) and implies that the field λ possesses asymptotically two (complex) degrees of freedom:- Let $\{X_{\underline{A}} : \underline{A} = 0,1\}$ be an asymptotically constant spin frame such that $\lim_{r\to\infty}(X_{\underline{0}A}\,X_{\underline{1}}{}^A) = 1$. Then for any asymptotically constant spinor field λ on H, the components $\{\overset{o}{\lambda}{}^{\underline{A}}\}$ of λ (with respect to $\{X_{\underline{A}}\}$) given by:

$$\overset{o}{\lambda}{}^{\underline{A}} = -\lim_{r\to\infty}(X^{\underline{A}}{}_A\,\lambda^A) \qquad 3.4.12,$$

are constant.

The space of asymptotically constant spinor fields is called the asymptotic spin space of S (this space depends only on S and not on the choice of null trivialization) and is denoted $\mathbb{S}(S)$. Thus, $\mathbb{S}(S)$ is a two-dimensional complex vector space associated with the embedded 2-surface S. Given any asymptotically constant spin frame $\{X_{\underline{A}}\}$, we may regard equation 3.4.12 giving the components of an element of $\mathbb{S}(S)$.

$\mathbb{S}(S)$ is equipped with a natural symplectic structure induced from ε. Let us denote the form on $\mathbb{S}(S)$ by $\overset{o}{\varepsilon}$, so we have:

$$\varepsilon(\lambda,\mu) = \lim_{r\to\infty} \varepsilon(\lambda,\mu) \qquad 3.4.13,$$

for all $\lambda,\mu \in \mathbb{S}(S)$.

Let Cone(M,g) denote the space of all two-dimensional, closed, spacelike submanifolds of (M,g) with the property that $S \in$ Cone(M,g) if and only if the null hypersurface H constructed from S (as in section 3.2) intersects $\mathcal{J}^+$ in a cut. Obviously, Cone(M,g) may be naturally mapped onto the abstract cone space defined in section 6.2. We now have the symplectic vector bundle $\mathbb{S}(M,g) \to$ Cone(M,g), where $\mathbb{S}(M,g) = \bigcup_S \mathbb{S}(S)$ with obvious projection onto Cone(M,g). In fact, an equivalent definition of the asymptotic spin space $\mathbb{S}(S)$ using the asymptotic twistor equation (see Bramson [B 19], [B 21] (although it should be noted that our notion of asymptotic constancy is referred to by Bramson as <u>alinement of frames on $\mathcal{J}^+$</u>) leads to the observation that $\mathbb{S}(M,g)$ is a trivializable bundle.

In order to relate the asymptotic spin space to symmetries, it is convenient to introduce the following definition:

<u>Definition (3.4)2</u>: A vector field ξ on H is said to be <u>asymptotically constant</u> if $\xi = \lambda \otimes \overline{\lambda}$ (via the Infeld-Van der Waerden isomorphisms) for some $\lambda \in \mathbb{S}(S)$.

The asymptotic limit of any asymptotically constant vector field is necessarily (the restriction to S_∞) of a generator of BMS translations. Thus, the four <u>real</u> dimensions of $\mathbb{S}(S)$ corresponds, in a very geometric way, to the four real dimensions of LT, the Lie algebra of the translation subgroup of BMS. For further discussion of asymptotically constant vector fields, we refer the reader to the Bramson references cited above and also to Geroch and Winicour [G 6] and to Walker [W 2]. Note, however, that Walker uses the term strong asymptotic constancy rather than asymptotic constancy.

Let us now return to the Ludvigsen-Vickers programme, the basic philosophy of which is to propagate the asymptotic spin space $\mathbb{S}(S)$ down the null hypersurface H to S in order to define quasi-local moments. For this reason, there is a strong similarity with the philosophy of the linkage framework of Geroch-Tamburino-Winicour (see above). This is to be contrasted with the Penrose approach in which the space $\mathbb{T}(S)$ is directly associated with S without any reference to future null infinity. We return to the propagation below, but first let us consider the "generalized linkage" integrand, i.e. which choice should we make for the 2-form $*F(\alpha;S)$ for insertion into 3.4.1?

We choose $*F(\alpha;S)$ by imposing certain reasonable requirements:- Firstly, since we wish to define a notion of quasi-local 4-momentum (see below for a possible definition of quasi-local angular momentum), the descriptor α should be related to a generator of translations in some suitable sense. Also, since moments are linear functionals, $*F(\alpha;S)$ should depend linearly on the translation generator (once the relationship between α and the generator has been imposed). Since the only translation group available is $T \triangleleft BMS$, we formulate our first requirement as follows:- (I) $*F(\alpha;S)$ should depend linearly on an asymptotically constant vector field on H.

The second requirement is based on dimensional grounds:- (II) $*F(\alpha;S)$ should involve no derivative of the asymptotically constant vector field of order treater than one. Note that Horowitz [H 11] has also discussed requirements (I) and (II).

If we allow ourselves only the use of vector fields, requirements (I) and (II) will give rise to only two possibilities for $*F \equiv *F(\alpha;S)$:

$$*F_1 = *d\xi^b \qquad 3.4.14,$$

$$*F_2 = d\xi^b \qquad 3.4.15,$$

where ξ is an asymptotically constant vector field on H (we omit ι_S^* for convenience). Since S possesses no boundary, we have $\int_S *F_2 = 0$, by Stokes' theorem, so we may discard $*F_2$. Unfortunately, $*F_1$ is not much use either, since $*F_1$ is just the Komar integrand (see equation 3.3.10) and we have discussed above the reasons preventing this from giving a useful definition of quasi-local momentum.

Since we have a spin structure, we may rescue the situation by considering 2-forms constructed from an asymptotically constant spinor field λ. The relationship between the descriptor $\alpha \equiv \lambda$ and translation generators is obtained by defining the asymptotically constant vector field $\xi = \lambda \otimes \bar{\lambda}$. There are now two further (complex) possibilities:

$$*F_3 = \lambda_A \nabla_{BA'} \bar{\lambda}_{B'} - \lambda_B \nabla_{AB'} \bar{\lambda}_{A'} \qquad 3.4.16,$$

$$*F_4 = F_3 \qquad 3.4.17.$$

Note that $\mathrm{Im}\ F_3 = F_1$ and $\mathrm{Im}\ F_4 = F_2$ and that the real parts are non-zero. The real part of $*F_3$ may be eliminated as a candidate on physical grounds (the asymptotic limit does not give the Bondi 4-momentum - see Horowitz [H 11]) or by imposing a third requirement: (III) $*F$ involves only derivatives tangent to S. Requirements (I), (II) and (III) lead to the unique integrand $\mathrm{Re}(*F_4)$ (up to a multiplicative constant of course). Thus, $\mathrm{Re}(*F_4)$ is the unique 2-form which is linearly dependent on the asymptotically constant vector field ξ, is of first order in the derivative of ξ and involves only derivatives tangential to S.

The uniquely defined 2-form $\mathrm{Re}(*F_4)$ is the integrand utilized by Ludvigsen and Vickers [L 10] and it has also appeared in the work of Nester [N 1].

Definition (3.4)3: The 2-form $F \equiv \mathrm{Re}\, F_4$ is called the Ludvigsen-Vickers-Nester (LVN) 2-form.

Proposition (3.4)4: The LVN 2-form is given by

$$F = \phi \otimes \bar{\varepsilon} + \bar{\phi} \otimes \varepsilon \qquad 3.4.18,$$

where the spinor field ϕ is defined by:

$$\phi_{AB} = \tfrac{1}{2}\lambda_{(A}\nabla_{B)}{}^{A'}\bar{\lambda}_{A'} - \tfrac{1}{2}\bar{\lambda}_{A'}\nabla_{(A}{}^{A'}\lambda_{B)} \qquad 3.4.19.$$

Proof: Let ψ be the unique symmetric rank two spinor field determined by F, so that $F = \psi \otimes \bar{\varepsilon} + \bar{\psi} \otimes \varepsilon$. Then $\psi_{AB} = \tfrac{1}{2}F_{AA'B}{}^{A'}$ (see Penrose and Rindler [P 11]).

We have $F = \mathrm{Re}\, F_4 = -\mathrm{Re}(*F_3)$, so that $F_{AA'BB'} = -\tfrac{1}{2}(\lambda_A\nabla_{BA'}\bar{\lambda}_{B'} - \lambda_B\nabla_{AB'}\bar{\lambda}_{A'} + \bar{\lambda}_{A'}\nabla_{AB'}\lambda_B - \bar{\lambda}_{B'}\nabla_{BA'}\lambda_A)$. Hence, $\psi_{AB} = -\tfrac{1}{4}(\lambda_A\nabla_{BA'}\bar{\lambda}^{A'} - \lambda_B\nabla_A{}^{A'}\bar{\lambda}_{A'} + \bar{\lambda}_{A'}\nabla_A{}^{A'}\lambda_B - \bar{\lambda}^{A'}\nabla_{BA'}\lambda_A) = \tfrac{1}{4}(\lambda_A\nabla_B{}^{A'}\bar{\lambda}_{A'} + \lambda_B\nabla_A{}^{A'}\bar{\lambda}_{A'} - \bar{\lambda}_{A'}\nabla_A{}^{A'}\lambda_B - \bar{\lambda}_{A'}\nabla_B{}^{A'}\lambda_A) = \tfrac{1}{2}(\lambda_{(A}\nabla_{B)}{}^{A'}\bar{\lambda}_{A'} - \bar{\lambda}_{A'}\nabla_{(A}{}^{A'}\lambda_{B)}) = \phi_{AB}$ □

The Ludvigsen-Vickers "generalized linkage" is now given by:

$$Q(\lambda;S_r) = \frac{1}{4\pi}\int_{S_r} *F(\lambda;S_r) \qquad 3.4.20,$$

for all $r \in [0,\infty)$ and $\lambda \in \mathbb{S}(S)$. Here, we have written the LVN 2-form F as $F(\lambda;S_r)$ to emphasize the dependence on the descriptor λ and to conform with the notation of 3.4.1. Once the asymptotically constant spinor field λ has been specified on the whole of H, the quantity $Q(\lambda;S_r)$ may be evaluated for any of the embedded 2-surfaces S_r, in addition to the original 2-surface $S \equiv S_o$. Before considering the use of Q as a quasi-local momentum, we first mention its asymptotic properties:-

In appendix (a) of [L10], it is demonstrated that $\lim_{r\to\infty} Q(\lambda;S_r)$ is precisely the Bondi 4-momentum (see equation 3.3.16) corresponding to the translation generator $\mathring{\xi} \equiv \mathring{\lambda} \otimes \overline{\mathring{\lambda}}$ given by the asymptotic limit of $\xi \equiv \lambda \otimes \overline{\lambda}$, and the cut S_∞. In particular, $\lim_{r\to\infty} Q(\lambda;S_r)$ depends only on $\mathring{\xi}$ and it is not necessary to impose any propagation equation on ξ (or, equivalently, on λ).

In order to utilize $Q(\lambda;S_r)$ as a measure of quasi-local momentum, it is necessary to determine the asymptotically constant spinor field λ on the entire null hypersurface H; up until now, λ has been unrestricted apart from the requirement that it satisfy the asymptotic constancy conditions 3.4.10, 3.4.11. In other words, we must propagate the asymptotic value $\mathring{\lambda}$ of λ using some propagation equation on H. We may think of such a propagation equation as providing a means of dragging down the asymptotic spin space $\mathbb{S}(S)$ to a particular embedded 2-surface S_r in the physical spacetime (M,g).

The question is, which propagation equation should we use? The choice is important because the value of the quasi-local momentum given by 3.4.20 will depend crucially upon the propagation equation. Three physically reasonable conditions to impose on the propagation equation are:- (i) it reduces to parallel propagation for (M,g) a flat spacetime; (ii) the corresponding quasi-local momentum satisfies the momentum-gain inequality, $Q(\lambda;S_{r'}) \geq Q(\lambda;S_r)$ for $r' > r$ provided that $\xi \equiv \lambda \otimes \overline{\lambda}$ is future directed and that the dominant energy condition holds in a neighbourhood of H; (iii) in the case of linearized gravity, $Q(\lambda;S_r)$ reduces to the usual expression for the total 4-momentum component linked by S_r.

Ludvigsen and Vickers [L10] write down the propagation equation:

$$o^A \bar{o}^{A'} \nabla_{BA'} \lambda_A = 0 \qquad 3.4.21,$$

and show that any asymptotically constant spinor field on H satisfying this equation is determined over the whole of H. Indeed, equation 3.4.21 may be written as:

$$o^A \text{þ} \lambda_A \equiv K\lambda \equiv \text{þ}\, \lambda_o + \kappa\lambda_1 = 0 \qquad 3.4.22,$$

$$o^A \eth' \lambda_A \equiv R\lambda \equiv \eth' \lambda_o + \rho\lambda_1 = 0 \qquad 3.4.23,$$

where the differential operators K, R are defined respectively by equations 6.2.24, 6.2.26, and κ, ρ are GHP spin coefficients (see Geroch et al. [G 7]). Equations 3.4.22 simplifies slightly after noting that the spin frame chosen corresponds to the Hawking gauge (see Hawking [H 4]), and hence, in particular, $\kappa = 0$. The Ludvigsen-Vickers propagation equation now reduces to:

$$\text{þ}\, \lambda_o = 0 \qquad 3.4.24,$$

$$\eth' \lambda_o + \rho\lambda_1 = 0 \qquad 3.4.25.$$

Note that $\text{þ}\, \lambda_o \equiv \dfrac{\partial \lambda_o}{\partial r}$, so that equation 3.4.24 determines the value of λ_o on H, given the asymptotic value $\mathring{\lambda}_o$. Equation 3.4.25 may then be used to define λ_1 on H. Thus, any asymptotically constant spinor field is uniquely determined by the Ludvigsen-Vickers propagation equation 3.4.21 or, equivalently, 3.4.24 and 3.4.25.

We refer the reader to [L 10] for a demonstration that equation 3.4.21 satisfies requirements (i), (ii) and (iii).

Given the propagation equation, the quantity $Q(\lambda; S_r)$ now defines a measure of quasi-local 4-momentum threading through the embedded 2-surface S_r, for any $r \in [0,\infty)$. The particular component of 4-momentum picked out will depend on the choice of descriptor $\lambda \in \mathbb{S}(S)$.

Given $\lambda \in \mathbb{S}(S)$, we have the uniquely defined asymptotically constant vector field $\xi \equiv \lambda \otimes \bar{\lambda}$, and ξ is determined by its asymptotic component $\mathring{\xi} \equiv \mathring{\lambda} \otimes \bar{\mathring{\lambda}}$, the restriction to S_∞ of a generator of BMS translations. Note also that $Q(\lambda;S_r)$ depends linearly on ξ. Therefore, we may define $Q_{LV}(\cdot;S_r) \in (LT)^*$ by $Q_{LV}(\alpha n;S_r) = Q(\lambda;S_r)$, where $\lambda \in \mathbb{S}(S)$ is such that the asymptotically constant vector field ξ corresponding to $\mathring{\xi} \equiv \alpha n|S_\infty$ is given by $\xi = \lambda \otimes \bar{\lambda}$, for all $\alpha n \in LT$.

<u>Definition (3.4)5</u>: Q_{LV} is called the <u>Ludvigsen-Vickers quasi-local momentum map</u>.

Note that an alternative expression for Q_{LV} is given by:

$$Q_{LV}(\alpha n;S_r) = -\frac{1}{4\pi}\int_{S_r} (\rho'|\lambda_o|^2 + \rho|\lambda_1|^2)dS_r \qquad 3.4.26.$$

This equation is obtained by using equation 3.4.25 and then using Stokes' theorem in the form $\int_{S_r} \eth(\bar{\lambda}_o, \lambda_1)dS_r = 0$.

Let us now return to the propagation equation 3.4.21 used in the definition of the Ludvigsen-Vickers quasi-local momentum map. As mentioned above, this equation satisfies the physically desirable conditions (i), (ii) and (iii). We now demonstrate that the equation arises in a natural geometric way, namely as the null limit of the Sen-Witten equation 2.3.12. Since the Sen-Witten equation is an essential ingredient in the Witten spinorial proof of the positive energy conjecture at spatial infinity (see [W 9] and section 3.3), our result gives a connection between the Ludvigsen-Vickers null framework on the one hand and the spatial framework on the other.

Another link is provided by the fact that the LVN 2-form may also be used in the definition of the ADM 4-momentum (see Israel and Nester [I 9]).

In fact, our result is slightly more general in that we allow the presence of a Maxwell field. There is also the possibility of introducing a general Yang-Mills field and considering the resulting propagation equation in the context of the quasi-local charge framework introduced by Tod [T 4]. Here, we restrict our attention to the Einstein-Maxwell case.

In what follows, our notation is that of sections 2.3 and 3.2. The first step is to generalize the Sen-Witten equation 2.3.12 so as to include an electromagnetic field Ω.

Let (M,g,Ω) be a spacetime satisfying the Einstein-Maxwell equations:

$$\mathrm{Ein}(g) = -8\pi\, T(g,\Omega) \qquad 3.4.27,$$

where

$$T(g,\Omega) = -\frac{1}{4\pi}\left(\Omega\cdot g^{-1}\cdot\Omega + \tfrac{1}{4}\|\Omega\|^2 g\right) \qquad 3.4.28,$$

is the stress-energy-momentum tensor field of the Maxwell field $\Omega \in \Omega^2(M)$. Let ψ denote the unique second rank symmetric spinor field determined by Ω. Then the spinorial version of 3.4.28 is:

$$T_{AA'BB'} = \frac{1}{2\pi}\,\psi_{AB}\,\overline{\psi}_{A'B'} \qquad 3.4.29,$$

and the Einstein-Maxwell equations reduce to:

$$\Phi_{AA'BB'} = 2\psi_{AB}\,\overline{\psi}_{A'B'} \qquad 3.4.30,$$

where -2Φ is the image of the trace-free part of $\mathrm{Ric}(g)$ under the Infeld-Van der Waerden isomorphism (see Penrose and Rindler [P 11]).

We now consider a generalization of the Sen-Witten equation which includes the Maxwell field ψ. This generalization is the Weyl spinor version of the equation introduced by Gibbons and Hull [G 8]:-

Let $f: N \hookrightarrow M$ be a codimension one spacelike embedding of N in (M,g) and let S_f denote the bundle of contravariant unprimed Weyl spinors over N (see definition (2.3)6).

Definition (3.4)6: The Maxwell-Sen-Witten operator $\tilde{D}^f: \Gamma(S_f \oplus \overline{S}^*_f) \longrightarrow \Gamma(f^*\Lambda^1(M) \otimes (S_f \oplus \overline{S}^*_f))$ is defined by

$\tilde{D}^f(\lambda^A, \mu_{A'}) = (D^f_{BB'}\lambda^A - h_{BB'}{}^{CC'}\psi_C{}^A\mu_{C'},\ D^f_{BB'}\mu_{A'} + h_{BB'\,CC'}\overline{\psi}_{A'}{}^{C'}\lambda^C)$,

for all $(\lambda,\mu) \in \Gamma(S_f \oplus \overline{S}^*_f)$.

By taking the ε-trace of $\tilde{D}^f$, we obtain the Maxwell-Sen-Witten equations:

$$D^f_{AA'}\lambda^A - h_{AA'}{}^{BB'}\psi_B{}^A\mu_{B'} = 0 \qquad 3.4.31,$$

$$D^f_{AA'}\mu^{A'} + h_{AA'BB'}\overline{\psi}^{A'B'}\lambda^B = 0 \qquad 3.4.32,$$

for $(\lambda,\mu) \in \Gamma(S \oplus \overline{S}^*_f)$. For convenience, we consider only equation 3.4.31; the second equation 3.4.2 may be treated in an entirely analogous fashion.

Now let S be an embedded, spacelike, closed 2-surface in (M,g) and let $\sigma: (0,\delta) \longrightarrow \mathrm{Emb}_S(S \times I, M)$ be the corresponding curve of codimension one spacelike embeddings. For each $s \in (0,\delta)$, we may construct the Maxwell-Sen-Witten equation 3.4.31 using the corresponding projection $(h_s)_{AA'}{}^{BB'} = \varepsilon_A{}^B\varepsilon_{A'}{}^{B'} - (t_s)_{AA'}(t_s)^{BB'}$. We have

$$t_s = \frac{a}{\sqrt{2}}(s^{-1}\,l + sn) \qquad 3.4.33,$$

where we have taken the embedding F_s to be an inclusion, for all

$s \in (0,\delta)$. Note that $a^2 = 1$ since t is the unit normal, but it is convenient to maintain a as an additional parameter; until the null limit is taken, a^2 must be taken as being equal to unity, but after the limit has been taken, there exist additional possibilities for the value of a^2.

For convenience, let us write $A = \dfrac{a}{s\sqrt{2}}$ and $B = \dfrac{as}{\sqrt{2}}$, so that $A = g(t,n)$ and $B = g(t,l)$. We consider each of the two terms in equation 3.4.31 separately and, in order to take the strain off the notation, we suppress mention of the parameter s:-

The first term is $D_{AA'}\lambda^A = h_{AA'}{}^{BB'}\nabla_{BB'}\lambda^A$

$$= (\varepsilon_A{}^B\varepsilon_{A'}{}^{B'} - t_{AA'}t^{BB'})\nabla_{BB'}\lambda^A$$

$$= ((o_A\iota^B - \iota_A o^B)(\bar{o}_{A'}\bar{\iota}^{B'} - \bar{\iota}_{A'}\bar{o}^{B'}) - (Ao_A\bar{o}_{A'} + B\iota_A\bar{\iota}_{A'})$$

$$(Ao_B\bar{o}_{B'} + B\iota_B\bar{\iota}_{B'}))\nabla_{BB'}\lambda^A$$

$$= -(o^A\bar{o}_{A'}\iota^B\bar{\iota}^{B'} - \iota^A\bar{o}_{A'}o^B\bar{\iota}^{B'} - o^A\bar{\iota}_{A'}\iota^B\bar{o}^{B'} + \iota^A\bar{\iota}_{A'}o^B\bar{o}^{B'}$$

$$- A^2\, o^A\bar{o}_{A'}o^B\bar{o}^{B'} - AB\,\iota^A\bar{\iota}_{A'}o^B\bar{o}^{B'} - ABo^A\bar{o}_{A'}\iota^B\bar{\iota}^{B'}$$

$$- B^2\,\iota^A\bar{\iota}_{A'}\iota^B\bar{\iota}^{B'})\nabla_{BB'}\lambda_A$$

$$= -((1 - AB)\bar{o}_{A'}o^A\,\text{þ}'\lambda_A - \bar{o}_{A'}\iota^A\,\eth\lambda_A - \bar{\iota}_{A'}o^A\,\eth'\lambda_A$$

$$+ (1 - AB)\bar{\iota}_{A'}\iota^A\,\text{þ}\,\lambda_A - A^2\bar{o}_{A'}o^A\,\text{þ}\,\lambda_A - B^2\,\bar{\iota}_{A'}\iota^A\,\text{þ}'\lambda_A)$$

$$= ((AB - 1)T + A^2K - iR')\bar{o}_{A'} - i((AB - 1)T' + B^2K' + iR)\bar{\iota}_{A'},$$

where we have written $K \equiv K\lambda$, etc. (see equations 6.2.24 - 6.2.27). Thus, we have proved equation 6.2.43.

The second term of equation 3.4.31 is $-h_{AA'}{}^{BB'}\psi_B{}^A\mu_{B'}$

$$= -t^A{}_{A'}t^B{}_{B'}\psi_{AB}\mu^{B'} = -(A^2o^Ao^B\bar{o}_{A'}\bar{o}_{B'} + AB\iota^Ao^B\bar{\iota}_{A'}\bar{o}_{B'}$$

$$+ ABo^A\iota^B\bar{o}_{A'}\bar{\iota}_{B'} + B^2\iota^A\iota^B\bar{\iota}_{A'}\bar{\iota}_{B'})\psi_{AB}\mu^{B'} = (A^2\psi_o\mu_{o'} + AB\,\psi_1\mu_{1'})\bar{o}_{A'}$$

$+ (AB\psi_1\mu_{o'} + B^2\psi_2\mu_{1'})\bar{\iota}_{A'}$, where $\psi_o = \psi_{AB}o^Ao^B$, $\psi_1 = \psi_{AB}o^A\iota^B$ and $\psi_2 = \psi_{AB}\iota^A\iota^B$.

Hence, equation 3.4.31 may be written:

$$(AB - 1)T + A^2K - iR' + A^2\psi_o\mu_{o'} + AB\psi_1\mu_{1'} = 0 \qquad 3.4.34,$$

$$i(AB - 1)T' + iB^2K' - R - AB\psi_1\mu_{o'} - B^2\psi_2\mu_{1'} = 0 \qquad 3.4.35.$$

Substituting for A, B and multiplying 3.4.34 throughout by s^2 gives:

$$s^2(\tfrac{1}{2}a^2-1)T + \tfrac{1}{2}a^2K - is^2R' + \tfrac{1}{2}a^2\psi_o\mu_{o'} + \tfrac{1}{2}s^2a^2\psi_1\mu_{1'} = 0 \qquad 3.4.36,$$

$$i(\tfrac{1}{2}a^2-1)T' + \tfrac{1}{2}is^2a^2K' - R - \tfrac{1}{2}a^2\psi_1\mu_{o'} - \tfrac{1}{2}s^2a^2\psi_2\mu_{1'} = 0 \qquad 3.4.37.$$

The null limit is now achieved by taking the limit as $s \to 0$. Substituting for T, etc., we obtain

$$a^2(\text{Þ}\,\lambda_o + \kappa\lambda_1 + \psi_o\mu_{o'}) = 0 \qquad 3.4.38,$$

$$(\tfrac{1}{2}a^2-1)(\text{Þ}\,\lambda_1 + \tau'\lambda_o) + (\eth'\lambda_o + \rho\lambda_1) + \tfrac{1}{2}a^2\psi_1\mu_{o'} = 0 \qquad 3.4.39,$$

and an analogous calculation of the null limit of equation 3.4.32 yields the following:

$$a^2(\text{Þ}\mu_{o'} + \bar{\kappa}\mu_{1'} - \bar{\psi}_{o'}\lambda_o) = 0 \qquad 3.4.40,$$

$$(\tfrac{1}{2}a^2-1)(\text{Þ}\,\mu_{1'} + \bar{\tau}'\mu_{o'}) + (\eth\,\mu_{o'} + \bar{\rho}\mu_{1'}) - \tfrac{1}{2}a^2\bar{\psi}_{1'}\lambda_o = 0 \qquad 3.4.41.$$

To summarize the above, we have proved the following:

<u>Proposition (3.4)7</u>: The null limit of the Maxwell-Sen-Witten equations is given by equations 3.4.38 - 3.4.41.

Now the null limit has been taken, we are at liberty to choose the value of the parameter a^2. The simplest and most natural choice is to take $a^2 = 2$ (and not $a^2 = 1$).

For applications to spinor propagation on the outgoing null hypersurface arising from an embedded 2-surface, it is appropriate to use the Hawking gauge (see Hawking [H 4] and section 6.2). Having made these two specializations, the null limit of the Maxwell-Sen-Witten equation becomes:

$$Þ \lambda_o + \psi_o \mu_{o'} = 0 \qquad 3.4.42,$$

$$ð' \lambda_o + \rho\lambda_1 + \psi_1 \mu_{o'} = 0 \qquad 3.4.43,$$

$$Þ \mu_{o'} - \overline{\psi}_{o'} \lambda_o = 0 \qquad 3.4.44,$$

$$ð \mu_{o'} + \rho\mu_{1'} - \overline{\psi}_{1'} \lambda_o = 0 \qquad 3.4.45.$$

Note that equations 3.4.42 - 3.4.45 are precisely those used by Ludvigsen and Vickers [L 11] in their proof of the mass-charge inequality $m \geqslant |e|$ in Einstein-Maxwell spacetimes.

A further specialization is obtained by putting $\psi = 0$, i.e. by taking a vacuum spacetime. Then we need only consider the first two (or the second two) of the four equations. We get:

$$Þ \lambda_o = 0 \qquad 3.4.46,$$

$$ð' \lambda_o + \rho\lambda_1 = 0 \qquad 3.4.47,$$

so we obtain the original Ludvigsen-Vickers propagation equations 3.4.24 and 3.4.25 (or, equivalently, 3.4.21). Thus, the spinor null propagation equations used in the Ludvigsen-Vickers treatment of positivity of the Bondi mass, inequalities relating mass-area and mass-charge, and also quasi-local 4-momentum, arise from the null limit of the (Maxwell-)Sen-Witten equation which is a natural spinor propagation equation on spacelike hypersurfaces.

Another use for the Ludvigsen-Vickers propagation equation is in a definition of quasi-local angular momentum (see [L 10]). The most convenient way to write the charge integral is:

$$Q(\lambda;S_r) = \frac{1}{4\pi}\int_{S_r} \lambda_{(A}\lambda_{B)}\ o^A \iota^A\ dS_r \qquad 3.4.48,$$

for all asymptotically constant spinor fields λ on the null hypersurface H. The propagation equation used is the same as the one used for defining quasi-local 4-momentum. Indeed, this propagation equation is the only one for which $\lim_{r\to\infty} Q(\lambda;S_r)$ exists. In fact, $\lim_{r\to\infty} Q(\lambda;S_r)$ is equal to the Bramson asymptotic angular momentum [B 20] given by:

$$Q(\lambda;S_\infty) = \frac{1}{4\pi}\int_{S_r} (\hat{\bar{\Psi}}_1 - 2\hat{\bar{\sigma}}\ \eth'_o\ \hat{\sigma} - \eth_o(\hat{\sigma}\ \hat{\bar{\sigma}}))\,|\mathring{\lambda}_o|^2\ dS_\infty \qquad 3.4.49,$$

where $\hat{\Psi}_1 = \hat{C}_{abcd}\ \hat{l}^a\ \hat{m}^b\ \hat{l}^c\ \hat{n}^d \equiv f^{-5}\Psi_1$. It may also be demonstrated that the quasi-local angular momentum expression 3.4.48 gives the expected value of the total angular momentum linked by the 2-surface S_r in the case of linearized gravity.

The evidence supporting the claim that the Ludvigsen-Vickers propagation equation is the natural one to use on a null hypersurface now seems quite strong:- From a physical viewpoint, this equation is an essential ingredient in a physically reasonable definition of both 4-momentum and angular momentum at the quasi-local level, and, from a geometric viewpoint, the equation is obtained by taking a null limit of the Sen-Witten equation.

To conclude our discussion on the Ludvigsen-Vickers approach to quasi-local moments, we mention the advantages and disadvantages compared with other approaches to the problem.

The major advantage of the Ludvigsen-Vickers approach is that it makes essential use of null hypersurfaces rather than spacelike/asymptotically null hypersurfaces. Null hypersurfaces gel especially well with spinor and conformal ideas and certain simplifications occur. For example, the propagation equations 3.4.24, 3.4.25 possess an extremely simple form; there is a decoupling into a radial equation relating λ_o to its asymptotic value and a 2-surface equation giving the value of λ_1. Moreover, it is easier to prove existence and uniqueness results for null propagation equations than it is for the elliptic equations defined on spacelike hypersurfaces.

Another advantage is that it is possible to <u>compare</u> the quasi-local momenta linking two distinct elements of Cone(M,g):- Suppose $S, S' \in \text{Cone}(M,g)$ with $H \cap \mathcal{I}^+ = S_\infty$ and $H' \cap \mathcal{I}^+ = S'_\infty$, where H,H' are the outgoing null hypersurfaces corresponding to S,S' respectively. The asymptotic spin spaces $\mathbb{S}(S)$, $\mathbb{S}(S')$ may be identified using Bramson's alinement of frames technique (see Bramson [B 19], [B 21]) and the Ludvigsen-Vickers propagation equation is used to propagate $\mathbb{S}(S) \equiv \mathbb{S}(S')$ to the embedded 2-surfaces S and S'. Once the spinor fields defining the quasi-local momenta linking S and S' have been identified in this way, the quasi-local momenta themselves may be related by flux integrals over H,H' and the submanifold H_∞ of $\mathcal{I}^+$ with $\partial H_\infty = S'_\infty \cup (-S_\infty)$. For any of the null hypersurfaces H, H', H_∞, the flux integrand is of course given by $*J = \frac{1}{4\pi} *\delta F$, where F is the LVN 2-form (see definition (3.4)3). Thus, for 4-momentum, we have:

$$Q(\lambda;S') - Q(\lambda;S) = \int_H *J + \int_{H_\infty} *J - \int_{H'} *J \qquad 3.4.50,$$

for the case in which S'_∞ lies entirely to the future of S_∞. The expression $Q(\lambda;S') - Q(\lambda;S)$ is the flux of 4-momentum between S and S' (see Swift and Vickers [S 25]).

Note that in the Penrose quasi-local framework, there is no analogous method of comparing the quasi-local moments linking two distinct embedded 2-surfaces in spacetime. We also remark that a connection may be made between the Ludvigsen-Vickers approach on the one hand and the Penrose approach on the other:- It may be demonstrated that the asymptotic constancy/alinement of frames condition used by Ludvigsen and Vickers is a special case of the Penrose superficial twistor equation applied on a cut of $\mathscr{I}^+$ (see Penrose [P 8]). Shaw [S 17] has discussed certain aspects of the relationship between Sen-Witten propagation on spacelike hypersurfaces and the Penrose superficial boundary conditions. However, the expressions written down by Shaw suffer from the disadvantage that they depend on the spacelike hypersurface spanning the embedded 2-surface under consideration. The use of <u>null</u> hypersurfaces as discussed in this section should lead to a clearer understanding of the links between the various quasi-local approaches. Any a priori spacelike equations may be applied on null hypersurfaces by taking the null limit as was done for the Maxwell-Sen-Witten equation above.

Let us now mention a couple of problems encountered with the Ludvigsen-Vickers approach. The first is the restriction to embedded 2-surfaces in Cone(M,g), i.e. to those 2-surfaces whose corresponding outgoing null hypersurface may be extended to $\mathscr{I}^+$ without the occurrence of caustics. For a general embedded 2-surface, caustics will occur. However, since the structure of the Ludvigsen-Vickers propagation equation is such that singularities of the spinor field λ

are, in general, integrable, the flux integrals in, for example, equation 3.4.50 may still be calculated provided that the cuts defined by the outgoing null hypersurfaces are sufficiently undistorted so as to allow the existence of asymptotically constant spinor fields.

Of course, the main use of the null hypersurface H is to allow the propagation to S of the asymptotic spin space $\mathbb{S}(S)$. Although the null hypersurface H is naturally associated with S, it would be better if a spin space could be constructed without the need of propagating in from null infinity. We could then construct quasi-local kinematical quantities associated with an embedded 2-surface in a general spacetime and not just in asymptotically flat spacetimes. However, in such an intrinsic framework, it would still be necessary to find a method of comparing the quasi-local moments linked by distinct 2-surfaces.

Another problem arising in the Ludvigsen-Vickers definition of quasi-local momentum has been pointed out by Shaw [S 19]. This paper discusses the application of quasi-local definitions to "large surfaces", i.e. surfaces constructed as cuts of an outgoing null hypersurface in a neighbourhood of future null infinity. Shaw's calculations demonstrate that in a stationary spacetime, the Ludvigsen-Vickers 4-momentum is physically reasonable to first order in the perturbation parameter, but at third order, unphysical contributions appear. These unphysical terms do not appear in the Penrose quasi-local 4-momentum, at least not up to third order in stationary spacetimes.

The discussion in this section has demonstrated that a certain amount of progress towards a good definition of quasi-local kinematical quantities in general relativity theory has been made over the past few years. The quasi-local framework seems to require a fundamental involvement of spinorial ideas and it appears that spinors will play an important rôle in any future development of this area. In any case, the theory of general relativity will not be complete until a good definition of quasi-local moments has been constructed, for it is only through such a definition that we can establish a relationship between the motion of the sources on the one hand and the asymptotic structure of spacetime on the other. Indeed, when gravitational wave astronomy becomes important, it will be necessary to utilize such definitions in order to deduce properties of astrophysical objects from (approximately) asymptotically detected gravitational radiation.

4.0 Introduction

This chapter concerns itself with various questions of invariance in geometry and in physics. The everywhere invariance of the title may be understood on two levels: On the one hand there is the concept of everywhere invariance arising in a group action situation which we introduced in our papers [S 26], [S 27]. This particular concept is discussed at length in this chapter. On the other hand, and on a meta-level, everywhere invariance may be regarded as the theme underlying studies in geometry and physics in which natural groups act on natural spaces - the useful quantities are then invariant under these actions. Indeed, when considering the influence and application of geometry in physics, we should perhaps be guided by the philosophy of Kobayashi, "All geometric structures are not created equal; some are creations of gods while others are products of lesser human minds" (see p. V in [K 6]). Only structures in the former category should be regarded as candidates for inclusion in physical theories, since then and only then may we hope for everywhere invariance.

The aims of this chapter may be divided into three, although there is overlap between these categories, as follows:

Firstly we discuss the idea of everywhere invariance - this concept arises when a G-action on a set S is given, and we wish to consider the behaviour of subsets of S under the group action. We expand and generalize the ideas contained in [S 26], [S 27], these papers themselves developing and geometrizing earlier work of d'Inverno and Smallwood [I 2] concerning functional form invariance. We present the algebraic framework of everywhere invariance (and of related concepts) in a unified fashion before applying the ideas to situations in geometry and general relativity. Further possible developments and applications are also mentioned.

Secondly we consider the general algebraic set up of everywhere invariance in the specific case of the action of the diffeomorphism group on the space of metrics on a given manifold. Everywhere invariance has both practical applications, for example in finding isometry groups of metrics, and also inter-relationships with other natural structures on manifolds. We demonstrate both these aspects of everywhere invariance.

Our third aim relates to the meta-meaning of everywhere invariance. We consider natural structures on a manifold M, in particular natural groups and their action on $Met(M)$, the space of metrics (of a given signature) on M, and we discuss the use of these structures in geometry and general relativity. This third aim also includes the desire to make links between this chapter and material contained in the rest of the thesis.

Having stated our aims, we now give a more detailed description of how this chapter is arranged:-

The space $Met(M)$ has made several important appearances in this thesis, notably in sections 1.4, 1.5, 1.6 and 2.2, and plays an important rôle in this chapter. We therefore devote section 4.1 to a more detailed study of this space. We also consider the action of natural groups on $Met(M)$ and these will provide us with some of our examples of the phenomenon of everywhere invariance. In these examples we consider embedded subspaces of $Met(M)$. These subspaces may be regarded as parameterized families of metrics or, equivalently, as metrics of a given "functional form". In this context, everywhere invariance may be used as a tool for investigating the relationship between the set of transformations leaving the given family invariant (i.e., those transformations leaving invariant the functional form of the metric) on the one hand and the isometry group of particular metrics in the family on the other. When considering these parameterized families of metrics we are re-emphasizing the importance of embeddings:- In Chapters Two and Three we discussed embeddings into M itself and, in the case when M was the manifold underlying spacetime, certain kinds of embeddings turned out to be very useful. Here we look at embeddings $F \hookrightarrow \mathrm{Met}(M)$, where F is the manifold parameterizing the family of metrics under consideration and we demonstrate the utility of considering a particular $g \in \mathrm{Met}(M)$, not as an individual metric, but as a member of such a parameterized family. Note that families of metrics often arise as solutions to the Einstein equations (e.g., the Kerr-Newman solutions are parameterized by a subspace of $\mathbb{R}^3$ and the pp-wave solutions possess an infinite dimensional parameter space - we return to pp-waves in section 4.5). For other contexts in which families of spacetime metrics have been studied, see Szekeres [S28] who considers solutions of the Einstein equations involving arbitrary functions and Geroch [G2] who looks at limits of spacetimes.

Returning now to the content of section 4.1, we should mention the analysis, or rather the lack of it, involved in our dealings with infinite dimensional spaces. We make no attempt to discuss in detail the differentiable structure on these spaces nor to consider other aspects of the global analysis involved. We note, however, that by working in the appropriate category of smooth manifolds and maps (inverse limit of Hilbert (ILH), or, better, the tame category of Nash-Moser-Hamilton), our formal algebro-geometrical results are valid at the analytical level also. For global analytical details we refer the reader to Ebin [E 4], Ebin and Marsden [E 6], Lang [L 2], and Michor [M 19], and also to the excellent reviews of Adams et al. [A 3], Hamilton [H 2] and Milnor [M 8] and references therein). We also refer the reader to references cited in section 2.2 where we discussed spaces of embeddings. To summarize our position on analysis - we proceed formally when working with infinite dimensional spaces, but any manifolds and maps (and hence group actions) will be smooth in the ILH (or tame) sense.

The main discussion of section 4.1 will concern Met(M), its submanifolds and its quotients. Note that in the case of positive definite metrics (i.e., Riemannian manifolds), the structure of Met(M) is topologically trivial, but in the case of indefinite metrics (e.g. Lorentzian manifolds) the topology of Met(M) may be much more complex. Since we are mainly interested in the geometry of Met(M), rather than in its topology, our discussion will be independent of the particular signature of the metrics, for example, Met(M) is always open in $S_2(M)$. However, we do make some brief remarks concerning the topology of Met(M) in the pseudo-Riemannian case.

A particularly important rôle is played by quotients of Met(M) under certain group actions. For example, the main group action in this chapter, and, indeed, in geometry in general, is that of the diffeomorphism group on the space of metrics. The quotient of this action, denoted Geom(M), may be regarded as the space of all geometries on the manifold M. Indeed, natural "nice" structures in (pseudo-)Riemannian geometry such as {Einstein manifolds}, {homogeneous spaces} and {space forms} are unions of orbits of Diff(M) acting on Met(M) and these structures may thus be regarded as subspaces of Geom(M).

In addition to being important in geometry itself, Geom(M) is also important in physics. In general relativity the spacetimes (M,g), $(M,\phi^* g)$ are regarded as physically equivalent for all $\phi \in \mathrm{Diff}(M)$, and so a spacetime is really given by $[g]_{\mathrm{Diff}(M)} \in \mathrm{Geom}(M)$. Various studies of Geom(M) have been made both for Lorentzian signature/4-manifolds (see, for example Isenberg and Marsden [I 7], Isenberg [I 6]) and for Riemannian signature/3-manifolds (when Met(M) is called superspace in general relativity (see Fischer [F 1]). Unfortunately, the space Geom(M) is not, in general, a manifold, since different metrics have isometry groups with different dimensions or different numbers of components. Geom(M) is, in fact, stratified by manifolds (see Fischer [F 1], Bourguignon [B 16]) but possesses singularities corresponding to metrics with symmetries. A natural resolution of the singularities of Geom(M) may be performed, see Fischer [F 2], [F 3], and in section 4.1 we describe these ideas and also propose a variant on Fischer's methods which utilizes the canonical bundle introduced in section 1.4.

A very important result of Ebin-Palais [E 4] is the existence of a slice of the action of Diff(M) on Met(M) (in the positive definite case). The slice theorem simplifies the study of the space Geom(M) considerably and may be generalized to other contexts such as the Lorentzian case (see Isenberg and Marsden [I 7]), the action of the conformorphism group (section 6.2) on Met(M) (see Fischer and Marsden [F 5]), and also to the action of Gau(P) on Conn(P) for some principal G-bundle (see Singer [S 20]).

Another important action for which there should exist a slice is that of Aut GL(M) acting on Met(M). This action is investigated and we show how the semi-direct product structure Aut GL(M) = Diff(M) $\ltimes$ Gau GL(M) (see 6.1.13) may be further decomposed. The action of the various natural subgroups of Aut GL(M) on Met(M) leads to useful geometrical structures, one of which is the generalized conformal structure which we first introduced in [S 26]. These geometrical structures all turn out to be everywhere invariant - in both senses of the phrase!

Having discussed Met(M) and related spaces, we turn to the algebraic theory of everywhere invariance. We present definitions and results in the full generality of a group G acting on a set S and define the concepts of everywhere invariance, inessential invariance and total invariance for subsets of S. The reason for introducing these concepts is that they provide a framework for discussing the interaction between stabilizers of particular elements of S on the one hand and the behaviour of subsets of S on the other. Although our eventual use of everywhere invariance is in geometry and general relativity, in this purely algebraic section we link everywhere invariance with other important ideas in algebra such as imprimitivity. Results given in this section lead to a partial characterization of invariant subsets of a G-set S.

Section 4.3 provides various examples of the ideas introduced in section 4.2. These are mainly related to Met(M) under the action of Diff(M). An example from general relativity is analyzed in section 4.5.

Finding diffeomorphisms leaving a given geometrical structure invariant is very difficult, and we often have to resort to calculating infinitesimal symmetries, i.e. vector fields which generate 1-parameter groups of symmetries. In the case of a single metric g, any infinitesimal symmetry must satisfy the Killing equation, and conversely, any vector field satisfying the Killing equation is an infinitesimal symmetry of g. Similarly for a conformal structure C, we have the conformal Killing equation whose solution space is precisely the Lie algebra of infinitesimal conformeomorphisms (i.e. conformal Killing vector fields). In the context of everywhere invariance, we have a corresponding equation called the invariance equation. We introduce this equation in section 4.4 and demonstrate how it may be used to find symmetries in section 4.5. The invariance equation may be regarded as a generalization of the Killing equation where, instead of a single metric, we are considering an entire family of metrics. Because of the extra degrees of freedom involved, the invariance equation may be decoupled and is thus easier to solve than the corresponding Killing equation.

The philosophy underlying the use of the invariance equation in finding the symmetries of a given metric $g \in \mathrm{Met}(M)$ is to first construct an everywhere invariant embedded submanifold of Met(M) of which g is a member. We then solve the corresponding invariance equation, thus finding all infinitesimal symmetries of the submanifold. The Killing vector fields of g are among these infinitesimal symmetries and may be integrated to give the corresponding isometries.

Since the frame bundle GL(M) often plays an important rôle in the study of natural structures on M, we also give a frame version of the invariance equation.

In section 4.5 we consider a specific parameterized family of solutions to the Einstein equations, namely the pp-wave solutions. We solve explicitly the invariance equation and integrate to give the full set of symmetries for the pp-wave solutions. This kind of calculation illustrates the practical use to which the ideas of everywhere invariance may be applied. Indeed, since the isometry group is a basic piece of information about a particular solution to Einstein's equations, we may regard everywhere invariance as a tool in the classification programme. Unlike other means of classification, such as the Petrov classification, there is no simple algorithm for finding the isometry group of a given solution. Indeed, solving the Killing equation for a complicated metric is very difficult. The simplifications introduced when we generalize to the invariance equation should therefore make the classification using isometry groups a much more tractable proposition.

The final section of this chapter contains suggestions for further investigations. These topics arise from both everywhere invariance and from the study of natural structures on manifolds. The latter include various maps and group actions related to Diff(M) and Met(M) and some of these have already been used in applications to physics.

The original ideas contained in this chapter consist of the development of the ideas of everywhere invariance in sections 4.2, 4.3, 4.4 and 4.5 and also the remarks concerning the use of the canonical bundle in the resolution of the singularities of Geom(M) which we discuss in section 4.1. We have not seen a consideration of the action of Aut GL(M) on Met(M) in the literature, nor a reference to the suggestions made in section 4.6 regarding natural maps. The suggestions for further investigations in everywhere invariance will be taken up elsewhere.

4.1 Metrics

This section is arranged as follows:- First, for convenience, we review the basic structure of the space of metrics on a manifold M. We then consider the structure of Aut GL(M), the group of automorphisms of the frame bundle of M and we describe the action of Aut GL(M) on Met(M), the space of metrics. This action leads to natural geometrical structures on M. The most important subgroup of Aut GL(M) is Diff(M), the group of diffeomorphisms of M, and we study the quotient Geom(M) of Met(M) by Diff(M). Geom(M) is not a manifold but we show how its singularities may be resolved using natural techniques. Finally, we consider certain other aspects of the space of metrics in relation to material contained elsewhere in this thesis. The ideas presented in this section will be used in the remainder of this chapter and some have already been utilized above. We refer the reader to Francaviglia [F 9] (and references therein) for details concerning the differentiable structure of the spaces considered in this section. Francaviglia also reviews various techniques of infinite dimensional differential geometry applied to general relativity.

Let M be a connected, oriented, smooth n-manifold without boundary. Let Diff(M) be the group of orientation preserving diffeomorphisms of M and $S_2(M) \equiv \Gamma(\odot^2 T^* M)$ the vector space of symmetric second rank covariant tensor fields on M. For a given fixed (non-degenerate) signature, let Met(M) denote the space of metrics of the given signature on M. The spaces $S_2(M)$, Diff(M) and Met(M) are all manifolds and Diff(M) is a Lie group (see the remarks in section 4.0). Note that below we sometimes use the spaces

$S_2(I)$, $\mathrm{Diff}(I)$ and $\mathrm{Met}(I)$, where I is an infinite dimensional manifold, but we make no claims concerning the differentiable structure in this case.

The topology of $\mathrm{Met}(M)$ depends on the signature chosen. In the positive (or negative) definite case, $\mathrm{Met}(M)$ is an open convex cone in $S_2(M)$. Thus, in this case $\mathrm{Met}(M)$ is connected and, indeed, contractible. If the signature is indefinite (e.g. Lorentzian), the topology of $\mathrm{Met}(M)$ is, in general, non-trivial. For example, the set of components of $\mathrm{Met}(M)$ is parameterized by certain homology groups and can certainly have more than one element. Also, each connected component may have non-trivial topology. For more details concerning the topology of $\mathrm{Met}(M)$ in the four dimensional Lorentzian case, see Shastri et al. [S14] and references therein.

Since the signature (definite or indefinite) is non-degenerate, $\mathrm{Met}(M)$ is an open submanifold of $S_2(M)$, so we have natural identifications $T\mathrm{Met}(M) = \mathrm{Met}(M) \times S_2(M)$, and $T_g\,\mathrm{Met}(M) = S_2(M)$, for all $g \in \mathrm{Met}(M)$. The appropriate (for applications) cotangent bundle to use is the L^2-cotangent bundle and this is constructed as follows; Fix $g \in \mathrm{Met}(M)$ and consider the embedding $h \longmapsto h^{\#} \otimes \mathrm{vol}(g)$ of $S_2(M)$ into its topological dual $(S_2(M))^*$. Here $h^{\#} \otimes \mathrm{vol}(g)$ acts on $S_2(M)$ by $k \longmapsto \int_M g(h,k)\mathrm{vol}(g)$, for all $k \in S_2(M)$. We now define $T_g^*\mathrm{Met}(M) = \{h^{\#} \otimes \mathrm{vol}(g) : h \in S_2(M)\} \cong \{p \otimes \mathrm{vol}(g) : p \in S^2(M)\}$, where $S^2(M)$ is the space of symmetric second rank contravariant tensor fields on M. We take the L^2-cotangent bundle of $\mathrm{Met}(M)$ to be $T^*\mathrm{Met}(M) = \bigcup_g T_g^*\,\mathrm{Met}(M) = \mathrm{Met}(M) \times S_d^2(M)$, where $S_d^2(M) = \Gamma((\otimes^2 TM) \otimes^* \Lambda^n M)$. Note that for this cotangent bundle to be defined, we must either take M compact or restrict attention to L^2-sections. The L^2-cotangent bundle $T^*\mathrm{Met}(M)$ is a sub-bundle of the true cotangent bundle and it carries a natural (weak) symplectic form ω. In the case of positive definite metrics in dimension three, Fischer and Marsden [F 6] use a reduction of the phase space $(T^*\mathrm{Met}(M), \omega)$ in their formulation of the 3+1 initial value problem in general relativity.

The most important group action on $\mathrm{Met}(M)$ is that by the diffeomorphism group $\mathrm{Diff}(M)$ by pullback. We prefer to have a left action and so we define (lower star) $\in \mathrm{Hom}(\mathrm{Diff}(M), \mathrm{Diff}(\mathrm{Met}(M)))$ by $\phi \longrightarrow \phi_* \equiv (\phi^{-1})^*$, for all $\phi \in \mathrm{Diff}(M)$. Thus

$$(\phi_* g)(x)(v,w) = g(\phi^{-1}(x))(D\phi^{-1}(x).v,\ D\phi^{-1}(x).w) \qquad 4.1.1,$$

for all $v,w \in T_xM$, $x \in M$, $g \in \mathrm{Met}(M)$, $\phi \in \mathrm{Diff}(M)$. We often identify $\mathrm{Diff}(M)$ with its image $\mathrm{Diff}_*(M)$ in $\mathrm{Diff}(\mathrm{Met}(M))$. The diffeomorphism group of M acts, via the action (lower star), on tensor bundles, e.g. the L^2-cotangent bundle, of $\mathrm{Met}(M)$. Indeed, $\mathrm{Diff}(M)$ leaves invariant natural structures on these bundles. For example, $\mathrm{Diff}(M)$ acts symplectomorphically on $(T^*\mathrm{Met}(M), \omega)$.

Another natural structure associated with $\mathrm{Met}(M)$ (M compact) is the map $G: \mathbb{R} \longrightarrow S_2(\mathrm{Met}(M))$; $t \longmapsto G_t$, where:

$$G_t(g)(h,k) = \int_M \{g(h,k) - t(\mathrm{trace}_g h)(\mathrm{trace}_g k)\}\mathrm{vol}(g) \quad 4.1.2,$$

for all $h,k \in T_g\mathrm{Met}(M) = S_2(M)$, $g \in \mathrm{Met}(M)$.

Suppose we take positive (or negative) definite metrics in $\mathrm{Met}(M)$ and let us assume $nt \neq 1$. Then the symmetric rank two covariant tensor field G_t on $\mathrm{Met}(M)$ is (weakly) non-degenerate; Suppose $g \in \mathrm{Met}(M)$ and $G_t(h,k) = 0$, for all $k \in S_2(M)$. Then, in particular, $G_t(h, h+s(\mathrm{trace}_g h)g) = 0$, where $s = {}^t/1-nt$, so

we have $0 = \int_M \{g(h,h) + s(\mathrm{trace}_g h)g(h,g) - t(\mathrm{trace}_g h)^2$
$- nst(\mathrm{trace}_g h)^2\}\mathrm{vol}(g) \;=\; \int_M \{g(h,h) + (s(1-nt)-t)(\mathrm{trace}_g h)^2\}\mathrm{vol}(g)$
$= \|h\|_g^2$, where $\|\cdot\|_g$ is the L^2-norm on $S_2(M)$ induced by g.
Hence, $G_t(g)(h,k) = 0$ for all $k \in S_2(M)$ implies $h = 0$, and thus G_t is (weakly) non-degenerate. So for $nt \neq 1$ and definite signature, G_t defines a (weak) metric on the manifold $\mathrm{Met}(M)$. The signature of G_t depends on the value of t.

Note that $g(h,k) = \mathrm{trace}(g^{-1}hg^{-1}k)$ and $\mathrm{trace}_g h = g(g,h)$ for all $h,k \in S_2(M)$, $g \in \mathrm{Met}(M)$. Now let $\phi \in \mathrm{Diff}(M)$. Then
$((\phi_*g)(\phi_*h,\ \phi_*k))(x) \;=\; (\mathrm{trace}((\phi_*g)^{-1}(\phi_*h)(\phi_*g)^{-1}(\phi_*k)))(x)$
$= \mathrm{trace}((Ag(\phi^{-1}(x))A^T)^{-1}Ah(\phi^{-1}(x))A^T(Ag(\phi^{-1}(x))A^T)^{-1}Ak(\phi^{-1}(x))A^T)$
(where $A = D\phi^{-1}(x)$)
$= \mathrm{trace}(g(\phi^{-1}(x))^{-1}h(\phi^{-1}(x))g(\phi^{-1}(x))^{-1}k(\phi^{-1}(x)))$
$= (\mathrm{trace}(g^{-1}hg^{-1}k))(\phi^{-1}(x))$
$= g(h,k)(\phi^{-1}(x))$. Hence $(\phi_*g)(\phi_*h,\phi_*k) = \phi_*(g(h,k))$, where $\phi_*f = (\phi^{-1})^*f \equiv f \circ \phi^{-1}$, for all $f \in C(M)$. Also, $\mathrm{trace}_{\phi_*g}(\phi_*h)$
$= (\phi_*g)(\phi_*g,\phi_*h) = \phi_*(g(g,h)) = \phi_*(\mathrm{trace}_g h)$. Using these formulae, we may now calculate the action of $\mathrm{Diff}(M)$ on $G_t \in S_2(\mathrm{Met}(M))$;
We have $((\phi_*)^*G_t)(g)(h,k) = G_t(\phi_*g)(D\phi_*(g).h,\ D\phi_*(g).k)$
$= G_t(\phi_*g)(\phi_*h,\phi_*k)$ (since $\phi_*: \mathrm{Met}(M) \to \mathrm{Met}(M)$ is the restriction of the linear map ϕ_* on $S_2(M)$)
$= \int_M \{(\phi_*g)(\phi_*h,\phi_*k) - t(\mathrm{trace}_{\phi_*g}(\phi_*h))(\mathrm{trace}_{\phi_*g}(\phi_*k))\}\mathrm{vol}(\phi_*g)$
$= \int_M \phi_*\{g(h,k) - t(\mathrm{trace}_g h)(\mathrm{trace}_g k)\}\phi_*\ \mathrm{vol}(g)$
$= \int_M \{g(h,k) - t(\mathrm{trace}_g h)(\mathrm{trace}_g k)\}\mathrm{vol}(g) = G_t(g)(h,k)$, for all $h,k \in S_2(M)$, $g \in \mathrm{Met}(M)$. Hence, the action of $\mathrm{Diff}(M)$ leaves the tensor field G_t invariant, for all $t \in \mathbb{R}$. In particular, in the

case when G_t is actually a metric (i.e. $nt \neq 1$, definite signature metrics in Met(M)), Diff(M) $\leq$ Isom(Met(M), G_t).

The metrics G_t on Met(M) have various useful applications. For example if we take $t = 0$, we obtain a positive definite metric on Met(M) (if the metrics in Met(M) are positive definite). The Riemannian metric G_o has been used by Ebin [E 4] in his construction of a slice of the action of Diff(M) on Met(M). Ebin also calculates the Levi-Cività connection of $G_o \in$ Met(Met(M)). The metric obtained by putting $t = 1$ $(n \neq 1)$ is called the De Witt metric and was introduced by De Witt [D 5] in connection with canonical quantum gravity. The metric G_1 may also be used in the initial value problem in general relativity (see Fischer and Marsden [F 4], who also calculate the geodesic spray of the De Witt metric). Note that G_1 is not positive definite. We refer the reader to Francaviglia [F 9] for a review of applications of the de Witt metric G_1 to topics such as the theory of superspace, the reduction of the Einstein-Hilbert action and the chronos principle.

We now make some further brief remarks concerning the structure of Met(M). We consider various group actions on this space below but, for the moment, let us restrict our attention to the action of Diff(M) on Met(M), where M is compact and the signature is definite. The most important result concerning the action 4.1.1 is that there exists a slice (see Ebin [E 4]) - this is a very powerful result, since the existence of a slice means that various structure theorems for Met(M) may be proved.

Let us recall the basic facts concerning slices (see Palais [P 1] for a survey of the ideas). Let $A \in$ Hom(G,Diff(X)) be a smooth action of the Lie group G on the manifold X. For $x \in X$

denote by Gx the orbit of x under G and by $st(x)$ the isotropy subgroup of x under G. A <u>slice</u> at x for the action A is a submanifold $S_x \subseteq X$ containing x such that (i) if $a \in st(x)$, then $aS_x = S_x$; (ii) if $a \in G$ and $(aS_x) \cap S_x \neq \emptyset$, then $a \in st(x)$; and (iii) there is a local cross-section $s: {}^{G}/st(x) \to G$ defined in a neighbourhood U of the identity coset such that the map $F: U \times S_x \to X$; $(u,y) \mapsto A(s(u),y)$ is a diffeomorphism onto a neighbourhood V of x.

If a slice exists for the action A, it may be regarded as an equivariant retraction of a neighbourhood of Gx (in X) onto Gx, and the action locally is completely determined. We may consider a slice S_x at x to be a submanifold transverse to the orbit Gx through x, which, together with a neighbourhood of the orbit, fills out an open neighbourhood of x in X. The action A is thus factored into a transitive action on Gx together with an action by the isotropy group $st(x)$ on the slice. If a slice exists, we may choose it to be an open ball in some topological vector space on which the isotropy group is represented as a group of bounded linear operators. If we regard the G-space X as a generalization of a principal G-bundle (see section 6.1), we should think of a slice as the analogue of a local section of the projection: $X \to {}^{X}/_{G}$. A slice is the best we can hope for because the action A is not, in general, free.

When G is compact, slices always exist, but this is not necessarily the case for non-compact G. On the other hand , if the existence of a slice can be proved for a non-compact action A, then many further results, analogous to those in the theory of compact transformation groups, may be derived.

The Ebin-Palais result for the existence of a slice of the (lower star) action of Diff(M) (not even <u>locally</u> compact) on Met(M) is thus a very important result and may be stated as follows:

Let Met(M) denote the space of positive definite metrics on the compact manifold M. Then, for each $g \in \mathrm{Met}(M)$, there exists a submanifold S_g of Met(M) containing g, which is diffeomorphic to a ball in (separable) Hilbert space, such that (i) if $\phi \in \mathrm{Isom}(M,g)$, then $\phi_* S_g = S_g$; (ii) if $\phi \in \mathrm{Diff}(M)$ and $(\phi_* S_g) \cap S \neq \emptyset$, then $\phi \in \mathrm{Isom}(M,g)$; and (iii) there exists a local cross-section $s\colon \mathrm{Diff}(M)/\mathrm{Isom}(M,g) \longrightarrow \mathrm{Diff}(M)$ defined on a neighbourhood U of the identity coset such that if $F\colon U \times S_g \to \mathrm{Met}(M)$ is defined by $F(u,g') = s(u)_* g'$, then F is a diffeomorphism onto a neighbourhood of g in Met(M).

The proof of this theorem is constructive and, as we have remarked above, uses the Diff(M)-invariant metric G_o on Met(M); the slice S_g is constructed by exponentiating a small disc in the G_o-orthogonal complement of the tangent space $T_g(\mathrm{Diff}_*(M).g)$ of the Diff(M) orbit through g. Hence, locally the slice S_g and the orbit $\mathrm{Diff}_*(M).g$ are orthogonal with respect to the metric G_o. As in the general case, the slice theorem implies that the slice S_g and the orbit $\mathrm{Diff}_*(M).g$ fill out a neighbourhood of g in Met(M), and the (lower star) action is factored into an action of Isom(M,g) on the slice and a transitive action on the orbit.

We now give some important consequences of the Ebin-Palais slice theorem. Let G be any Lie group and let $\mathrm{Met}_G(M) = \{g \in \mathrm{Met}(M)\colon \mathrm{Isom}(M,g) \cong G\}$. Then, $\mathrm{Met}_G(M)$ is an open dense subspace of $\{g \in \mathrm{Met}(M)\colon G$ is isomorphic to some subgroup of $\mathrm{Isom}(M,g)\}$. This gives us the local decreasing property of the isometry group. In particular, $\mathrm{Met}_1(M)$ (the space of metrics with trivial isometry group) is an open dense subspace of Met(M) itself. We say that metrics in $\mathrm{Met}_1(M)$ are generic. Another application of the slice theorem is the stratification of Geom(M) = Met(M)/Diff(M) - this stratification is into manifolds of geometries of particular symmetry, the geometries of high symmetry being contained in the boundary of manifolds containing geometries of lower symmetry. We refer the reader to Bourguignon [B 16], Ebin [E 4] and Fischer [F 1] for further details concerning the consequences of the slice theorem. We return to Geom(M) below in this section.

Fischer [F 1] has made interesting remarks concerning the interaction between the coupled actions (lower star) of Diff(M) on Met(M) on the one hand and Isom(M,g) on M (some $g \in$ Met(M)) on the other. By exploiting this interaction it is possible to determine the topological implications (for M) of M admitting a Riemannian structure with particular symmetry. The general idea is that a non-generic metric cannot be supported by an arbitrary manifold - the global rigidity inherent in the symmetry of the geometry must be reflected in the underlying topology. Fischer investigates the questions (i) which Lie groups can occur as isometry groups for Riemannian metrics on a given manifold M?; and (ii) Which topologies are compatible with a Riemannian metric whose isometry group is isomorphic to a given Lie group? Fischer restricts his attention to the case dim M = 3 (since he is interested in applications to the Wheeler superspace in general relativity) and he proves a classification theorem for isometry groups arising from metrics on closed 3-manifolds.

We use the following terminology (see [F 1]):- The manifold M is said to be <u>symmetric</u> if there exists $g \in \mathrm{Met}(M)$ with $\dim \mathrm{Isom}(M,g) > 0$, <u>random</u> if $\dim \mathrm{Isom}(M,g) = 0$, for all $g \in \mathrm{Met}(M)$, and <u>wild</u> if $\mathrm{Met}_1(M) = \mathrm{Met}(M)$. If M is random, then all metrics on M possess discrete isometry groups and the action (lower star) is almost free (i.e., it has discrete isotropy). If M were wild, then (lower star) would be free and Geom(M) would be a manifold. M wild is equivalent to Diff(M) having no finite subgroups, but no such manifold is known in any dimension. Thus, the set of wild manifolds is possibly empty. Fischer's classification theorem demonstrates that "most" manifolds are random (in dimension three at least) and that the only symmetric closed 3-manifolds are S^3, $\mathbb{R}\mathbb{P}^3$, $S^1 \times S^1 \times S^1$, $S^2 \times S^1$, $L(p,q)$ (Lens spaces) and polyhedral manifolds (and various connected sums of these 3-manifolds). A classification theorem of this kind for dimension ≥ 4 would be difficult to achieve since there does not even exist a classification theorem for closed n-manifolds for $n \geq 4$.

We now turn our attention to other group actions on Met(M). In particular we study the group Aut GL(M) of automorphisms of the frame bundle GL(M) of M (see section 6.1 for the definition of Aut GL(M)). In what follows we do not assume M orientable unless explicitly stated and so Diff(M) denotes the group of all (not just orientation preserving) diffeomorphisms.

In section 6.1, we demonstrate that $\mathrm{Aut}\, GL(M) \cong \mathrm{Diff}(M) \ltimes \mathrm{Gau}\, GL(M)$, the latter with group structure given by $(\phi_1,\Psi_1)(\phi_2,\Psi_2) = (\phi_1 \circ \phi_2, \Psi_1 \circ \hat{\phi}_1 \circ \Psi_2 \circ \hat{\phi}_1^{-1})$, for all $(\phi_1,\Psi_1), (\phi_2,\Psi_2) \in \mathrm{Diff}(M) \times \mathrm{Gau}\, GL(M)$, and where $\hat{\phi} \equiv \ell(\phi)$ is the lift of $\phi \in \mathrm{Diff}(M)$ to an automorphism of GL(M) (see definition (6.1)27 and 6.1.13). The explicit isomorphism q of

Aut GL(M) onto Diff(M) $\ltimes$ GauGL(M) is given by $q(\Psi) = (\overline{\Psi}, \Psi \circ \hat{\overline{\Psi}}^{-1})$, for all $\Psi \in$ AutGL(M), and its inverse is given by $q^{-1}(\phi,\Psi) = \Psi \circ \hat{\phi}$, for all $(\phi,\Psi) \in$ Diff(M) $\ltimes$ GauGL(M). Recall that the two groups Γ(ConjGL(M)) and $C_{conj}(GL(M), GL(n,\mathbb{R}))$ are natural isomorphs of GauGL(M). There is also another group to which GauGL(M) is naturally isomorphic. This is the automorphism group of the tangent bundle, Aut(TM) = $\{F \in \text{Diff(TM)}: \pi_{TM} \circ F = \pi_{TM}$ and $F|T_xM \in GL(T_xM)$, for all $x \in M\} \cong \{F \in \Gamma(T^*M \otimes TM): \det F \neq 0\}$. Let us define α: Aut(TM) $\longrightarrow$ GauGL(M) by:

$$(\alpha(F))(u) = u(\kappa_u^{-1} \circ F(\pi(u)) \circ \kappa_u) \qquad 4.1.3,$$

for all $u \in GL(M)$, $F \in$ Aut(TM). Note that α does indeed take its values in GauGL(M) (since $\alpha(F) \circ R_a = R_a \circ \alpha(F)$, $\pi \circ \alpha(F) = \pi$, for all $a \in GL(n,\mathbb{R})$, $F \in$ Aut(TM)). The inverse is

$$(\alpha^{-1}(\Psi))(x) = \kappa_{\Psi(u)} \circ \kappa_u^{-1} \qquad 4.1.4,$$

for any $u \in \pi^{-1}(x)$, for all $x \in M$, $\Psi \in$ Gau GL(M). The definition of $(\alpha^{-1}(\Psi))(x)$ does not depend on the choice of $u \in \pi^{-1}(x)$, since $\kappa_{\Psi(ua)} \circ \kappa_{ua}^{-1} = \kappa_{\Psi(u)a} \circ \kappa_{ua}^{-1} = (\kappa_{\Psi(u)} \circ a) \circ (\kappa_u \circ a)^{-1} = \kappa_{\Psi(u)} \circ \kappa_u^{-1}$, for all $a \in GL(n,\mathbb{R})$. The maps α, α^{-1} are mutually inverse homomorphisms as is easily checked, so that α is an isomorphism of Aut(TM) onto GauGL(M).

There exists a natural action A (push forward) of Diff(M) on Aut(TM). This is given by:

$$(A_\phi(F))(x) = D\phi(\phi^{-1}(x)) \circ F(\phi^{-1}(x)) \circ D\phi^{-1}(x) \qquad 4.1.5,$$

for all $x \in M$, $F \in$ Aut(TM) and $\phi \in$ Diff(M). It is straightforward to show (see similar calculations below) that A $\in$ Hom(Diff(M), Aut(Aut(TM))). Hence we may form $\mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$.

Proposition (4.1)1: The isomorphism α is equivariant with respect to the actions A, $\mathrm{conj} \circ \ell$ of $\mathrm{Diff}(M)$ on $\mathrm{Aut}(TM)$, $\mathrm{GauGL}(M)$ respectively.

Proof: We must demonstrate that $\alpha \circ A_\phi = (\mathrm{conj} \circ \ell)_\phi \circ \alpha$, for all $\phi \in \mathrm{Diff}(M)$. Let $\phi \in \mathrm{Diff}(M)$, $F \in \mathrm{Aut}(TM)$ and $u \in GL(M)$. We have $((\alpha \circ A_\phi)(F))(u) = u(\kappa_u^{-1} \circ (A_\phi(F))(\pi(u)) \circ \kappa_u)$

$= u(\kappa_u^{-1} \circ D\phi(\phi^{-1}(\pi(u))) \circ F(\phi^{-1}(\pi(u))) \circ D\phi^{-1}(\pi(u)) \circ \kappa_u)$

$= u(\kappa_u^{-1} \circ D\phi^{-1}(\pi(u))^{-1} \circ F(\phi^{-1}(\pi(u))) \circ D\phi^{-1}(\pi(u)) \circ \kappa_u)$

$= u(\kappa^{-1}_{\hat{\phi}^{-1}(u)} \circ F(\pi(\hat{\phi}^{-1}(u))) \circ \kappa_{\hat{\phi}^{-1}(u)})$

(using $\kappa_{\hat{\psi}(u)} = D\psi(\pi(u)) \circ \kappa_u$, $\pi \circ \hat{\psi} = \psi \circ \pi$, for all $\psi \in \mathrm{Diff}(M)$)

$= \hat{\phi}(\hat{\phi}^{-1}(u)(\kappa^{-1}_{\hat{\phi}^{-1}(u)} \circ F(\pi(\hat{\phi}^{-1}(u))) \circ \kappa_{\hat{\phi}^{-1}(u)})) = \hat{\phi}((\alpha(F))(\hat{\phi}^{-1}(u)))$

$= (\hat{\phi} \circ \alpha(F) \circ \hat{\phi}^{-1})(u) = ((\mathrm{conj} \circ \ell)_\phi(\alpha(F)))(u)$. Hence $\alpha \circ A_\phi = (\mathrm{conj} \circ \ell)_\phi \circ \alpha$, for all $\phi \in \mathrm{Diff}(M)$ □

Corollary (4.1)2: $\mathrm{AutGL}(M)$ is naturally isomorphic with $\mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$.

Proof: Define $\alpha' : \mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM) \longrightarrow \mathrm{AutGL}(M) = \mathrm{Diff}(M) \ltimes \mathrm{GauGL}(M)$ by $(\phi, F) \longmapsto (\phi, \alpha(F))$. Then α' is obviously a bijection. We now show that α' is a homomorphism; Let (ϕ_1, F_1), (ϕ_2, F_2) $\in \mathrm{Diff}(M) \times \mathrm{Aut}(TM)$. Then

$\alpha'((\phi_1, F_1)(\phi_2, F_2)) = \alpha'(\phi_1 \circ \phi_2,\ F_1 A_{\phi_1}(F_2)) = (\phi_1 \circ \phi_2,\ \alpha(F_1 A_{\phi_1}(F_2)))$

$= (\phi_1 \circ \phi_2,\ \alpha(F_1)(\alpha \circ A_{\phi_1})(F_2)) = (\phi_1 \circ \phi_2,\ \alpha(F_1((\mathrm{conj} \circ \ell)_\phi \circ \alpha)(F_2))$

$= (\phi_1 \circ \phi_2,\ \alpha(F_1)(\mathrm{conj} \circ \ell_\phi)(\alpha(F_2))) = (\phi_1, \alpha(F_1))(\phi_2, \alpha(F_2))$

$= \alpha'(\phi_1, F_1)\alpha'(\phi_2, F_2)$.

Thus α' is an isomorphism of $\mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$ onto $\mathrm{AutGL}(M)$ □

We now wish to define an action of AutGL(M) on Met(M). To do this, we first define an action of Aut(TM) on Met(M), thence an action of Diff(M) $\ltimes$ Aut(TM), and finally, using the isomorphism α', an action of AutGL(M).

<u>Definition (4.1)3</u>: Define $C \in \mathrm{Hom}(\mathrm{Aut}(TM), \mathrm{Diff}(\mathrm{Met}(M)))$ by $C_F(g)(x)(v,w) = g(x)(F^{-1}(x).v,\ F^{-1}(x).w)$, for all $v,w \in T_xM$, $x \in M$, $g \in \mathrm{Met}(M)$ and $F \in \mathrm{Aut}(TM)$. C is called the <u>generalized conformal action</u>.

The reason for the term generalized conformal action is that the (pointwise) conformal action of $C^+(M)$ on Met(M) (see section 6.2) is the restriction of C to $C^+(M)$ regarded as a subgroup of Aut(TM) via the monomorphism: $f \longmapsto f^{-\frac{1}{2}}\, \mathbb{1}_{TM}$, where $\mathbb{1}_{TM}$ is the identity automorphism of TM.

<u>Definition (4.1)4</u>: A <u>generalized conformal structure</u> on M is an orbit of the group Aut(TM).

In other words, a generalized conformal structure is an equivalence class of the relation $\sim$, where $g_1 \sim g_2$ if and only if there exists $F \in \mathrm{Aut}(TM)$ such that $g_1 = C_F(g_2)$. We return to generalized conformal structures below.

<u>Proposition (4.1)5</u>: The actions (lower star) and C are compatible with the action A in that together they define an action B of Diff(M) $\ltimes$ Aut(TM) on Met(M).

<u>Proof</u>: Let $\phi \in \mathrm{Diff}(M)$, $F \in \mathrm{Aut}(TM)$. Then

$$(\phi_* \circ C_F)(g)(x)(v,w) = C_F(g)(\phi^{-1}(x))(D\phi^{-1}(x).v,\ D\phi^{-1}(x).w)$$
$$= g(\phi^{-1}(x))(F^{-1}(\phi^{-1}(x))\ D\phi^{-1}(x).v,\ F^{-1}(\phi^{-1}(x))D\phi^{-1}(x).w)$$
$$= g(\phi^{-1}(x))(D\phi(\phi^{-1}(x))^{-1}(A_\phi(F^{-1}))(x).v,\ D\phi(\phi^{-1}(x))^{-1}(A_\phi(F^{-1}))(x).w)$$
$$= g(\phi^{-1}(x))(D\phi^{-1}(x)(A_\phi(F)^{-1})(x).v,\ D\phi^{-1}(x)(A_\phi(F)^{-1})(x).w)$$

$= (\phi_* g)(x)(A_\phi(F)^{-1}(x).v,\ A_\phi(F)^{-1}(x).w) = C_{A_\phi(F)}(\phi_* g)(x)(v,w)$

$= (C_{A_\phi(F)} \circ \phi_*)(g)(x)(v,w)$, for all $v,w \in T_xM$, $x \in M$, $g \in Met(M)$.

Hence $\phi_* \circ C_F = C_{A_\phi(F)} \circ \phi_*$.

Now define, for each $(\phi,F) \in Diff(M) \ltimes Aut(TM)$, the map $B_{(\phi,F)} = C_F \circ \phi_*$: $Met(M) \longrightarrow Met(M)$. We have

$$B_{(\phi_1,F_1)(\phi_2,F_2)} = B_{(\phi_1\circ\phi_2,F_1(A_{\phi_1}(F_2)))} = C_{F_1(A_{\phi_1}(F_2))} \circ (\phi_1\circ\phi_2)_*$$

$$= C_{F_1} \circ C_{A_{\phi_1}(F_2)} \circ \phi_{1*} \circ \phi_{2*} = C_{F_1} \circ \phi_{1*} \circ C_{F_2} \circ \phi_{2*} = B_{(\phi_1,F_1)} \circ B_{(\phi_2,F_2)},$$

for all (ϕ_1,F_1), $(\phi_2,F_2) \in Diff(M) \times Aut(TM)$. Obviously $B_{(id_M,\ \mathbb{1}_{TM})} = id_{Met(M)}$, so $(\phi,F) \longmapsto B_{(\phi,F)}$ defines a map $B \in Hom(Diff(M) \ltimes Aut(TM),\ Diff(Met(M)))$. B is the required action of $Diff(M) \times Aut(TM)$ on $Met(M)$. □

Now, using the isomorphism α': $Diff(M) \ltimes Aut(TM) \rightarrow AutGL(M)$ (see (4.1)2), we obtain the action $B' = B \circ (\alpha')^{-1}$ of $AutGL(M)$ on $Met(M)$. Thus, B' is given by:

$$B'_{(\phi,\Psi)} = B_{(\phi,\alpha^{-1}(\Psi))} \equiv C_{\alpha^{-1}(\Psi)} \circ \phi_* \qquad 4.1.6,$$

for all $(\phi,\Psi) \in Diff(M) \ltimes GauGL(M) \cong Aut\ GL(M)$ (From now on, we identify $Aut\ GL(M)$ with $Diff(M) \ltimes GauGL(M)$ using the isomorphism q, but we maintain a distinction between $Aut\ GL(M)$ and $Diff(M) \ltimes Aut(TM)$).

Note that it is occasionally convenient to regard metrics on M, not as tensor fields on M, but as equivariant maps on the frame bundle. Let $S(p,q;\mathbb{R}) \cong GL(n,\mathbb{R})/O(p,q)$ denote the space of real symmetric $n \times n$ matrices of signature (p,q) (where (p,q) is the signature of the metrics in $Met(M)$). Define an action σ

of $GL(n,\mathbb{R})$ on $S(p,q;\mathbb{R})$ by $\sigma_a(s) = (a^{-1})^T s a^{-1}$, for all $s \in S(p,q;\mathbb{R})$, $a \in GL(n,\mathbb{R})$, and denote, as usual, the space of equivariant maps from $GL(M)$ into $S(p,q;\mathbb{R})$ by $C_\sigma(GL(M), S(p,q;\mathbb{R}))$. Now define $\beta\colon Met(M) \to C_\sigma(GL(M), S(p,q;\mathbb{R}))$ by:-

$$\beta(g)(u)(\underline{x},\underline{y}) = g(\pi(u))(\kappa_u(\underline{x}), \kappa_u(\underline{y})) \qquad 4.1.7,$$

for all $\underline{x}, \underline{y} \in \mathbb{R}^n$, $u \in GL(M)$, $g \in Met(M)$. Note that $\kappa_u\colon \mathbb{R}^n \to T_{\pi(u)}M$ is the linear isomorphism defined for each $u \in GL(M)$ (see 6.1.1), and we regard elements of $S(p,q;\mathbb{R})$ as bilinear forms on $\mathbb{R}^n$. The map β does take its values in $C_\sigma(GL(M), S(p,q;\mathbb{R}))$ as we now demonstrate:- Let $g \in Met(M)$, $u \in GL(M)$, $a \in GL(n,\mathbb{R})$ and $\underline{x},\underline{y} \in \mathbb{R}^n$, then $\beta(g)(ua)(\underline{x},\underline{y}) = g(\pi(ua))(\kappa_{ua}(\underline{x}),\kappa_{ua}(\underline{y})) = g(\pi(u))(\kappa_u(a\underline{x}),\kappa_u(a\underline{y}))$ $= \beta(g)(u)(a\underline{x}, a\underline{y}) = (a\underline{x})^T(\beta(g)(u))(a\underline{y}) = \underline{x}^T a^T(\beta(g)(u))a\underline{y}$ $= \sigma_{a^{-1}}(\beta(g)(u))(\underline{x},\underline{y})$. Hence, $\beta(g)(ua) = \sigma_{a^{-1}}(\beta(g)(u))$, so $\beta(g) \in C_\sigma(GL(M), S(p,q;\mathbb{R}))$, for all $g \in Met(M)$.

The map β is actually a diffeomorphism of smooth manifolds with inverse given by:-

$$(\beta^{-1}(s)(x)(v,w) = s(u)(\kappa_u^{-1}(v), \kappa_u^{-1}(w)) \qquad 4.1.8,$$

for all $v,w \in T_xM$, any $u \in \pi^{-1}(x)$, for all $x \in M$ and $s \in C_\sigma(GL(M), S(p,q;\mathbb{R}))$. We may use β to transfer the actions of $Diff(M)$ and $Aut(TM)$ on $Met(M)$ to actions on $C_\sigma(GL(M), S(p,q;\mathbb{R}))$. Define, for each $\phi \in Diff(M)$ and $F \in Aut(TM)$ the diffeomorphisms $\phi' = \beta \circ \phi_* \circ \beta^{-1}$, $C'_F = \beta \circ C_F \circ \beta^{-1}$ of $C_\sigma(GL(M), S(p,q;\mathbb{R}))$. It is straightforward to verify that $\phi' = \hat{\phi}_*$ and $C'_F = \alpha(F)_*$ (so that $\phi'(s) = s \circ \hat{\phi}^{-1}$, $C'_F(s) = s \circ \alpha(F^{-1})$, for all $s \in C_\sigma(GL(M), S(p,q;\mathbb{R})))$. Also, by definition, $\phi' \circ C'_F = C'_{A_\phi(F)} \circ \phi'$, so that the actions of $Diff(M)$, $Aut(TM)$ together give an action

$D_{(\phi,F)} = C_F' \circ \phi'$ of $\mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$ (thence Aut GL(M)) on $C_\sigma(GL(M), S(p,q;\mathbb{R}))$ (Cf. the proof of proposition (4.1)5). We use these results below (see proposition (4.1)11).

We now consider the structure of Aut GL(M) in more detail. In particular, we discuss decompositions of the group $\mathrm{Aut}(TM) \cong \mathrm{GauGL}(M)$. Let us first consider the group $GL(n,\mathbb{R})$.

We have the epimorphism $\det\colon GL(n,\mathbb{R}) \to \mathbb{R}^*$ and the inverse images under det of subgroups of $\mathbb{R}^*$ constitute important subgroups of $GL(n,\mathbb{R})$; Consider 1, $\mathbb{Z}_2$ and $\mathbb{R}^+ \leqslant \mathbb{R}^*$, then $\det^{-1}(1) = SL(n,\mathbb{R})$, $\det^{-1}(\mathbb{Z}_2) = OL(n,\mathbb{R})$ (our notation) and $\det^{-1}(\mathbb{R}^+) = GL^+(n,\mathbb{R})$.

We also have the epimorphism $\det\colon \mathrm{Aut}(TM) \to C^*(M)$ (we use the same notation for the two determinant maps) and corresponding subgroups of Aut(TM). Thus, $\mathrm{SAut}(TM) = \{F \in \mathrm{Aut}(TM)\colon \det F = 1\}$, $\mathrm{OAut}(TM) = \{F \in \mathrm{Aut}(TM)\colon \det F \in \mathbb{Z}_2 \leqslant C^*(M)\}$ and $\mathrm{Aut}^+(TM) = \{F \in \mathrm{Aut}(TM)\colon \det F \in \mathbb{R}^+ \leqslant C^*(M)\}$.

Proposition (4.1)6: The subgroups SAut(TM), OAut(TM) and $\mathrm{Aut}^+(TM)$ are each invariant under the action A of Diff(M) on Aut(TM).

Proof: First note that $\det\colon \mathrm{Aut}(TM) \to C^*(M)$ is equivariant with respect to the actions A, (lower star) on Aut(TM), $C^*(M)$ respectively; For $\phi \in \mathrm{Diff}(M)$, $F \in \mathrm{Aut}(TM)$ and $x \in M$, we have $(\det\circ A_\phi)(F)(x) = \det(A_\phi(F)(x)) = \det(D\phi(\phi^{-1}(x))\circ F(\phi^{-1}(x))\circ D\phi^{-1}(x))$ (see 4.1.5) $= \det(F(\phi^{-1}(x)))$ (since $D\phi(\phi^{-1}(x)) = D\phi^{-1}(x)^{-1}$) $= (\det F)(\phi^{-1}(x)) = (\phi_* \det F)(x)$. Hence $\det\circ A_\phi = \phi_*\circ\det$, for all $\phi \in \mathrm{Diff}(M)$.

Now note that $\phi_* G = G$ for any subgroup G of $\mathbb{R}^*$ regarded as a subgroup of $C^*(M)$, and hence $F \in \det^{-1}(G) \iff \det F \in G$

$\Longleftrightarrow \phi_* \det F \in \phi_* G = G \Longleftrightarrow \det(A_\phi(F)) \in G \Longleftrightarrow A_\phi(F) \in \det^{-1}(G)$.
i.e., $\det^{-1}(G)$ is invariant under the action A for all subgroups G of $\mathbb{R}^* \leq C^+(M)$. In particular, $SAut(TM)$, $OAut(TM)$ and $Aut^+(TM)$ are each invariant under $Diff(M)$. □

We have defined a generalized conformal structure on M to be an orbit of Aut(TM) acting on Met(M) (see definition (4.1)4). This definition may be generalized slightly in the following way:

Definition (4.1)7: Let K be any subgroup of $Aut(TM)$. Then K acts on $Met(M)$ by restricting the generalized conformal action C (see definition (4.1)3) to K. We define a K-conformal structure on M to be an orbit of the group K acting on $Met(M)$.

In particular a $C^+(M)$-conformal structure is just a conformal structure in the usual sense (see section 6.2) and an Aut(TM)-conformal structure is a generalized conformal structure (as defined by (4.1)4). The subgroups $SAut(TM)$, $OAut(TM)$ and $Aut^+(TM)$ also give rise to K-conformal structures on M. The Diff(M) invariance of all these subgroups implies that the corresponding K-conformal structures are each everywhere invariant (see section 4.3).

Let us now discuss a decomposition of Aut(TM). First we consider the subgroup $OL(n,\mathbb{R})$ of $GL(n,\mathbb{R})$. Let ι be the inclusion of $OL(n,\mathbb{R})$ in $GL(n,\mathbb{R})$ and, for fixed $s \in \mathbb{R}^*$, let $\lambda_s = |\det|^s : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^+$; $a \longmapsto |\det a|^s$, for each $a \in GL(n,\mathbb{R})$. The map λ_s is an epimorphism with $\operatorname{Ker} \lambda_s = \{a \in GL(n,\mathbb{R}): |\det a|^s = 1\} = \{a \in GL(n,\mathbb{R}): |\det a| = 1\} = \{a \in GL(n,\mathbb{R}): \det a \in \mathbb{Z}_2\} = OL(n,\mathbb{R})$. We thus have the short exact sequence:

$$1 \longrightarrow OL(n,\mathbb{R}) \xhookrightarrow{\iota} GL(n,\mathbb{R}) \xrightarrow{\lambda_s} \mathbb{R}^+ \longrightarrow 1 \qquad 4.1.9.$$

For each $s \in \mathbb{R}^*$, this sequence splits; define $\gamma_s : \mathbb{R}^+ \longrightarrow GL(n,\mathbb{R})$

by

$$\gamma_s(r) \quad = \quad r^{1/ns}\ \mathbb{1}_n \qquad 4.1.10,$$

for all $r \in \mathbb{R}^+$. The map γ_s is a homomorphism and $\lambda_s \circ \gamma_s = id_{\mathbb{R}^+}$, so γ_s defines a splitting of 4.1.9. The splitting γ_s gives rise to an action θ_s of $\mathbb{R}^+$ on $OL(n,\mathbb{R})$ by automorphisms (see the discussion of semi-direct products in section 6.3) given by $\theta_s(t,a) = \gamma_s(t)a\gamma_s(t^{-1}) = a$, for all $(t,a) \in \mathbb{R}^+ \times OL(n,\mathbb{R})$. Thus, the action is trivial and so the corresponding semi-direct product $\mathbb{R}^+ \ltimes_{\theta_s} OL(n,\mathbb{R})$ is, in fact, direct. We thus have an isomorphism ξ_s of $\mathbb{R}^+ \times OL(n,\mathbb{R})$ (direct product) onto $GL(n,\mathbb{R})$ given by:

$$\xi_s(r,a) \quad = \quad a\ \gamma_s(r) \quad = \quad r^{1/ns}\ a \qquad 4.1.11,$$

for all $(t,a) \in \mathbb{R}^+ \times OL(n,\mathbb{R})$, with inverse given by:

$$\xi_s^{-1}(a) \quad = \quad (|\det a|^s,\ |\det a|^{-1/n}\ a) \qquad 4.1.12,$$

for all $a \in GL(n,\mathbb{R})$. The parameter s may be chosen for convenience with $s = 1/n$ perhaps a natural choice.

This decomposition for $GL(n,\mathbb{R})$ may be used to obtain a corresponding decomposition of $Aut(TM) \cong C_{conj}(GL(M),\ GL(n,\mathbb{R}))$. The decomposition of $Aut(TM)$ relies on the behaviour of the conjugation action under the isomorphism $\xi_s^{-1}\colon GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^+ \times OL(n,\mathbb{R})$. Define, for each $s \in \mathbb{R}^*$, $conj_s \in Hom(GL(n,\mathbb{R}),\ Aut(\mathbb{R}^+ \times OL(n,\mathbb{R})))$ by $conj_s(a) = \xi_s^{-1} \circ conj(a) \circ \xi_s$, for all $a \in GL(n,\mathbb{R})$. Then, under the action of $a \in GL(n,\mathbb{R})$, $(r,b) \in \mathbb{R}^+ \times OL(n,\mathbb{R})$ is mapped to $\xi_s^{-1}(a\ \xi_s(r,b)\ a^{-1}) = \xi_s^{-1}(a(r^{1/ns}\ b)a^{-1}) = \xi_s^{-1}(r^{1/ns}\ aba^{-1}) = (r,\ aba^{-1})$. Thus, $GL(n,\mathbb{R})$ acts trivially on the $\mathbb{R}^+$ factor and by conjugation on the $OL(n,\mathbb{R})$ factor. Note that this action is independent of s, and so we write conj' rather than conj_s (all $s \in \mathbb{R}^*$). The map $\Phi \mapsto \xi_s^{-1} \circ \Phi$ is an isomorphism of $C_{conj}(GL(M), GL(n,\mathbb{R}))$ onto $C_{conj'}(GL(M), \mathbb{R}^+ \times OL(n,\mathbb{R}))$ and, because of the triviality of conj' on the $\mathbb{R}^+$ factor, we have an isomorphism of $C_{conj'}(GL(M), \mathbb{R}^+ \times OL(n,\mathbb{R}))$ onto $C^+(M) \times C_{conj}(GL(M), OL(n,\mathbb{R}))$. Finally, using the isomorphisms $C_{conj}(GL(M), GL(n,\mathbb{R})) \cong Aut(TM)$ and $C_{conj}(GL(M), OL(n,\mathbb{R})) \cong OAut(TM)$ (given by $\Phi \mapsto F_\Phi$; $x \mapsto \kappa_u \circ \Phi(u) \circ \kappa_u^{-1}$, any $u \in \pi^{-1}(x)$), we obtain an isomorphism δ_s of $Aut(TM)$ onto $C^+(M) \times OAut(TM)$, and δ_s is given by:

$$\delta_s(F) = (|\det F|^s, |\det F|^{-1/n} F) \qquad 4.1.13,$$

for all $F \in Aut(TM)$. The inverse is given by:

$$\delta_s^{-1}(f,F) = f^{1/ns} F \qquad 4.1.14,$$

for all $(f,F) \in C^+(M) \times OAut(TM)$. Note that the choice $s = -{}^2/n$ leads to the monomorphism: $C^+(M) \hookrightarrow Aut(TM)$; $f \mapsto f^{-\frac{1}{2}} 1\!\!1_{TM}$ referred to above.

We have proved the following:

Proposition (4.1)8: There exists a natural 1-parameter family $\{\delta_s : s \in \mathbb{R}^*\}$ of isomorphisms of $Aut(TM)$ onto $C^+(M) \times OAut(TM)$, where δ_s is given by 4.1.13.

We have the actions (lower star) and A of $Diff(M)$ by automorphisms on the groups $C^+(M)$ and $OAut(TM)$ respectively (see proposition (4.1)6 for the latter) and hence the action $A' \equiv (\text{lower star}) \times A$ of $Diff(M)$ by automorphisms on the group

$C^+(M) \times OAut(TM)$. Thus, we may construct the semi-direct product $Aut(M) \equiv Diff(M) \ltimes (C^+(M) \times OAut(TM))$.

Proposition (4.1)9: There exists a natural 1-parameter family $\{\omega_s' : s \in \mathbb{R}^*\}$ of isomorphisms of $Aut\, GL(M)$ onto $Aut(M)$.

Proof: First define $\delta_s' : Diff(M) \ltimes Aut(TM) \longrightarrow Aut(M)$ by $\delta_s'(\phi,F) = (\phi,\delta_s(F))$, for all $(\phi,F) \in Diff(M) \ltimes Aut(TM)$. By an argument similar to that given in the proof of corollary (4.1)2, δ_s' is an isomorphism if δ_s is equivariant with respect to the actions A, A' of $Diff(M)$ on $Aut(TM)$, $C^+(M) \times OAut(TM)$ respectively. We now demonstrate this equivariance; Let $F \in Aut(TM)$, $\phi \in Diff(M)$. Then $(\delta_s \circ A_\phi)(F) = (|\det A_\phi(F)|^s, |\det A_\phi(F)|^{-1/n} A_\phi(F))$ $= (|\phi_*(\det F)|^s, |\phi_*(\det F)|^{-1/n} A_\phi(F))$ (using the equivariance of det shown in the proof of proposition (4.1)6)
$= |\phi_* \det F|^s, |\phi_* \det F|^{-1/n} A_\phi(F)) = (\phi_*|\det F|^s, A_\phi(|\det F|^{-1/n} F))$ $= A_\phi'(|\det F|^s, |\det F|^{-1/n} F) = (A_\phi' \circ \delta_s)(F)$. Thus, $\delta_s \circ A_\phi = A_\phi' \circ \delta_s$, for all $\phi \in Diff(M)$, and so δ_s' is an isomorphism.

We now define $\omega_s' : Aut\, GL(M) \longrightarrow Aut(M)$ by $\omega_s' = \delta_s' \circ (\alpha')^{-1}$, and this gives us the required 1-parameter family of isomorphisms □

The action B' of $Aut\, GL(M)$ on $Met(M)$ (see equation 4.1.6) may be transferred to an action of $Aut(M) \equiv Diff(M) \ltimes (C^+(M) \times OAut(TM))$ on $Met(M)$ using the isomorphisms ω_s'. We define, for each $s \in \mathbb{R}^*$, $E^s \in Hom(Aut(M), Diff(Met\,(M)))$ by $E^s_{(\phi,f,F)} = B'^{-1}_{\omega_s}(\phi,f,F)$ $= C_{\delta_s^{-1}(f,F)} \circ \phi_*$. Let $C^s \in Hom(C^+(M) \times OAut(TM), Diff(Met(M)))$ be defined by $C^s_{(f,F)} = C_{\delta_s^{-1}(f,F)}$. Then $(C^s_{(f,F)}(g))(x)(v,w)$
$= (C_{f^{1/ns}F}(g))(x)(v,w) = g(x)((f^{1/ns}F)^{-1}(x).v, (f^{1/ns}F)^{-1}(x).w)$
$= g(x)(f^{-1/ns}(x)F^{-1}(x).v, f^{-1/ns}(x)F^{-1}(x).w)$
$= f^{-2/ns}(x)g(x)(F^{-1}(x).v, F^{-1}(x).w) = (f^{-2/ns}C_F(g))(x)(v,w)$, for all

$v,w \in T_xM$, $x \in M$ and $g \in Met(M)$. Hence $C^s_{(f,F)} = f^{-2/ns} \circ C_F$, where the map $f^{-2/ns}: Met(M) \longrightarrow Met(M)$ is the (pointwise) conformal action of $C^+(M)$ on Met (M) (see section 6.2). The action E^s of Aut(M) on Met(M) is therefore given by:

$$E^s_{(\phi,f,F)} = f^{-2/ns} \circ C_F \circ \phi_* \qquad 4.1.15,$$

for all $(\phi,(f,F)) \in Aut(M)$.

Equation 4.1.15 demonstrates that the action of the natural group Aut(M) factors into a product (in Diff(Met(M))) of the three important geometric actions, namely pointwise conformal, OAut(TM)-conformal and the (lower star) action by diffeomorphisms. By "turning off" one or more of these actions, we obtain the usual actions on Met(M), e.g. putting $F = 1\!\!1_{TM}$ gives us the action of the conformorphism group $Conf(M) \equiv Diff(M) \ltimes C^+(M)$ on Met(M) (see section 6.2) (to conform with convention we should use the parameter value $s = -{}^2/n$ in this case).

The relation with the conformorphism group is made more explicit by the following:

<u>Proposition (4.1)10</u>: Define $J \in Hom(Conf(M), Aut(OAut(TM)))$ by $J_{(\phi,f)}(F) = A_\phi(F)$, for all $F \in OAut(TM)$, $(\phi,f) \in Conf(M)$. Then the corresponding semi-direct product $Conf(M) \ltimes OAut(TM)$ is isomorphic with Aut(M).

<u>Proof</u>: Define μ: $Conf(M) \ltimes OAut(TM) \longrightarrow Aut(M)$ $\equiv Diff(M) \ltimes (C^+(M) \times OAut(TM))$ by $\mu((\phi,f),F) = (\phi,(f,F))$, for all $((\phi,f),F) \in Conf(M) \ltimes OAut(TM)$. Then $\mu(((\phi_1,f_1),F_1)(\phi_2,f_2),F_2))$
$= \mu((\phi_1,f_1)(\phi_2,f_2),\ F_1 J_{(\phi_1,f_1)}(F_2))$
$= \mu((\phi_1 \circ \phi_2, f_1(\phi_1)_* f_2),\ F_1 A_{\phi_1}(F_2)) = (\phi_1 \circ \phi_2,(f_1(\phi_1)_* f_2,\ F_1\ A_{\phi_1}(F_2)))$

$= (\phi_1 \circ \phi_2, (f_1,F_1)((\phi_1)_* f_2, A_{\phi_1}(F_2))) = (\phi_1,(f_1,F_1))(\phi_2,(f_2,F_2))$

(since ϕ_1 maps (f_2,F_2), via A', to $((\phi_1)_* f_2, A_{\phi_1}(F_2)))$

$= \mu((\phi_1,f_1),F_1)\mu((\phi_2,f_2),F_2)$, for all $((\phi_1,f_1),F_1),((\phi_2,f_2), F_2)$ $\in \mathrm{Conf}(M) \ltimes \mathrm{OAut}(TM)$. Hence μ is a homomorphism, obviously a bijection, and hence $\mathrm{Conf}(M) \ltimes \mathrm{OAut}(TM) \cong \mathrm{Aut}(M)$ □

We summarize the above discussion. The group $\mathrm{Aut}(M)$ $\equiv \mathrm{Diff}(M) \ltimes (C^+(M) \times \mathrm{OAut}(TM))$ is a natural isomorph of the group $\mathrm{Aut}\, GL(M)$ of automorphisms of the frame bundle. $\mathrm{Aut}(M)$ acts on $\mathrm{Met}(M)$ in a manner which unifies the action by the conformorphism group $\mathrm{Conf}(M)$ with the group of $\mathrm{OAut}(TM)$-conformal transformations. In fact we have a family of actions on $\mathrm{Met}(M)$, this family parameterized by $\mathbb{R}^*$.

We now consider how $\mathrm{Aut}(M)$ interacts with other natural structures on M. First consider the canonical bundle introduced in section 1.4 (see definition (1.4)1). For ease of exposition, we restrict out attention to positive definite signature metrics, but the discussion goes through for arbitrary signature. In section 1.4, we dealt with oriented manifolds, but here we do not assume that M is orientable. We define the <u>canonical principal $O(n)$-bundle of M</u> to be the following fibration (Cf. 1.4.1):

$$O(n) \hookrightarrow O(M) \xrightarrow{\sigma} \mathrm{Met}(M) \times M \qquad 4.1.16,$$

where $O(M) = \{(g,u) \in \mathrm{Met}(M) \times GL(M) : u \in O(M,g)\}$, and $\sigma(g,u) = (g,\pi(u))$, for all $(g,u) \in O(M)$. As in 1.4.1, we have the free right action of $O(n)$ on $O(M)$ given by $((g,u),a) \mapsto (g,ua)$, for all $(g,u) \in O(M)$, $a \in O(n)$, and then σ is the corresponding quotient map making 4.1.16 a principal $O(n)$-bundle.

Note that if we identify $\mathrm{Met}(M)$ with the space $C_\sigma(GL(M), S(n;\mathbb{R}))$ (see 4.1.7), we may define the evaluation map:

$$\mathrm{ev}\colon \mathrm{Met}(M) \times GL(M) \longrightarrow S(n;\mathbb{R});\ \ (g,u) \longmapsto \beta(g)(u) \qquad 4.1.17,$$

for all $(g,u) \in \mathrm{Met}(M) \times GL(M)$. The total space $O(M)$ of the canonical bundle is then just $\mathrm{ev}^{-1}(\mathrm{can})$, where $\mathrm{can} \in S(n;\mathbb{R})$ is the standard Euclidean inner-product on $\mathbb{R}^n$.

Proposition (4.1)11: The group $\mathrm{Aut}(M)$ may be naturally identified with a subgroup of $\mathrm{Aut}\, O(M)$, i.e. we may realize $\mathrm{Aut}(M)$ as a group of automorphisms of the canonical $O(n)$-bundle of M.

Proof: Let $s \in \mathbb{R}^*$. It is convenient to utilize the isomorph $\mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$ of $\mathrm{Aut}(M)$. We first construct an action of $\mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$ on $O(M)$, and then use the isomorphism δ'_s ((4.1)9) to transfer this to an action of $\mathrm{Aut}(M)$.

Define, for each $(\phi,F) \in \mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$, the map $Q_{(\phi,F)} = B_{(\phi,F)} \times (\alpha(F) \circ \hat{\phi})\colon O(M) \longrightarrow \mathrm{Met}(M) \times GL(M)$, where the action $B \in \mathrm{Hom}(\mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM), \mathrm{Diff}(\mathrm{Met}(M)))$ is defined in proposition (4.1)5, and $\alpha\colon \mathrm{Aut}(TM) \longrightarrow \mathrm{Gau}\, GL(M)$ is the isomorphism given by equation 4.1.3.

We first show that $Q_{(\phi,F)}$ maps $O(M)$ into itself. Equivalently, we demonstrate that $\mathrm{ev} \circ Q_{(\phi,F)} = \text{constant}\ (\mathrm{can})\colon O(M) \longrightarrow S(n,\mathbb{R})$. Note that $B_{(\phi,F)} = C_F \circ \phi_* = \beta^{-1} \circ C'_F \circ \phi' \circ \beta = \beta^{-1} \circ \alpha(F)_* \circ \hat{\phi}_* \circ \beta = \beta^{-1} \circ (\alpha(F) \circ \hat{\phi})_* \circ \beta$, where β is the diffeomorphism of $\mathrm{Met}(M)$ onto $C_\sigma(GL(M), S(n;\mathbb{R}))$ defined by 4.1.7. Let $(g,u) \in O(M)$. Then, $(\mathrm{ev} \circ Q_{(\phi,F)})(g,u)$ $= \mathrm{ev}(B_{(\phi,F)}(g), (\alpha(F) \circ \hat{\phi})(u)) = (\beta \circ B_{(\phi,F)})(g)((\alpha(F) \circ \hat{\phi}(u))$ $= ((\alpha(F) \circ \hat{\phi}_*) \circ \beta)(g)((\alpha(F) \circ \hat{\phi})(u)) = \beta(g)((\alpha(F) \circ \hat{\phi})^{-1} \circ (\alpha(F) \circ \hat{\phi}))(u)$ $= \beta(g)(u) = \mathrm{can}$. Hence $Q_{(\phi,F)}$ maps $O(M)$ into itself.

We next show that $Q_{(\phi,F)}$ is an automorphism of the canonical principal $O(n)$-bundle, $\sigma\colon O(M) \longrightarrow \mathrm{Met}(M)\times M$. By inspection, $Q_{(\phi,F)}$ is a diffeomorphism of $O(M)$ onto itself, so we must demonstrate that $R_a \circ Q_{(\phi,F)} = Q_{(\phi,F)} \circ R_a$, where $R_a \in \mathrm{Diff}(O(M))$ is the right action of $O(n)$ on $O(M)$, for all $a \in O(n)$. We have $(R_a \circ Q_{(\phi,F)})(g,u) = R_a(B_{(\phi,F)}(g),\ (\alpha(F)\circ\hat{\phi})(u))$ $= (B_{(\phi,F)}(g),\ (\alpha(F)\circ\hat{\phi})(u)a) = (B_{(\phi,F)}(g),\ (\alpha(F)\circ\hat{\phi})(ua))$ (since $\alpha(F)\circ\hat{\phi} \in \mathrm{Aut}\,GL(M)) = Q_{(\phi,F)}(g,ua) = (Q_{(\phi,F)}\circ R_a)(g,u)$, for all $(g,u) \in O(M)$, $a \in O(n)$. Hence, $Q_{(\phi,F)} \in \mathrm{Aut}\,O(M)$.

Note that $(\sigma \circ Q_{(\phi,F)})(g,u) = \sigma(B_{(\phi,F)}(g),\ (\alpha(F)\circ\hat{\phi})(u))$ $= (B_{(\phi,F)}(g), (\pi\circ\alpha(F)\circ\hat{\phi})(u)) = (B_{(\phi,F)}(g),\ (\phi\circ\pi)(u))$ (since $\alpha(F) \in \mathrm{Gau}\,GL(M)$ and $\pi\circ\hat{\phi} = \phi\circ\pi) = (\overline{Q}_{(\phi,F)}\circ\sigma)(g,u)$, for all $(g,u) \in O(M)$. Here, $\overline{Q}_{(\phi,F)} \in \mathrm{Diff}(\mathrm{Met}(M)\times M)$ is the projection of the automorphism $Q_{(\phi,F)}$ and is given by $\overline{Q}_{(\phi,F)}(g,x)$ $= (B_{(\phi,F)}(g),\ \phi(x))$, for all $(g,x) \in \mathrm{Met}(M)\times M$.

Now let $(\phi_1,F_1), (\phi_2,F_2) \in \mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$. Then $Q_{(\phi_1,F_1)(\phi_2,F_2)}$

$$= B_{(\phi_1,F_1)(\phi_2,F_2)} \times (\alpha(F_1 A_{\phi_1}(F_2))\circ(\phi_1\circ\phi_2)^{\wedge})$$

$$= (B_{(\phi_1,F_1)}\circ B_{(\phi_2,F_2)}) \times (\alpha(F_1)\circ(\alpha\circ A_{\phi_1})(F_2)\circ\hat{\phi}_1\circ\hat{\phi}_2)$$

$$= (B_{(\phi_1,F_1)}\circ B_{(\phi_2,F_2)}) \times (\alpha(F_1)\circ((\mathrm{conj}\circ\ell)_{\phi_1}\circ\alpha)(F_2)\circ\hat{\phi}_1\circ\hat{\phi}_2)$$

$$= (B_{(\phi_1,F_1)}\circ B_{(\phi_2,F_2)}) \times (\alpha(F_1)\circ(\hat{\phi}_1\circ\alpha(F_2)\circ\hat{\phi}_1^{-1})\circ\hat{\phi}_1\circ\hat{\phi}_2$$

$$= (B_{(\phi_1,F_1)}\circ B_{(\phi_2,F_2)}) \times ((\alpha(F_1)\circ\hat{\phi}_1)\circ(\alpha(F_2)\circ\hat{\phi}_2))$$

$$= (B_{(\phi_1,F_1)} \times (\alpha(F_1)\circ\hat{\phi}_1))\circ(B_{(\phi_2,F_2)} \times (\alpha(F_2)\circ\phi_2)) = Q_{(\phi_1,F_1)}\circ Q_{(\phi_2,F_2)}.$$

Thus, $(\phi,F) \longrightarrow Q_{(\phi,F)}$ defines a homomorphism of $\mathrm{Diff}(M)\ltimes\mathrm{Aut}(TM)$ into $\mathrm{Aut}\ O(M)$. This homomorphism is, by inspection, injective so that $\mathrm{Diff}(M)\ltimes\mathrm{Aut}(TM)$ is isomorphic to the subgroup

$Q(\mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)) \leqslant \mathrm{Aut}\, O(M)$.

For each (ϕ,F), the automorphism $Q_{(\phi,F)}$ projects to the diffeomorphism $\overline{Q}_{(\phi,F)}$ given above. Note that $\overline{Q}_{(\phi,F)} = \mathrm{id}_{\mathrm{Met}(M)\times M}$ if and only if $(\phi,F) = (\mathrm{id}_M, 1\!\!1_{TM})$, so that the image of Q has trivial intersection with $\mathrm{Gau}\, O(M)$.

We now define $P^s \in \mathrm{Hom}(\mathrm{Aut}(M), \mathrm{Aut}\, O(M))$ by $P^s = Q \circ (\delta_s')^{-1}$, and this is the required monomorphism of $\mathrm{Aut}(M)$ into $\mathrm{Aut}\, O(M)$ □

The explicit form of P^s is given by $P^s_{(\phi,f,F)}(g,u)$ $= Q_{(\phi,f^{1/ns}F)}(g,u) = (C_{f^{1/ns}F}(\phi_* g), (\alpha(f^{1/ns}F))(\hat{\phi}(u))) = E^s_{(\phi,f,F)}(g)$, $f^{1/ns}(\phi(\pi(u)))(\alpha(F))(\hat{\phi}(u)))$. Let us put $u = \{e_a\}$, $x = \pi(u)$, then:

$$P^s_{(\phi,f,F)}(g,u) = (f^{-2/ns}\, C_F(\phi_* g),\ \{f(\phi(x))^{1/ns}F(\phi(x))D\phi(x).e_a\}) \qquad 4.1.18.$$

Now put $s = -\frac{2}{n}$ and fix g. Then equation 4.1.18 may be regarded as a unification of equations 1.5.1 and 1.6.2 with the $O\mathrm{Aut}(TM)$ action; We obtain an isomorphism of the principal $O(n)$-bundle $O(M,g)$ onto the principal $O(n)$-bundle $O(M,f\, C_F(\phi_* g))$ given by $u = \{e_a\} \longmapsto \{f(\phi(x))^{-\frac{1}{2}}\, F(\phi(x))D\phi(x).e_a\}$. The bundles $O(M,g)$, $O(M,f\, C_F(\phi_* g))$ are the fibres of the fibration, $pr_1 \circ \sigma$: $O(M) \longrightarrow \mathrm{Met}(M)$ above g, $fC_F(\phi_* g)$ respectively and the first is mapped onto the second under the action of $(\phi,f,F) \in \mathrm{Aut}(M)$.

We return to the canonical bundle $O(M)$ below, but first let us consider the action of $\mathrm{Aut}(M)$ on other natural structures. From now on, we assume that M is oriented and we denote by $\mathrm{Diff}(M)$ the group of orientation preserving diffeomorphisms.

Let us first consider the symplectic manifold $(T^*\mathrm{Met}(M),\omega)$ where ω is the canonical (weak) symplectic form on the L^2-cotangent bundle $T^*\mathrm{Met}(M)$. Recall (see above) that $T_g^*\mathrm{Met}(M) = \{p \otimes \mathrm{vol}(g) : p \in S^2(M)\}$, with L^2-pairing $\langle p_d, h\rangle = \int_M p\cdot h\ \mathrm{vol}(g)$, for all $h \in T_g\mathrm{Met}(M) = S_2(M)$, $p_d \equiv p \otimes \mathrm{vol}(g) \in T_g^*\mathrm{Met}(M)$. Since $T^*\mathrm{Met}(M)$ is open in $S_2(M) \times S_d^2(M)$, the tangent space to $T^*\mathrm{Met}(M)$ at any point may be naturally identified with $S_2(M) \oplus S_d^2(M)$, and we write $T_{(g,p_d)}(T^*\mathrm{Met}(M)) = S_2(M) \oplus S_d^2(M)$. The symplectic structure ω is given by:

$$\omega(g,p_d)((h,q_d),(h',q_d')) = \int_M (q'\cdot h - q\cdot h')\mathrm{vol}(g) \qquad 4.1.19,$$

for all $(h,q_d),(h',q_d') \in T_{(g,p_d)}(T^*\mathrm{Met}(M))$, $(g,p_d) \in T^*\mathrm{Met}(M)$ (see Fischer and Marsden [F 6]).

Before considering the action of $\mathrm{Aut}(M)$ on $(T^*\mathrm{Met}(M),\omega)$, let us recall the general situation in which a group action $I \in \mathrm{Hom}(G, \mathrm{Diff}(X))$ on a manifold X lifts up to a symplectic action $\hat{I}$ on (T^*X,ω), where ω is the canonical symplectic form on T^*X:

Let $\tau : T^*X \to X$ denote projection, then $\omega = -d\eta$, where η is the canonical 1-form on T^*X given by $\eta(\alpha) = \alpha \circ D\tau(\alpha)$ for all $\alpha \in T^*X$. The lifted action is given by $\hat{I}_a(\alpha) = (DI_a(\tau(\alpha))^{-1})^*$, for all $\alpha \in T^*X$, $a \in G$. This action leaves η, and hence ω, invariant and the corresponding moment mapping, $j : T^*X \to LG^*$ is given by $\langle j(\alpha), \xi\rangle = \langle \alpha, \xi_X(\tau(\alpha))\rangle$, for all $\xi \in LG$, $\alpha \in T^*X$. Here ξ_X is the infinitesimal generator of I corresponding to the Lie algebra element ξ, so that $\xi_X(x) = DL_x(1).\xi$, where $L_x : G \to X;\ a \mapsto L_a(x)$, for all $x \in X$. The map $\xi \longmapsto \xi_X$ defines a Lie algebra homomorphism of LG into $\mathrm{Vect}(X)$. We refer the reader to Abraham and Marsden [A 2] for more details.

We now apply this general framework to the action E^s of $\mathrm{Aut}(M)$ on $\mathrm{Met}(M)$ given by equation 4.1.15. It will be convenient to define an action C^* of $\mathrm{Aut}(TM)$ on $S^2(M)$ which is dual to the action C (see definition (4.1)3) extended to an action on $S_2(M)$:

<u>Definition (4.1)12</u>: Define $C^* \in \mathrm{Hom}(\mathrm{Aut}(TM), GL(S^2(M)))$ by $(C_F^*q)(x)(\alpha,\beta) = q(x)(F(x)^*.\alpha,\ F(x)^*.\beta)$, for all $\alpha,\beta \in T_x^*M$, $x \in M$, $q \in S^2(M)$ and $F \in \mathrm{Aut}(TM)$. Here $F(x)^*: T_x^*M \dashrightarrow T_x^*M$ is the dual of the automorphism $F(x)$ of T_xM, for each $x \in M$.

Note that $q\cdot(C_{F^{-1}}h) = (C_F^*q)\cdot h$ for all $h \in S_2(M)$, $q \in S^2(M)$ and $F \in \mathrm{Aut}(TM)$. There exists a similar formula for the (lower star) actions of $\mathrm{Diff}(M)$ on $S_2(M)$, $S^2(M)$, namely $q\cdot((\phi^{-1})_*h) = (\phi^{-1})_*((\phi_*q)\cdot h)$, for all $h \in S_2(M)$, $q \in S^2(M)$ and $\phi \in \mathrm{Diff}(M)$, as is easily verified.

<u>Proposition (4.1)13</u>: Consider the action E^s of $\mathrm{Aut}(M)$ on $\mathrm{Met}(M)$. The lifted action $\hat{E}^s$ of $\mathrm{Aut}(M)$ on $T^*\mathrm{Met}(M)$ is given by:

$$\hat{E}^s_{(\phi,f,F)}(p\otimes \mathrm{vol}(g)) = f^{2/ns}\, C_F^*(\phi_*p)\otimes \mathrm{vol}(\phi_*g) \qquad 4.1.20,$$

for all $p \otimes \mathrm{vol}(g) \in T_g^*\mathrm{Met}(M)$, $g \in \mathrm{Met}(M)$ and $(\phi,f,F) \in \mathrm{Aut}(M) = \mathrm{Diff}(M) \ltimes (C^+(M) \times O\mathrm{Aut}(TM))$.

<u>Proof</u>: First note that for $(\phi,f,F) \in \mathrm{Aut}(M)$, $h \in T_g\mathrm{Met}(M) = S_2(M)$, $DE^s_{(\phi,f,F)}(g).h = (f^{-2/ns} \circ C_F \circ \phi_*)(h)$, since $E^s_{(\phi,f,F)} \in \mathrm{Diff}(\mathrm{Met}(M))$ is the restriction to $\mathrm{Met}(M)$ of a linear map on $S_2(M)$. The lifted action is given by $\hat{E}^s_{(\phi,f,F)}(p\otimes \mathrm{vol}(g)) = (DE^s_{(\phi,f,F)}(g)^{-1})^*(p\otimes \mathrm{vol}(g))$, so that $\langle \hat{E}^s_{(\phi,f,F)}(p\otimes \mathrm{vol}(g)),h\rangle = \langle p\otimes \mathrm{vol}(g),\ DE^s_{(\phi,f,F)}(g)^{-1}.h\rangle$
$= \langle p\otimes \mathrm{vol}(g),\ ((\phi^{-1})_* \circ C_{F^{-1}} \circ f^{2/ns})(h)\rangle$

$$= \int_M p\cdot((\phi^{-1})_*(f^{2/ns}\, C_{F^{-1}}(h)))\mathrm{vol}(g) = \int_M (\phi^{-1})_*((\phi_*p)\cdot(f^{2/ns}C_{F^{-1}}(h))\mathrm{vol}(g)$$

$$= \int_M (\phi_*p)\cdot(f^{2/ns}\, C_{F^{-1}}(h))\phi_*\mathrm{vol}(g) \quad \text{(by change of variables formula)}$$

$$= \int_M (f^{2/ns}\phi_* p)\cdot(C_{F^{-1}}(h))\mathrm{vol}(\phi_* g) = \int_M C_F^*(f^{2/ns}\ \phi_* p)\cdot h\ \mathrm{vol}(\phi_* g)$$

$= \langle f^{2/ns}\ C_F^*(\phi_* p) \otimes \mathrm{vol}(\phi_* g), h\rangle$, for all $h \in T_g\mathrm{Met}(M)$.

Hence $\hat{E}^s_{(\phi,f,F)}(p \otimes \mathrm{vol}(g)) = f^{2/ns}\ C_F^*(\phi_* p) \otimes \mathrm{vol}(\phi_* g)$, for all $p \otimes \mathrm{vol}(g) \in T_g^*\mathrm{Met}(M)$ and $(\phi,f,F) \in \mathrm{Aut}(M)$ □

By the general results for lifted actions, we have $(\hat{E}^s_{(\phi,f,F)})^*\omega = \omega$, for all $(\phi,f,F) \in \mathrm{Aut}(M)$, so $\mathrm{Aut}(M)$ is naturally isomorphic to a subgroup of the symplectomorphism group of $(T^*\mathrm{Met}(M),\omega)$. We now calculate the moment mapping (or momentum) for the Aut(M)-action. Note that the Lie algebra of Aut(M) is given by $\mathrm{LAut}(M) = \mathrm{Vect}(M) \oplus\!\!\!\!\leftarrow (C(M) \oplus \mathrm{LOAut}(TM))$, where $\mathrm{LOAut}(TM) \equiv \mathrm{LSAut}(TM) = \{H \in \Gamma(T^*M \otimes TM)\colon \mathrm{trace}\ H = 0\}$. Another description of LAut(M) is obtained if we regard Aut(M) as an isomorph of Aut GL(M) (by proposition (4.1)9); The Lie algebra of $\mathrm{Aut}\ GL(M) \equiv \{\Psi \in \mathrm{Diff}\ GL(M)\colon \Psi \circ R_a = R_a \circ \Psi$, for all $a \in GL(n,\mathbb{R})\}$ is given by $\mathrm{LAut}\ GL(M) = \{X \in \mathrm{Vect}\ GL(M)\colon (R_a)_*X = X$, for all $a \in GL(n,\mathbb{R})\}$, the Lie algebra of $GL(n,\mathbb{R})$-invariant vector fields on GL(M). In what follows we use the former description of LAut(M). The dual of LAut(M) is given by $\mathrm{LAut}(M)^* = \Omega^1_d(M) \oplus\!\!\!\!\leftarrow ({}^*\Omega^n(M) \oplus \mathrm{LOAut}_d(T^*M))$, where $\Omega^1_d(M) = \Gamma(\Lambda^1(M) \otimes {}^*\Lambda^n(M))$ and $\mathrm{LOAut}_d(T^*M) = \{H_d \equiv H\otimes\mathrm{vol} \in \Gamma(TM \otimes T^*M \otimes {}^*\Lambda^n(M))\colon \mathrm{trace}\ H = 0\}$.

<u>Proposition (4.1)14</u>: The moment for the action by symplectomorphisms of Aut(M) on $(T^*\mathrm{Met}(M),\omega)$ is given by:

$$j\colon T^*\mathrm{Met}(M) \longrightarrow \Omega^1_d(M) \oplus\!\!\!\!\leftarrow ({}^*\Omega^n(M) \oplus \mathrm{LOAut}_d(T^*M)),$$

where

$$j(p \otimes \mathrm{vol}(g)) = -2((\mathrm{div}_g p)^b, \tfrac{1}{ns}\,\mathrm{trace}_g p,\ p^o) \otimes \mathrm{vol}(g) \qquad 4.1.21,$$

for all $p \otimes \mathrm{vol}(g) \in T_g^*\mathrm{Met}(M)$. Here $\mathrm{div}_g \equiv -\mathrm{trace} \circ \nabla_g$: $S^2(M) \longrightarrow \mathrm{Vect}(M)$ is the divergence operator associated with the metric g, and $(\cdot)^o$: $S^2(M) \longrightarrow \Gamma(TM \otimes T^*M)$ is given by taking the trace-free part of the (1,1)-form of an element of $S^2(M)$ (note that $(\cdot)^o$ is a g-dependent map).

<u>Proof</u>: We must first calculate the infinitesimal generators of the action E^s of $\mathrm{Aut}(M)$ on $\mathrm{Met}(M)$. Let $g \in \mathrm{Met}(M)$ and define $e : \mathrm{Aut}(M) \longrightarrow \mathrm{Met}(M)$ by $e(\phi,f,F) = E^s_{(\phi,f,F)}(g) = (f^{-2/ns} \circ C_F \circ \phi_*)(g)$, for all $(\phi,f,F) \in \mathrm{Aut}(M)$. Then, $De(id_M, 1, 1\!\!1_{TM})$: $L\mathrm{Aut}(M) \longrightarrow T_g\mathrm{Met}(M)$ is given by $(X,h,H) \longmapsto De_1(id_M).X + De_2(1).h + De_3(1\!\!1_{TM}).H$, where e_1: $\mathrm{Diff}(M) \longrightarrow \mathrm{Met}(M)$; $\phi \longmapsto \phi_* g$, e_2: $C^+(M) \longrightarrow \mathrm{Met}(M)$; $f \longmapsto f^{-2/ns}g$, e_3: $\mathrm{OAut}(TM) \longrightarrow \mathrm{Met}(M)$; $F \longmapsto C_F(g)$ are the partial maps of e at the identity.

We have $De_1(id_M).X = \frac{d}{dt} \exp(tX)_* g\Big|_{t=0}$, where $\exp$: $\mathrm{Vect}(M) \longrightarrow \mathrm{Diff}(M)$ is the Lie group exponential, given by $tX \longmapsto \phi_t$, where $\{\phi_t\}$ is the local 1-parameter group of local diffeomorphisms generated by $X \in \mathrm{Vect}(M)$. Hence

$$De_1(id_M).X = \frac{d}{dt}(\phi_t)_* g\Big|_{t=0} = \frac{d}{dt}\ \phi^*_{-t} g\Big|_{t=0} = -\frac{d}{dt}\ \phi^*_t\ g\Big|_{t=0} = -L_X\ g$$

(see equation 1.6.1).

The derivative of e_2 is given by $De_2(f).h = -\frac{2}{ns}\ f^{-2/ns-1}hg$, for all $h \in C(M)$, $f \in C^+(M)$, so that $De_2(1).h = -\frac{2}{ns}\ hg$.

In matrix notation we may write $C_F(g) = (F^{-1})^T g(F^{-1})$, so that $e_3 = e_3' \circ \mathrm{inv}$, where e_3': $\mathrm{OAut}(TM) \longrightarrow \mathrm{Met}(M)$; $F \longmapsto F^T gF$, and inv: $\mathrm{OAut}(TM) \longrightarrow \mathrm{OAut}(TM)$; $F \longmapsto F^{-1}$. We have $De_3'(F).H = H^T gF + F^T gH$, and $D\,\mathrm{inv}(F).H = -F^{-1}HF^{-1}$, for all $H \in T_F\mathrm{OAut}(TM)$, $F \in \mathrm{OAut}(TM)$. Thus $De_3(1\!\!1_{TM}).H = De_3'(\mathrm{inv}(1\!\!1_{TM})).\,D\mathrm{inv}(1\!\!1_{TM}).H = -(H^T g + gH)$, for all $H \in L\mathrm{OAut}(TM)$.

The infinitesimal generator of E^s is now given by

$$\xi_{Met(M)}(g) = - L_X g - \frac{2}{ns} hg - (H^T g + gH) \qquad 4.1.22,$$

for all $g \in Met(M)$, where $\xi = (X,h,H) \in LAut(M)$.

Using the general formula for the moment j, we obtain

$$<j(p \otimes vol(g)), \xi> = <p \otimes vol(g), -L_X g - \frac{2}{ns} hg - (H^T g + gH)>.$$

We consider each of the three terms in turn:

Firstly, $<p \otimes vol(g), L_X g> = \int_M p \cdot L_X g \; vol(g)$

$= 2 \int_M p \cdot \nabla X^b \; vol(g)$ (using $L_X g = 2 \; symm(\nabla X^b)$)

$= 2 \int_M (-div_g(p(X^b,\cdot)) + <X^b, div_g p>) vol(g)$

$= 2 \int_M <(div_g p)^b, X> vol(g)$ (since the first term vanishes by the divergence theorem) $= 2<(div_g p)^b \otimes vol(g), X>$.

Secondly, $<p \otimes vol(g), hg> = \int_M P \cdot (hg) vol(g)$

$= \int_M (trace_g p) h \; vol(g) = <(trace_g p) vol(g), h>$.

Finally, we have $<p \otimes vol(g), H^T g + gH>$

$= \int_M p \cdot (H^T g + gH) vol(g) = 2 \int_M p \cdot (gH) vol(g) = 2 \int_M p^o \cdot H \; vol(g)$

(since $trace \; H = 0$) $= 2<p^o \otimes vol(g), H>$.

Putting these results together, we obtain $<j(p \otimes vol(g)), \xi>$

$= <-2(div_g p)^b \otimes vol(g), X> + <-\frac{2}{ns}(trace_g p) vol(g), h>$

$+ <-2 \; p^o \otimes vol(g), H> = <-2((div_g p)^b, \frac{1}{ns} \; trace_g p, \; p^o) \otimes vol(g), \xi>$,

for all $\xi \in LAut(M)$. Hence $j(p \otimes vol(g)) = -2((div_g p)^b, \frac{1}{ns} \; trace_g p, p^o)$ $\otimes vol(g)$, for all $p \otimes vol(g) \in T^* Met(M)$, $g \in Met(M)$, as required $\square$

By restricting $j(p \otimes vol(g))$ to subalgebras of $LAut(M)$, we obtain particular conserved quantities. For example, if we consider $LConf(M) \leq LAut(M)$, we obtain the moment calculated by Fischer and Marsden [F 5]. This moment leads to a reduced phase space $(C_{div} \cap C_{trace})/Conf(M)$ which parameterizes the space of the true gravitational degrees of freedom (in the case of positive definite metrics in dimension three). Here $C_{div} = \{p \otimes vol(g): div_g p = 0\}$ and $C_{trace} = \{p \otimes vol(g): \ trace_g p = 0\}$. Note that C_{div} is one of the constraint spaces in the 3+1 formalism of the dynamics of general relativity (see Fischer and Marsden [F 6]).

By a general theorem on actions on cotangent bundles, the moment j is equivariant with respect to the co-adjoint action of $Aut(M)$ on $LAut(M)^*$, i.e. $j \circ E^s_{(\phi,f,F)} = Ad^*_{(\varphi^{-1},f^{-1},F^{-1})} \circ j$, for all $(\phi,f,F) \in Aut(M)$. A more detailed investigation into the symplectic action of $Aut(M)$ on $(T^*Met(M), \omega)$ would be interesting and provides an avenue for future work, but now we return to the interaction of $Aut(M)$ with other natural structures on M.

Having discussed the cotangent bundle of $Met(M)$ together with its natural symplectic structure, we turn now to the tangent bundle of $Met(M)$. As remarked above, there exists a 1-parameter family $\{G_t: t \in \mathbb{R}\}$ of symmetric, rank two, covariant tensor fields on $Met(M)$ (see equation 4.1.2). For definite signature metrics in $Met(M)$ and $nt \neq 1$, G_t defines a (weak) metric in the vector bundle $TMet(M)$. In order to consider the behaviour of G_t under the action of $Aut(M)$, we require the following lemma:

<u>Lemma (4.1)15</u>: The map $vol: Met(M) \longrightarrow {}^*\Omega^n(M)$ is invariant under the action of $OAut(TM)$ on $Met(M)$.

<u>Proof</u>: Let $F \in OAut(TM)$. Using the diffeomorphism β (equation

4.1.7), we may consider metrics as matrix valued functions on the frame bundle. We then have $\mathrm{vol}(g) = \sqrt{|\det\beta(g)|}$ when considered as an equivariant function on $GL(M)$. It is straightforward to check that $\beta(C_F(g)) = (\tilde{F}^{-1})^T\ \beta(g)\tilde{F}^{-1}$ (where $\tilde{F} \in C_{conj}(GL(M), OL(n,\mathbb{R}))$ corresponds to $F \in OAut(TM)$), so that $\mathrm{vol}(C_F(g))$
$= \sqrt{(\det \tilde{F})^2|\det\beta(g)|} = \sqrt{|\det\beta(g)|} = \mathrm{vol}(g)$, since $\det \tilde{F} \in \mathbb{Z}_2$ □

<u>Proposition (4.1)16</u>: For each $t \in \mathbb{R}$, the tensor field $G_t \in S_2(\mathrm{Met}(M))$ is invariant under the action of $\mathrm{Diff}(M) \ltimes OAut(TM) \leqslant Aut(M)$.

<u>Proof</u>: Let $(\phi,F) \in \mathrm{Diff}(M) \ltimes OAut(TM)$, $g \in \mathrm{Met}(M)$ and $h,k \in T_g\mathrm{Met}(M)$. We have $(C_F \circ \phi_*)^* G_t = (\phi_*)^* C_F^* G_t$. Consider $C_F^* G_t$; we have $(C_F^* G_t)(g)(h,k) = G_t(C_F(g))(DC_F(g).h,\ DC_F(g).k)$
$= G_t(C_F(g))(C_F(h),\ C_F(k))$ (since $C_F \in \mathrm{Diff}(\mathrm{Met}(M))$ is the restriction of a linear automorphism of $S_2(M)$)

$$= \int_M (C_F(g)(C_F(h),\ C_F(k)) - t(\mathrm{trace}_{C_F(g)}\ C_F(h))(\mathrm{trace}_{C_F(g)} C_F(k)))\mathrm{vol}(C_F(g)).$$

Now recall that $g(h,k) = \mathrm{trace}\ (g^{-1}\ hg^{-1}k)$ and $\mathrm{trace}_g h = g(g,h)$.
Thus $C_F(g)(C_F(h),\ C_F(k))$
$= \mathrm{trace}(((F^{-1})^T g\ F^{-1})^{-1}(F^{-1})^T hF^{-1}(((F^{-1})^T_g F^{-1})^{-1}(F^{-1})^T kF^{-1})$
$= \mathrm{trace}\ (g^{-1}hg^{-1}k) = g(h,k)$. Similarly, $\mathrm{trace}_{C_F(g)}\ C_F(h) = \mathrm{trace}_g h$.
Hence, $(C_F^* G_t)(g)(h,k) = \int_M (g(h,k) - t(\mathrm{trace}_g h)(\mathrm{trace}_g k))\mathrm{vol}(g)$
(using lemma (4.1)15) $= G_t(g)(h,k)$, so that $C_F^* G_t = G_t$.

Now we have $(C_F \circ \phi_*)^* G_t = (\phi_*)^* G_t = G_t$, by the result above regarding the Diff(M)-invariance of G_t. Thus, G_t is invariant under the action of $\mathrm{Diff}(M) \ltimes OAut(TM)$ □

<u>Corollary (4.1)17</u>: For $nt \neq 1$ and definite signature metrics in $\mathrm{Met}(M)$, $\mathrm{Diff}(M) \ltimes OAut(TM)$ is isomorphic to a subgroup of $\mathrm{Isom}(\mathrm{Met}(M),\ G_t)$.

As we have mentioned above in this section, the (weak) (pseudo-) Riemannian manifold $(\mathrm{Met}(M), G_t)$ is used in various applications. It is obviously useful to consider the isometry group of $(\mathrm{Met}(M), G_t)$ and corollary (4.1)17 gives a subgroup of the isometry group which properly contains (an isomorphic image of) Diff(M). Note that $C^+(M)$ does <u>not</u> act by isometries so we can't obtain an inclusion of the whole of Aut(M) in $\mathrm{Isom}(\mathrm{Met}(M), G_t)$.

Although we do not prove it here, it seems likely that a slice theorem may be proved for the action E^s of Aut(M) on Met(M). Fischer and Marsden [F 5] have proved such a theorem for the action of $\mathrm{Conf}(M) \leq \mathrm{Aut}(M)$ on Met(M). Also, using the metric G_t on which $\mathrm{Diff}(M) \ltimes \mathrm{OAut}(TM) \leq \mathrm{Aut}(M)$ acts by isometries, it should be possible to proceed as in Ebin [E 4] to construct a slice of the action of $\mathrm{Diff}(M) \times \mathrm{OAut}(TM)$ on Met(M). Consequences of such a slice theorem would be locally decreasing generalized conformal groups, and generically trivial generalized conformal groups. (Here, the <u>generalized conformal group</u> of $g \in \mathrm{Met}(M)$ is just $\{(\phi,f,F) \in \mathrm{Aut}(M): E^s_{(\phi,g,F)}(g) = g\} \cong \{\phi \in \mathrm{Diff}(M):$ there exists $(f,F) \in C^+(M) \times \mathrm{OAut}(TM)$ with $\phi^* g = fC_F(g)\}$. We could also consider the <u>generalized conformal superspace</u>, Met(M)/Aut(M). A slice theorem would imply that this was stratified into manifolds, and the singularities could be resolved in the standard manner (see below).

We now turn to another structure with which the groups discussed above interact. This is the space of volume elements on the manifold M. Recall that M is an oriented n-manifold which, for ease of exposition, we assume compact (without boundary). Let $V(M) = \{\omega \in {}^*\Omega^n(M): \omega$ is positively oriented$\}$ be the space of volume elements on the oriented manifold M. Note that, under the (lower star) action of the group of orientation preserving diffeomorphisms Diff(M), the stabilizer, $\mathrm{Diff}_\omega(M)$, of any element $\omega \in V(M)$ is a closed subgroup of Diff(M) with Lie algebra given by $\mathrm{LDiff}_\omega(M) = \{X \in \mathrm{Vect}(M)) : \mathrm{div}_\omega X = 0\}$ (see Ebin and Marsden [E 6] for technical details). We have the natural map $I \in C^+(V(M))$ given by $I(\omega) = \int_M \omega$ and a theorem of Moser (see [E 6]) implies that Diff(M) acts transitively on $I^{-1}(r)$ for each $r \in \mathbb{R}^+$, i.e. $I(\omega) = I(\omega')$ if and only if there exists $\phi \in \mathrm{Diff}(M)$ with $\phi_*\omega = \omega'$. Using Moser's result, Ebin and Marsden [E 6] demonstrate that Diff(M) is diffeomorphic to $\mathrm{Diff}_\omega(M) \times I^{-1}(I(\omega))$, for any $\omega \in V(M)$. In particular, since $I^{-1}(\omega)$ is convex, and hence contractible, $\mathrm{Diff}_\omega(M)$ is a deformation retract of Diff(M). The importance of the group $\mathrm{Diff}_\omega(M)$ for physics emerges fron the fact that it is the appropriate configuration space for the hydrodynamics of a homogeneous incompressible fluid (see Adams et al [A 3]). In fact, given $g \in \mathrm{Met}(M)$, there exists a right invariant metric on $\mathrm{Diff}_\omega(M)$ whose spray may be used to obtain existence and uniqueness of solutions of the classical Euler equations for a perfect fluid (see [E 6]). (Note that in section 4.5 below, we discuss various natural maps, one of which gives rise to the metric on $\mathrm{Diff}_\omega(M)$).

If a discussion of compressible hydrodynamics is required, then the appropriate configuration space may be obtained from a reduction of the semi-direct product $\mathrm{Diff}(M) \ltimes C(M)$ (see Marsden et al. [M 18]).

Let us now consider the interaction of V(M) with Met(M). We have the <u>volume fibration</u>, given by the surjection:

$$\mathrm{vol}\colon\ \mathrm{Met}(M) \longrightarrow V(M) \tag{4.1.23,}$$

which associates with each metric its (oriented) volume element. Note that $D\mathrm{vol}(g).h = \frac{1}{2}(\mathrm{trace}_g h)\mathrm{vol}(g)$, for all $h \in T_g\mathrm{Met}(M)$, so that the vertical subspace at g is given by $\{h \in T_g\mathrm{Met}(M)\colon \mathrm{trace}_g h = 0\}$.

We also have the <u>total volume map</u>, $\mathrm{Vol} = I\circ\mathrm{vol} \in C^+(\mathrm{Met}(M))$ which associates to each metric g, the volume of the corresponding Riemannian manifold (M,g). By Moser's theorem, we see that two metrics g, g' have the same total volume if and only if there exists a diffeomorphism ϕ such that $\phi_* g$, g' lie in the same fibre of vol.

Each fibre of 4.1.23 is an embedded submanifold of $\mathrm{Met}(M)$. Suppose that we equip $\mathrm{Met}(M)$ with the (weak) Riemannian metric G_o (see equation 4.1.2). Then, for each $\omega \in V(M)$, we let $G_o^\omega \in \mathrm{Met}(\mathrm{vol}^{-1}(\omega))$ be the Riemannian metric induced by the embedding of the fibre $\mathrm{vol}^{-1}(\omega)$ into $\mathrm{Met}(M)$. Ebin [E 4] has demonstrated that $(\mathrm{vol}^{-1}(\omega), G_o^\omega)$ is a symmetric space and he calculates the corresponding flip map and also the geodesics. The groups $\mathrm{Diff}_\omega(M)$, $\mathrm{Diff}(M)$ act isometrically on the Riemannian manifolds $\mathrm{vol}^{-1}(\omega)$, $\mathrm{Vol}^{-1}(I(\omega))$ respectively and Ebin shows that the corresponding quotients are homeomorphic topological spaces. Ebin remarks that a study of the geometrically attractive $\mathrm{vol}^{-1}(\omega)$ should shed light on the structure of $\mathrm{Vol}^{-1}(I(\omega))/\mathrm{Diff}(M)$, and therefore also on that of $\mathrm{Geom}(M)$ if ω is allowed to run over $V(M)$. Another method of examining the structure of $\mathrm{Geom}(M)$ is by resolving its singularities and we discuss this shortly. First let us say a few words concerning the action of $Aut(M)$ on the volume fibration:

We may regard $Conf(M) \equiv Diff(M) \ltimes C^+(M)$ as a subgroup of $Diff(V(M))$ via the monomorphism; $(\phi,f) \longmapsto \overline{E}^s_{(\phi,f)}$, where $\overline{E}^s_{(\phi,f)}(\omega) = f^{-1/s}\phi_*\omega$, for all $\omega \in V(M)$ (here, we are fixing $s \in \mathbb{R}^*$). Since $vol \circ E^s_{(\phi,f,F)} = \overline{E}^s_{(\phi,f)} \circ vol$, for all $(\phi,f,F) \in Aut(M)$, we may regard $Aut(M)$ as a subgroup of the group of (weak) automorphisms of the fibration vol. The group $OAut(TM)$ acts by "gauge transformations" whilst $C^+(M)$ acts transitively and freely on the orbit space. The group $Diff(M)$ intertwines the $OAut(TM)$ and $C^+(M)$ actions and its orbits project to the spaces $I^{-1}(r)$ for $r \in \mathbb{R}^+$, by Moser's theorem.

We now consider the important geometrical action of $Diff(M) \leq Aut(M)$ on $Met(M)$, in particular the orbit space $Geom(M) = Met(M)/Diff(M)$, the space of geometries on the manifold M.

We have noted above that the space $Geom(M)$ is very important in geometry and in classical and quantum gravity theory. Unfortunately, $Geom(M)$ does not have a manifold structure due to the presence of singularities corresponding to metrics with symmetries. We first give a brief description of the structure of $Geom(M)$ (for more details, see Fischer [F 1] and Bourguignon [B 16]) and then show how the singularities may be resolved or unfolded. As elsewhere in this section, M is a connected smooth n-manifold. For certain technical results, the details of which we do not go into, compactness of M is also required. We do not assume that M is orientable, so that $Diff(M)$ denotes the group of all diffeomorphisms of M and $GL(M)$ is the bundle of all frames on M. For ease of exposition, we deal only with positive definite metrics on M, but the results may be extended to include the indefinite case as well.

The basic topological result concerning Geom(M) is the following (see Fischer [F 1]); Geom(M) (with the quotient topology) is a connected, second countable, metrizeable space. This result shows that Geom(M) possesses the strongest separation and countability properties that a space can exhibit, and corollaries include the following; Geom(M) is Hausdorff, separable and paracompact.

Topologically, therefore, Geom(M) possesses nice structure. We turn now to a review of the natural differential properties of this space. Unless M is wild, some metrics on M possess non-trivial isometry groups and then the associated symmetric geometries do not have neighbourhoods homeomorphic to neighbourhoods of geometries with trivial isometry group (i.e., those geometries in the projection of $\mathrm{Met}_1(M)$). Thus, Geom(M) cannot admit a manifold structure based on the quotient topology; the differences in dimension (and number of components) of isometry groups cause the orbit space Geom(M) to have singularities. It can be shown, however, that Geom(M) is partitioned into manifolds of geometries such that the geometries of high symmetry are contained in the boundary of manifolds made up of geometries of lower symmetry. The manifolds constituting Geom(M) are called strata, and the decomposition into manifolds of geometries is called a stratification. The basic idea behind this decomposition is to collect together all geometries which have the same symmetry type. Then it must be shown that these strata, the set of which is indexed by the conjugacy classes in Diff(M) of the isometry groups of metrics, fit together in a regular fashion so as to give a bona fide stratification (it is possible that the strata might wind around one another in a complex way with transversal or self intersections, but it may be shown that these phenomena do not occur). For details of the stratification theorem, see Bourguignon [B 16], Fischer [F 1].

The stratification provides the framework for passing smoothly from one stratum to another, thereby allowing a generalized dynamics to take place on the stratified topological space Geom(M). We refer the reader to Francaviglia [F 9] for applications of this concept of generalized dynamics to general relativity.

Since it is the singularities themselves which complicate the structure of Geom(M), it is important to have ways of resolving or unfolding them:

Definition (4.1)18: A resolution of Geom(M) is a continuous, open, surjective map $\rho: X \to \mathrm{Geom}(M)$, where X is a manifold, and such that for each $[g] \in \mathrm{Geom}(M)$, the space $\rho^{-1}([g])$ is a finite dimensional closed submanifold of X.

Since ρ is not, in general, a covering map, we refer to each $\rho^{-1}([g])$, $[g] \in \mathrm{Geom}(M)$, as a pseudo-fibre (see Fischer [F 2]).

The simplest approach to resolution is to remove the non-free aspect of the action (lower star) of Diff(M) on Met(M); We restrict our attention to $\mathrm{Met}_1(M)$ and then (the restriction of) (lower star) is free. The Ebin-Palais slice theorem discussed above implies that the resulting quotient space $\mathrm{Geom}_1(M) \equiv \mathrm{Met}_1(M)/\mathrm{Diff}(M)$ is a smooth manifold. This approach is not very useful since the singular points have just been thrown out. Indeed, in general relativity, the symmetric metrics which give rise to the singularities are the ones in which we are often interested. Note, however, that this method does give a manifold structure to a very large subspace of Geom(M).

A second approach which actually resolves the singularities in Geom(M) itself, albeit in an unnatural manner, is the following; Let $x \in M$ and consider the group of diffeomorphisms fixing a frame at x. This is:

$$\mathrm{Diff}_x(M) = \{\phi \in \mathrm{Diff}(M): \phi(x) = x \text{ and } D\phi(x) = \mathrm{id}_{T_x M}\} \tag{4.1.24.}$$

The action of Diff(M) on Met(M) restricts to an action of $\mathrm{Diff}_x(M)$. Since we wish to consider principal bundles, we use the pullback action of diffeomorphisms, rather than (lower star) which is a left action. $\mathrm{Diff}_x(M)$ is a closed subgroup of Diff(M) and its action on Met(M) is free; for suppose $\phi^* g = g$ for $\phi \in \mathrm{Diff}_x(M)$. Then ϕ is an isometry of the connected Riemannian manifold (M,g) that fixes a point and whose derivative at that point is the identity. Therefore, by a classical theorem (see Helgason [H 8], lemma 11.2), ϕ is the identity diffeomorphism. Hence the action is free.

Again using the slice theorem, it can be demonstrated that the quotient space $\mathrm{Geom}_x(M) = \mathrm{Met}(M)/\mathrm{Diff}_x(M)$ is a manifold, and the projection π_x: $\mathrm{Met}(M) \to \mathrm{Geom}_x(M)$ is a principal $\mathrm{Diff}_x(M)$-bundle over $\mathrm{Geom}_x(M)$. We have the surjective map ρ_x: $\mathrm{Geom}_x(M) \to \mathrm{Geom}(M)$; $[g]_x \mapsto [g]$, mapping the $\mathrm{Diff}_x(M)$-orbits to the Diff(M)-orbits, and ρ_x is a resolution of Geom(M). Fischer [F 3] shows that the pseudo-fibre $\rho_x^{-1}([g])$ is the finite dimensional closed submanifold $\mathrm{Diff}^*(M).g/\mathrm{Diff}_x(M)$ of $\mathrm{Geom}_x(M)$. This pseudo-fibre is naturally diffeomorphic to the double coset manifold $\mathrm{Isom}(M,g)\backslash\mathrm{Diff}(M)/\mathrm{Diff}_x(M)$ $= \{\mathrm{Isom}(M,g)\circ\phi\circ\mathrm{Diff}(M): \phi \in \mathrm{Diff}(M)\}$.

This second approach does give a resolution of the singularities of Geom(M) itself, but a point $x \in M$ must be chosen. Also we have to restrict our attention to a proper subgroup of Diff(M). A more natural resolution due to Fischer is the following:

Let S be the right action of Diff(M) on Met(M) × GL(M) given by:

$$S_\phi(g,u) = (\phi^* g, \hat{\phi}^{-1}(u)) \qquad 4.1.25,$$

for all $(g,u) \in \mathrm{Met}(M) \times GL(M)$, $\phi \in \mathrm{Diff}(M)$. Recall that $\phi^* = (\phi_*)^{-1}$, so that the orbit spaces of Met(M) under the action of Diff(M) by upper star (i.e., pullback) and lower star (i.e., push forward) are equivalent qua stratified topological spaces. We shall denote both orbit spaces by $\mathrm{Geom}(M) = \mathrm{Met}(M)/\mathrm{Diff}(M)$.

The action S is free; for suppose $S_\phi(g,u) = (g,u)$. Then $\phi^* g = g$ and $\hat{\phi}(u) = u$. By the theorem quoted above concerning fixed points of isometries, we see that $\phi = id_M$. We therefore have a free action without restricting either the diffeomorphism group or the space of metrics. Fischer proves the following result:

Theorem (4.1)18 (Fischer, [F 3]): The action S of Diff(M) on Met(M) × GL(M) is smooth, free and proper, and the orbit space $\mathrm{Geom}_F(M) = (\mathrm{Met}(M) \times GL(M))/\mathrm{Diff}(M)$ is a smooth manifold. Moreover, the projection map $\pi_F\colon \mathrm{Met}(M) \times GL(M) \longrightarrow \mathrm{Geom}_F(M)$ is a submersion, and has the structure of a principal Diff(M)-bundle over the manifold $\mathrm{Geom}_F(M)$.

This result is proved by considering local equivariant cross sections of the action S. These local sections are constructed au Palais using the S-invariant metric $\Lambda \in (\mathrm{Met}(M) \times GL(M))$ given by:

$$\Lambda(g,u) = G_o(g) \oplus \hat{g}(u) \qquad 4.1.26,$$

where, for each $g \in \mathrm{Met}(M)$, $\hat{g} \in \mathrm{Met}(GL(M))$ is given by:

$$\hat{g} = \pi^* g \oplus \mathrm{can}(LC(g) \otimes LC(g)) \qquad 4.1.27,$$

where $LC(g)\colon TGL(M) \longrightarrow gl(n,\mathbb{R})$ is the Levi-Cività connection of g, and can is the standard Euclidean inner-product on $gl(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$.

The relationship of $\mathrm{Geom}_F(M)$ to $\mathrm{Geom}(M)$ may be described as follows: Let us regard the orbit map:

$$\pi_G\colon \mathrm{Met}(M) \longrightarrow \mathrm{Geom}(M) \qquad 4.1.28,$$

as a "pseudo" principal $\mathrm{Diff}(M)$-bundle (pseudo because $\mathrm{Geom}(M)$ is not a manifold and because the $\mathrm{Diff}(M)$ (right) action is not free - only if M were wild, would 4.1.28 be a bona fide principal fibration). We have the map $\rho_F\colon \mathrm{Geom}_F(M) \longrightarrow \mathrm{Geom}(M)$ given by:

$$\rho_F([(g,u)]) = [g] \qquad 4.1.29,$$

for all $[(g,u)] \in \mathrm{Geom}_F(M)$. The map ρ_F gives the pseudo fibre bundle with standard fibre $GL(M)$ associated with the pseudo principal $\mathrm{Diff}(M)$-bundle π_G via the action "hat" of $\mathrm{Diff}(M)$ on $GL(M)$. For $g \in \mathrm{Met}(M)$, we have the usual diffeomorphism $\kappa_g\colon GL(M) \longrightarrow \rho_F^{-1}(\pi_G(g)) = \rho_F^{-1}([g])$; $u \longmapsto [(g,u)]$, and κ_g coincides with the map $\pi_F(g,\cdot)$.

The map ρ_F is a resolution of the singularities of $\mathrm{Geom}(M)$. The pseudo-fibre $\rho_F^{-1}([g])$ is easily shown to be diffeomorphic to $GL(M)/\mathrm{Isom}(M,g)$. Since, in $\mathrm{Met}(M) \times GL(M)$, all metrics are initially crossed with $GL(M)$, the deviation $\mathrm{Isom}(M,g)$ of the pseudo-fibre from $GL(M)$ measures the degree of unfolding of $\mathrm{Geom}(M)$ at the geometry $[g]$ necessary to give a manifold structure to $\mathrm{Geom}_F(M)$. As expected, the symmetry group of g parameterizes the degree of unfolding at $[g]$.

The canonical Fischer resolution ρ_F is, in fact, a natural unification of the point dependent resolutions ρ_x introduced above; First note that for fixed $x \in M$, and for each frame $u \in \pi^{-1}(x)$, we have the map $d_u\colon \mathrm{Geom}_x(M) \to \mathrm{Geom}_F(M)$; $[g]_x \mapsto [(g,u)]$. Assuming either that M is non-orientable (so that $\mathrm{Diff}(M)$ acts transitively on the connected manifold $GL(M)$), or that M is oriented and possesses an orientation reversing diffeomorphism (so that $\mathrm{Diff}(M)$ acts transitively on the two component manifold $GL(M)$), the map d_u is a diffeomorphism. Moreover, d_u maps the pseudo-fibres of the resolution ρ_x to those of the resolution ρ_F (i.e. $\rho_F \circ d_u = \rho_x$), so that the two resolutions ρ_x, ρ_F are equivalent. The resolution ρ_x is not canonical, and to pass to the canonical resolution ρ_F, it is necessary to utilize the frame dependent diffeomorphism d_u.

We have a family ρ_x of non-canonical resolutions parameterized by the manifold M. These can be collected together in a natural way as we now demonstrate. For more details, see Fischer [F 3]:

The group $GL(n,\mathbb{R})$ acts on the manifold $\mathrm{Met}(M) \times GL(M)$ by $((g,u),a) \mapsto (g,ua)$, for all $(g,u) \in \mathrm{Met}(M) \times GL(M)$ and $a \in GL(n,\mathbb{R})$. Since $\hat{\phi}(ua) = \hat{\phi}(u)a$, for all $\phi \in \mathrm{Diff}(M)$ and $a \in GL(n,\mathbb{R})$, this action passes to an action of $GL(n,\mathbb{R})$ on $(\mathrm{Met}(M) \times GL(M))/\mathrm{Diff}(M)$ $\equiv \mathrm{Geom}_F(M)$, given by $([(g,u)],a) \mapsto [(g,ua)]$. Note that this is a right action on $\mathrm{Geom}_F(M)$, but we can still form the bundle associated to the frame bundle $GL(n,\mathbb{R}) \hookrightarrow GL(M) \to M$ with standard fibre $\mathrm{Geom}_F(M)$. We denote this associated bundle by $\pi_E\colon E \to M$, so that $E = GL(M) \times_{GL(n,\mathbb{R})} \mathrm{Geom}_F(M)$, and $\pi_E([(u,[(g,u')])]) = \pi(u)$, for all $[(u,[(g,u')])] \in E$. As usual (see 6.1.1), we have the diffeomorphisms

κ_u: $\mathrm{Geom}_F(M) \longrightarrow \pi_E^{-1}(\pi(u))$ $(u \in GL(M))$ given by:

$$\kappa_u([(g,u')] = [(u,[(u,[(g,u')])] \qquad 4.1.30,$$

for all $[(g,u')] \in \mathrm{Geom}_F(M)$.

Now, for fixed $x \in M$, choose a frame $u \in \pi^{-1}(x)$, and define $\lambda_u = \kappa_u \circ d_u$: $\mathrm{Geom}_x(M) \longrightarrow \pi_E^{-1}(x)$, so that $\lambda_u([g]_x) = [(u, [(g,u)])]$, for all $[g]_x \in \mathrm{Geom}_x(M)$. Note that $[(ua,[(g,ua)])] = [(ua,[(g,u)]a)] = [(u,[(g,u)])]$, for all $a \in GL(n,\mathbb{R})$, so that λ_u is independent of the choice of frame u at x. Therefore, for each $x \in M$, we may define the diffeomorphism λ_x $(\equiv \lambda_u$, any $u \in \pi^{-1}(x))$: $\mathrm{Geom}_x(M) \longrightarrow \pi_E^{-1}(x)$ given by:

$$\lambda_x([g]_x) = [(u,[(g,u)])] \qquad 4.1.31,$$

for all $[g]_x \in \mathrm{Geom}_x(M)$. We may regard the bundle E as the grand resolution space of Geom(M); the standard fibre is the canonical resolution space $\mathrm{Geom}_F(M)$, whilst, via the diffeomorphism λ_x, the fibre above x can be identified with the particular resolution space $\mathrm{Geom}_x(M)$ in a frame independent manner. With the identifications $\{\lambda_x: x \in M\}$, we may write $E = \bigcup_{x \in M} \mathrm{Geom}_x(M)$, showing explicitly the fact that E ties together the canonical resolution space with the family (parameterized by M) of particular resolution spaces.

The Fischer approach to the resolution of the singularities of Geom(M) just described is very natural and elegant. The unfolding of $\mathrm{Geom}_F(M)$ at each geometry $[g] \in \mathrm{Geom}(M)$ is parameterized by the isometry group of g, so the Fischer construction gives complete knowledge of the unfolding at each geometry necessary to make $\mathrm{Geom}_F(M)$ a manifold. The grand resolution space E

provides a bundle theoretic framework for studying the relationships between the canonical resolution on the one hand and the particular x dependent resolutions on the other. The Fischer approach may also be used to resolve the singularities in other infinite dimensional stratified spaces of interest in geometry and physics. For example, given any principal G-bundle P, the orbit space $C(P) \equiv \mathrm{Conn}(P)/\mathrm{Aut}(P)$, is the moduli space of connections (the action of $\mathrm{Aut}(P)$ on $\mathrm{Conn}(P)$ is discussed in appendix 6.1). The space $C(P)$ is a stratified topological space with singularities due to the existence of non-isomorphic isotropy groups at different connections in P. To obtain a free action, we consider the space $\mathrm{Conn}(P) \times P$ with $\mathrm{Aut}(P)$ acting (on the right) in an obvious fashion. It can be shown that this action is free, smooth and proper and it then follows that the orbit space $C_F(P) \equiv (\mathrm{Conn}(P) \times P)/\mathrm{Aut}(P)$ is a smooth manifold and that the natural projection $\mathrm{Conn}(P) \times P \longrightarrow C_F(P)$ has the structure of a principal $\mathrm{Aut}(P)$-bundle over $C_F(P)$. We also have the projection ρ_F: $C_F(P) \longrightarrow C(P)$; $[(\omega,u)] \longmapsto [\omega]$, and this is a resolution of the singularities of $C(P)$. The projection ρ_F may be regarded as the bundle associated with the pseudo principal $\mathrm{Aut}(P)$-bundle, $\mathrm{Conn}(P) \longrightarrow C(P)$, via the natural left action (evaluation) of $\mathrm{Aut}(P)$ on P. A grand resolution space may also be constructed (see Fischer [F 3]).

We now propose a slight variant of the Fischer resolution of Geom(M). This approach again uses a bundle framework; in particular, we utilize the canonical bundle introduced in section 1.4. As usual, we proceed in a formal geometrical manner. More details will appear elsewhere.

Consider the canonical principal $O(n)$-bundle $\sigma\colon O(M) \longrightarrow Met(M) \times M$ (see 1.4.1, 4.1.16). We have shown above (proposition (4.1)11) that $Aut(M)$ acts on $O(M)$ by principal bundle automorphisms. In particular, the subgroup $Diff(M)$ of $Aut(M)$ acts on $O(M)$. We now restrict our attention to the action of the diffeomorphism group on the canonical $O(n)$-bundle. Consider the action given by $\phi \longmapsto Q_{(\phi^{-1},1\!\!1_{TM})}$; $(g,u) \longmapsto (\phi^{*}g, \hat{\phi}^{-1}(u))$, for all $(g,u) \in O(M)$ and $\phi \in Diff(M)$. We will denote this action by S since it is the restriction of the action S (4.1.25) to the submanifold $O(M)$ of $Met(M) \times GL(M)$. Again, by the classical theorem on isometries, the action $S\colon O(M) \times Diff(M) \longrightarrow O(M)$ is free.

The methods used in proving theorem (4.1)18 may now be applied to the submanifold $O(M)$. In particular, the metric Λ (4.1.26) induces an S-invariant Riemannian structure on $O(M)$, and this may be used to construct local equivariant cross sections for the action of S on $O(M)$. The existence of such sections implies the following:

The orbit space $Geom_o(M) \equiv O(M)/Diff(M)$ admits the structure of a smooth manifold and we have the following principal $Diff(M)$-bundle over $Geom_o(M)$:

$$Diff(M) \hookrightarrow O(M) \xrightarrow{\pi_o} Geom_o(M) \qquad 4.1.32,$$

where π_o is the orbit projection map; $(g,u) \longmapsto [(g,u)]$, for all $(g,u) \in O(M)$. We also have the resolution ρ_o of the singularities of $Geom(M)$ given by:

$$\rho_o\colon Geom_o(M) \longrightarrow Geom(M);\quad [(g,u)] \longmapsto [g] \qquad 4.1.33,$$

for all $[(g,u)] \in Geom_o(M)$. The pseudo bundle ρ_o may be regarded as a sub-bundle of the pseudo bundle ρ_F (see 4.1.29).

Although the results contained in the previous paragraph have not been rigorously proved, we now derive some simple consequences of the formal results 4.1.32, 4.1.33:

Proposition (4.1)19: The pseudo fibre $\rho_o^{-1}([g])$ is naturally diffeomorphic to $O(M,g)/Isom(M,g)$, for each geometry $[g] \in Geom(M)$.

Proof: $\rho_o^{-1}([g]) = \{[(g',u)] \in Geom_o(M): [g'] = [g]\}$
$= \{[(g',u)] \in Geom_o(M):$ there exists $\phi \in Diff(M)$ with $\phi^* g' = g\}$
$= \{[(g,u)] \in Geom_o(M)\}$. Now define $k: O(M,g)/Isom(M,g) \longrightarrow \rho_o^{-1}([g])$ by $k([u]_{Isom(M,g)}) = [(g,u)]$ for all $[u]_{Isom(M,g)} \in O(M,g)/Isom(M,g)$. Then k is well defined, since $[u]_I = [u']_I$ $(I \equiv Isom(M,g))$ implies that there exists $\phi \in Isom(M,g)$ with $\hat{\phi}(u) = u'$. Then $[(g,u')] = [(g,\hat{\phi}(u))] = [(\phi^* g,u)] = [(g,u)]$. The map k is also a bijection; Surjectivity is obvious, and to prove injectivity, suppose $k([u]_I] = k([u']_I)$. Then $[(g,u)] = [(g,u')]$, so that there exists $\phi \in Diff(M)$ with $\phi^* g = g$ and $\hat{\phi}(u') = u$. Since $\phi^* g = g$ implies $\phi \in Isom(M,g)$, then $[u]_I = [\hat{\phi}(u')]_I = [u']_I$. Since k, $k^{-1}(;[(g,u)] \longmapsto [u]_I)$ are smooth, k is the required diffeomorphism of the pseudo-fibre $\rho_o^{-1}([g])$ onto $O(M,g)/Isom(M,g)$ □

Since, in $O(M)$, each metric g is extended by its orthonormal frame bundle $O(M,g)$ (via the map $pr_1 \circ \sigma$ - see section 1.4), the deviation of the pseudo-fibre $\rho_o^{-1}([g])$ from $O(M,g)$ is a measure of the degree of unfolding of $Geom(G)$ at $[g]$ necessary to give a manifold structure to $Geom_o(M)$. Hence, as expected, the unfolding at the geometry $[g]$ is parameterized by the isometry group of g, as in the Fischer resolution using the entire space $Met(M) \times GL(M)$, rather than the submanifold $O(M)$.

The space $O(M)$ is the total space of two natural principal bundles. By definition, we have the principal $O(n)$-fibration, $\sigma\colon O(M) \longrightarrow Met(M) \times M$, and we also have the principal $Diff(M)$-fibration, $\pi_o\colon O(M) \longrightarrow Geom_o(M)$, the base space of the latter bundle being a resolution space for $Geom(M)$. Note that the $O(n)$ and $Diff(M)$ actions on $O(M)$ commute. We summarize the various bundles and resolutions in the following diagram:

$$
\begin{tikzcd}
 & Diff(M) \arrow[d, hook] & O(n) \arrow[dl, hook'] & Diff(M) \arrow[dd, hook] & & GL(n,\mathbb{R}) \arrow[dd, hook] \\
 & O(M) \arrow[dl, "\sigma"'] \arrow[dd, "\pi_o"'] \arrow[drr, hook] & & & & \\
Met(M)\times M \arrow[dd, "pr_1"'] & & & Met(M)\times GL(M) \arrow[rr, "pr_2"] \arrow[dd, "\pi_F"] & & GL(M) \arrow[dd, "\pi"] \\
 & Geom_o(M) \arrow[drr, hook] \arrow[dd, "\rho_o"'] & & & & \\
Met(M) \arrow[dr, "\pi_G"] & & & Geom_F(M) \arrow[r, hook] \arrow[dll, "\rho_F"'] & E \arrow[r, "\pi_E"] & M \\
 & Geom(M) & & & &
\end{tikzcd}
$$

4.1.34.

A construction analogous to the Fischer grand resolution space does not appear so naturally in the $O(n)$-approach, but we do have the following metric dependent construction:

Recall that the $O(n)$, $Diff(M)$-actions on $O(M)$ commute, so that the $O(n)$-action passes to the quotient $O(M)/Diff(M) \equiv Geom_o(M)$.

The (right) action on $\mathrm{Geom}_o(M)$ is given by $([(g,u)],a) \mapsto [(g,ua)]$, for all $[(g,u)] \in \mathrm{Geom}_o(M)$ and $a \in O(n)$. Now fix $g \in \mathrm{Met}(M)$ and consider the principal $O(n)$-bundle of g-orthonormal frames, $O(n) \hookrightarrow O(M,g) \to M$. Let $E_g = O(M,g) \times_{O(n)} \mathrm{Geom}_o(M)$ be the bundle associated to $O(M,g)$ via the action of $O(n)$ on $\mathrm{Geom}_o(M)$. Thus, E_g has standard fibre $\mathrm{Geom}_o(M)$ and projection $\pi_g: E_g \to M$; $[(u, [(g,u')])] \mapsto \pi(u)$. We have the usual diffeomorphism κ_u: $\mathrm{Geom}_o(M) \to \pi_g^{-1}(\pi(u))$, but here there is no analogue of the map d_u for $u \in GL(M)$. Thus, there is no direct analogue of the grand resolution space E. We may, however, regard the bundle $\mathrm{Geom}_o(M) \hookrightarrow E_g \to M$ as a sub-bundle of the Fischer bundle $\mathrm{Geom}_F(M) \hookrightarrow E \to M$.

Before leaving the topic of the resolution of the singularities in $\mathrm{Geom}(M)$, we make a further remark concerning the manifold $O(M)$ as the total space of the two natural principal fibrations $O(n) \hookrightarrow O(M) \xrightarrow{\sigma} \mathrm{Met}(M)\times M$ and $\mathrm{Diff}(M) \hookrightarrow O(M) \xrightarrow{\pi_o} \mathrm{Geom}_o(M)$. Just as the action of $O(n)$ on $O(M)$ passes to an action on $\mathrm{Geom}_o(M)$, the action of $\mathrm{Diff}(M)$ passes to an action on $\mathrm{Met}(M) \times M$. Let $\mathrm{Geom}_H(M) = \mathrm{Geom}_o(M)/O(n)$. Note that the space $\mathrm{Geom}_H(M)$ is <u>not</u> a manifold (the $O(n)$-action is not free). We have the homeomorphism $\psi: \mathrm{Geom}_H(M) \to (\mathrm{Met}(M) \times M)/\mathrm{Diff}(M)$ given by:

$$\psi([[(g,u)]]) \quad = \quad [(g,\pi(u))]_{\mathrm{Diff}(M)} \qquad 4.1.35,$$

for all $[[(g,u)]] \in \mathrm{Geom}_H(M)$. We also have the continuous projection, $\rho_H : \mathrm{Geom}_H(M) \to \mathrm{Geom}(M)$; $[[(g,u)]] \mapsto [g]$, with pseudo fibres $\rho_H^{-1}([g])$ homeomorphic to ${}^M/\mathrm{Isom}(M,g)$, for each $g \in \mathrm{Met}(M)$ (Cf. proposition (4.1)19, but here ρ_H is <u>not</u> a resolution of $\mathrm{Geom}(M)$).

The above discussion indicates the usefulness of $O(M)$ and its quotients in the study of the space of geometries on a manifold M. Since $\mathrm{Geom}_o(M)$ is a manifold, and, moreover, a resolution space for $\mathrm{Geom}(M)$, it is an obvious candidate for the configuration space for a dynamical theory of classical or quantum geometry (for applications in gravity theory or other areas of physics). In particular, it should be the domain of natural functionals. In this context, we mention the natural tensors of Epstein (see [E 14]). In our language, these may be described as follows:

Let B be a natural bundle over M, i.e. there exists a pull-back action of $\mathrm{Diff}(M)$ on B, and hence on the space $\Gamma(B)$ of sections of B. Let us denote this action by $(s,\phi) \mapsto \phi^* s$, for all $(s,\phi) \in \Gamma(B) \times \mathrm{Diff}(M)$. A typical example of a natural bundle is a bundle associated to a G-structure (see section 6.1). A <u>natural tensor</u> is an equivariant map $n\colon \mathrm{Met}(M) \to \Gamma(B)$, i.e. $n \circ \phi^* = \phi^* \circ n$, for all $\phi \in \mathrm{Diff}(M)$. To be more explicit, let F be a manifold and $\rho \in \mathrm{Hom}(GL(n,\mathbb{R}), \mathrm{Diff}(F))$ a left action of $GL(n,\mathbb{R})$ on F. Let $B = GL(M) \times_\rho F$ be a bundle associated to the frame bundle via the action ρ, so that there exists a natural diffeomorphism $\beta : \Gamma(B) \to C_\rho(GL(M),F)$ (Cf. 4.1.7, and see section 6.1). The natural action of $\mathrm{Diff}(M)$ on $C_\rho(GL(M),F)$ is given by $\phi' = \beta \circ \phi^* \circ \beta^{-1}$; $S \mapsto S \circ \hat{\phi}$, for all $S \in C_\rho(GL(M),F)$, $\phi \in \mathrm{Diff}(M)$ (see the remarks following 4.1.7).

Now suppose n is a natural tensor. Define $\hat{n}\colon O(M) \to F$ by:

$$\hat{n}(g,u) = \beta(n(g))(u) \qquad 4.1.36,$$

for all $(g,u) \in O(M)$. Let $\phi \in \mathrm{Diff}(M)$. Then $(\hat{n} \circ S_\phi)(g,u)$

$= \hat{n}(\phi^* g, \hat{\phi}^{-1}(u)) = \beta(n(\phi^* g))(\hat{\phi}^{-1}(u)) = \beta(\phi^*(n(g)))(\hat{\phi}^{-1}(u))$
$= \phi'(\beta(n(g)))(\hat{\phi}^{-1}(u)) = \beta(n(g))(\hat{\phi}(\hat{\phi}^{-1}(u))) = \beta(n(g))(u) = \hat{n}(g,u)$,
for all $(g,u) \in O(M)$. Hence $\hat{n} \circ S_\phi = \hat{n}$, and so $\hat{n}$ projects to a map $\overset{\Delta}{n}: \mathrm{Geom}_o(M) \to F$. Moreover, $\overset{\Delta}{n}([(g,u)]a) = \overset{\Delta}{n}([(g,ua)])$
$= \beta(n(g))(ua) = \rho(a^{-1}).\beta(n(g))(u) = \rho(a^{-1})\overset{\Delta}{n}([(g,u)])$, for all $[(g,u)] \in \mathrm{Geom}_o(M)$, $a \in O(n) \leq GL(n,\mathbb{R})$, so $\overset{\Delta}{n} \in C_\rho(\mathrm{Geom}_o(M),F)$ ($\rho \equiv \rho|O(n)$). Thus, each natural tensor induces an $O(n)$-equivariant map on the resolution space $\mathrm{Geom}_o(M)$. Similarly, any natural tensor induces a $GL(n,\mathbb{R})$-equivariant map on $\mathrm{Geom}_F(M)$. A reverse construction yields natural maps on $\mathrm{Met}(M)$ starting from equivariant maps on $\mathrm{Geom}_o(M)$ and $\mathrm{Geom}_F(M)$ in an obvious manner. We make further comments on natural maps in section 4.6.

We conclude this section by referring to another aspect of the structure of the space of metrics which relates to ideas of everywhere invariance discussed below. This is the so called YATS decomposition (see Isenberg [I 6]) and is another idea which has found recent application in general relativity:

A natural question to ask is that of whether a given metric $g \in \mathrm{Met}(M)$ is necessarily pointwise conformal to a metric of constant scalar curvature. This problem is known as the Yamabe problem because it was first formulated by Yamabe in 1960. Subsequent work by Trudinger, Aubin and Schoen lead to a complete solution of this problem in the case of positive definite metrics on a compact manifold. We refer to Schoen [S 7] and references therein for details, but here we just state the result:

Theorem (4.1)20 (Yamabe-Aubin-Trudinger-Schoen): Let M be a compact manifold of dimension not less than three. For each $g \in \mathrm{Met}(M)$ (positive definite metrics), there exists $f \in C^+(M)$

such that $\mathrm{Scal}(fg)$ is either -1, 0 or $+1$. The function f is unique in the ± 1 case and defined up to $\mathbb{R}^+ \leq C^+(M)$ in the 0 case.

Following Isenberg, we refer to theorem (4.1)20 as the YATS theorem. The proof of the theorem involves showing the existence of a positive solution u of the non-linear partial differential equation $Y_g u + \frac{\lambda(n-2)}{4(n-1)} u^{\frac{n+2}{n-2}} = 0$, where Y_g is the Yamabe operator associated with $g \in \mathrm{Met}(M)$ referred to in appendix 6.2, and $\lambda = -1, 0$ or $+1$.

An important consequence of the YATS theorem is that the space $\mathrm{Met}(M)$ is naturally partitioned into three subspaces $Y_\lambda(M)$, $\lambda \in \{-1,0,+1\}$, where $Y_\lambda(M) = \{g \in \mathrm{Met}(M)\colon$ there exists $f \in C^+(M)$ with $\mathrm{Scal}(fg) = \lambda\}$. Note that $Y_\lambda(M)$ could be empty for some M and λ. Given a metric g, the sign of the lowest eigenvalue $\lambda_1(g)$ of the Yamabe operator Y_g determines which of the three classes g is in; g is pointwise conformal to a metric with scalar curvature λ, the sign of λ being the same as that of $\lambda_1(g)$.

Since $\phi_* \circ \mathrm{Scal} = \mathrm{Scal} \circ \phi_*$ and $\mathrm{Scal}(fg) = \mathrm{Scal}(f_o^{-1} f(f_o g))$, for all $(\phi, f) \in \mathrm{Conf}(M)$, $g \in \mathrm{Met}(M)$, we see that each of the three classes $Y_\lambda(M)$ is a union (possibly empty) of orbits of the conformorphism group, $\mathrm{Conf}(M) \leq \mathrm{Aut}(M)$. Thus the partition given by the YATS theorem is natural in the sense of everywhere invariance (see section 4.2).

The YATS theorem has played a key rôle in a parameterization of the space of solutions of Einstein's equations due to Isenberg [I 6]. Isenberg considers globally hyperbolic spacetimes admitting a compact embedded spacelike hypersurface of constant mean curvature and obtains a parameterization of such spacetimes which are solutions of the (vacuum) Einstein equations. The parameterization is based on the conformal treatment of the constraint equations of the 3 + 1 formalism developed by Lichnerowicz, Choquet-Bruhat, York and O'Murchadha. Since this treatment is conformally invariant, it is possible to simplify it by choosing the hypersurface metric to have constant scalar curvature -1, 0 or +1 according to the YATS theorem. Once such spatial metrics have been chosen, an analysis of the constraint equations (in the form of the scale equation) may be made using the method of sub and super solutions. Isenberg thus completes the Lichnerowicz-Choquet-Bruhat-York-O'Murchadha programme and obtains a natural parameterization of the class of spacetimes under consideration. We refer to [I 6] for details of the parameterization, although we note here that the conformorphism group of the spatial hypersurface plays an important rôle. The Isenberg parameterization is a useful framework for studying issues like the stability and genericity of certain properties of spacetimes, such as the existence of Cauchy horizons.

This completes our remarks on the structure of the space of metrics on a manifold M. The space Met(M) has rich structure and many applications in geometry and gravity, some of which we have mentioned above. We return to some of our ideas concerning the action of Aut(M) and natural maps below, but first we introduce the concept of everywhere invariance.

4.2 Algebraic Framework

This section is purely algebraic and in it we set up the basic definitions of everywhere invariance and related ideas in the general setting of a group G acting on a set X. The ideas will be applied to geometrical and physical situations in the remaining sections of this chapter. In particular, we will consider everywhere invariance in the context of the action of the diffeomorphism group on the space of metrics on a manifold.

In the usual study of group actions, particular emphasis is put on individual elements in the G-set X. For example, the stabilizer (isotropy subgroup) of a particular element is often considered. In this section, we wish to generalize the discussion to a study of the behaviour of subsets under the group action. The interaction of the stabilizer of a subset U of X with the stabilizers of the elements of U is of particular interest in applications to geometry and general relativity.

In this section, we give only the basic definitions and some simple results which will be used below in this chapter. We do not give a thorough algebraic discussion of everywhere invariance.

Let X be any set and denote by $B(X)$ the group of bijections (permutations) of X. Let G be a group and $A \in \mathrm{Hom}(G,B(X))$ a (left) action of G on X. The action A induces an action $\tilde{A}$ on the power set of X; $\tilde{A} \in \mathrm{Hom}(G,B(\mathrm{Power}(X)))$ is given by $\tilde{A}_a(U) = \{A_a(x): x \in U\} \equiv A_a(U)$, for all non-empty $U \subseteq X$, and $\tilde{A}_a(\emptyset) = \emptyset$, for all $a \in G$. For $U \subseteq X$, $H \subseteq G$, U, H non-empty, let $A_H(U) = \{A_a(x): a \in H, x \in U\}$ denote the h-orbit of U. If $U = \{x\}$ for some $x \in X$, we write $A_H(U) = Hx$ if A is understood.

We now introduce the stabilizers under the actions A and $\tilde{A}$. For $U \subseteq X$, let $st(U) = \{a \in G: A_a(U) \subseteq U\}$ denote the set of elements stabilizing U under the action A. If $U = \{x\}$ for some $x \in X$, then $st(x) \equiv st(U)$ is a subgroup of G. In general, however, $st(U)$ possesses only the structure of a semigroup (A semigroup is a non-empty set equipped with a unital associative binary operation. Note that some authors refer to such a structure as a monoid). The total stabilizer, $tst(U)$, of U is defined to be $\bigcup_{x \in U} st(x)$. The set $tst(U)$ contains the unit element of G, but, in general, possesses no algebraic structure. We also have the bona fide stabilizer of U under the action $\tilde{A}$. We denote this by $\tilde{st}(U)$, so that $\tilde{st}(U) = \{a \in G: A_a(U) = U\}$, and $\tilde{st}(U)$ is the largest subgroup of G leaving U invariant under the action A. Note that $\bigcap_{x \in U} \tilde{st}(x) \leqslant \tilde{st}(U) \subseteq st(U)$.

In this chapter, we are particularly interested in the interaction between stabilizers of elements of a subset U on the one hand and the stabilizer (under A) of the subset U itself on the other. We will therefore be mainly concerned with the relationship between $st(U)$ and $tst(U)$. In what follows, we always assume sets are non-empty unless stated otherwise.

Definition (4.2)1: Let X,G,A be as above and let U be a proper subset of X containing at least two elements. U is said to be everywhere A-invariant if $tst(U) \subseteq st(U)$, inessentially A-invariant if $st(U) \subseteq tst(U)$, and totally A-invariant if $st(U) = tst(U)$. If A is understood, we use the abbreviation EI to denote both the adjective everywhere invariant and also the concept of everywhere invariance. Similarly for II and TI.

Obviously, there exist actions for which there are subsets U which are neither EI nor II (so certainly not TI). The examples we shall give in section 4.3 demonstrate that the two concepts EI and II are non-coincidental.

Our main interest is in EI, and we shall indicate uses of this idea in general relativity and in geometry. Note that U is EI if and only if $st(x) \subseteq st(U)$, for all $x \in U$, so that an EI subset is "everywhere invariant" in the sense that no element of U leaves U under the action of the point stabilizers.

We remark that proving a subset U is II often involves the use of some kind of fixed point theorem, and thus is generally more difficult than proving EI. Some geometrical examples of II are given in section 4.3.

Before deriving some consequences of definition (4.2)1, let us briefly consider another concept in algebra which involves the idea of a group acting on subsets; this is the concept of primitivity:

Consider the action $\tilde{A}$ of G on Power(X) induced by the action A of G on X. A partition p of X is said to be stable under A if $\tilde{A}_a(p) \subseteq p$, for all $a \in G$. Any action admits two stable partitions, namely $p_o = \{\{x\}: x \in X\}$ and $p_1 = \{X\}$. The action A is called primitive if the set of stable partitions is precisely $\{p_o, p_1\}$. Consider the partition p_A of X into G-orbits; this partition is stabilized by G since $A_a(Gx) = Gx$ for any G-orbit Gx. If $p_A = p_o$, then each orbit is a single point, so the action A is trivial. If $p_A = p_1$, then there is just one orbit, so the action A is transitive. Hence a non-trivial intransitive action cannot be primitive, so that the interesting case is that of a transitive action on a set X containing at least two elements. In the latter case, the following criterion may be proved (see Jacobson [J 1]): If A is a transitive action on a set X containing at least two elements, then

A is privitive if and only if $st(x)$ is a maximal subgroup of G, for all $x \in X$.

In the non-primitive case, we have the following simple result indicating a link with EI:

Proposition (4.2)2: Suppose $A \in Hom(G,B(X))$ is a non-primitive action on the set X, and suppose X contains at least three elements. Then X contains an EI subset.

Proof: A is non-primitive, therefore there exists a stable partition p of X with $p \neq p_o, p_1$. Any such partition contains an element U which is a proper subset of X and which contains at least two elements. The partition p is stable, so, for any $a \in G$, either $A_a(U) = U$ or $A_a(U) \cap U = \emptyset$. Now, let $x \in U$. and $a \in st(x)$. Then $A_a(U) \cap U \neq \emptyset$, so we must have $A_a(U) = U$. This implies that $a \in \tilde{st}(U) \subseteq st(U)$, so $st(x) \subseteq st(U)$, for all $x \in U$. The subset U is a proper subset of X containing at least two elements and, moreover, $tst(U) \subseteq st(U)$. Hence U is EI. □

We now give a result which gives a method of constructing EI subsets. In section 4.3, we will use this result to tie together EI with the group actions introduced in section 4.1. First recall that, given any subset S of a group L, the normalizer of S in L, denoted $Norm_L(S)$, is defined to be $\{a \in L : aSa^{-1} = S\}$. If S is a subgroup of L, then $Norm_L(S)$ is the largest subgroup of L having S as a normal subgroup.

Proposition (4.2)3: Suppose $A \in Hom(G,B(X))$, $C \in Hom(K,B(X))$ are actions on the set X by the groups G, K respectively. Assume that C is intransitive and that $A(G) \leqslant Norm_{B(X)}(C(K))$. Then any C-orbit (with at least two elements) is everywhere

A-invariant.

Proof: Let $U = Kx_o \subseteq X$ be any C-orbit and let $x \in U$. Hence, there exists $k \in K$ with $x = C_k(x_o)$. Suppose $a \in st(x)$ (all stabilizers are with respect to A). Then $A_a(C_k(x_o)) = C_k(x_o)$. Since $A(G) \leqslant Norm_{B(X)}(C(K))$, there exists $k_1 \in K$ such that $A_a \circ C_k = C_{k_1} \circ A_a$. Hence $C_{k_1}(A_a(x_o)) = C_k(x_o)$, so $A_a(x_o) = C_{k_2}(x_o)$, where $k_2 = k_1^{-1}k$.

Now, let $y \in U$, so that there exists $k_3 \in K$ with $y = C_{k_3}(x_o)$. Then $A_a(y) = A_a(C_{k_3}(x_o)) = C_{k_4}(A_a(x_o))$ for some $k_4 \in K$. Hence $A_a(y) = C_{k_4}(C_{k_2}(x_o)) = C_{k_5}(x_o) \in U$, where $k_5 = k_4k_2$. We have shown that, for all $x \in U$, $a \in st(x)$ implies $a \in st(U)$, so $tst(U) \subseteq st(U)$. Also U is a proper subset of X (since C is intransitive), so, assuming U contains at least two elements, U is everywhere A-invariant □

Corollary (4.2)4: Suppose $A \in Hom(G,B(X))$, $C \in Hom(K,B(X))$ (with C intransitive) with $C(K) \lhd A(G) \leqslant B(x)$. Then, any C-orbit (containing at least two elements) is everywhere A-invariant.

Proof: $C(K) \lhd A(G)$ implies that $A(G) \leqslant Norm_{B(X)}(C(K))$ □

A useful situation in which proposition (4.2)3 may be used is when we have a semi-direct product structure. Suppose $\theta \in Hom(G,Aut(K))$, $A \in Hom(G,B(X))$ and $C \in Hom(K,B(X))$. The actions θ,A,C are said to be compatible if:

$$A_a \circ C_k \circ A_a^{-1} = C_{\theta_a(k)} \qquad 4.2.1,$$

for all $a \in G$ and $k \in K$ (Cf. proposition (4.1)5). If θ,A,C are compatible, then the map $(a,k) \rightarrow C_k \circ A_a$ defines an action of the semi-direct product $G \ltimes_\theta K$ (see appendix 6.3) on X. Obviously, if θ, A, C are compatible, then $C(K) \lhd A(G)$. We may now use corollary (4.2)4 to deduce the following:

Corollary (4.2)5: Suppose $\theta \in \mathrm{Hom}(G, \mathrm{Aut}(K))$, $A \in \mathrm{Hom}(G, B(X))$, $C \in \mathrm{Hom}(K, B(X))$ are compatible and that C is intransitive. Then any C-orbit (containing at least two elements) is everywhere A-invariant.

When we deal with families of metrics (see below), we often consider an embedding $\gamma : F \hookrightarrow \mathrm{Met}(M)$, where F is some manifold. In the purely algebraic setting, we may consider an injection $\gamma : F \rightarrow X$, where F is a set. Questions of A-invariance ($A \in \mathrm{Hom}(G, B(X))$) may be transferred to F using the pullback of A; Let $U = \gamma(F) \subseteq X$. Then we may define $A^\gamma : \mathrm{st}(U) \rightarrow B(F)$; $a \mapsto \gamma^{-1} \circ A_a \circ \gamma$. We return to maps such as γ in section 4.4 when we consider the invariance equation.

Having introduced the relevant definitions, we turn now to some examples. Some applications of everywhere invariance will be given below in sections 4.4 and 4.5.

4.3 Examples

In this section, we give various geometrical examples illustrating the concepts introduced in the previous section. Since we are interested in the space of metrics and isometries, we take $X = \mathrm{Met}(M)$, $G = \mathrm{Diff}(M)$ and $A =$ (lower star). Here, as usual, M is a connected, smooth manifold, not necessarily orientable. We also assume that M is not wild (see section 4.1), so that $\mathrm{Met}_1(M) \neq \mathrm{Met}(M)$; i.e. there exist non-generic metrics on M.

The signature of the metrics in $Met(M)$ will be unimportant unless specified otherwise. To conform with the notation introduced in section 4.2, we let $st(g) \equiv Isom(M,g)$ denote the stabilizer of g under the action (lower star).

The first four examples are fairly trivial, but they do illustrate the three ideas given by definition (4.2)1.

4.3.1: Let $U = \{g,g'\}$ where g is generic and g' is non-generic. Then, $tst(U) = \{id_M\} \cup st(g') = st(g') \neq \{id_M\}$. Since g, g' do not lie in the same Diff(M)-orbit, we see that $st(U) = \{id_M\} \subsetneq tst(U)$. Hence, U is II but not TI.

4.3.2: Let $U = Met_1(M)$. Then $tst(U) = \{id_M\}$ and $st(U) = Diff(M)$. Hence, $tst \subsetneq st(U)$, so that U is EI but not TI.

4.3.3: Let $g_o \in Met(M)$, and let $U = S_{g_o}$ be a slice of the (lower star) action through g_o (see section 4.1 for a discussion of slices). The two properties of a slice we require are (i) $\phi \in st(g_o)$ implies $\phi \in st(U)$ and (ii) If $\phi \in Diff(M)$ and $\phi_*(U) \cap U \neq \emptyset$, then $\phi \in st(g_o)$. We now demonstrate that U is TI. Let $g \in U$ and $\phi \in st(g)$. Then $\phi_*(U) \cap U \neq \emptyset$, so $\phi \in st(g_o)$, by (ii). Property (i) now implies that $\phi \in st(U)$. Hence $st(g) \subseteq st(U)$, for all $g \in U$, so U is EI. Now suppose $\phi \in st(U)$. Then $\phi_*(U) \cap U = \emptyset$, so $\phi \in st(g_o) \subseteq tst(U)$. Hence $st(U) \subseteq tst(U)$. Hence $st(U) \subseteq tst(U)$, so U is also II. We conclude that U is TI.

Another example of a TI subset of $Met(M)$ is the following: Let $g \in Met(M)$ and $\lambda \in \mathbb{R}^+ - \{1\}$ and put $U = \{g, \lambda g\}$. Then, since $st(\lambda g) = st(g)$, $tst(U) = st(g)$. Also, as is easily checked, $\phi \in st(U)$ if and only if $\phi \in st(g)$. Hence, $st(U) = tst(U)$, and so U is TI.

4.3.4: Let $g \in \mathrm{Met}(M)$ be non-generic and $\psi \in \mathrm{Diff}(M) - \mathrm{Norm}_{\mathrm{Diff}(m)}(\mathrm{st}(g))$ such that $(\psi\mathrm{st}(g)) \cap \mathrm{st}(g)\psi^{-1} = \emptyset$. Let $U = \{g, \psi_* g\}$ and, for convenience, denote $\mathrm{st}(g)$ by I. Then $\mathrm{tst}(U) = I \cup (\psi I \psi^{-1}) \supsetneqq I \supsetneqq \{\mathrm{id}_M\}$. Suppose $\phi \in \mathrm{st}(U)$. Then, either (a) $\phi_* g = g$ and $\phi_*\psi_* g = \psi_* g$, or (b) $\phi_* g = \psi_* g$ and $\phi_*\psi_* g = g$. Case (a) implies that $\phi \in I \cap (\psi I \psi^{-1})$ and case (b) implies that $\phi \in (\psi I) \cap (I\psi^{-1})$. Thus, $\mathrm{st}(U) = (I \cap (\psi I \psi^{-1})) \cup ((\psi I) \cap (I\psi^{-1}))$.

Now consider the interaction between $\mathrm{tst}(U) \equiv I \cup (\psi I \psi^{-1})$ and $\mathrm{st}(U) \equiv (I \cap (\psi I \psi^{-1})) \cup ((\psi I) \cap (I\psi^{-1}))$. We will show that $\mathrm{tst}(U) \nsubseteq \mathrm{st}(U)$ and $\mathrm{st}(U) \nsubseteq \mathrm{tst}(U)$.

Suppose $\phi \in \mathrm{tst}(U) - (\psi I \psi^{-1}) \subseteq \mathrm{tst}(U)$ (Note that $\mathrm{tst}(U) - (\psi I \psi^{-1})$ is non-empty because $\psi \notin \mathrm{Norm}_{\mathrm{Diff}(M)}(I))$. Then $\phi \in I$ and $\phi \notin \psi I \psi^{-1}$. In particular, $\phi \notin I \cap (\psi I \psi^{-1})$. Now suppose $\phi \in (\psi I) \cap (I\psi^{-1})$. Then there exist $\phi_1, \phi_2 \in I$ with $\psi \circ \phi_1 = \phi = \phi_2 \circ \psi^{-1}$. But $\phi \in I$ and $\psi \in I$, so this leads to a contradiction. Hence $\phi \notin (\psi I) \cap (I\psi^{-1})$. We have shown that there exists $\phi \in \mathrm{tst}(U)$ with $\phi \notin \mathrm{st}(U)$. Hence $\mathrm{tst}(U) \nsubseteq \mathrm{st}(U)$.

Conversely, let $\phi \in (\psi I) \cap (I\psi^{-1}) \subseteq \mathrm{st}(U)$. Then there exist $\phi_1, \phi_2 \in I$ with $\psi \circ \phi_1 = \phi = \phi_2 \circ \psi^{-1}$. Since $\psi \in I$, we must have $\phi \notin I$ and $\phi \notin \psi I \psi^{-1}$. Hence, $\phi \notin \mathrm{tst}(U)$, so that $\mathrm{st}(U) \nsubseteq \mathrm{tst}(U)$.

The subset U is therefore neither EI nor II.

4.3.5: Let us now discuss how the group actions introduced in section 4.1 interact with the concept of EI. Recall that $\mathrm{Aut}(M) \cong \mathrm{Aut}\, GL(M) \cong \mathrm{Diff}(M) \ltimes \mathrm{Aut}(TM)$ (see proposition (4.1)9 and corollary (4.1)2). It will be convenient to work with the isomorph Diff(M) $\ltimes$ Aut(TM) in this section. The group Aut(TM) acts on Met(M) via the generalized conformal action C (see definition (4.1)3) and any subgroup K of Aut(TM) acts on Met(M) by restricting C to K. We also have the push forward action A of Diff(M) on Aut(TM) by automorphisms (4.1.5), and A leaves invariant certain natural subgroups of Aut(TM).

Recall that a K-conformal structure on M, K $\leq$ Aut(TM), is an orbit of K acting on Met(M).

Proposition (4.3)1: Suppose K is an A-invariant subgroup of Aut(TM). Then a K-conformal structure is EI.

Proof: K is A-invariant, so we have the action (also denoted by A) A $\in$ Hom(Diff(M), Aut(K)). Let us denote the action C restricted to K also by the letter C. Then, by a trivial modification of proposition (4.1)5, the actions A, (lower star), C are compatible. We may now use corollary (4.2)5 to deduce that any C-orbit (i.e. any K-conformal structure) is EI (recall that EI means everywhere (lower star)-invariant in this section) □

Corollary (4.3)2: Conformal structures (section 6.2), generalized conformal structures (definition (4.1)4), SAut(TM) -, OAut(TM) - and Aut^+(TM)-conformal structures are each EI.

Proof: Put K = C^+(M), Aut(TM), SAut(TM), OAut(TM) and Aut^+(TM). Each of these subgroups is invariant under the A action of Diff(M) (see proposition (4.1)6 for the last three. The invariance of the first two is trivial) □

We may also consider orbits of subgroups of Diff(M):

Proposition (4.3)3: Let K $\lhd$ Diff(M). Then any K-orbit is EI.

Proof: K is a normal subgroup of $\mathrm{Diff}(M)$, so that the conjugation action of $\mathrm{Diff}(M)$ on itself restricts to an action $\mathrm{conj} \in \mathrm{Hom}(\mathrm{Diff}(M), \mathrm{Aut}(K))$. Trivially, the actions conj, (lower star), (lower Star)$|K$ are compatible, so, by corollary (4.2)5, we deduce that any K-orbit is EI □

In connection with the above proposition, we should make a remark concerning the normal subgroup structure of $\mathrm{Diff}(M)$. In fact, if we restrict our attention to $\mathrm{Diff}_o(M)$, the connected component containing id_M, then there don't exist any (non-trivial, proper) normal subgroups at all; Mather [M 6] has demonstrated that $\mathrm{Diff}_o(M)$ is simple. This is proved by first showing that the commutator subgroup of $\mathrm{Diff}_o(M)$ is simple, and then showing that $\mathrm{Diff}_o(M)$ is perfect (i.e. equal to its own commutator subgroup). Typically, therefore, we do not expect proposition (4.3)3 to generate many EI subspaces of $\mathrm{Met}(M)$.

4.3.6: Let us now consider conformal structures in more detail. Let $g \in \mathrm{Met}(M)$ and $C_g = \{fg: f \in C^+(M)\}$, the conformal structure on M containing g. As usual (see section 6.2), $\mathrm{Conf}(M,g) \equiv \mathrm{Conf}(M,C_g)$ denotes the conformal group (group of conformeomorphisms of (M,C_g)).

Let $U = C_g$. Then $st(U) = \mathrm{Conf}(M,g)$, and $tst(U) =$

$$= \bigcup_{f \in C^+(M)} st(fg).$$

We have already demonstrated (see corollary (4.3)2) that U is EI. We now consider the II of U. For this, we use some results from the theory of essential conformeomorphisms. We use the definition of Obata [O 1].

Definition (4.3)4: The subgroup G of $\mathrm{Conf}(M,g)$ is said to be

<u>inessential</u> if there exists $f \in C^+(M)$ such that $G \leq st(fg)$. Otherwise G is said to be <u>essential</u>.

It can be shown that any compact subgroup of $Conf(M,g)$ is inessential (see Ishihara [I 10]). In particular, if M is compact, then $st(g)$ is compact, so that a (closed) subgroup G of $Conf(M,g)$ is inessential if and only if G is compact.

Another sufficient condition for inessentiality involves the Weyl tensor $Weyl(g)$. Consider the pointwise norm $\|Weyl(g)\|_g$. Under a conformal rescaling, $g \to fg$, this norm changes by a factor f^{-1}, so that $\|Weyl(g)\|_g g$ is invariant under conformal rescalings. Suppose $Weyl(g) \neq 0$. Then $\|Weyl(g)\|_g g$ is a metric in the conformal class of g which is invariant under $Conf(M,g)$. Hence, the non-vanishing of the Weyl tensor of g implies the inessentiality of $Conf(M,g)$. Note that here we assume $\dim M \geq 4$. For $\dim M = 3$, the Weyl-Schouten tensor may be used to obtain a similar result.

Suppose $Conf(M,g) \equiv st(U)$ is inessential. Then there exists $f \in C^+(M)$ such that $st(U) = st(fg) \subseteq tst(U)$, so that U is II, and hence TI. Therefore, the above results may be utilized to deduce the following:

For $\dim M \geq 4$ $(\dim M = 3)$, the non-vanishing of $Weyl(g)$ (Weyl-Schouten(g)) for any (and hence all) $g \in U$ implies that U is TI. For M compact, if $st(U)$ is compact, then U is TI.

For completeness, we mention the following result concerning inessentiality (see Obata [O 2]); $Conf_o(M,g)$ is essential if and only if (M,g) is conformally equivalent to (S^n, can) (M compact) or $(\mathbb{R}^n, can)$ (M non-compact). Here, $Conf_o(M,g)$

denotes the identity component of Conf(M,g).

4.3.7: In general relativity, we are often interested in a family of metrics parameterized by some manifold F. Often, F is some function space, since, for example, solutions of the Einstein field equations sometimes depend on arbitrary functions. We exhibit the parameterization by an embedding $\gamma: F \hookrightarrow \mathrm{Met}(M)$. The metrics in $U \equiv \gamma(F)$ are then of a given "functional form". The diffeomorphisms in st(U) are those which preserve the functional form of the family of metrics, i.e. $\phi \in$ st(U) if and only if, for all $f \in F$, there exists $f' \in F$ such that $\phi_*\gamma(f) = \gamma(f')$. The total stabilizer of U is just the amalgamation of the isometry groups of all the metrics in U. Elements of st(U) were originally called FFI (functionally form invariant) transformations by d'Inverno and Smallwood ([I 2], [I 3]). The examples given in these two papers, for instance, the generalized Schwarzschild (given by $\gamma: C^+(M) \hookrightarrow \mathrm{Met}(M)$; $f \longmapsto f dt^2 - f^{-1} dr^2 - r^2$ can, $M = N \times S^2$ with $N \subseteq \mathbb{R}^2$) and the type {3,1} vacuum solutions with twist, illustrate the idea of a functionally parameterized family of metrics. Another example is analyzed in section 4.5 where we calculate st(U) for U = {pp-waves} using the invariance equation introduced in section 4.4.

Other interesting families of spacetimes have been discussed in the literature. Moncrief [M 13] considers the space of (generalized) Taub-NUT metrics on $\mathbb{R} \times S^3$. This space is infinite dimensional and contains as a two dimensional subspace the space of Taub-NUT solutions. In a more recent paper [M 14], Moncrief discusses vacuum metrics on the manifold $\mathbb{R} \times B_n$ (B_n the total space of an arbitrary S^1-bundle over S^2) which admit a spacelike isometry group isomorphic to S^1. Again, an infinite dimensional family of metrics is obtained. We should also mention the work of Szekeres [S 28] in which he discusses solutions of Einstein's equations involving arbitrary functions, and he derives necessary and sufficient conditions for a solution of the vacuum Einstein equations to depend on an "arbitrary" functional on $C(M)$.

Having given several examples exhibiting EI and related properties, we turn now to a consideration of infinitesimal symmetries of a family of metrics. In particular, we discuss the so called invariance equation, which is a generalization of the Killing equation.

4.4 The Invariance Equation

In this section, we shall study certain infinitesimal aspects of the action of the diffeomorphism group on the space of metrics. In order to find symmetries of a family of metrics, in particular the isometries of a single metric, we often have to resort to finding infinitesimal symmetries (i.e. vector fields which generate local symmetries), and then integrating to find the corresponding global symmetries. In the particular case of a single metric, the procedure involves solving the Killing equation to find infinitesimal isometries or Killing vector fields. In this section, we generalize the Killing equation in order to deal with an entire family of metrics. In the following section, we use this "invariance equation" to find the symmetries of a particular family of metrics, namely the pp-wave solutions of Einstein's equations.

Before considering the invariance equation, we give a brief discussion of the infinitesimal structure of the diffeomorphism group. For a thorough discussion, we refer the reader to Adams et al. [A 3], Hamilton [H 2], Milnor [M 8] and Ratiu and Schmid [R 2]. As is explained in these references, for technical reasons, we should assume that M is compact. In the non-compact case, there exist incomplete vector fields, so that the space of vector fields is too large to be the Lie algebra of the diffeomorphism group - incomplete vector fields cannot be globally integrated. On the other hand, the set of complete vector fields isn't even closed under addition, and so it certainly can't be regarded as the Lie algebra of Diff(M). One solution is to utilize $\mathrm{Diff}_c(M)$, the group of diffeomorphisms of compact support, with corresponding Lie algebra $\mathrm{Vect}_c(M)$, although this appears to be rather a restriction.

Let M be a smooth, connected, (compact), n-manifold without boundary. Diff(M) is an open subspace of C(M,M) and thus possesses tangent space:

$$T_\phi \mathrm{Diff}(M) \;=\; \mathrm{Vect}_\phi(M) \equiv \{X \in C(M,TM)\colon \pi\circ X = \phi\} \qquad 4.4.1,$$

for all $\phi \in \mathrm{Diff}(M)$. Here, $\pi\colon TM \to M$ is the tangent bundle of M. In particular, the tangent space at id_M is just Vect(M), and this is the vector space underlying the Lie algebra, L Diff(M). To calculate the Lie algebra product, we need to consider the space of left invariant vector fields, $\mathrm{Vect}_L(\mathrm{Diff}(M))$, on Diff(M):

First let us consider the composition map given by

$$\mathrm{comp}\colon \mathrm{Diff}(M) \times \mathrm{Diff}(M) \to \mathrm{Diff}(M);\; (\phi,\psi) \mapsto \phi\circ\psi \qquad 4.4.2,$$

for all $\phi,\psi \in \mathrm{Diff}(M)$. This map is differentiable (see Irwin [I 5]) with derivative given by:

$$(D\mathrm{comp}(\phi,\psi).(V,W))(x) = D\phi(\psi(x)).W(x) + V(\psi(x)) \qquad 4.4.3,$$

for all $x \in M$, $(V,W) \in T_{(\phi,\psi)}(\mathrm{Diff}(M) \times \mathrm{Diff}(M)) = \mathrm{Vect}_{\phi}(M) \oplus \mathrm{Vect}_{\psi}(M)$ and $(\phi,\psi) \in \mathrm{Diff}(M) \times \mathrm{Diff}(M)$. The partial maps of comp are just left and right multiplication in the Lie group $\mathrm{Diff}(M)$: $L_{\phi} = \mathrm{comp}(\phi,\cdot)$ and $R_{\psi} = \mathrm{comp}(\cdot,\psi)$, for all $\phi,\psi \in \mathrm{Diff}(M)$. Extracting the partial derivatives from equation 4.4.3, we obtain

$$(DL_{\phi}(\psi).W)(x) = D\phi(\psi(x)).W(x) \qquad 4.4.4,$$

for all $x \in M$, $W \in T_{\psi}\,\mathrm{Diff}(M) \equiv \mathrm{Vect}_{\psi}(M)$ and $\psi,\phi \in \mathrm{Diff}(M)$, and

$$DR_{\psi}(\phi).V = V \circ \psi \qquad 4.4.5,$$

for all $V \in T_{\phi}\mathrm{Diff}(M) \equiv \mathrm{Vect}_{\phi}(M)$ and $\phi,\psi \in \mathrm{Diff}(M)$.

The usual isomorphism of the tangent space at the identity onto the space of left invariant vector fields is given by $\lambda\colon \mathrm{Vect}(M) \longrightarrow \mathrm{Vect}_L(\mathrm{Diff}(M))$; $X \mapsto X^L \equiv \lambda(X)\colon \phi \longmapsto DL_{\phi}(\mathrm{id}_M).X$. Using equation 4.4.4, we see that $(X^L(\phi))(x) = D\phi(x).X(x)$ $= (\phi_* X)(\phi(x))$. Hence, we obtain:

$$X^L(\phi) = (\phi_* X) \circ \phi \qquad 4.4.6,$$

for all $\phi \in \mathrm{Diff}(M)$, $X \in \mathrm{Vect}(M)$.

The Lie algebra of $\mathrm{Diff}(M)$ is the vector space $T_{\mathrm{id}_M}\mathrm{Diff}(M)$ $\equiv \mathrm{Vect}(M)$ equipped with the Lie algebra product obtained by pulling back the Lie bracket on $\mathrm{Vect}_L(\mathrm{Diff}(M)) \leqslant \mathrm{Vect}(\mathrm{Diff}(M))$ using the (vector space) isomorphism λ. This product may be calculated as follows:

Fix $X \in \mathrm{Vect}(M)$ and let $\sigma \in \mathrm{Hom}(\mathbb{R}, \mathrm{Diff}(M))$ be the unique 1-parameter subgroup of $\mathrm{Diff}(M)$ satisfying $\frac{d\sigma(t)}{dt} = X^L(\sigma(t))$. Then, by analogy with the theory of finite dimensional Lie groups, we put $\exp(X) = \sigma(1)$, so that $\exp(tX) = \sigma(t)$.

We may write the differential equation for σ as

$$\dot{\sigma}(t) = (\sigma(t)_* X) \circ \sigma(t) \qquad 4.4.7,$$

where we have used equation 4.4.6. The ordinary differential equation 4.4.7 possesses a unique solution satisfying $\sigma(0) = id_M$ (just evaluate 4.4.7 at an arbitrary point in M and use the standard existence and uniqueness theorem of Picard). Now let $\{\phi_t\}$ be the 1-parameter group of diffeomorphisms generated by X, i.e. $\dot{\phi}_t = X \circ \phi_t$ and $\phi_o = id_M$. Since $[X,X] = 0$, we have $\phi_{t*} X = X$, so that $\dot{\phi}_t = (\phi_{t*} X) \circ \phi_t$. Hence, we must have $\sigma(t) = \phi_t$. We have demonstrated that the 1-parameter subgroup corresponding to X (qua element of $\mathrm{LDiff}(M)$) coincides with the 1-parameter subgroup of diffeomorphisms generated by X (qua vector field on M). We deduce that:

$$\exp(tX) = \phi_t \qquad 4.4.8.$$

Now consider the adjoint representation of $\mathrm{Diff}(M)$ on $\mathrm{Vect}(M)$. This is given by $\mathrm{Ad}_\phi = \mathrm{Dconj}_\phi(id_M) = D(L_\phi \circ R_{\phi^{-1}})(id_M) \in GL(\mathrm{Vect}(M))$. Hence, $(\mathrm{Ad}_\phi(X))(x) = (DL_\phi(\phi^{-1}).DR_{\phi^{-1}}(id_M).X)(x)$ $= (DL_\phi(\phi^{-1}).(X \circ \phi^{-1}))(x) = D\phi(\phi^{-1}(x)).X(\phi^{-1}(x))$ (using equations 4.4.4, 4.4.5) $= (\phi_* X)(x)$, for all $x \in M$, $X \in \mathrm{Vect}(M)$. Thus, we have:

$$\mathrm{Ad}_\phi = \phi_* \qquad 4.4.9,$$

for all $\phi \in \mathrm{Diff}(M)$.

The mapping $Ad: Diff(M) \to GL(Vect(M))$ is a homomorphism of Lie groups, and its derivative at the identity is therefore a homomorphism of Lie algebras. Regarding Ad as an action, this homomorphism is just the infinitesimal generator, and is denoted by $ad \equiv DAd(id_M) \in Hom(Vect(M), gl(Vect(M)))$. We have $ad_X(Y)$ $= \frac{d}{dt} Ad_{\exp(tX)}(Y)\Big|_{t=0} = \frac{d}{dt} \phi_{t*}Y\Big|_{t=0}$ (where $\{\phi_t\}$ is the 1-parameter group of diffeomorphisms generated by X, and we have used equation 4.4.9) $= - L_X Y$ (see equation 1.6.1) $= - [X,Y]$, for all $X,Y \in Vect(M)$. Here $[,]$ denotes the usual Lie bracket of vector fields. However, the Lie algebra product on $Vect(M)$ induced by the isomorphism $\lambda ; X \longmapsto X^L$ (4.4.6) is given by $[X,Y]_{Diff(M)} = \lambda^{-1}([X^L,Y^L]) = ad_X(Y)$, where the last equality follows from a calculation identical to the one in finite-dimensional theory. Thus, we have demonstrated that $[X,Y]_{Diff(M)} = - [X,Y]$, for all $X,Y \in LDiff(M) \equiv Vect(M)$, i.e. the Lie algebra of $Diff(M)$ is $Vect(M)$ equipped with the Lie algebra product given by the <u>negative</u> of the usual Lie bracket of vector fields. Note that we have followed convention by using <u>left</u>-invariant vector fields on the Lie group. Had we utilized right-invariant vector fields, there would have been no sign difference between the Lie bracket coming from the Lie group structure of $Diff(M)$ and the Lie bracket of vector fields in $Vect(M)$.

We remark that the structure of $Diff(M)$ is much more complex than that of other infinite dimensional Lie groups such as $C(M,G)$ (G a finite dimensional Lie group) or $Gau(P)$ (P a principal bundle over M). For example, in the former case, the exponential map is given by $\exp: L(C(M,G)) \equiv C(M,LG) \to C(M,G);\ \xi \mapsto \exp_G \circ \xi$, for all $\xi \in C(M,LG)$. Here, $\exp_G$ denotes the exponential map of the Lie group G. The map exp possesses a local inverse (constructed using the inverse of $\exp_G$) and is a local homeomorphism near the identity of $C(M,G)$. Using this natural local coordinate system, the group $C(M,G)$ can be given an analytic structure. This structure comes entirely from G and therefore exists even if M is only smooth.

In contrast, Diff(M) possesses no canonical chart about the identity and is certainly not analytic. We have the exponential map, $\exp: \mathrm{Vect}(M) \to \mathrm{Diff}(M)$, given by equation 4.4.8, but this is far from being a local homeomorphism. Indeed, there exists no neighbourhood of id_M onto which exp maps surjectively, so that there are diffeomorphisms arbitrarily close to the identity which are not on any 1-parameter subgroup. There are also diffeomorphisms which are on many 1-parameter subgroups. A demonstration that $\exp: \mathrm{Vect}(S^1) \to \mathrm{Diff}(S^1)$ is neither locally injective nor locally surjective is given on p. 28 of Pressley and Segal [P17]. A consequence of the fact that exp is not locally surjective is that it is not C^1 (by the inverse function theorem); it is only continuous.

Having discussed certain aspects of the Lie group structure of the diffeomorphism group, we return now to its action on Met(M) by (lower star). For fixed $g \in \mathrm{Met}(M)$, the stabilizer of g under the action is $st(g) \equiv \mathrm{Isom}(M,g)$, the isometry group of g, and this is a Lie subgroup of Diff(M) of dimension at most $\frac{1}{2}n(n+1)$. A diffeomorphism ϕ is an isometry of g if and only if $\hat{\phi} \equiv \hat{\phi}|O(M,g) \in \mathrm{Aut}\, O(M,g)$. Indeed, any automorphism of $O(M,g)$ which leaves invariant (the restriction of) the canonical

1-form θ (and hence the Levi-Cività connection 1-form $LC(g)$) is the lift of an isometry of g. For each $u \in O(M,g)$ we have the embedding l_u: $\mathrm{Isom}(M,g) \longrightarrow O(M,g)$; $\phi \longmapsto \hat{\phi}(u)$, so we may regard $\mathrm{Isom}(M,g)$ as a (closed) submanifold of $O(M,g)$.

At the infinitesimal level, we have the concept of a Killing vector field or infinitesimal isometry. A Killing vector field X is a vector field whose corresponding local 1-parameter group $\{\phi_t\}$ of local diffeomorphisms consists of local isometries. If M is compact, then every vector field is complete and generates a global 1-parameter group of diffeomorphisms. If M is non-compact, then there exist incomplete vector fields whose local 1-parameter group cannot be extended to a (global) 1-parameter group of diffeomorphisms. However, if (M,g) is a complete Riemannian manifold, then every Killing vector field is complete.

In any case, the Lie algebra of $\mathrm{Isom}(M,g)$ is naturally isomorphic with the Lie algebra of all complete Killing vector fields. In particular, if (M,g) is complete, then $L\mathrm{Isom}(M,g)$ can be identified with the Lie algebra of all Killing vector fields.

The differential equation characterizing Killing vector fields is $L_X g = 0$ (obtained by differentiating $\phi_{t*} g = g$, where $\{\phi_t\}$ is the 1-parameter group of diffeomorphisms generated by X). Using the relation $\frac{d}{dt} \phi_{t*} + \phi_{t*} \circ L_X \equiv 0$ (on any space of tensor fields), we see that $L_X g = 0$ implies that X is a Killing vector field.

Another characterization of Killing vector fields is obtained if we consider the lifted vector fields on the frame bundle; Let $X \in \mathrm{Vect}(M)$ with (local) 1-parameter group of (local) diffeomorphisms $\{\phi_t\}$. We define $\hat{X} \in \mathrm{Vect}\ GL(M)$ by:

$$\hat{X}(u) = \frac{d}{dt}\hat{\phi}_t(u)\Big|_{t=0} \qquad 4.4.10,$$

for all $u \in GL(M)$. Using the fact that $\hat{\phi}_t \in \ell(\mathrm{Diff}(M)) = \mathrm{stab}(\theta) \leqslant \mathrm{Aut}\,GL(M)$ (see appendix 6.1), it is straightforward to show that (i) $(R_a)_* \hat{X} = \hat{X}$, for all $a \in GL(n,\mathbb{R})$; (ii) $L_{\hat{X}}\theta = 0$; and (iii) $D\pi(u).\hat{X}(u) = X(\pi(u))$, for all $u \in GL(M)$ (Note that we are using π to denote both the tangent bundle and the frame bundle in this section). Conversely, given $\hat{X} \in \mathrm{Vect}\,GL(M)$ satisfying (i), (ii), there exists a unique $X \in \mathrm{Vect}(M)$ satisfying (iii). The vector field $\hat{X}$ is called the natural lift of X to the frame bundle. Let us also denote by $\hat{X}$ the restriction of $\hat{X}$ to any submanifold, in particular any sub-bundle, of $GL(M)$.

It can be easily seen that $X \in \mathrm{Vect}(M)$ is a Killing vector field of (M,g) if and only if $\hat{X}$ is tangent to $O(M,g)$ at every point of $O(M,g)$, i.e. the restriction of $\hat{X}$ actually defines an element of $\mathrm{Vect}\,O(M,g)$. These, and other, results concerning isometries and Killing vector fields are given in Kobayashi and Nomizu [K 7]. Similar results may be shown for a conformal structure; here, we have conformal Killing vector fields generating (local) conformeomorphisms which satisfy the conformal Killing equation, $L_X g + hg = 0$ $(h \in C(M))$, and whose lifts are tangent to the conformal frame bundle (see section 6.2 and Poor [P 16]).

We now consider families of metrics more general than $\{g\}$ and $C_g \equiv \{fg\colon f \in C^+(M)\}$. First, we introduce some notation:

Definition (4.4)1: Let U be a submanifold of $\mathrm{Met}(M)$, and, as usual, let $\mathrm{st}(U) = \{\phi \in \mathrm{Diff}(M)\colon \phi_* g \in U,$ for all $g \in U\}$. Let

$K(U) = \{X \in \mathrm{Vect}(M)\colon \{\phi_t^X\} \subseteq \mathrm{st}(U)\}$ and

$L(U) = \{X \in \mathrm{Vect}(M)\colon L_X g \in T_g U,$ for all $g \in U\}$. We refer to

$K(U)$ as the space of generalized Killing vector fields.

Proposition (4.4)2: $K(U) \subseteq L(U)$.

Proof: Let $X \in K(U)$, and fix $g \in U$. We define the curve: $I \subseteq \mathbb{R} \to U$; $t \mapsto g_t = (\phi_t^X)_* g$. Then

$L_X g \equiv -\frac{d}{dt}(\phi_t^X)_* g\Big|_{t=0} = -\frac{dg_t}{dt}\Big|_{t=0} \in T_g U$, so that $X \in L(U)$.

Hence, $K(U) \subseteq L(U)$ □

Particularly important submanifolds U of $Met(M)$ arise when we consider parameterized families of metrics. Let F be a manifold, and $\gamma \in Emb(F, Met(M))$, so that $U = \gamma(F)$ is a family of metrics parameterized by F. As in section 4.2, we have the map (lower star)$^\gamma$: $st(U) \to Diff(F)$ defined by $\phi \mapsto \gamma_\phi = \gamma^{-1} \circ \phi_* \circ \gamma$ (where γ^{-1} denotes the inverse of γ, mapping U onto F). In this case, we have $L(U) = \{X \in Vect(M)\colon L_X(\gamma(f)) \in T_{\gamma(f)}U$, for all $f \in F\} \equiv \{X \in Vect(M)$: for all $f \in F$, there exists $\mu \in T_f F$ such that $L_X(\gamma(f)) = D\gamma(f).\mu\}$, using the fact that $T_{\gamma(f)}U = D\gamma(f)(T_f F)$.

Definition (4.4)3: Let γ be as above. We say that $X \in Vect(M)$ satisfies the invariance equation for γ if, for all $f \in F$, there exists $\mu \in T_f F$ such that

$$L_X(\gamma(f)) = D\gamma(f).\mu \qquad 4.4.11.$$

We see that $L(\gamma) \equiv L(\gamma(F))$ is the solution space for the invariance equation for γ. In particular, since $K(U) \subseteq L(U)$ (proposition (4.4)2), any vector field X in $K(U)$ must satisfy the invariance equation 4.4.11. In fact, suppose $X \in K(U)$. Then $\{\phi_t^X\} \subseteq st(U)$, so we have $\gamma_t \equiv \gamma_{\phi_t^X} \in Diff(F)$. Let $f \in F$ and

$\mu = -\frac{d}{dt}\gamma_t(f)\Big|_{t=0} \in T_f F$. Then, $L_X(\gamma(f)) = -\frac{d}{dt}\phi_{t*}(\gamma(f))\Big|_{t=0} =$

$= -\frac{d}{dt}\gamma(\gamma_t(f))\Big|_{t=0} = -D\gamma(f).(\frac{d}{dt}\gamma_t(f)\Big|_{t=0}) = D\gamma(f).\mu$, showing explicitly that $K(U) \subseteq L(U)$ in this case.

Note that, in the case of F being an open subspace of a (topological) vector space, the invariance equation reduces to the FFI equation of d'Inverno and Smallwood [I 2].

In general, $K(U) \subsetneq L(U)$, i.e. there will exist vector fields on M satisfying the invariance equation which do not generate symmetries of U. The two cases mentioned above, namely, $U = \{g\}$ (so $F = \{*\}$) and $U = C_g \equiv \{fg: f \in C^+(M)\}$ (an orbit of the group $C^+(M) = F$), do satisfy the condition $K(U) = L(U)$. In fact, these two cases are particular instances of a more general class of embeddings γ for which $K(U) = L(U)$ $(U = \gamma(F))$:

Proposition (4.4)4: Let F be an abelian Lie group and $C \in \mathrm{Hom}(F,\mathrm{Diff}(\mathrm{Met}(M)))$ a free action of F on $\mathrm{Met}(M)$. Suppose $\theta \in \mathrm{Hom}(\mathrm{Diff}(M), \mathrm{Aut}(F))$ is such that θ, (lower star), C are compatible (see 4.2.1). Then, any C-orbit U is such that $K(U) = L(U)$.

Proof: Fix $g \in \mathrm{Met}(M)$ and let $U = Fg$ be the C-orbit containing g. Define $\gamma: F \hookrightarrow \mathrm{Met}(M)$ by $\gamma(f) = C_f(g)$, so that $\gamma \in \mathrm{Emb}(F,\mathrm{Met}(M))$ and $\gamma(F) = U$.

We have $K(U) \subseteq L(U)$ by proposition (4.4)2, so we must prove that $L(U) \subseteq K(U)$. Suppose $X \in L(U)$. Then X satisfies the invariance equation for γ (4.4.11), so that, for all $f \in F$, there exists $\mu \in T_fF$ such that $L_X(\gamma(f)) = D\gamma(f).\mu$. We must now integrate this equation.

Let $\{\phi_t\}$ be the 1-parameter group of diffeomorphisms generated by X and fix $f \in F$. Now we apply $D\phi_{t*}(\gamma(f))$: $T_{\gamma(f)}\mathrm{Met}(M) \to T_{\phi_{t*}(\gamma(f))}\mathrm{Met}(M)$ to the invariance equation. We obtain

$D\phi_{t*}(\gamma(f)).(L_X(\gamma(f))) = D\phi_{t*}(\gamma(f)).D\gamma(f).\mu$. Now use the fact that $\phi_{t*} : Met(M) \to Met(M)$ is the restriction of a linear map on $S_2(M)$, so that $D\phi_{t*}(g') = \phi_{t*}$, for all $g' \in Met(M)$, to obtain; $\phi_{t*}(L_X(\gamma(f))) = D(\phi_{t*}\circ\gamma)(f).\mu$, so that $\frac{d}{dt}((\phi_{t*}\circ\gamma)(f)) + D(\phi_{t*}\circ\gamma)(f).\mu = 0$. Now note that, for all $f' \in F$, $(\phi_{t*}\circ\gamma)(f') = \phi_{t*}(C_{f'}(g)) = (\phi_{t*}\circ C_{f'f^{-1}})(C_f(g)) = (\phi_{t*}\circ C^{\gamma(f)})(f'f^{-1})$ (where $C^{g'} : F \to Met(M)$; $f' \mapsto C_{f'}(g')$, for all $g' \in Met(M)$) $= (\phi_{t*}\circ C^{\gamma(f)}\circ R_{f^{-1}})(f')$, where $R_{f^{-1}}$ is right multiplication by f^{-1} (in F).

The compatibility of θ,(lower star),C means that $\phi_* \circ C_{f'} \circ \phi_*^{-1} = C_{\theta_\phi(f')}$, for all $f' \in F$ and $\phi \in Diff(M)$. Let $g' \in Met(M)$. Then $\phi_*(C_{f'}(g')) = C_{\theta_\phi(f')}(\phi_* g)$, so that $\phi_*(C^{g'}(f')) = C^{\phi_* g'}(\theta_\phi(f'))$, for all $f' \in F$. Hence, $\phi_*\circ C^{g'} = C^{\phi_* g'}\circ\theta_\phi$, for all $\phi \in Diff(M)$, $g' \in Met(M)$.

Returning now to our calculation, we see that $(\phi_{t*}\circ\gamma)(f') = (C^{\phi_{t*}(\gamma(f))}\circ\theta_{\phi_t}\circ R_{f^{-1}})(f')$, so $\phi_{t*}\circ\gamma = C^{\sigma_f(t)}\circ\theta_{\phi_t}\circ R_{f^{-1}}$, where σ_f; $t \to (\phi_{t*}\circ\gamma)(f) \in Met(M)$. Therefore the application of $D\phi_{t*}(\gamma(f))$ to the invariance equation results in $\frac{d}{dt}\sigma_f(t) + D(C^{\sigma_f(t)}\circ\theta_{\phi_t}\circ R_{f^{-1}})(f).\mu = 0$, which we write as:

$$\frac{d}{dt}\sigma_f(t) - DC^{\sigma_f(t)}(1).(\frac{d}{dt}\alpha_f(t)) = 0 \qquad 4.4.12,$$

where

$$\alpha_f;\ t \mapsto -\int_0^t D(\theta_{\phi_s}\circ R_{f^{-1}})(f).\mu\, ds \in LF \qquad 4.4.13,$$

where LF denotes the Lie algebra of F.

To summarize our proof so far; $X \in K(U)$ implies that, for all $f \in F$, there exists $\mu \in T_fF$ such that the curve σ_f in $Met(M)$ given by:

$$\sigma_f(t) = (\phi_{t*} \circ \gamma)(f) \qquad 4.4.14,$$

satisfies the ordinary differential equation 4.4.12. The usual existence and uniqueness theorem for solutions of ordinary differential equations (Picard's theorem) shows that 4.4.12 possesses a unique solution σ_f satisfying $\sigma_f(0) = \gamma(f)$.

We now obtain a solution of 4.4.12; Let the curve τ_f in Met(M) be defined by $\tau_f = \gamma \circ R_f \circ \exp \circ \alpha_f$, where $\exp\colon LF \longrightarrow F$ is the exponential map of the Lie group F. Note that $\tau_f(0) = \gamma(f)$. We now demonstrate that τ_f satisfies equation 4.4.12:

We have $DC^{\tau_f(t)}(1).(\frac{d}{dt}\alpha_f(t)) = \frac{d}{dt} C^{\tau_f(t)}(\exp(s\dot{\alpha}_f(t)))\Big|_{s=0}$

$= \frac{d}{ds} C_{\exp(s\dot{\alpha}_f(t))}(\tau_f(t))\Big|_{s=0} = \frac{d}{dt}(C_{\exp(s\dot{\alpha}_f(t))} \circ \gamma \circ R_f \circ \exp \circ \alpha_f)(t)\Big|_{s=0}$

$= \frac{d}{ds}(\gamma \circ L_{\exp(s\dot{\alpha}_f(t))} \circ R_f \circ \exp \circ \alpha_f)(t)\Big|_{s=0}$ (using the fact that $C_{f'} \circ \gamma = \gamma \circ L_{f'}$, for all $f' \in F$, where L is left multiplication in F) $= \frac{d}{ds}(\gamma \circ R_f \circ L_{\exp(s\dot{\alpha}_f(t))} \circ \exp \circ \alpha_f)(t)\Big|_{s=0}$

$= D(\gamma \circ R_f)(\exp(\alpha_f(t))) \cdot \frac{d}{ds} L_{\exp(s\dot{\alpha}_f(t))}(\exp(\alpha_f(t)))\Big|_{s=0}$

$= D(\gamma \circ R_f)(\exp(\alpha_f(t))) \cdot \frac{d}{ds} R_{\exp(\alpha_f(t))}(\exp(s\dot{\alpha}_f(t)))\Big|_{s=0}$

$= D(\gamma \circ R_f)(\exp(\alpha_f(t))).DR_{\exp(\alpha_f(t))}(1).\ \dot{\alpha}_f(t).$

Now, recall that for any Lie group F, the derivative of the exponential map, $\exp\colon LF \longrightarrow F$, is given by $D\exp(\xi)$ $= DL_{\exp(\xi)}(1) \circ e(\mathrm{ad}(\xi))\colon LF \longrightarrow T_{\exp(\xi)}F$, for all $\xi \in LF$. Here $e(x) \equiv x^{-1}(1 - e^{-x})$ denotes the power series $\sum_{r=0}^{\infty} \frac{(-x)^r}{(r+1)!}$ (see Helgason [H 8], p. 105). In our case, F is abelian, so that $L = R$ and $\mathrm{ad} = 0$. Therefore, $D\exp(\xi) = DR_{\exp(\xi)}(1)$, for all

$\xi \in LF$. In particular, $DR_{\exp(\alpha_f(t))}(1) = D\exp(\alpha_f(t))$, so that

$$DC^{\tau_f(t)}(1).\,\dot{\alpha}_f(t) = D(\gamma\circ R_f)(\exp(\alpha_f(t))).D\exp(\alpha_f(t)).\,\dot{\alpha}_f(t)$$

$$= \frac{d}{dt}(\gamma\circ R_f \circ \exp \circ \alpha_f)(t) = \dot{\tau}_f(t).$$

We have shown that τ_f satisfies the equation $\dot{\tau}_f(t) - DC^{\tau_f(t)}(1).\,\dot{\alpha}_f(t) = 0$. But this is precisely the equation 4.4.12. Therefore, since $\tau_f(0) = \gamma(f) = \sigma_f(0)$, we must have $\sigma_f = \tau_f$, i.e. $\phi_{t_*}(\gamma(f)) = \gamma(f\exp(\alpha_f(t)))$, for all $t \in I \subseteq \mathbb{R}$, $f \in F$. Hence, $\{\phi_t\} \subseteq st(U)$ $(U \equiv \gamma(F))$, so that $X \in K(U)$.

This concludes the proof that $L(U) \subseteq K(U)$, and so we have that $K(U) = L(U)$ □

The infinitesimal aspects of EI and II (section 4.2) may be stated as follows:

Proposition (4.4)5: Let U be a submanifold of $Met(M)$. If U is EI, then $K(U)$ contains all Killing vector fields of all metrics in U. If U is II, then for each vector field in $K(U)$, there exists a metric in U for which the given vector field is Killing.

Proof: Suppose that U is EI, and let X be a Killing vector field for some metric $g \in U$. Then $\{\phi^X_t\} \leqslant st(g) \subseteq tst(U) \subseteq st(U)$, so that $X \in K(U)$.

Now suppose that U is II, and let $X \in K(U)$. Then, $\{\phi^X_t\} \subseteq st(U) \subseteq tst(U)$, so that there exists $g \in U$ with $\{\phi^X_t\} \leqslant st(g)$. i.e. there exists a metric in U for which X is a Killing vector field □

The EI part of the above proposition is especially useful, for, since $K(U) \subseteq L(U)$, it means that by completely solving the invariance equation (4.4.11) for an EI submanifold $U = \gamma(F)$, we can find all Killing vector fields of all individual elements of U. These vector fields may then be exponentiated to give the identity components of the isometry group of all metrics in U. Since the invariance equation is often easier to solve than an individual Killing equation (due to the extra degrees of freedom available), the idea of embedding a particular metric in an EI submanifold of $\mathrm{Met}(M)$ provides a practical method for finding symmetries of particular metrics on M. An example illustrating the use of the invariance equation for finding the isometries of a particular family of metrics is given in the following section.

To conclude this section we consider the frame aspects of the invariance equation since, in applications, it is often more convenient to consider frame components of a metric rather than components with respect to local coordinates.

Note that, for U a submanifold of $\mathrm{Met}(M)$, we may define a submanifold $O(M,U)$ of $O(M)$ by:

$$O(M,U) = \{(g,u) \in O(M) : g \in U\} \qquad 4.4.15,$$

so that $O(M,U)$ may be regarded as the restriction of the canonical $O(n)$-bundle to the submanifold $U \times M$ of $\mathrm{Met}(M) \times M$. Regarded in this way, $O(M,U) \equiv O(M)|(U \times M)$ is a principal $O(n)$-bundle over $U \times M$.

The semigroup $st(U) \subseteq \mathrm{Diff}(M) \leqslant \mathrm{Aut}(M)$ acts on $O(M,U)$ by restricting the action given by proposition (4.1)11 (this action is just the left version of the restriction of S (4.1.25) regarded as a semigroup action, and is given by $(\phi,(g,u)) \longmapsto (\phi_* g, \hat{\phi}(u))$, for all $(\phi, (g,u)) \in st(U) \times O(M,U))$.

If $U = \{g\}$, then $O(M,U) = \{g\} \times O(M,g) \cong O(M,g)$ and is acted upon by $st(U) \equiv Isom(M,g)$. If $U = C_g = \{fg: f \in C^+(M)\}$, then $O(M,U) \cong \coprod_{f \in C^+(M)} O(M,fg)$, and is acted upon by $st(U)$ $\equiv Conf(M,g)$ (Cf. 4.1.18 with $F = 1\!\!1_{TM}$). By collapsing the disjoint union $O(M,C_g)$ to the set union, we obtain the conformal frame bundle of (M,C_g) (see definition (6.2)2), which has the structure of a $CO(n)$-bundle over M.

By generalizing C_g to an arbitrary submanifold U of $Met(M)$ we may obtain an analogue of the conformal frame bundle: Let $Q(U) = \bigcup_{g \in U} O(M,g) \subseteq GL(M)$ and define a surjection by

$$pr_2 : O(M,U) \longrightarrow Q(U); \quad (g,u) \mapsto u \qquad 4.4.16,$$

for all $(g,u) \in O(M,U)$. Note that $pr_2^{-1}(u) \cong \{g \in U: u \in O(M,g)\}$ and that the action of the semigroup $st(U)$ projects to an action on $Q(U)$.

Returning now to infinitesimal symmetries, we have the following result:

<u>Proposition (4.4)6</u>: Let U be any submanifold of $Met(M)$. Then the lift of $K(U)$ is a space of vector fields contained in $Vect\ O(U)$. Suppose U is EI, then $\hat{X} \in Vect\ O(M,g)$ for some $g \in U$ implies that $X \in K(U)$. Now suppose U is II, then $X \in K(U)$ implies that there exists $g \in U$ such that $X \in Vect\ O(M,g)$.

<u>Proof</u>: Let $X \in K(U)$. Then $\{\phi_t^X\} \subseteq st(U)$. Fix $u \in Q(U)$ and let $g \in U$ such that $u \in O(M,g)$. Then, putting $u_t = \hat{\phi}_t^X(u), g_t = \phi_{t*}^X g$, we have $u_t \in O(M,g_t)$ for all $t \in I$. But $\phi_t^X \in st(U)$, so that $g_t \in U$. Hence, $t \mapsto u_t$ defines a curve in $Q(U)$. Now,

$$\hat{X}(u) = \frac{d}{dt}\hat{\phi}_t(u)\Big|_{t=0} \text{ (see equation 4.4.10) } = \frac{du_t}{dt}\Big|_{t=0} \in T_uQ(U).$$

Hence $\hat{X} \in VectQ(U)$.

Now suppose U is EI, so that $tst(U) \subseteq st(U)$. Let $g \in U$ and $\hat{X} \in \mathrm{Vect}\, O(M,g)$. This implies that X is a Killing vector field for g and hence $X \in K(U)$ (by proposition (4.4)5).

Finally, suppose U is II, so $st(U) \subseteq tst(U)$. Let $X \in K(U)$. Then, by proposition (4.4)5, there exists $g \in U$ such that $\{\phi_t^X\} \subseteq st(g)$. Hence $X \in \mathrm{Vect}\, O(M,g)$ □

We now obtain a frame version of the invariance equation for $\gamma\colon F \hookrightarrow \mathrm{Met}(M)$. For $U = \gamma(F)$, let $q : Q(U) \to M$ be the restriction of $\pi\colon GL(M) \to M$. Note that q, in general, is not a bundle, but, nevertheless, we shall refer to maps $s\colon W \to Q(U)$ (W open in M) such that $q \circ s = id_M$, as local sections of q, and denote the space of such maps by $\Gamma(Q(U)|W)$.

<u>Definition (4.4)7</u>: Let $\gamma \in \mathrm{Emb}(F,\mathrm{Met}(M))$ and $U = \gamma(F)$. We say that γ is <u>frame friendly</u> if, for each W open in M, γ induces a differentiable map $\bar{\gamma} : F \to \Gamma(Q(U)|W)$ such that $(\gamma(f),\ \bar{\gamma}(f)(x)) \in O(M,U)$, for all $x \in W$, $f \in F$. We call $\bar{\gamma}$ a <u>frame map associated with γ</u>.

We remark that there may exist more than one such $\bar{\gamma}$ for a frame friendly γ. Embeddings γ which arise as orbit maps of natural groups are frame friendly, and admit a unique natural frame map. For example, for γ an orbit embedding of $\mathrm{Aut}(M)$, $\bar{\gamma}$ is essentially given by equation 4.1.18.

Note that, if γ is frame friendly, then we may transfer all F-dependence to the local n-bein and write:

$$\gamma(f)|W \;=\; \eta_{ab}\theta^a(f) \otimes \theta^b(f) \qquad 4.4.17,$$

where $\eta_{ab} = \mathrm{diag}(1,\ldots,1,\ -1,\ldots,-1) \in S(p,q;\ \mathbb{R})$ and $\{\theta^a(f)\}$ is the n-bein dual to $\{e_a(f)\} \equiv \bar{\gamma}(f) \in \Gamma(O(M,\gamma(f))|W)$, for all

$f \in F$.

Now, suppose γ is frame friendly, and fix a frame map $\bar{\gamma}: F \rightarrow \Gamma(Q(U)|W)$ for some open W in M. Let $X \in L(U)$, so that X satisfies the invariance equation (4.4.11). Putting $\gamma(f) \equiv \gamma(f)|W$, the invariance equation implies that $L_X(\eta_{ab}\theta^a(f) \otimes \theta^b(f)) = D(f' \mapsto \eta_{ab}\theta^a(f') \otimes \theta^b(f'))(f).\mu$, so that $\eta_{ab}(L_X\theta^a(f)) \otimes \theta^b(f) + \eta_{ab}\theta^a(f) \otimes (L_X\theta^b(f)) = \eta_{ab}(D\theta^a(f).\mu) \otimes \theta^b(f) + \eta_{ab}\theta^a(f) \otimes (D\theta^b(f).\mu)$. Now define $\Omega^a = L_X\theta^a(f) - D\theta^a(f).\mu$, and then the invariance equation implies that:

$$\eta_{ab}\Omega^a \otimes \theta^b + \eta_{ab}\theta^a \otimes \Omega^b = 0 \qquad 4.4.18,$$

where we have suppressed the f-dependence. If we now define $\Omega^a{}_b = \langle\Omega^a, e_b\rangle$, equation 4.4.18 becomes

$$\eta_{bc}\Omega^c{}_a + \Omega^c{}_b\,\eta_{ca} = 0 \qquad 4.4.19,$$

so that $(\Omega^a{}_b)$ takes its values in $LO(p,q)$.

Conversely, if, for all $f \in F$, there exists $\mu \in T_fF$ and $X \in Vect(M)$, such that $\Omega^a{}_b \equiv \langle L_X\theta^a(f) - D\theta^a(f).\mu,\ e_b\rangle$ satisfies equation 4.4.19, then X satisfies the invariance equation for γ and therefore is an element of $L(U)$.

We refer to equation 4.4.19 as the <u>frame equation</u> for γ (and $W \subseteq M$), and this equation is equivalent to the invariance equation for γ. Note that the frame equation is satisfied if $L_X\theta^a(f) = D\theta^a(f).\mu$ for some $\mu \in T_fF$, $f \in F$.

Having considered various aspects of the invariance equation in this section, we now use this equation to derive the isometries of a family of solutions of the Einstein equations.

4.5 The Invariance Equation and Isometries

In this section, we consider a specific example of a parameterized family of metrics arising in general relativity, namely the pp-wave solutions of Einstein's equations. After a brief discussion of this family, we utilize the invariance equation of section 4.4 to derive its symmetries. The calculation we perform is a good illustration of how the invariance equation provides a simple method of finding the Killing vector fields of a family of metrics, and also of how such vector fields may be exponentiated to give isometries.

The class of metrics that we wish to investigate is the plane-fronted gravitational waves with parallel rays, or pp-waves for short. The example has been chosen both for its important physical significance and for its geometrical simplicity. The class also fits into a more general framework of pure radiation fields in general relativity. For more details concerning the relationships between pp-waves and other exact solutions, and for generalizations (for example the charged case and the Siklos-Lobachevsky plane waves), see Ehlers and Kundt [E17] and Kramer et al. [K13].

Definition (4.5)1: A pp-wave is a vacuum solution of the Einstein equations admitting a covariantly constant null vector field. i.e. A pp-wave is a spacetime (M,g) such that $\mathrm{Ric}(g) = 0$ and there exists $k \in \mathrm{Vect}(M)$ such that $\|k\|_g^2 = 0$ and $\nabla_g k = 0$.

The null congruence generated by the vector field k is normal, non-shearing and non-expanding; the reason for the term plane-fronted. Furthermore, this congruence is also rotation free (since $\nabla_g k = 0$) and this is the reason for the term parallel rays. We refer to Ehlers and Kundt [E17] for several other characterizations of pp-waves.

We remark that the work of Brinkman [B 28] on conformal transformations of Einstein spaces also leads to a characterization of pp-waves. This may be stated as follows: Suppose (M,g), (M,fg) $(f \in C^+(M))$ are nonflat, vacuum spacetimes. Then, either $f \in \mathbb{R}^+ \leq C^+(M)$, or g, fg are both pp-waves. In other words, nonflat, non-homothetically related vacuum spacetimes in the same conformal class are necessarily each pp-waves. A similar result is given by Eardley et al. [E 2]. A generalization of the Brinkman theorem to the non-vacuum case is given by Hall and Rendall [H 17].

It can be shown (see Ehlers and Kundt [E 17]) that any pp-wave metric must be (locally) of the form

$$g \;=\; 2(dzd\bar{z} - dudv - Hdu^2) \qquad 4.5.1,$$

where H is a v-independent function such that, for each u, $H(z,u)$ is the real part of an arbitrary holomorphic function of z. Let F denote the manifold of such functions H, and define $\gamma\colon F \longrightarrow \mathrm{Met}(M)$ such that the image of H under γ is the metric 4.5.1. Let $U = \gamma(F)$ denote the space of pp-wave solutions - U is a space harameterized by the arbitrary functions H.

We now solve the invariance equation for γ (see equation 4.4.11). An equivalent approach would be to solve the corresponding frame equation (4.4.19) using the frame map $H \mapsto \bar{\gamma}(H)$ $= (\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial u} - H\frac{\partial}{\partial v}, \frac{\partial}{\partial v})$ (note that $k = \frac{\partial}{\partial v}$), but here we shall utilize the "coordinate" version.

Suppose $X \in L(U)$, so that for all $H \in F$, there exists $\mu \in T_HF$ such that $L_X(\gamma(H)) = D\gamma(H).\mu$. We label the coordinates $(x^1,x^2,x^3,x^4) \equiv (x,y,u,v)$, where $z = \frac{1}{\sqrt{2}}(x+iy)$, so that

$\gamma(H) = (dx^1)^2 + (dx^2)^2 - 2dx^3dx^4 - 2H(dx^3)^2$. Recall that, for all $g \in \mathrm{Met}(M)$, $X \in \mathrm{Vect}(M)$, $(L_X g)_{ab} \equiv X^c g_{ab,c} + g_{bc} X^c{}_{,a} + g_{ac} X^c{}_{,b}$, where comma denotes partial coordinate derivative. Letting $L \equiv L_X g$, we obtain:

$$L_{11} \equiv 2X^1{}_{,1} = 0 \qquad 4.5.2,$$

$$L_{12} \equiv X^2{}_{,1} + X^1{}_{,2} = 0 \qquad 4.5.3,$$

$$L_{13} \equiv -2HX^3{}_{,1} - X^4{}_{,1} + X^1{}_{,3} = 0 \qquad 4.5.4,$$

$$L_{14} \equiv - X^3{}_{,1} + X^1{}_{,4} = 0 \qquad 4.5.5,$$

$$L_{22} \equiv 2X^2{}_{,2} = 0 \qquad 4.5.6,$$

$$L_{23} \equiv -2HX^3{}_{,2} - X^4{}_{,2} + X^2{}_{,3} = 0 \qquad 4.5.7,$$

$$L_{24} \equiv - X^3{}_{,2} + X^2{}_{,4} = 0 \qquad 4.5.8,$$

$$L_{33} \equiv - 2XH - 4HX^3{}_{,3} - 2X^4{}_{,3} = -2\mu \qquad 4.5.9,$$

$$L_{34} \equiv - X^3{}_{,3} - 2HX^3{}_{,4} - X^4{}_{,4} = 0 \qquad 4.5.10,$$

$$L_{44} \equiv - 2X^3{}_{,4} = 0 \qquad 4.5.11.$$

We require that 4.5.2 - 4.5.11 hold for all $H \in F$, so in order to solve these equations, we use the decoupling method; $AH + B = 0$, for all $H \in F$, implies $A = B = 0$. Now, letting $X \equiv (X^1,X^2,X^3,X^4) = (\alpha,\beta,\gamma,\delta)$, we find:

4.5.2 implies $\alpha \equiv \alpha(y,u,v)$; 4.5.6 implies $\beta \equiv \beta(x,u,v)$; 4.5.11 implies $\gamma \equiv \gamma(x,y,u)$; 4.5.4 implies $\gamma \equiv \gamma(y,u)$ and $\frac{\partial\delta}{\partial x} = \frac{\partial\alpha}{\partial u}$; 4.5.7 implies $\gamma \equiv \gamma(u)$ and $\frac{\partial\delta}{\partial y} = \frac{\partial\beta}{\partial u}$; 4.5.10 implies $\frac{\partial\delta}{\partial v} + \frac{d\gamma}{du} = 0$; 4.5.5 implies $\alpha \equiv \alpha(y,u)$; 4.5.8 implies $\beta \equiv \beta(x,u)$; and 4.5.3 implies $\frac{\partial\beta}{\partial x} = -\frac{\partial\alpha}{\partial y}$.

The above may be summarized as follows: $\alpha \equiv \alpha(y,u)$, $\beta \equiv \beta(x,u)$, $\gamma \equiv \gamma(u)$ and $\delta \equiv \delta(x,y,u,v)$ satisfy $\frac{\partial\delta}{\partial x} = \frac{\partial\alpha}{\partial u}$ $\frac{\partial\delta}{\partial y} = \frac{\partial\beta}{\partial u}$, $\frac{\partial\delta}{\partial v} + \frac{d\gamma}{du} = 0$, $\frac{\partial\beta}{\partial x} = -\frac{\partial\alpha}{\partial y}$. These equations lead directly to the solutions $\alpha(y,u) = - by + c(u)$, $\beta(x,u) = bx + d(u)$, $\delta(x,y,u,v) = - \gamma'(u)v + c'(u)x + d'(u)y + a(u)$, where $b \in \mathbb{R}$ and a,c,d

are $\mathbb{R}$-valued. The function γ is, as yet, undetermined.

We now transform back to $(z,\bar{z},u,v)$-coordinates and define $e(u) = \frac{1}{\sqrt{2}}(c(u) + id(u))$, so e is $\mathbb{C}$-valued. The equation 4.5.9 for μ becomes $\mu = XH + 2\gamma'(u)H + \overline{e''(u)}z + e''(u)\bar{z} - \gamma''(u)v + a'(u)$, and noting that $\frac{\partial H}{\partial v} = 0$, and so $\frac{\partial \mu}{\partial v} = 0$, we have $\gamma(u) = cu + d$ for some $c,d \in \mathbb{R}$.

The solution of the invariance equation is then:

$$X = (ibz + e(u))\frac{\partial}{\partial z} + (-ib\bar{z} + \overline{e(u)})\frac{\partial}{\partial \bar{z}}$$

$$+ (cu + d)\frac{\partial}{\partial u} + (\overline{e'(u)}z + e'(u)\bar{z} - cv + a(u))\frac{\partial}{\partial v} \quad 4.5.12,$$

and

$$\mu = XH + 2cH + \overline{e''(u)}z + e''(u)\bar{z} + a'(u) \quad 4.5.13.$$

We now exponentiate the vector field X given by equation 4.5.12. Fix $x_o \in M$ with local coordinates $(z_o,\bar{z}_o,u_o,v_o)$, and let $t \mapsto \phi_t(x_o)$ be the integral curve of X with $\phi_o(x_o) = x_o$. We need to solve $\frac{d}{dt}\phi_t(x_o) = X(\phi_t(x_o))$ with $\phi_o(x_o) = x_o$. Let $x(t) \equiv \phi_t(x_o)$ with local coordinates $(z(t), \bar{z}(t), u(t), v(t))$, and then we have:

$$\dot{z} = ibz + e(u) \quad 4.5.14,$$

$$\dot{\bar{z}} = -ib\bar{z} + \overline{e(u)} \quad 4.5.15,$$

$$\dot{u} = cu + d \quad 4.5.16,$$

$$\dot{v} = -cv + \overline{e'(u)}\,z + e'(u)\bar{z} + a(u) \quad 4.5.17,$$

where $z(0) = z_o$, $\bar{z}(0) = \bar{z}_o$, $u(0) = u_o$ and $v(0) = v_o$.

Equation 4.5.16 implies $u(t) = (u_o + \frac{d}{c})e^{ct} - \frac{d}{c}$, so that, using 4.5.14, we have $\frac{d}{dt}(e^{-ibt}z) = e^{-ibt}e((u_o + \frac{d}{c})e^{ct} - \frac{d}{c})$, and so

$$z(t) = e^{ibt}(z_o + \int_o^t e^{-ibs}\, e((u_o + \frac{d}{c})e^{cs} - \frac{d}{c})ds).$$ Equation 4.5.17

now gives $\frac{d}{dt}(e^{ct}v) = e^{ct}(\overline{e'((u_o + \frac{d}{c})e^{ct} - \frac{d}{c})}z(t) + e'((u_o + \frac{d}{c})e^{ct} - \frac{d}{c})\overline{z(t)} + a((u_o + \frac{d}{c})e^{ct} - \frac{d}{c}))$, and so

$$v(t) = e^{-ct}(v_o + \int_o^t e^{cr}\, (\overline{e'((u_o + \frac{d}{c})e^{cr} - \frac{d}{c})}z(r) + e'((u_o + \frac{d}{c})e^{cr} - \frac{d}{c})\overline{z(r)} + a((u_o + \frac{d}{c})e^{cr} - \frac{d}{c}))dr).$$

Let us now make a change of notation; let $a = e^{-ct}$;

$w = \frac{d}{c}(1-e^{-ct})$; $h(u_o) = \int_o^t e^{-ibs}\, e((u_o + \frac{d}{c})e^{cs} - \frac{d}{c})ds$; $\alpha = bt$; and

$$g(u_o) = \int_o^t (e^{(c+ib)r}\, \overline{e'((u_o + \frac{d}{c})e^{cr} - \frac{d}{c})} \int_o^r e^{-ibs} e((u_o + \frac{d}{c})e^{cs} - \frac{d}{c})ds$$
$$+ e^{(c-ib)r}\, e'((u_o + \frac{d}{c})e^{cr} - \frac{d}{c}) \int_o^r e^{ibs}\, \overline{e((u_o + \frac{d}{c})e^{cs} - \frac{d}{c})}ds$$
$$+ e^{cr}\, a((u_o + \frac{d}{c})e^{cr} - \frac{d}{c}))dr.$$

Note that we have suppressed the t-dependence of a, w, h, α and g. We now obtain:

$$z = e^{i\alpha}(z_o + h(u_o)) \qquad 4.5.18,$$

$$\overline{z} = e^{-i\alpha}(\overline{z}_o + \overline{h(u_o)}) \qquad 4.5.19,$$

$$u = a^{-1}(u_o + w) \qquad 4.5.20,$$

$$v = a(v_o + h'(u_o)\overline{z}_o + \overline{h'(u_o)}z_o + g(u_o)) \qquad 4.5.21.$$

Allowing the point $x_o = (z_o, \overline{z}_o, u_o, v_o) \in M$ to vary, equations 4.5.18 - 4.5.21 give the diffeomorphism ϕ_t^X generated by the vector field $X \in L(U)$, i.e., for $x = (z, \overline{z}, u, v) \in M$, we have

$$\phi_t^X(x) = (e^{i\alpha}(z+h(u)),\ e^{-i\alpha}(\overline{z}+\overline{h(u)}),\ a^{-1}(u+w),\ a(v+h'(u)\overline{z} + \overline{h'(u)}z + g(u)).$$

Let us now calculate the change in the metric $\gamma(H')$ (some

$H' \in F$) under the action of the diffeomorphism ϕ_t^X (fixed t) generated by $X \in L(U)$. The easiest way to do this is to re-interpret 4.5.18 - 4.5.21 as giving a passive coordinate transformation $(z, \bar{z}, u, v) \mapsto (z', \bar{z}', u', v')$ (where we have written z for z_o and z' for z, etc.). We have $dz' = e^{i\alpha}(dz + h'(u)du)$, $d\bar{z}' = e^{-i\alpha}(d\bar{z} + \overline{h'(u)}\, du)$, $du' = a^{-1}du$, and $dv' = a(dv + h''(u)\bar{z}du + h'(u)d\bar{z} + \overline{h''(u)}\, zdu + \overline{h'(u)}dz + g'(u)du)$. Hence, $\gamma(H') \equiv 2dz'd\bar{z}' = 2du'dv' - 2H'(du')^2 = 2dzd\bar{z} - 2dudv - 2(-|h'(u)|^2 + h''(u)\bar{z} + \overline{h''(u)}z + g'(u) + a^{-2}H')du^2$, since the terms involving $dzdu$ and $d\bar{z}du$ cancel. We now re-adopt an active viewpoint, and we see that $(\phi_t^X)^*\gamma(H') = \gamma(H)$, where

$$H' = a^2(H - h''(u)\bar{z} - \overline{h''(u)}z + |h'(u)|^2 - g'(u)) \qquad 4.5.22.$$

To summarize the above; Any vector field $X \in L(U)$ (i.e. any vector field X satisfying the invariance equation for $\gamma: F \to \{\text{pp-waves}\}$) generates a 1-parameter group of diffeomorphisms $\{\phi_t^X\}$ given by equations 4.5.18 - 4.5.21. Moreover, each ϕ_t^X is actually a symmetry of the space of pp-waves, i.e. an element of $st(U)$, mapping $\gamma(H)$ to $\gamma(H')$, where $H' \in F$ is given by equation 4.5.22. In other words, we have shown that $L(U) \subseteq K(U)$, and, since by proposition (4.4)2, $K(U) \subseteq L(U)$, we have proved the following:

Proposition (4.5)1: Let U denote the space of pp-waves (since we are essentially working locally, the underlying manifold may be taken to be an open submanifold of $\mathbb{R}^4$). Then $K(U) = L(U)$, i.e. a vector field X generates a local 1-parameter group of local diffeomorphisms mapping any pp-wave to another pp-wave if and only if X satisfies the invariance equation for $\gamma: F = \{H\} \to U$.

We now make several remarks concerning the above result:

(i) In solving the invariance equation for $X \in L(U)$, we have made essential use of the decoupling method. This method works only because we are considering an entire parameterized family of metrics. Solving the Killing equation for a given particular pp-wave is much more difficult than solving the invariance equation for the entire space of pp-waves; (ii) The symmetries 4.5.18 - 4.5.22 have been derived by alternative methods elsewhere in the literature. For example, see Ehlers and Kundt [E 17] and Kramer et al. [K13] (our notation is essentially that of Kramer et al.); (iii) For given $H \neq 0$, the equation $H' = H$ may be solved to give the isometry group of $\gamma(M)$. This group always contains the diffeomorphisms with $a = 1$, $w = \alpha = h = 0$ and $g =$ constant. Thus, the generic pp-wave admits a one dimensional isometry group (generated by $k = \frac{\partial}{\partial v}$); (iv) In order to investigate the symmetries of the pp-wave solutions in more detail, it is useful to restrict γ to various subspaces of F by considering functions H of a particular form. These various specializations are listed in table 21.1 on p. 235 of Kramer et al. [K13], and show that for $H \neq 0$, all isometries of $\gamma(H)$ are contained in the semigroup of symmetries given by 4.5.18 - 4.5.21, i.e. the space of nonflat pp-waves is EI. The maximal dimension of isometry group for $\gamma(M)$ ($H \neq 0$) is six, and there exists a three parameter family of pp-waves, each of whose elements admits a six dimensional isometry group. Of course, the isometry group of $\gamma(0)$ has dimension ten.

The main point of this section is to demonstrate the usefulness of considering a <u>space</u> of metrics, rather than just a particular metric, when finding isometry groups. The advantage of this approach is reflected in the fact that the corresponding invariance equation may be decoupled and therefore may be solved more easily. We also obtain the semigroup of diffeomorphisms leaving an entire family of metrics invariant. We have solved this problem for pp-waves, but obviously any other parameterized space of solutions of the Einstein equations may be treated in the same manner. This technique of using the invariance equation for finding isometries should therefore be a tool in the classification programme; if we ensure that the space of metrics U into which we embed the particular metric (or family of metrics) under consideration is EI, then proposition (4.4)5 implies that the space of solutions $L(U)$ of the invariance equation contains all Killing vector fields of all metrics in U. In particular, by exponentiating $L(U)$, we obtain the (identity component of) the isometry group of our original metric (or of the elements in a family of metrics).

Having given a practical application of the invariance equation and of EI, we end this chapter with some suggestions for further investigations into these topics, and more generally into other natural aspects of the space of metrics on a manifold M.

4.6 Further Investigations and Conclusions

In this chapter, we have made various remarks concerning the structure of the space of metrics on a manifold M. Our underlying theme has been the idea of everywhere invariance, both in the sense of natural group actions leaving certain canonical structures invariant, and also in the sense of the specific concept introduced in section 4.2. We have given examples of how everywhere invariance arises, and also how it may be applied to find the symmetries of a given family of metrics in general relativity. The topics discussed are therefore of interest not only from an abstract viewpoint, but also from a computational one.

In this final section, we introduce several more ideas relating to natural aspects of the structure of Met(M) and to everywhere invariance. These ideas are not yet fully developed, and they provide avenues for further investigation:

4.6.1: In this chapter, when applying the ideas of EI and II (definition (4.2)1) to Met(M), we refer always to the action of Diff(M) on Met(M), i.e. to everywhere (lower star)-invariance. This is because we are particularly interested in $st_{Diff(M)}(g) \equiv Isom(M,g)$, $g \in Met(M)$, and also in the manner in which the action of Diff(M) interacts with that of other groups acting on Met(M). Another possibility would be to consider everywhere A-invariance, etc., where A is another action on Met(M). For example, we have the natural action of Aut(M) (4.1.15) on Met(M), and also the restriction of this to subgroups of Aut(M) (one of which is, of course, (lower star)). Given A, we could ask whether or not parameterized families of metrics arising in geometry and general relativity are EI. Using an equation analogous to the invariance equation (4.4.11), we could then find stabilizers under the A-action as we did for the (lower star)-action in section 4.5.

4.6.2: Rather than considering a group other than Diff(M), as in 4.6.1, we could consider an alternative Diff(M)-space. For example, we could consider the everywhere A-invariance, etc., of (parameterized) submanifolds of $\Gamma(B)$, where B is a bundle over

M on which $\mathrm{Diff}(M)$ acts naturally, e.g. a tensor bundle. An embedding $\gamma: F \hookrightarrow \Gamma(B)$ gives us a parameterized space of sections of B and a corresponding invariance equation. Particular cases to consider would be families of maps from M into some other manifold, families of vector fields and families of k-forms; in particular families of non-degenerate closed 2-forms in relation to symplectic geometry, and hence families of Hamiltonian systems.

4.6.3: Returning now to the case studied in this chapter, namely the (lower star)-action of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$, a complete characterization of EI and II submanifolds of $\mathrm{Met}(M)$ would be useful for practical calculations such as that given in section 4.5. Such a characterization might involve a detailed study of how EI and II interact with the structure of $\mathrm{Geom}(M)$ (or one of its resolutions).

Further examples of invariant submanifolds may be found by considering interesting differential geometric ideas such as G-structures. A Riemannian structure and a conformal structure are both EI, and these are both examples of G-structures. Do there exist other EI spaces of metrics arising from G-structures?

4.6.4: Parameterized families of metrics arise in areas of physics other than classical general relativity theory, and the ideas of EI may also be applied here. For example, such spaces of metrics arise naturally in the Kaluza-Klein models: Let $\pi: P \to M$ be a principal G-bundle, and fix an Ad-invariant inner product k on LG. We then have the Kaluza-Klein map, $KK: \mathrm{Met}(M) \times \mathrm{Conn}(P) \to \mathrm{Met}(P)$; $(g,\omega) \mapsto \pi^* g \oplus \omega^* k$ (of course, we could also allow k to vary). We may regard the image of KK as a parameterized family of metrics on P.

For fixed $(g,\omega) \in \mathrm{Met}(M) \times \mathrm{Conn}(P)$, the geometrical properties of the Riemannian submersion $\pi: (P, KK(g,\omega)) \longrightarrow (M,g)$ have been given by Wood [W11]. From a physical viewpoint, the Euler-Lagrange equation for the functional $EH \circ KK$ ($EH \equiv$ Einstein-Hilbert; see 1.6.19) constitutes the Einstein-Yang-Mills system on M, and the geodesics of $(P, KK(g,\omega))$ project down to paths of (Yang-Mills) charged particles on M.

4.6.5: Given $\gamma: F \hookrightarrow \mathrm{Met}(M)$, the invariance equation 4.4.11 may be regarded as a generalized Killing equation. It would be useful to be able to write this equation as the Killing equation for some Riemannian manifold (E,G) related in some natural way to (M,F,γ). One procedure is to take E to be the total space of a bundle over M with typical fibre F, and G a metric on E related to γ. The simplest case is to take $E = M \times F$ with $G \in \mathrm{Met}(E)$ defined by:

$$G(x,f) = (\pi_1^{*}(\gamma(f)) \oplus \pi_2^{*}k)(x,f) \qquad 4.6.1,$$

for all $(x,f) \in E$. Here, $k \in \mathrm{Met}(F)$ is to be specified subject to the requirement that $\gamma_\phi^{*}k = k$ $(\gamma_\phi \equiv \gamma^{-1} \circ \phi_* \circ \gamma)$ for all $\phi \in st(U)$ $(U \equiv \gamma(F))$.

For $\phi \in st(U)$, let us define $\bar{\phi} \in \mathrm{Diff}(E)$ by $\bar{\phi} = (\phi, \gamma_\phi)$. Then it is easily seen that $\bar{\phi} \in \mathrm{Isom}(E,G)$. The infinitesimal version of this is as follows: Let $X \in K(U)$ generate $\{\phi_t\} \subseteq st(U)$, and define $\gamma_X \in \mathrm{Vect}(F)$ by $\gamma_X = \frac{d}{dt} \gamma_{\phi_t}\Big|_{t=0}$ and $\bar{X} \in \mathrm{Vect}(E)$ by $\bar{X}(x,f)(h) = X(x)(h(\cdot,f)) + \gamma_X(f)(h(x,\cdot))$, for all $h \in C(E)$, $(x,f) \in E$. Then $L_{\bar{X}}G = 0$. In particular, if $L(U) = K(U)$, then every solution X of the invariance equation for γ induces a Killing vector field of (E,G).

To illustrate this idea, let us consider the conformal case: Fix $g \in \mathrm{Met}(M)$, and define $\gamma: C^+(M) \longrightarrow U \equiv C_g \subseteq \mathrm{Met}(M)$ by $\gamma(f) = fg$. Note that $\gamma^{-1}: U \longrightarrow C^+(M)$ is given by $\gamma^{-1}(g') = \frac{1}{n}\,\mathrm{trace}_g g'$, for all $g' \in U$, so that:

$$\gamma_\phi(f) = \frac{1}{n}\,\mathrm{trace}_g(\phi_* g)\phi_* f \qquad 4.6.2,$$

for all $f \in C^+(M)$, $\phi \in st(U) \equiv \mathrm{Conf}(M,g)$. Let us write $t(\phi) = \frac{1}{n}\,\mathrm{trace}_g(\phi_* g)$, and note that $\phi^{-1}{}_* g = \phi^{-1}{}_*(t(\phi)^{-1})g$.

Let us define $k \in \mathrm{Met}(C^+(M))$ by:

$$k(f)(h_1,h_2) = \int_M f^{\frac{n-4}{2}} h_1 h_2 \,\mathrm{vol}(g) \qquad 4.6.3,$$

for all $h_1, h_2 \in T_f C^+(M) \cong C(M)$, $f \in C^+(M)$.

<u>Lemma (4.6)1</u>: $\gamma_\phi \in \mathrm{Isom}(C^+(M),k)$, for all $\phi \in \mathrm{Conf}(M,g)$.

<u>Proof</u>: Let $\phi \in \mathrm{Conf}(M,g)$; $f \in C^+(M)$; $h_1,h_2 \in T_f C^+(M)$. Then,

$(\gamma_\phi{}^* k)(f)(h_1,h_2) = k(t(\phi)\phi_* f)(t(\phi)\phi_* h_1,\ t(\phi)\phi_* h_2)$ (since $D\gamma_\phi(f) = t(\phi)\phi_*$) $= \int_M (t(\phi)\phi_* f)^{\frac{n-4}{2}}(t(\phi)\phi_* h_1)(t(\phi)\phi_* h_2)\mathrm{vol}(g)$

$= \int_M t(\phi)^{\frac{n}{2}}\phi_*(f^{\frac{n-4}{2}} h_1 h_2)\mathrm{vol}(g) = \int_M \phi^{-1}{}_*(t(\phi)^{\frac{n}{2}})\, f^{\frac{n-4}{2}} h_1 h_2 \,\mathrm{vol}(\phi^{-1}{}_*(t(\phi)^{-1})g)$

$= \int_M \phi^{-1}{}_*(t(\phi)^{\frac{n}{2}}) f^{\frac{n-4}{2}} h_1 h_2 \phi^{-1}{}_*(t(\phi)^{-\frac{n}{2}})\mathrm{vol}(g) = \int_M f^{\frac{n-4}{2}} h_1 h_2 \,\mathrm{vol}(g)$

$= k(f)(h_1,h_2)$. Hence, $\gamma_\phi{}^* k = k$, for all $\phi \in \mathrm{Conf}(M,g)$ □

The above remarks now lead us to consider the Riemannian manifold $(M \times C^+(M), G)$, with G defined by 4.6.1 and k given by 4.6.3. Since $L(U) = K(U)$ (the conformal Killing equation can be integrated - see proposition (4.4)4) we deduce that every conformal Killing vector field of (M,g) induces a Killing vector field of $(M \times C^+(M), G)$.

More generally, suppose F is a Lie group and we have the following data:- (i) $\theta \in \mathrm{Hom}(\mathrm{Diff}(M), \mathrm{Aut}(F))$; (ii) For each $g \in \mathrm{Met}(M)$, an inner product K_g on LF such that $D\theta_\phi(1)_* K_g = K_{\phi_* g}$, for all $\phi \in \mathrm{Diff}(M)$, and (iii) $\gamma \in \mathrm{Emb}(F, \mathrm{Met}(M))$, such that $L_{\gamma_\phi(f)^{-1}} \circ \gamma_\phi = L_{\theta_\phi(f)^{-1}} \circ \theta_\phi$, for all $f \in F$ and $\phi \in st(\gamma(F))$. We then define $k \in \mathrm{Met}(F)$ by:

$$k(f)(h_1,h_2) = K_{\gamma(f)}(DL_{f^{-1}}(f).h_1, DL_{f^{-1}}(f).h_2) \qquad 4.6.4,$$

for all $h_1,h_2 \in T_fF$, $f \in F$. Then $\gamma_\phi^* k = k$, for all $\phi \in st(U)$ $(U \equiv \gamma(F))$:- We have $(\gamma_\phi^* k)(f)(h_1,h_2) = k(\gamma_\phi(f))(D\gamma_\phi(f).h_1, D\gamma_\phi(f).h_2)$
$= K_{\gamma(\gamma_\phi(f))}(DL_{\gamma_\phi(f)^{-1}}(\gamma_\phi(f)).D\gamma_\phi(f).h_1, DL_{\gamma_\phi(f)^{-1}}(\gamma_\phi(f)).D\gamma_\phi(f).h_2)$
$= K_{\phi_*(\gamma(f))}(D(L_{\gamma_\phi(f)^{-1}} \circ \gamma_\phi)(f).h_1, D(L_{\gamma_\phi(f)^{-1}} \circ \gamma_\phi)(f).h_2)$
$= K_{\gamma(f)}(D\theta_{\phi^{-1}}(1).D(L_{\theta_\phi(f)^{-1}} \circ \theta_\phi)(f).h_1, D\theta_{\phi^{-1}}(1).D(L_{\theta_\phi(f)^{-1}} \circ \theta_\phi)(f).h_2)$
$= K_{\gamma(f)}(D(\theta_{\phi^{-1}} \circ L_{\theta_\phi(f)^{-1}} \circ \theta_\phi)(f).h_1, D(\theta_{\phi^{-1}} \circ L_{\theta_\phi(f)^{-1}} \circ \theta_\phi)(f).h_2)$.

Note that $L_{\theta_\phi(f)^{-1}} \circ \theta_\phi = \theta_\phi \circ L_{f^{-1}}$, so that $(\gamma_\phi^* k)(f)(h_1,h_2)$
$= K_{\gamma(f)}(DL_{f^{-1}}(f).h_1, DL_{f^{-1}}(f).h_2) = k(f)(h_1,h_2)$, for all $h_1, h_2 \in T_fF$, $f \in F$. Hence $\gamma_\phi^* k = k$, for all $\phi \in st(U)$.

Again, we conclude that if $L(U) = K(U)$, then any vector field satisfying the invariance equation for γ induces a Killing vector field of $(M \times F, G)$ with G defined by 4.6.1 and k by 4.6.4. We obtain the conformal case by setting $F = C^+(M)$, θ = push forward, $K_{g'}(h_1,h_2) = \int_M h_1 h_2 \, \mathrm{vol}(g')$, for all $h_1,h_2 \in LF$, $g' \in \mathrm{Met}(M)$, and $\gamma(f) = fg$ for all $f \in C^+(M)$, and g some fixed element of $\mathrm{Met}(M)$.

Given a Riemannian manifold (E,G) such that $X \in L(U)$ gives rise to a Killing vector field of (E,G), we may also ask the converse question; do Killing vector fields of (E,G) project to elements of $L(U)$ (or even of $K(U)$)?

More generally, we would like to relate the isometries of some Riemannian manifold (E,G) (into which M is embedded) to the diffeomorphisms preserving the parameterized family $U = \gamma(F)$. A knowledge of the symmetries of (E,G) would enable us to investigate $st(U)$, and vice-versa.

4.6.6: We remark that there exist various natural maps relating Met and Diff. We have already used some of these in this chapter and others have been applied in various situations in the literature. For convenience, we give a brief discussion of these maps here.

We have already utilized (lower star) $\in \mathrm{Hom}(\mathrm{Diff}(M),\mathrm{Diff}(\mathrm{Met}(M)))$ and conj $\in \mathrm{Hom}(\mathrm{Diff}(M),\mathrm{Aut}(\mathrm{Diff}(M)))$. We also have the map $\mu\colon \mathrm{Met}(M) \longrightarrow \mathrm{Met}(\mathrm{Diff}(M))$ given by:

$$\mu(g)(\phi)(X_1, X_2) = \int_M (g \circ \phi)(X_1,X_2)\ \mathrm{vol}(g) \qquad 4.6.5,$$

for all $X_1,X_2 \in T_\phi\mathrm{Diff}(M) \equiv \mathrm{Vect}_\phi(M)$, $\phi \in \mathrm{Diff}(M)$ and $g \in \mathrm{Met}(M)$, and the map $\nu\colon \mathrm{Met}(M) \longrightarrow \mathrm{Met}(\mathrm{Met}(M))$ given by:

$$\nu(g)(k)(h_1,h_2) = \tfrac{1}{2}\int_M \mathrm{trace}(g^{-1}h_1\ k^{-1}h_2 + k^{-1}h_1\ g^{-1}h_2)\mathrm{vol}(k) \qquad 4.6.6,$$

for all $h_1,h_2 \in T_k\mathrm{Met}(M)$, $k,g \in \mathrm{Met}(M)$. Note that $\nu(g)(g) = G_o(g)$, for all $g \in \mathrm{Met}(M)$ (see 4.1.2), so that the restriction of ν to the diagonal is the metric used by Ebin in the proof of the slice theorem (see section 4.1).

For fixed $g \in \mathrm{Met}(M)$, the metric $\mu(g)$ on $\mathrm{Diff}(M)$ restricts to the closed subgroup $\mathrm{Diff}_{\mathrm{vol}(g)}(M)$, and the Riemannian manifold

$(\mathrm{Diff}_{\mathrm{vol}(g)}(M),\ \mu(g))$ is the configuration space for the hydrodynamics of an incompressible fluid. The motion of a perfect incompressible fluid is a geodesic of $\mu(g)$ (see Adams et al [A 3] and Ebin and Marsden [E 6].

The maps μ,ν are each equivariant with respect to the action of the diffeomorphism group:

Proposition (4.6)2: (i) $\mu\circ\phi_* = (\mathrm{conj}_\phi)_*\circ\mu$, and (ii) $\nu\circ\phi_* = (\phi_*)_*\circ\nu$, for all $\phi\in\mathrm{Diff}(M)$.

Proof: (i) Let $\phi\in\mathrm{Diff}(M)$, $g\in\mathrm{Met}(M)$. Then

$$(\mu\circ\phi_*)(g)(\psi)(X_1,X_2) = \int_M ((\phi_*g)\circ\psi)(X_1,X_2)\mathrm{vol}(\phi_*g)$$

$= \int_M f\ \mathrm{vol}(g)$, where $f = (\phi^{-1})_*(((\phi_*g)\circ\psi)(X_1,X_2))$. We have

$$f(x) = (\phi_*g)(\psi(\phi(x)))(X_1(\phi(x)),X_2(\phi(x)))$$

$$= g((\phi^{-1}\circ\psi\circ\phi)(x))(D\phi^{-1}(\psi(\phi(x))).X_1(\phi(x)),\ D\phi^{-1}(\psi(\phi(x))).X_2(\phi(x))).$$

Now note that $(D\,\mathrm{conj}_{\phi^{-1}}(\psi).X)(x) = (D(L_{\phi^{-1}}\circ R_\phi)(\psi).X)(x)$

$= (DL_{\phi^{-1}}(\psi\circ\phi).DR_\phi(\psi).X)(x) = (DL_{\phi^{-1}}(\psi\circ\phi).(X\circ\phi))(x)$ (using 4.4.5)

$= D\phi^{-1}(\psi(\phi(x))).X(\phi(x))$ (using 4.4.4). Hence,

$$f(x) = g(\mathrm{conj}_{\phi^{-1}}(\psi)(x))((D\mathrm{conj}_{\phi^{-1}}(\psi).X_1)(x),\ D\mathrm{conj}_{\phi^{-1}}(\psi).X_2)(x)),$$

for all $x\in M$. Therefore, $(\mu\circ\phi_*)(g)(\psi)(X_1,X_2)$

$$= \int_M (g\circ\mathrm{conj}_{\phi^{-1}}(\psi))(D\mathrm{conj}_{\phi^{-1}}(\psi).X_1,\ D\mathrm{conj}_{\phi^{-1}}(\psi).X_2)\ \mathrm{vol}(g)$$

$$= \mu(g)(\mathrm{conj}_\phi^{-1}(\psi))(D\mathrm{conj}_\phi^{-1}(\psi).X_1,\ D\mathrm{conj}_\phi^{-1}(\psi).X_2)\mathrm{vol}(g)$$

$= ((\mathrm{conj}_\phi)_*\circ\mu)(g)(\psi)(X_1,X_2)$, for all $X_1,X_2\in T_\psi\mathrm{Diff}(M)$ and $\psi\in\mathrm{Diff}(M)$. Hence, $\mu\circ\phi_* = (\mathrm{conj}_\phi)_*\circ\mu$, for all $\phi\in\mathrm{Diff}(M)$.

(ii) Let $\phi\in\mathrm{Diff}(M)$, $g\in\mathrm{Met}(M)$. Then

$$(\nu\circ\phi_*)(g)(k)(h_1,h_2) = \tfrac{1}{2}\int_M \operatorname{trace}((\phi_*g)^{-1}h_1k^{-1}h_2 + k^{-1}h_1(\phi_*g)^{-1}h_2)\operatorname{vol}(k)$$

$$=\tfrac{1}{2}\int_M \phi_* \operatorname{trace}(g^{-1}(\phi^{-1}{}_*h_1)(\phi^{-1}{}_*k)^{-1}(\phi^{-1}{}_*h_2) + (\phi^{-1}{}_*k)^{-1}(\phi^{-1}{}_*h_1)g^{-1}(\phi^{-1}{}_*h_2))\operatorname{vol}(k)$$

$$=\tfrac{1}{2}\int_M \operatorname{trace}(g^{-1}(\phi^{-1}{}_*h_1)(\phi^{-1}{}_*k)^{-1}(\phi^{-1}{}_*h_2) + (\phi^{-1}{}_*k)^{-1}(\phi^{-1}{}_*h_1)g^{-1}(\phi^{-1}{}_*h_2))\operatorname{vol}(\phi^{-1}{}_*k)$$

$= \nu(g)(\phi^{-1}{}_*k)(D\phi^{-1}{}_*(k).h_1,\ D\phi^{-1}{}_*(k).h_2)$ (since $D\phi^{-1}{}_*(k) = \phi^{-1}{}_*$)

$= ((\phi_*)_*\circ\nu)(g)(k)(h_1,h_2)$, for all $h_1,h_2 \in T_k\operatorname{Met}(M)$ and $k \in \operatorname{Met}(M)$.

Hence, $\nu\circ\phi_* = (\phi_*)_*\circ\nu$, for all $\phi \in \operatorname{Diff}(M)$ □

The following result is important in the application of μ to the study of the motion of incompressible fluids:

Proposition (4.6)3: Let $g \in \operatorname{Met}(M)$ and let the restriction of $\mu(g)$ to $\operatorname{Diff}_{\operatorname{vol}(g)}(M)$ also be denoted by $\mu(g)$. Then $\mu(g)$ is a right invariant metric on $\operatorname{Diff}_{\operatorname{vol}(g)}(M)$.

Proof: Let $\phi \in \operatorname{Diff}(M)$. Then $R_\phi^*(\mu(g))(\psi)(X_1,X_2)$
$= \mu(g)(\psi\circ\phi)(DR_\phi(\psi).X_1, DR_\phi(\psi).X_2) = \mu(g)(\psi\circ\phi)(X_1\circ\phi,\ X_2\circ\phi)$
(using 4.4.5) $= \int_M (g\circ\psi\circ\phi)(X_1\circ\phi,\ X_2\circ\phi)\operatorname{vol}(g)$
$= \int_M \phi^*((g\circ\psi)(X_1,X_2))\operatorname{vol}(g) = \int_M (g\circ\psi)(X_1,X_2)\operatorname{vol}(g)$
(since $\phi^*\operatorname{vol}(g) = \operatorname{vol}(g)$) $= \mu(g)(\psi)(X_1,X_2)$, for all
$X_1,X_2 \in T_\psi\operatorname{Diff}_{\operatorname{vol}(g)}(M)$, $\psi \in \operatorname{Diff}_{\operatorname{vol}(g)}(M)$. Hence, $R_\phi^*(\mu(g)) = \mu(g)$, so $\mu(g)$ is right invariant □

Note that the maps μ,ν have only found applications under the restrictions referred to above. It would be interesting to consider further uses for the full maps, especially since μ,ν, together with conj and (lower star), essentially exhaust the possibilities for canonical maps : Met $\longrightarrow$ Met$\circ$Diff, Met$\circ$Met and Diff $\longrightarrow$ Diff$\circ$Diff, Diff$\circ$Met.

4.6.7: Our final remark concerns the interaction between Chapters Four and One. We have already discussed certain aspects of how conformal structures and spinors come together in section 1.5, and similarly for diffeomorphisms and spinors in section 1.6. In other words, we have considered the interaction of spinors and the conformorphism group, $\mathrm{Conf}(M) \equiv \mathrm{Diff}(M) \ltimes C^{+}(M)$. To include the effect of generalized conformal transformations on spin structure, we should consider the entire group $\mathrm{Aut}(M) \equiv$ $\mathrm{Diff}(M) \ltimes (C^{+}(M) \times \mathrm{OAut}(TM))$. In section 4.1, we discussed how $\mathrm{Aut}(M)$ acts on the base space $\mathrm{Met}(M)$ of the metric-spinor field configuration bundle (see 1.4.4); the next step is to lift this action to the total space, thereby introducing a notion of generalized conformal spin structure.

CHAPTER 5 CONCLUSIONS AND FUTURE DIRECTIONS

The main purpose of this chapter is to present general conclusions and possibilities for future avenues of research. More specific conclusions and suggestions for further work may be found in particular chapters.

Our main conclusion is that a geometric approach to problems in the theory of general relativity is very useful for clarifying the situation and for indicating possibilities for new investigations. Especially useful is the study of naturally arising group actions and infinite dimensional manifolds - a study of the symmetries of the space of all geometric objects of a particular type often sheds light on problems involving a single geometric object. We have applied this "everywhere invariance" approach to all the main themes of the thesis; to spinors in sections 1.4 and 1.6, to embeddings in section 2.2, and to metrics in Chapter Four. Further examples may be found in sections 6.1 and 6.2. The infinite-dimensional approach also leads to very useful applications in physics and we have indicated examples of such applications to general relativity theory in the sections mentioned above. A particularly interesting application is to the study of the space of metrics on a given manifold - this has an impact on both classical and quantum gravity theory, as we indicated in section 4.1.

We hope that this thesis has also demonstrated how the basic geometric notions of spin structures and embeddings play an important, if not essential, rôle in the theory of general relativity. The important uses for spinors were outlined in sections 1.0, 1.5, 1.7, 1.8, 1.9, 2.3, 3.3 and 3.4 whilst applications of the theory of embeddings was given in each of the sections of Chapter Two and in sections 3.1 and 3.4. The seemingly fundamental nature of thsse ideas is manifested in their essential appearance in the formulation of a notion of general relativistic moment - a basic ingredient in any physical theory.

Two new practical techniques which we have discussed are the null limit approach to obtaining useful null equations (see sections 3.2, 3.4) and also the method of finding isometries of a metric using the invariance equation (see sections 4.4, 4.5). We anticipate that these techniques will prove useful in future investigations in general relativity theory and in other areas of geometry and physics.

The geometrization of certain earlier ideas concerning the interaction between the Lorentz group and the 2-sphere has resulted in the projective null bundle framework of section 1.9. This framework ties together spinor, conformal and null ideas in a four-dimensional Lorentzian context and should prove to be a useful tool in the study of the structure of field theories on spacetime. Another application would be to the spinor null propagation used in investigations of gravitational momentum.

Let us now give a few possibilities for further work based on the ideas discussed in this thesis:-

There is obviously more scope for further study and applications of the manifold of embeddings described in section 2.2. In particular, after bringing in the spinorial ideas of section 2.3, it should be possible to construct a spinor-metric-embedding configuration space for application in, for example, the study of quasi-local momentum. In this context, it is worth mentioning a remark of

Richard Newman [N 5], namely that the Einstein tensor field, fundamental to the theory of general relativity, does not arise naturally from many purely geometric considerations:- One case is the natural appearance of the Einstein tensor field in the proofs of the positivity of mass conjectures (both null and spatial) using spinor techniques (see section 3.3), and a second case is provided by the study of variations of codimension two spacelike embeddings in a spacetime (see [N 5]). Thus, the Einstein tensor field provides a link between spinors/gravitational energy and embedded 2-surfaces. Perhaps a more direct link would shed light on the fundamental importance of spinors in the definition of quasi-local gravitational energy.

There is certainly much work to be done in the area of quasi-local momentum, as we indicated in section 3.4. A better definition would be a good start, and then it will be necessary to apply this definition; to physically interesting spacetimes, to the proof of isoperimetric inequalities, to the cosmic censorship conjecture and, most importantly (for physics), to obtain a better understanding of the relationship between the motion of the sources of the gravitational field on the one hand and the asymptotic structure of spacetime on the other. Only when this latter problem has been resolved, will it be possible to consider equations of motion in general relativity theory and to relate observational data to the structure of the spacetime fields.

Hopefully, the null techniques introduced in Chapter Three, based as they are on natural structures arising in four-dimensional Lorentzian geometry, will be useful in any future work on gravitational momentum.

The work of Chapter Four is also open ended. The theory of everywhere invariance is certainly useful, and a further development of the ideas is desirable. In particular, we should search for a more complete characterization of everywhere invariant spaces of metrics and for alternative applications to geometry and physics. Rigorous proofs of our statements involving the space of metrics also need to be given; in particular, the use of the canonical O(n)-bundle in resolving the singularities of superspace. We anticipate that these proofs will follow from the global analytical techniques developed in the papers cited in section 4.1. We refer the reader to section 4.6 for further suggestions for future research.

6.1 Bundles

Many of the ideas within the main body of this thesis are expressed within the framework of fibre bundles. For completeness, therefore, this section gives a brief exposition of the main definitions and results concerning bundles, connections in bundles and infinite dimensional groups associated with bundles. As well as establishing our notation, this section also serves to present the main properties of the frame bundle of a manifold, the frame bundle being used extensively in Chapters One and Four. For more details we refer the reader to Kobayashi and Nomizu [K7] and Poor [P16], and for information concerning analysis on infinite dimensional manifolds, to the references cited in sections 4.0, 4.1. As usual, all manifolds and maps are smooth (in the appropriate category).

The central idea is that of a principal fibre bundle:

Definition (6.1)1: Let M be a manifold and G a Lie group. A principal fibre bundle over M with group G consists of a manifold P together with a free right action of G on P such that $M = {}^{P}/G$ and P is locally trivial, i.e. $\forall x \in M$, $\exists$ a neighbourhood U of x and $\psi\colon \pi^{-1}(U) \overset{\cong}{\to} U \times G$ such that $\psi(u) = (\pi(u), \phi(u))$ where $\phi(ua) = \phi(u)a$, $\forall u \in \pi^{-1}(U)$, $a \in G$. (Here we denote by π the orbit map, $\pi\colon P \to M$, and by $(u,a) \longmapsto R_a(u) \equiv ua$, the right action of G on P).

We call P the total space, M the base space, π the projection and G the structure group of the principal bundle $G \hookrightarrow P \xrightarrow{\pi} M$.

For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P, called the <u>fibre over x</u>. If $u \in P$, then $\pi^{-1}(x) \equiv \{ua: a \in G\}$ is called the <u>fibre through u</u>. Every fibre is diffeomorphic (though not canonically) to G. Local triviality of P implies that $\pi^{-1}(W)$ is also a principal bundle over M for any submanifold W of M. We call $\pi^{-1}(W)$ the <u>restriction of P to W</u> and denote it $P|W$

The action of G on P induces a Lie algebra homomorphism: $LG \to \mathrm{Vect}(P)$; $\xi \mapsto \xi_P$, where $\xi_P(u) = DR_u(1).\xi$, for all $u \in P$, $\xi \in LG$. Here $R_u: G \to P$; $a \mapsto R_a(u) = ua$. ξ_P is called the fundamental vector field corresponding to $\xi \in LG$. Since G maps each fibre of P onto itself, $\xi_P(u)$ is tangent to the fibre through $u \in P$. Also, G acts freely, so, for $\xi \neq 0$, ξ_P never vanishes on P. We have $\dim(\pi^{-1}(x)) = \dim LG$, $\forall x \in M$, so $\xi \mapsto \xi_P(u)$ is a linear isomorphism of LG onto the tangent space at u of the fibre $\pi^{-1}(\pi(u))$ through u. It is easily shown that $(R_a)_*\xi_P = (\mathrm{Ad}(a^{-1}).\xi)_P$, $\forall \xi \in LG$, $a \in G$, where $\mathrm{Ad} \in \mathrm{Hom}(G, GL(LG))$ is the adjoint representation of G on LG.

Given a principal G-bundle $\pi: P \to M$, we may construct associated bundles over M. These arise from actions of G on other manifolds. Note that all actions (apart from the right actions defining principal bundles) are <u>left actions</u>:

<u>Definition (6.1)2</u>: Let F be a manifold and $\rho \in \mathrm{Hom}(G,\mathrm{Diff}(F))$ a left action of G on F. We define a right action on $P \times F$ by $((u,f),a) \to (ua, \rho(a^{-1}).f)$, $\forall (u,f) \in P \times F$, $a \in G$. Let us denote the quotient space by $E = P \times_\rho F$ (or, sometimes by $P \times_G F$ if the particular action ρ is understood), and define $\pi_E: E \to M$; $[(u,f)] \mapsto \pi(u)$. $\pi_E^{-1}(x)$ is called the <u>fibre of E above x</u> and E is called the <u>fibre bundle associated with P via the action ρ</u>. F is

called the <u>standard fibre</u> of E. A differentiable structure may be defined on E in a natural way so that π_E is smooth, E is locally trivial and $\pi_E^{-1}(x)$ is diffeomorphic to F for each $x \in M$. In fact, for each $u \in P$, we have the diffeomorphism

$$\kappa_u: \quad F \to \pi_E^{-1}(\pi(u)); \quad f \mapsto [(u,f)] \qquad 6.1.1,$$

which satisfies $\kappa_{ua} = \kappa_u \circ \rho(a)$, $\forall u \in P$, $a \in G$. If the manifold F carries a particular algebraic structure [e.g. F is a vector space] and if $\rho(G)$ is a group of automorphisms of this algebraic structure, then 6.1.1 may be used to endow each fibre of E with a similar structure by requiring that κ_u be an isomorphism for each $u \in P$. In the case that F is a vector space and each κ_u is a linear isomorphism of F onto $\pi_E^{-1}(\pi(u))$, we say π_E: $E \to M$ is <u>a vector bundle over M</u>.

Given any principal G-bundle P, there exist various natural associated bundles:

Firstly, take $F = G$ with G acting on itself by left translation, so that $\rho(a)b = ab$, $\forall a, b \in G$. The resulting associated bundle is bundle isomorphic (see below) to P itself.

Secondly, take $F = G$ and G acting on itself by conjugation (inner automorphisms), so that $\rho(a)b = aba^{-1}$, $\forall a, b \in G$. The associated bundle in this case is called the <u>conjugation bundle</u>, denoted $\mathrm{Conj}(P)$.

Finally, take $F = LG$ and $\rho = \mathrm{Ad} \in \mathrm{Hom}(G, GL(LG))$ the adjoint representation of G on its Lie algebra LG. The associated bundle is called the <u>Lie algebra bundle</u> and denoted $\mathrm{Ad}(P)$.

Note that $\mathrm{Conj}(P)$ and $\mathrm{Ad}(P)$ have algebraic structure defined on their fibres: Conjugation is an automorphism of G and hence $\mathrm{Conj}(P)$ is a bundle of groups, and, similarly, G acts by Lie algebra automorphisms via Ad and so $\mathrm{Ad}(P)$ is a bundle of Lie algebras.

Another useful example of associated bundle arises as follows: Let H be a subgroup of G and $G \hookrightarrow P \xrightarrow{\pi} M$ a principal G-bundle. G acts on $^G/H$ in a natural way, namely $(a,bH) \longmapsto abH$, $\forall a \in G$, $bH \in {}^G/H$. Let P_H denote the corresponding associated bundle with standard fibre $^G/H$. We also have the right action of H on P arising because $H \leq G$. Let $^P/H$ be the quotient of P by this H-action. Then P_H may be identified with $^P/H$ in a natural way.

We now consider maps between bundles. Let $\pi_i \colon P_i \to M_i$ be principal G_i-bundles $(i = 1,2)$.

<u>Definition (6.1)3:</u> A <u>homomorphism</u> Ψ of P_1 into P_2 consists of a pair (Ψ', Ψ'') where $\Psi' \colon P_1 \to P_2$ and $\Psi'' \in \mathrm{Hom}(G_1,G_2)$ such that $\Psi'(u_1a_1) = \Psi'(u_1)\Psi''(a_1)$ $\forall u_1 \in P_1$, $a_1 \in G_1$. We often denote Ψ', Ψ'' by the same letter Ψ.

Every homomorphism Ψ of P_1 into P_2 maps each fibre of P_1 into a fibre of P_2 and hence induces a map of M_1 into M_2 which we shall denote by $\overline{\Psi}$. The homomorphism Ψ of P_1 into P_2 is called an <u>embedding</u> if $\Psi' \in \mathrm{Emb}(P_1,P_2)$ and if $\Psi'' \in \mathrm{Hom}(G_1,G_2)$ is a monomorphism. Note that $\Psi' \in \mathrm{Emb}(P_1,P_2)$ implies $\overline{\Psi} \in \mathrm{Emb}(M_1,M_2)$, and by identifying P_1 with $\Psi'(P_1)$, G_1 with $\Psi''(G_1)$ and M_1 with $\overline{\Psi}(M_1)$, we say that P_1 is a <u>subbundle</u> of P_2. If, moreover, $M_1 = M_2$ and the induced map $\overline{\Psi}$ is the identity on M, then the embedding Ψ is called a <u>reduction</u> of the structure group G_2 of P_2 to G_1. The subbundle P_1 is called a <u>reduced bundle</u>.

Given $H \leqslant G$, we say that the G-bundle P is reducible to H if there exists such a reduced bundle with structure group H.

The homomorphism Ψ of P_1 into P_2 is said to be an isomorphism if Ψ' is a diffeomorphism of P_1 onto P_2 and Ψ'' is an isomorphism of G_1 onto G_2. If such an isomorphism exists, then we say the principal bundles $G_1 \hookrightarrow P_1 \to M_1$, $G_2 \hookrightarrow P_2 \to M_2$ are bundle isomorphic.

An important case for us is that of bundle maps from a given principal G-bundle P into itself:

Definition (6.1)4: Let $\pi: P \to M$ be a principal G-bundle. An automorphism Ψ of P is an isomorphism of P onto itself (with $\Psi'' = id_G$). Let $Aut(P)$ denote the set of all automorphisms of P.

Hence $Aut(P) = \{\Psi \in Diff(P): \Psi \circ R_a = R_a \circ \Psi, \forall a \in G\}$. $Aut(P)$ has the structure of an infinite dimensional Lie group, and we have the projection $b \in Hom(Aut(P), Diff(M))$; $\Psi \mapsto \overline{\Psi}$, so that $\overline{\Psi}(x) = \pi(\Psi(u))$ for any $u \in \pi^{-1}(x)$, and for all $x \in M$. Let $Gau(P) = Ker\ b = \{\Psi \in Aut(P): \pi \circ \Psi = \pi\}$ denote the normal subgroup of $Aut(P)$ consisting of all automorphisms of P which project to the identity diffeomorphism of M. $Aut(P)$ is called the automorphism group of P and $Gau(P)$ is called the gauge group of P (sometimes the vertical automorphism group or the group of gauge transformations).

We have the following exact sequence of groups:

$$1 \longrightarrow Gau(P) \longrightarrow Aut(P) \xrightarrow{b} Diff(M) \qquad 6.1.2,$$

together with the corresponding sequence of Lie algebras. [As usual, $LDiff(M)$ is just $Vect(M)$, whilst $LGau(P)$, $LAut(P)$ are respectively the spaces of vector fields on P generating local

1-parameter groups of (local) diffeomorphisms in Gau(P), Aut(P).]

If we fix a connection (see below) in P, then vector fields on M may be lifted to vector fields on P. Hence b(Aut(P)) $\supseteq \mathrm{Diff}_o(M)$ (the identity component of Diff(M)), so we have the exact sequence

$$1 \longrightarrow \mathrm{Gau}(P) \longrightarrow \mathrm{Aut}_o(P) \longrightarrow \mathrm{Diff}_o(M) \longrightarrow 1 \qquad 6.1.3,$$

where $\mathrm{Aut}_o(P) = b^{-1}(\mathrm{Diff}_o(M))$. A natural question to ask is whether or not the exact sequence 6.1.2 (or 6.1.3) splits, i.e. does there exist $\ell \in \mathrm{Hom}(\mathrm{Diff}(M), \mathrm{Aut}(P))$ such that $b \circ \ell = \mathrm{id}_M$? If such an ℓ exists we call ℓ a lift of Diff(M) to Aut(P). An important case for which a lift does exist is that of the frame bundle of a manifold. This is discussed below and in section 4.1. Lifts also exist for other canonical bundles and obviously for any principal G-bundle P isomorphic to $M \times G$ (such a bundle is called trivializable, and in this case the structure group G is reducible to the trivial group 1). The existence of lifts for general non-trivial principal bundles is not known, although a necessary condition may be given (see Lecomte [L15]).

We now introduce the notion of section of a bundle and thence to an alternative description of the gauge group of a principal bundle:

Definition (6.1)5: Let $\pi_E\colon E \longrightarrow M$ be a bundle associated to the principal G-bundle P. A section of E is a map $s\colon M \longrightarrow E$ such that $\pi_E \circ s = \mathrm{id}_M$. The space of sections of E is denoted $\Gamma(E)$.

A useful result relating sections and reduction is the following: The structure group G of the principal G-bundle P is reducible to a subgroup H if and only if the associated bundle P_H admits a section. In particular, for P itself, a section exists if and only if P is <u>trivializable</u> (i.e. P is isomorphic to $M \times G$), so that G is reducible to 1. In general, the correspondence between $\Gamma(P_H)$ and reductions of G to H is one-to-one.

<u>Definition (6.1)6</u>: Let $\rho \in \mathrm{Hom}(G, \mathrm{Diff}(F))$ and let $\pi_E\colon E \to M$ be the bundle associated to the principal G-bundle P via the action ρ. A map $S\colon P \to F$ is said to be <u>equivariant</u> if $S \circ R_a = \rho(a^{-1}) \circ S$, $\forall a \in G$. Let $C_\rho(P,F)$ (or $C_G(P,F)$ if ρ is understood) denote the space of equivariant maps of P into F.

Note that there is a one-to-one correspondence between $\Gamma(E)$ and $C_\rho(P,F)$; for $s \in \Gamma(E)$, define $S \in C(P,F)$ by $S(u) = \kappa_u^{-1}(s(\pi(u)))$, $\forall u \in P$. Then $S(ua) = \kappa_{ua}^{-1}(s(\pi(ua)))$ $= (\rho(a^{-1}) \circ \kappa_u^{-1})(s(\pi(u))) = \rho(a^{-1}).S(u)$, so that S is indeed an element of $C_\rho(P,F)$. Conversely, given $S \in C_\rho(P,F)$, define $s \in C(M,E)$ by $s(x) = \kappa_u(S(u))$ for any $u \in \pi^{-1}(x)$. Note that $\kappa_{ua}(S(ua)) = (\kappa_u \circ \rho(a))(\rho(a^{-1}).S(u))$, so that s is well defined. Also $(\pi_E \circ s)(x) = \pi_E(\kappa_u(S(u))) = \pi(u) = x$, so that $s \in \Gamma(E)$. It is easily seen that the two maps just defined are inverses, and these provide the desired bijection between $\Gamma(E)$ and $C_\rho(P,F)$. This bijection is, in fact, a diffeomorphism of smooth manifolds.

If E is a vector bundle over M, then $\Gamma(E)$ has the structure of both a vector space over $\mathbb{K}$ and of a $C(M,\mathbb{K})$-module [Here, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ for real, complex vector bundles respectively]. We may also consider maps of vector bundles:

<u>Definition (6.1)7</u>: Let $\pi_i\colon E_i \to M_i$ $(i = 1,2)$ be vector bundles (over the same field). A map $\Psi\colon E_1 \to E_2$ is said to be a <u>vector bundle homomorphism</u> of E_1 into E_2 if the restriction

of Ψ to any fibre of E_1 is a linear map into a fibre of E_2.

Obviously a vector bundle homomorphism Ψ induces a map $\overline{\Psi} \in C(M_1,M_2)$ such that $\pi_2 \circ \Psi = \overline{\Psi} \circ \pi_1$, and we say that the homomorphism Ψ is <u>along the map $\overline{\Psi}$</u>. If $M_1 = M_2 = M$ and the projected map $\overline{\Psi} = id_m$, we say that Ψ is a <u>strong bundle homomorphism</u> of E_1 into E_2. When dealing with a pair of vector bundles E_1, E_2 over the same base manifold M we will consider only strong bundle homomorphisms of E_1 into E_2 and usually omit the word strong in this case. If $\Psi: E_1 \longrightarrow E_2$ is a strong bundle homomorphism such that the restriction of Ψ to any fibre of E_1 is a linear isomorphism onto the corresponding fibre of E_2, then Ψ is said to be a <u>vector bundle isomorphism</u> and we say that E_1 and E_2 are <u>isomorphic vector bundles</u>. Note that a strong vector bundle homomorphism from E_1 to E_2 is a vector bundle isomorphism if and only if it is a diffeomorphism. Similarly we have the idea of <u>vector bundle isomorphism along the map $\overline{\Psi}$</u>.

Let $\pi: E \longrightarrow M$ be a vector bundle over M. An isomorphism of E onto itself is called an automorphism of E. Let Aut(E) denote the group of all automorphisms of E, so that we have $\mathrm{Aut}(E) = \{\Psi \in \mathrm{Diff}(E): \pi \circ \Psi = \pi \text{ and } \Psi|\pi^{-1}(x) \in GL(\pi^{-1}(x)), \forall x \in M\}$.

Definition (6.1)7 refers to vector bundles but there is an obvious analogue for bundles whose fibres are endowed with algebraic structures other than that of a vector space - we just require that homomorphisms, etc. preserve the algebraic structure fibrewise.

We now return to the natural groups associated with a given principal G-bundle $\pi: P \longrightarrow M$, in particular to the gauge group

$\mathrm{Gau}(P) = \{\Psi \in \mathrm{Diff}(P)\colon \pi\circ\Psi = \pi$ and $\Psi\circ R_a = R_a\circ\Psi,\ \forall a \in G\}$ (see definition (6.1)4). We now demonstrate that Gau(P) is isomorphic to $\Gamma(\mathrm{Conj}(P))$ (and hence to $C_{\mathrm{conj}}(P,G)$ by the remarks following definition (6.1)6). For $\Psi \in \mathrm{Gau}(P)$, define $s\colon M \to \mathrm{Conj}(P)$ by $s(x) = \kappa_u(a_u)$ for any $u \in \pi^{-1}(x)$. Here a_u is the unique element of G defined by $ua_u = \Psi(u)$, and $\kappa_u\colon G \to \pi^{-1}_{\mathrm{conj}}(\pi(u))$ is the isomorphism defined by 6.1.1. Note that s is well defined since choosing $ub \in \pi^{-1}(x)$ leads to $\kappa_{ub}(a_{ub}) = \kappa_u(ba_{ub}\,b^{-1})$, but $(ub)a_{ub} = \Psi(ub) = \Psi(u)b = ua_u b$, so $a_{ub} = b^{-1}a_u b$ and hence $\kappa_{ub}(a_{ub}) = \kappa_u(a_u)$. We have $\pi_{\mathrm{conj}}(s(x)) = \pi(u) = x$, so that $s \in \Gamma(\mathrm{Conj}(P))$. Conversely, given $s \in \Gamma(\mathrm{Conj}(P))$, define $\Psi\colon P \to P$ by $\Psi(u) = u\,\kappa_u^{-1}(s(\pi(u)))$, $\forall u \in P$. Then $\pi(\Psi(u)) = \pi(u)$ and $\Psi(ua) = ua\kappa_{ua}^{-1}(s(\pi(ua)))$ $= ua\,a^{-1}\,\kappa_u^{-1}(s(\pi(u)))a = u\kappa_u^{-1}(s(\pi(u)))a = \Psi(u)a$. Hence $\Psi \in \mathrm{Gau}(P)$. The two maps just defined are obviously inverse to each other and are both homomorphisms of groups (the group multiplication in Gau(P) is composition of diffeomorphisms and in $\Gamma(\mathrm{Conj}(P))$ it is defined pointwise in the fibres). Hence Gau(P) $\cong$ (Conj(P)) ($\cong C_{\mathrm{conj}}(P,G)$). These isomorphisms are useful in our discussion of the frame bundle and everywhere invariance in Chapter Four.

In section 1.1 use was made of principal bundle extensions and prolongations. We now define these related concepts:

Definition (6.1)8: Let $\pi\colon P \to M$ be a principal G-bundle. Let $\lambda \in \mathrm{Hom}(K,G)$ where K is a Lie group. A λ-prolongation of P to the group K is a pair $(\tilde{P},\eta)$ where $\tilde{P}$ is a principal K-bundle over M and $\eta\colon \tilde{P} \to P$ is a principal bundle homomorphism over id_M such that $\eta(\tilde{u}k) = \eta(\tilde{u})\lambda(k)$, $\forall \tilde{u} \in \tilde{P}$, $k \in K$.

Two λ-prolongations of P, $(\tilde{P}_1,\eta_1)$ and $(\tilde{P}_2,\eta_2)$, are said to be <u>equivalent</u> if there exists a principal K-bundle isomorphism $\beta\colon \tilde{P}_1 \to \tilde{P}_2$ (over id_M) such that $\eta_2 \circ \beta = \eta_1$.

Now let $\mu \in \mathrm{Hom}(G,K)$ so that we have a left action of G on K given by $(a,k) \mapsto \mu(a)k$, $\forall a \in G$, $k \in K$. Let $P_\mu = P \times_G K$ denote the associated bundle with standard fibre K. There is a free right action of K on P_μ given by $([(u, k)],k') \mapsto [(u,kk')]$, $\forall [(u,k)] \in P_\mu$, $k' \in K$ (this is obviously well defined), under which $K \hookrightarrow P_\mu \to M$ is a principal K-bundle over M. P_λ is called the <u>μ-extension</u> of P.

Suppose F is a manifold and $\rho \in \mathrm{Hom}(K,\mathrm{Diff}(F))$. Then $\rho \circ \mu \in \mathrm{Hom}(G, \mathrm{Diff}(F))$ and we have an isomorphism of the corresponding associated bundles: $P_\mu \times_K F \to P \times_G F$ given by $[([(u,k)],f)] \mapsto [(u,\rho(k).f)]$, $\forall\ [([(u,k)],f)] \in P_\mu \times_K F$. It is straightforward to verify that this map is a well defined bundle isomorphism.

The relationship between extensions and reductions may be described as follows. Suppose we are given a principal K-bundle P' over M. Then $\mu \in \mathrm{Hom}(G,K)$ defines a class of reductions of P' to the group G; We define a <u>μ-reduction</u> of P' to be a principal G-bundle P over M together with an embedding $\Psi\colon P \hookrightarrow P'$ over id_M satisfying $\Psi(ua) = \Psi(u)\mu(a)$, $\forall u \in P$, $a \in G$. A μ-reduction of P' is certainly a reduction of the structure group K of P' to the group G in the sense of definition (6.1)3, so long as μ is a monomorphism (otherwise we have a slightly more general concept). Such a reduction induces an obvious isomorphism from the μ-extension of P onto P'. Conversely, if $\pi\colon P \to M$ is any principal G-bundle with μ-extension P_μ, then the homomorphism $\Psi_\mu\colon P \to P \times_G K$; $u \mapsto [(u,1)]$ (1 is the unit of K) is a μ-reduction of the structure group of P_μ from K to G.

When considering embeddings we often need the idea of an induced or pullback bundle. The general situation is as follows (Here we use the word <u>bundle</u> to mean any locally trivial submersion and, although it is not necessary, we may regard this as associated to some principal G-bundle).

<u>Definition (6.1)9</u>: Let $f \in C(M,N)$ and $\pi: E \longrightarrow N$ a bundle with standard fibre F. Define $f^*E = (f \times \pi)^{-1}(N \times N)$ $= \{(x,e) \in M \times E : f(x) = \pi(e)\}$ and $f^*\pi: f^*E \longrightarrow M$; $(x,e) \longmapsto x$. Then $f^*\pi: f^*E \longrightarrow M$ is a bundle over M with standard fibre F called the <u>pullback of E by the map f</u>.

Given E and f, there is a natural bundle homomorphism, $\pi^*f: f^*E \longrightarrow E$; $(x,e) \longmapsto e$, along f, i.e. $\pi \circ (\pi^*f) = f \circ (f^*\pi)$.

Suppose E is a vector bundle over N. Then f^*E is a vector bundle over M which is isomorphic to E along the map f. Also, f^*E is unique up to isomorphism in the sense that a vector bundle E' over M is isomorphic to f^*E if and only if it is isomorphic to E along f.

Differential forms are, of course, extremely useful in geometry and physics. The special structure of a principal G-bundle enables certain spaces of differential forms to be specified:

<u>Definition (6.1)10</u>: Let $\pi: P \longrightarrow M$ be a principal G-bundle and $\rho \in \mathrm{Hom}(G, GL(V))$ a representation of G on the vector space V. Let $\Omega^k(P,V)$ denote the space of all V-valued k-forms on P (so that $\Omega^k(P,V) \equiv \Gamma(\Lambda^k P \otimes (P \times V)))$. The form $\alpha \in \Omega^k(P,V)$ is said to be <u>ρ-equivariant</u> if $R_a^*\alpha = \rho(a^{-1})\alpha$, $\forall a \in G$, and α is said to be <u>horizontal</u> if $\alpha(u)(v_1, \dots, v_k) = 0$ if $v_i \in V_u$ for some $i \in \{1, \dots, k\}$, $\forall u \in P$ ($V_u = \{v \in T_uP : D\pi(u).v = 0\} \equiv T_u\pi^{-1}(\pi(u))$ is the <u>vertical subspace</u> at $u \in P$). If $\alpha \in \Omega^k(P,V)$ is both

ρ-equivariant and horizontal, then α is said to be <u>ρ-tensorial</u>. Let $\Omega^k_\rho(P,V)$ denote the space of all ρ-tensorial k-forms on P.

Forms in $\Omega^k_\rho(P,V)$ project to forms on the base space M in the following sense: there is a linear isomorphism between $\Omega^k_\rho(P,V)$ on the one hand and $\Omega^k(E)$ on the other. Here $\Omega^k(E) = \Gamma(\Lambda^k M \otimes E)$ is the space of k-forms on M taking their values in the vector bundle $E = P \times_\rho V$. Define $\tau\colon \Omega^k_\rho(P,V) \to \Omega^k(E)$ by

$$\tau(\alpha)(x)(w_1,\dots, w_k) \;=\; \kappa_u(\alpha(u)(v_1,\dots, v_k)) \qquad 6.1.4,$$

$\forall w_1,\dots, w_k \in T_xM$, $x \in M$, $\alpha \in \Omega^k_\rho(P,V)$. Here $u \in \pi^{-1}(x)$ and $v_i \in T_uP$ such that $D\pi(u).v_i = w_i$, $i \in \{1,\dots, k\}$. The right-hand side of 6.1.4 is independent of the choice of $u \in \pi^{-1}(x)$ and $v_i \in T_uP$ (projecting onto w_i) and so τ is well defined and does indeed map $\Omega^k_\rho(P,V)$ linearly into $\Omega^k(E)$, since $\kappa_u\colon V \to \pi_E^{-1}(\pi(u))$ from equation 6.1.1. Moreover, τ possesses an inverse given by

$$\tau^{-1}(\bar\alpha)(u)(v_1,\dots, v_k) \;=\; \kappa_u^{-1}(\bar\alpha(\pi(u))(D\pi(u).v_1,\dots, D\pi(u).v_k)) \qquad 6.1.5,$$

$\forall v_1,\dots, v_k \in T_uP$, $u \in P$, $\bar\alpha \in \Omega^k(E)$ (τ^{-1} is just pullback to P). The map τ provides the required linear isomorphism of $\Omega^k_\rho(P,V)$ onto $\Omega^k(E)$.

For example, if $\rho(a) = id_V$, $\forall a \in G$, then a ρ-tensorial form α on P is <u>basic</u> in the sense that $\alpha = \pi^*\bar\alpha$ for some V-valued form $\bar\alpha$ on M ($E = M\times V$ if ρ is trivial). Another important example is the case $k = 0$ - then $\Omega^k_\rho(P,V) = C_\rho(P,V)$, $\Omega^k(E) = \Gamma(E)$ and τ reduces to the isomorphism of the space of (ρ-) equivariant maps of P into V onto the space of sections of the associated bundle E (see the remarks following definition (6.1)6).

We remark that the automorphism group $Aut(P)$ (and, a fortiori, the gauge group $Gau(P)$) acts on $\Omega^k(P,V)$ by pullback.

In order to make full use of principal bundles in physics and geometry we must consider the notion of connection. In a principal G-bundle $\pi : P \longrightarrow M$, each fibre $\pi^{-1}(x)$ is diffeomorphic to the standard fibre G. However, the identification is not canonical; it depends on the covering $\{U_\alpha\}$ of M and on the choice of local trivializations $\Psi_\alpha: \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$. Thus there is no natural way to identify different fibres of P. A connection on P is an extra piece of structure introduced in order to be able to give a correspondence between any two fibres $\pi^{-1}(x)$, $\pi^{-1}(y)$ of P (assuming M is connected) - this is parallel transport along a curve in M from x to y. We define a connection as a particular kind of 1-form on the total space of the principal bundle:

<u>Definition (6.1)11</u>: A <u>connection</u> in the principal G-bundle $\pi: P \longrightarrow M$ is an Ad-equivariant 1-form ω on P such that $\omega(\xi_P) = \xi$, $\forall \xi \in LG$. Let $Conn(P)$ denote the space of all connections in P,

i.e. $Conn(P) = \{\omega \in \Omega^1(P,LG): R_a^*\omega = Ad(a^{-1})\omega, \ \forall a \in G,$ and $\omega(\xi_P) = \xi, \ \forall \xi \in LG\}$.

It is straightforward to show that $\omega - \omega_o \in \Omega^1_{Ad}(P,LG) \cong \Omega^1(Ad(P))$, $\forall \omega,\omega_o \in Conn(P)$, so that $Conn(P)$ is the affine space associated with the vector space $\Omega^1_{Ad}(P,LG)$. In particular $T_\omega Conn(P)$ may be identified with $\Omega^1_{Ad}(P,LG)$ for each connection ω in P. The groups $Aut(P)$ and $Gau(P)$ act on $Conn(P)$ by pullback.

Given a connection ω in P, we may define a dimM-dimensional distribution H^{ω} on P by $H^{\omega}_u = \mathrm{Ker}(\omega(u) \subseteq T_uP$. Then H^{ω} is complementary to the vertical distribution V, i.e. $T_uP = V_u \oplus H^{\omega}_u$, $\forall u \in P$, and $H^{\omega}_{ua} = DR_a(u).H^{\omega}_u$, $\forall u \in P$, $a \in G$. Conversely, given a distribution H on P which is complementary to V and equivariant with respect to the G-action on P we may define a unique connection ω in P by $\omega(u).v = \xi$, where ξ is the unique element of LG satisfying $\xi_P(u) = \mathrm{ver}(v)$ (where $v = \mathrm{ver}(v)+\mathrm{hor}(v)$ is the unique decomposition of $v \in T_uP$ given by the distribution H).

Definition (6.1)12: Let $\omega \in \mathrm{Conn}(P)$ with associated projections ver, hor : $TP \to V, H^{\omega}$. $v \in T_uP$ is said to be horizontal (vertical) if ver $v = 0$ (hor $v = 0$). $\alpha \in \Omega^k(P,V)$ is said to be vertical (horizontal) if α vanishes when one or more of its arguments is horizontal (vertical).

Note that the concept of a horizontal vector (and thence of a vertical form) depends on the choice of a connection in P, whereas that of a vertical vector (horizontal form) is associated naturally with the principal bundle P. By definition any connection ω is vertical. Given $\omega \in \mathrm{Conn}(P)$ we have the linear isomorphism $h^{\omega}_u = D\pi(u)|H^{\omega}_u$ of the horizontal subspace H^{ω}_u onto $T_{\pi(u)}M$ and hence a means of lifting vector fields on M up to P:

Definition (6.1)13: Let $\omega \in \mathrm{Conn}(P)$. The (horizontal)-lift of $X \in \mathrm{Vect}(M)$ is the unique vector field $X^{\omega} \in \mathrm{Vect}(P)$ which is both horizontal (i.e. $X^{\omega}(u)$ is horizontal, $\forall u \in P$) and which projects onto X (i.e. $D\pi(u).X^{\omega}(u) = X(\pi(u))$, $\forall u \in P$).

It can be shown that the lift X^{ω} of any vector field X on M is G-invariant, and conversely, every G-invariant horizontal vector field on P is the lift of some vector field on M. The lift map: $\mathrm{Vect}(M) \longrightarrow \mathrm{Vect}(P)$; $X \longmapsto X^\omega$ is a homomorphism of Lie algebras and satisfies $(fX)^\omega = (\pi^* f)X^\omega$, $\forall f \in C(M)$, $X \in \mathrm{Vect}(M)$.

Definition (6.1)14: Let $c: I \longrightarrow M$ be a curve on M. A (horizontal) lift of c is a horizontal curve $c^\omega: I \longrightarrow P$ such that $\pi \circ c^\omega = c$ (c^ω horizontal means $\frac{dc^\omega}{dt}(t)$ is horizontal, $\forall t \in I$).

If X^ω is the lift of $X \in \mathrm{Vect}(M)$, then the integral curve of X^ω through $u_o \in P$ is a lift of the integral curve of X through $\pi(u_o)$.

Using the local triviality of P it may also be shown that for each curve $c: I \longrightarrow M$ and for each $u_o \in \pi^{-1}(c(0))$ there exists a unique lift $c^\omega_{u_o}$ of P with $c^\omega_{u_o}(0) = u_o$. This result now enables us to define parallel transport of fibres:

Definition (6.1)15): Let $\omega \in \mathrm{Conn}(P)$, $u_o \in P$ and $c^\omega_{u_o}: I \rightarrow P$ the unique lift of $c: I \longrightarrow M$ through the point u_o. Then the end-point of $c^\omega_{u_o}$ is $u_1 = c^\omega_{u_o}(1) \in \pi^{-1}(c(1))$. The map $c_\omega: \pi^{-1}(c(0)) \longrightarrow \pi^{-1}(c(1))$; $u_o \longrightarrow c^\omega_{u_o}(1)$ is called parallel transport along the curve c.

Parallel transport is equivariant with respect to the G-action on P, i.e. $c_\omega \circ R_a = R_a \circ c_\omega$, $\forall a \in G$, and hence is a diffeomorphism of $\pi^{-1}(c(0))$ onto $\pi^{-1}(c(1))$. Parallel transport is also parameterization invariant and $(c^{-1})_\omega = c_\omega^{-1}$ if c^{-1} denotes the (curve) inverse of c.

A connection in a principal bundle also induces notions of horizontality in associated bundles and also the very important idea of covariant derivative acting on sections of vector bundles. The covariant derivative may also be introduced directly (au Koszul) but we prefer to start with a connection in a principal bundle, and we prefer to start out with a connection in a principal bundle, and the two approaches are indeed equivalent.

Definition (6.1)16: Let $E = P\times_\rho F$ be the bundle with standard fibre F associated with the principal G-bundle P via the action ρ. Let $V_e = \mathrm{Ker}\ D\pi_E(e)$ denote the vertical subspace at $e \in E$, so that V_e is the tangent space to the fibre of E through e. Let $e = [(u_o,f)]$ for some $(u_o,f) \in P\times F$ and define $m_f: P \to E$; $u \mapsto [u,f)]$, $\forall u \in P$. Then we define the horizontal subspace at e to be $H^\omega_e = Dm_f(u_o).H^\omega_{u_o}$.

Note that $Dm_{\rho(a^{-1}).f}(u_oa).H^\omega_{u_oa} = Dm_f(u).H^\omega_u$, so that H^ω_e is defined independently of the choice of representative (u_o,f) of e. We also have that $T_eE = V_e \oplus H^\omega_e$.

Given a curve $c: I \longrightarrow M$, a (horizontal) lift c^ω of c is a horizontal curve in E such that $\pi_E \circ c^\omega = c$. Given $e_o \in \pi_E^{-1}(c(0))$, there exists a unique lift $c^\omega_{e_o}$ starting from e_o - this is constructed using the existence and uniqueness result for lifts to P. Parallel transport $c_\omega: \pi_E^{-1}(c(0)) \longrightarrow \pi_E^{-1}(c(1))$; $e_o \longmapsto c^\omega_{e_o}(1)$ may now be defined in E. Let U be an open subset of M and $s \in \Gamma(E|U)$ a local section of E. Then s is said to be parallel if the parallel transport of $s(c(0))$ along any curve $c: I \longrightarrow U$ is equal to $s(c(1))$, i.e. s is parallel if and only if $Ds(x).T_xM \subseteq H^\omega_{s(x)}$, $\forall x \in U$.

An important case is when $E = P\times_G(G/H)$, where H is a closed subgroup of G. Let $s \in \Gamma(E)$ and $H \hookrightarrow Q_s \longrightarrow M$ the reduced principal H-bundle corresponding to s (see the remarks following definition (6.1)5). Then $\omega \in \mathrm{Conn}(P)$ is reducible (see below) to a connection in Q_s if and only if s is parallel with respect to ω.

We now focus our attention on any vector bundle E associated to P.

<u>Definition (6.1)17</u>: Let $\omega \in \mathrm{Conn}(P)$, $c: I \longrightarrow M$ and $s \in \Gamma(c^* E)$ a section of E along the curve c. The <u>covariant derivative</u> $\nabla^{\omega}_{\dot{c}}\, s$ of the section s in the direction $\dot{c}$ $(\equiv \frac{dc}{dt} \in \Gamma(c^* TM))$ is defined by

$$(\nabla^{\omega}_{\dot{c}}\, s)(t) = \lim_{h \to 0} \frac{1}{h} \{c^{t,h}_{\omega}(s(t+h)) - s(t)\} \qquad 6.1.6,$$

where $c^{t,h}_{\omega}: \pi_E^{-1}(c(t+h)) \longrightarrow \pi_E^{-1}(c(t))$ denotes parallel transport along c.

The covariant derivative in the direction $\dot{c}$ maps $\Gamma(c^* E)$ into itself. We may define a covariant derivative of sections of E itself as follows: Let $v \in T_x M$ and s a section of E defined in a neighbourhood of x. Then the <u>covariant derivative of s in the direction v</u> is defined by:

$$\nabla^{\omega}_{v}\, s = (\nabla^{\omega}_{\dot{c}}\, (s \circ c))(0) \qquad 6.1.7,$$

where $c: I \longrightarrow M$ is a curve with $c(0) = x$, $\dot{c}(0) = v$. Using 6.1.6 it can be easily shown that $\nabla^{\omega}_{v} s$ depends only on $v \in T_x M$ and not on the choice of c. A section $s \in \Gamma(E|U)$, U an open subset of M, is parallel if $\nabla^{\omega}_{v}\, s = 0$ $\forall v \in T_x M$, $x \in U$.

If $X \in \mathrm{Vect}(M)$ we define <u>covariant differentiation along X</u>, $\nabla^{\omega}_{X}: \Gamma(E) \longrightarrow \Gamma(E)$, by

$$(\nabla^{\omega}_{X}\, s)(x) = \nabla^{\omega}_{X(x)}\, s \qquad 6.1.8,$$

$\forall x \in M$, $s \in \Gamma(E)$. It is the 6.1.8 version of covariant derivative that we usually consider. The covariant derivative has the following four properties $\forall X,Y \in \mathrm{Vect}(M)$, $f \in C(M,\mathbb{K})$, $s \in (E)$: ∇^{ω}_{X} is $\mathbb{K}$-linear, $\nabla^{\omega}_{X}(fs) = \langle df,X\rangle s + f\nabla^{\omega}_{X}s$, $\nabla^{\omega}_{X+Y}s = \nabla^{\omega}_{X}s + \nabla^{\omega}_{Y}s$ and $\nabla^{\omega}_{fX}s = f\nabla^{\omega}_{X}s$. These properties enable us to define a linear map $\nabla^{\omega}: \Gamma(E) = \Omega^{o}(E) \longrightarrow \Gamma(T^{*}M \otimes E) = \Omega^{1}(E)$ by

$$(\nabla^{\omega}\, s)(X) \quad = \quad \nabla^{\omega}_{X}\, s \qquad\qquad 6.1.9,$$

$\forall X \in \mathrm{Vect}(M)$, $s \in \Gamma(E)$. A derivative mapping $\Omega^{k}(E)$ to $\Omega^{k+1}(E)$ for $k > 0$ will be given below when we introduce the covariant exterior derivative. The covariant derivative ∇^{ω}_{X} corresponds to Lie differentiation in the following sense: Let X^{ω} be the lift of X and let $s \in \Gamma(E)$. Then

$$L_{X^{\omega}}(\tau^{-1}(s)) \quad = \quad \tau^{-1}(\nabla^{\omega}_{X}\, s) \qquad\qquad 6.1.10,$$

where τ is the canonical isomorphism of $\Omega^{k}_{\rho}(P,V)$ onto $\Omega^{k}(E)$ (for $k = 0$ in this case) - see 6.1.4, 6.1.5.

We now turn to a discussion of covariant exterior derivative:

Definition (6.1)18: Let $\omega \in \mathrm{Conn}(P)$ and $\rho \in \mathrm{Hom}(G, GL(V))$. Denote by $\Omega^{k}_{hor}(P,V)$, $\Omega^{k}_{G}(P,V)$ the space of all horizontal, ρ-equivariant V-valued k-forms on P respectively (see definition (6.1)10), so that $\Omega^{k}_{\rho}(P,V) = \Omega^{k}_{hor}(P,V) \cap \Omega^{k}_{G}(P,V)$. Let $\Omega(P,V) = \bigoplus_{k\geq 0} \Omega^{k}(P,V)$ denote the graded left $\Omega(P)$-module of V-valued differential forms on P. Similarly we have $\Omega_{hor}(P,V)$, $\Omega_{G}(P,V)$ and $\Omega_{\rho}(P,V)$. Define a horizontal projection $h^{\omega}: \Omega(P,V) \longrightarrow \Omega(P,V)$ by

$$(h^{\omega}\alpha)(u)(v_{1},\ldots, v_{k}) \quad = \quad \alpha(u)(\mathrm{hor}(v_{1}),\ldots, \mathrm{hor}(v_{k})) \qquad 6.1.11,$$

$\forall v_1, \ldots, v_k \in T_uP$, $u \in P$, $\alpha \in \Omega^k(P,V)$, $k \geq 0$.

The operator h^ω is linear and satisfies $h^\omega(\alpha \wedge \beta) = h^\omega\alpha \wedge h^\omega\beta$, $\forall \alpha \in \Omega(P)$, $\beta \in \Omega(P,V)$. The image of h^ω is $\Omega_{hor}(P,V)$ and $(h^\omega)^2 = h^\omega$, $h^\omega \circ R_a = R_a \circ h^\omega$, $\forall a \in G$. Hence h^ω is an equivariant projection onto the space of horizontal forms, so we may restrict to obtain the map $h^\omega \equiv h^\omega|\Omega^k_G(P,V)$: $\Omega^k_G(P,V) \to \Omega^k_\rho(P,V)$, for each $k \geq 0$. Note also that $\omega \in \ker h^\omega$ (for $V = LG$, $\rho = Ad$).

We also have the exterior derivative $d: \Omega^k(P,V) \to \Omega^{k+1}(P,V)$ and hence:

<u>Definition (6.1)19</u>: Let $\omega \in Conn(P)$ and V a vector space. The <u>covariant exterior derivative</u> associated with ω is the linear map $d^\omega = h^\omega \circ d: \Omega^k(P,V) \to \Omega^{k+1}(P,V)$, $k \geq 0$.

Using the properties of h^ω listed above together with results for d (e.g. $d \circ R^*_a = R^*_a \circ d$, $\forall a \in G$), we have the following satisfied by d^ω: $d^\omega(\alpha \wedge \beta) = d^\omega\alpha \wedge h^\omega\beta + (-1)^k h^\omega\alpha \wedge d^\omega\beta$, $\forall \alpha \in \Omega^k(P)$, $\beta \in \Omega(P,V)$, $d^\omega \circ R^*_a = R^*_a \circ d^\omega$, $\forall a \in G$, $d^\omega \circ \pi^* = \pi^* \circ d^\omega$ and $\iota_{\xi_P} \circ d^\omega = 0$, $\forall \xi \in LG$. We therefore have the restrictions $d^\omega: \Omega_{hor}(P,V) \to \Omega_{hor}(P,V)$, $d^\omega: \Omega_G(P,V) \to \Omega_\rho(P,V)$ and, in particular, $\Omega_\rho(P,V)$ is invariant under d^ω. A useful formula for d^ω acting on tensorial forms is given by:

$$d^\omega\alpha = d\alpha + \omega(\alpha) \qquad 6.1.12,$$

$\forall \alpha \in \Omega_\rho(P,V)$. Here $\omega(\cdot): \Omega^k(P,V) \to \Omega^{k+1}(P,V)$, $k \geq 0$, is defined in the following manner: We have $\rho: G \to GL(V)$, so that $D\rho(1): LG \to g\ell(V)$ ($\equiv L(GL(V))$) and we may define a bilinear map: $LG \times V \to V$; $(\xi,v) \mapsto (D\rho(1).\xi)v$. This bilinear map induces a $C(P,\mathbb{K})$-module homomorphism: $\Omega^{k_1}(P,LG) \times \Omega^{k_2}(P,V) \to \Omega^{k_1+k_2}(P,V)$

in the standard way. Since $\omega \in \Omega^1(P,LG)$, we may define the linear map $\omega: \Omega^k(P,V) \to \Omega^{k+1}(P,V)$ written $\alpha \mapsto \omega(\alpha)$, $\forall\alpha \in \Omega(P,V)$. An important case of 6.1.12 is when $V = LG$ and $\rho = Ad$. Then, since $DAd(1): LG \to g\ell(LG)$ is just the adjoint action of LG on itself; $\xi \mapsto Ad(\xi)$; $\eta \mapsto [\xi,\eta]$, we write $\omega(\alpha) = [\omega,\alpha]$, $d^\omega\alpha = d\alpha + [\omega,\alpha]$, $\forall\alpha \in \Omega^k(P,LG)$.

The covariant exterior derivative may be applied to any V-valued k-form on P, in particular to the connection form ω itself. Indeed, since ω is Ad-equivariant, $d^\omega\omega$ is Ad-tensorial.

Definition (6.1)20: The curvature form Ω^ω of the connection ω is the Ad-tensorial 2-form $\Omega^\omega = d^\omega\omega$. The curvature mapping on P is the map $\Omega: Conn(P) \to \Omega^2_{Ad}(P,LG)$; $\omega \mapsto \Omega^\omega \equiv d^\omega\omega$. The curvature tensor of ω is $R^\omega = 2\tau(\Omega^\omega) \in \Omega^2(Ad(P))$.

The important results concerning the curvature of a connection are the Cartan structural equation, $\Omega^\omega = d\omega + \frac{1}{2}[\omega,\omega]$, and the Bianchi identity, $d^\omega\Omega^\omega = 0$. We remark that since $\omega \notin \Omega^1_{Ad}(P,LG)$, the result 6.1.12 cannot be used to prove the Cartan structural equation. However, $\Omega^\omega \in \Omega^2_{Ad}(P,LG)$, so 6.1.12 may be used to give $d^\omega\Omega^\omega = d\Omega^\omega + [\omega,\Omega^\omega]$ and the right-hand side of this vanishes identically. For any $\alpha \in \Omega^k_\rho(P,V)$, we have $d^\omega d^\omega\alpha = \Omega^\omega(\alpha)$ (where $\Omega^\omega(\alpha)$ is defined in an analogous fashion to $\omega(\alpha)$), so, in particular, $d^\omega d^\omega \neq 0$ in general.

We relate the covariant exterior derivative acting on $\Omega^k_\rho(P,V)$ (definition (6.1)19) to the covariant derivative (definition (6.1)17) using the isomorphism τ. We have $d^\omega: \Omega^k_\rho(P,V) \to \Omega^{k+1}_\rho(P,V)$ and we define $d^\omega_M: \Omega^k(E) \to \Omega^{k+1}(E)$ by $d^\omega_M = \tau\circ d^\omega\circ\tau^{-1}$. For $k = 0$, we obtain $d^\omega_M: \Gamma(E) \to \Gamma(T^*M \otimes E)$ and this coincides with ∇^ω defined by 6.1.9.

The behaviour of connections under principal bundle homomorphisms is very important especially in discussions of embeddings, etc. Suppose $G_i \hookrightarrow P_i \longrightarrow M_i$ $(i = 1,2)$ are principal bundles and Ψ a homomorphism of P_1 into P_2 with corresponding $\Psi'' \in \mathrm{Hom}(G_1,G_2)$ and such that $\overline{\Psi}$ is a diffeomorphism. Let $\omega_1 \in \mathrm{Conn}(P_1)$ with curvature $\Omega_1 \in \Omega^2_{Ad}(P_1,LG_1)$. Then it can be shown that there exists a unique $\omega \in \mathrm{Conn}(P_2)$ such that ω_1-horizontal subspaces are mapped by $D\Psi$ into ω_2-horizontal subspaces. In addition, $\Psi^*\Omega_2 = D\Psi''(1)\circ\Omega_1$ and $\Psi^*\omega_2 = D\Psi''(1)\circ\Omega_1$, where $D\Psi''(1)\colon\ LG_1 \longrightarrow LG_2$ in the Lie algebra homomorphism induced by Ψ''.

<u>Definition (6.1)21</u>: Let $P_1P_2,\Psi,\omega_1,\omega_2$ be as just described. We say that ω_2 is the <u>image of ω_1 under Ψ</u>. In particular, in the case when $H \hookrightarrow Q \longrightarrow M$ is a reduction of $G \hookrightarrow P \longrightarrow M$ (so we are taking H a closed subgroup of G, Ψ'' inclusion of H in G and $\overline{\Psi} = id_M$), we say that the connection ω_2 in P is <u>reducible</u> to the connection ω_1 in Q.

We have noted above that $\mathrm{Aut}(P)$ acts on $\mathrm{Conn}(P)$ by pullback (this action arises as a special case of the result given preceding definition (6.1)21). An automorphism Ψ is called an <u>automorphism of the connection</u> ω if $\Psi^*\omega = \omega$, and in this case, ω is said to be <u>Ψ-invariant</u>.

The notion dual to that of definition (6.1)21 is that of induced or pullback connection. We make use of this in Chapters Two and Three. Let $G_i \hookrightarrow P_i \longrightarrow M_i$ $(i = 1,2)$ be principal bundles and Ψ a homomorphism of P_1 into P_2 such that $\Psi'' \in \mathrm{Hom}(G_1,G_2)$ induces an isomorphism $D\Psi''(1)$ of LG_1 onto LG_2. Let $\omega_2 \in \mathrm{Conn}(P_2)$ with curvature $\Omega_2 \in \Omega^2_{Ad}(P_2,LG_2)$. Then it can be shown that there exists a unique $\omega_1 \in \mathrm{Conn}(P_1)$ such that ω_1-horizontal subspaces are mapped by D into ω_2-horizontal subspaces. In addition, $\omega_1 = D\Psi''(1)^{-1} \circ \Psi^*\omega_2$ and $\Omega_1 = D\Psi''(1)^{-1} \circ \Psi^*\Omega_2$.

Definition (6.1)22: Let P_1, P_2, Ψ, ω_1, ω_2 be as just described. We say that ω_1 is the connection induced by Ψ from ω_2.

If $G_1 = G_2 = G$ and $\Psi'' = id_G$, then the induced connection is simply $\omega_1 = \Psi^*\omega_2$. In particular, given $G \hookrightarrow P \rightarrow N$ together with $f \in C(M,N)$, we have a map f^*: Conn $(P) \rightarrow \mathrm{Conn}(f^*P)$, so that we may pullback connections to pullback bundles.

It is convenient to translate (6.1)21, (6.1)22 into definitions concerning covariant derivatives and vector bundles, since covariant derivatives are used in calculations in Chapters Two and Three. Suppose $G_i \hookrightarrow P_i \rightarrow M_i$ $(i = 1,2)$ are principal bundles, Ψ is a homomorphism of P_1 into P_2, $\rho_i \in \mathrm{Hom}(G_i, GL(V_i))$ $(i=1,2)$ are representations and $\Psi_V \in L(V_1,V_2)$ such that $\Psi_V \circ \rho_1(a) = \rho_2(\Psi''(a)) \circ \Psi_V$, $\forall a \in G_1$. Then in an obvious manner, we have the vector bundle homomorphism Ψ_E: $P_1 \times_{\rho_1} V_1 \rightarrow P_2 \times_{\rho_2} V_2$; $[(u_1,v_1)] \rightarrow [(\Psi(u_1), \Psi_V(v_1))]$, $\forall [(u_1,v_1)] \in P_1 \times_{\rho_1} V_1$. Using this construction together with the definition of covariant derivatives (6.1.7), we have analogues of definitions (6.1)21 and (6.1)22. We discuss here only the analogue of (6.1)22.

Let π_i: $E_i \rightarrow M_i$ $(i = 1,2)$ be vector bundles (over the same field) and Ψ: $E_1 \rightarrow E_2$ a vector bundle homomorphism. Let ∇^2 be a connection in E_2 (i.e. ∇^2 is the covariant derivative operator associated with a connection in the principal bundle to which E_2 is associated). Then it can be shown that there exists a unique connection ∇^1 in E_1 such that, for any $x \in M_1$, $v \in T_xM_1$,

$s_i \in \Gamma(E_i)$ $(i = 1,2)$ satisfying $s_2 \circ \overline{\Psi} = \Psi \circ s_1$, we have $\Psi(\nabla^1_v\, s_1) = \nabla^2_{D\overline{\Psi}(x).v}\, s_2$.

Definition (6.1)23: Let E_1, E_2, Ψ, ∇^1, ∇^2 be as above. ∇^1 is said to be the connection induced in E_1 by the homomorphism Ψ from ∇^2.

In particular, given a vector bundle $\pi\colon E \longrightarrow N$ together with $f \in C(M,N)$, any connection ∇ in E gives rise to an induced connection ∇^f in the pullback vector bundle $f^*\pi\colon f^*E \longrightarrow M$ (see definition (6.1)9). Note that $\Gamma(f^*E)$ is $C(M)$-module isomorphic to $\Gamma_f(E) \equiv \{s \in C(M,E)\colon \pi \circ s = f\}$. An element of $\Gamma_f(E)$ is called a section of E along f.

To conclude this appendix we turn from principal bundles in general to the frame bundle of a manifold M. In fact any principal G-bundle may be regarded as a reduction of a bundle of frames in the following sense: Let $\pi_E\colon E \longrightarrow M$ be a vector bundle and let $GL(E) = \{u\colon u \text{ is a basis of } \pi_E^{-1}(x) \text{ for some } x \in M\}$. Let the rank ($\equiv$ fibre dimension) of E be p. Then there is a free right action of $GL(p, \mathbb{F})$ on $GL(E)$ defined by $(u,a) \longmapsto ua$ where $ua = \{e_i a^i{}_j\}$, $\forall\, u = \{e_j\} \in GL(E)$, $a = (a^i{}_j) \in GL(p, \mathbb{F})$. Under this action $GL(E)$ is the total space of a principal $GL(p, \mathbb{F})$-bundle over M.

Definition (6.1)24: Let E be a vector bundle. The principal bundle $GL(E)$ is called the frame bundle of E.

Suppose E is associated to the principal G-bundle P over M. Then, since E is a vector bundle, $G \leqslant GL(p, \mathbb{F})$, and in fact, P is a reduction of $GL(E)$ to the group G.

An example of the frame bundle is the bundle of (g, s_g)-spin frames $Spin(n) \hookrightarrow \widetilde{\widetilde{SO}}(M,g) \longrightarrow M$ for some g-spin structure

$s_g \in \tilde{\Sigma}(M,g)$, $g \in Met(M)$ (see section 1.1). The most important case, however, is when $E = TM$, the tangent bundle of the manifold M. We may also consider the complexified tangent bundle $T^{\mathbb{C}}M$, but we deal only with the real case here:

<u>Definition (6.1)25</u>: The <u>frame bundle of M</u>, denoted $GL(M)$, is the frame bundle of TM.

Obviously $GL(M)$ is a principal $GL(n, \mathbb{R})$-bundle naturally arising from the n-dimensional manifold M. The tensor bundles over M are vector bundles associated to $GL(M)$ via tensor products of the defining representation $\rho^{(1,0)}$ (and its dual $\rho^{(0,1)}$) of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$. The defining representation itself gives rise to the tangent bundle of M and vector bundle automorphisms (see (6.1)7)) of TM on the one hand and principal bundle automorphisms (see (6.1)4) of $GL(M)$ on the other are tied together in a satisfying algebraic fashion as we saw in section 4.1.

The two important phenomena which occur by virtue of the fact that $GL(M)$ is canonical are firstly the existence of a natural 1-form on $GL(M)$, and secondly the splitting of the exact sequences 6.1.2 and 6.1.3. These two phenomena are related as we now discuss.

<u>Definition (6.1)26</u>: Let $\pi_M\colon GL(M) \to M$ be the frame bundle of the n-manifold M. The <u>canonical 1-form</u> (or <u>soldering form</u>) θ_M (or just θ if M is understood) is defined by $\theta_M(u) = \kappa_u^{-1} \circ D\pi_M(u)$, $\forall u \in GL(M)$ (Here, $\kappa_u\colon \mathbb{R}^n \longrightarrow T_{\pi_m(u)}M$; $\underline{x} \longmapsto [(u,\underline{x})] \equiv x^a e_a$, $\forall \underline{x} = (x^a) \in \mathbb{R}^n$, $u = \{e_a\} \in GL(M)$, is the linear isomorphism defined by 6.1.1).

It is straightforward to demonstrate that θ_M is $\rho^{(1,0)}$-tensorial and that $\tau(\theta_M) \in \Omega^1(TM) = \Gamma(T^*M \otimes TM) \cong End(Vect(M))$ is the identity endomorphism. The canonical 1-form θ_M "solders" $GL(M)$ to M, so that the frame bundle has more intimate interaction with M than do other principal $GL(n,\mathbb{R})$-bundles over M. In fact it may be shown that a principal $GL(n,\mathbb{R})$- bundle over M is isomorphic to $GL(M)$ if and only if there exists a $\rho^{(1,0)}$-tensorial 1-form θ on the total space such that $\mathrm{Ker}\theta$ is equal to the vertical distribution. Thus, the pair $(GL(M),\theta_M)$ is unique.

Now consider the natural groups associated with the frame bundle (see definition (6.1)4). We have $\mathrm{Aut}\, GL(M)$, the group of automorphisms of the frame bundle, and also its normal subgroup $\mathrm{Gau}\, GL(M)$, the group of gauge transformations of the frame bundle. The latter may be identified with $\Gamma(\mathrm{Conj}\, GL(M))$, the space of sections of the conjugation bundle, and also with $C_{\mathrm{conj}}(GL(m), GL(n,\mathbb{R}))$, the space of equivariant maps of the frame bundle into the general linear group, as described above. We also have a splitting of the exact sequences 6.1.2 and 6.1.3:

<u>Definition (6.1)27</u>: Define $\ell \in \mathrm{Hom}(\mathrm{Diff}(M), \mathrm{Aut}\, GL(M))$ by $\ell(\phi) = \hat{\phi}$, where $\hat{\phi}(u) = \{D\phi(\pi_m(u))e_a\}$, $\forall u = \{e_a\} \in GL(M)$, $\phi \in \mathrm{Diff}(M)$. The automorphism $\hat{\phi}$ is called the <u>(natural) lift</u> of the diffeomorphism ϕ.

(We have $(\hat{\phi} \circ R_a)(u) = \{D\phi(\pi_m(ua)).e_b a^b_a\} = \{D\phi(\pi_m(u)).e_a\}a$ $= (R_a \circ \hat{\phi})(u)$, so $\hat{\phi}$ is indeed an automorphism of $GL(M)$. Similarly, $\ell(\phi_1 \circ \phi_2) = \ell(\phi_1) \circ \ell(\phi_2)$, so ℓ is a homomorphism (smooth) of (Lie) groups). Moreover, ℓ is a section of the projection $b\colon \mathrm{Aut}\, GL(M) \to \mathrm{Diff}(M)$, i.e. $b \circ \ell = \mathrm{id}_M$, and hence a splitting of the sequences 6.1.2 and 6.1.3. We therefore have a semi-direct product structure for $\mathrm{Aut}\, GL(M)$. Indeed, let $\mathrm{Diff}(M)$ act on $\mathrm{Gau}\, GL(M)$ via the homomorphism $\mathrm{conj} \circ \ell\colon \mathrm{Diff}(M) \to \mathrm{Aut}(\mathrm{Gau}\, GL(M))$, so that $\Psi \mapsto \hat{\phi} \circ \Psi \circ \hat{\phi}^{-1}$ under the action of $\phi \in \mathrm{Diff}(M)$,

$\forall\Psi \in \text{Gau } GL(M)$. Then $\text{Aut } GL(M) \cong \text{Diff}(M) \ltimes \text{Gau } GL(M)$ with group structure given by:

$$(\phi_1,\Psi_1)(\phi_2,\Psi_2) = (\phi_1\circ\phi_2,\ \Psi_1\circ\hat{\phi}_1\circ\Psi_2\circ\hat{\phi}_1^{-1}) \qquad 6.1.13,$$

$\forall(\phi_1,\psi_1)$, $(\phi_2,\psi_2) \in \text{Diff}(M)\ltimes\text{Gau } GL(M)$, in the standard way. The explicit isomorphism $q\colon \text{Aut } GL(M) \longrightarrow \text{Diff}(M) \propto \text{Gau } GL(M)$ is given by $q(\Psi) = (b(\Psi),\ \Psi\circ(\ell\circ b)(\Psi^{-1}))$. $\forall\Psi \in \text{Aut } GL(M)$, and $q^{-1}(\phi,\Psi) = \Psi\circ\ell(\phi)$, $\forall(\phi,\Psi) \in \text{Diff}(M)\times\text{Gau } GL(M)$. The structure of $\text{Aut } GL(M)$ is explored further in Chapter Four.

We now show that the stabilizer of the canonical 1-form is precisely the lift of the diffeomorphism group of M. Suppose $\phi \in \text{Diff}(M)$. Then $(\hat{\phi}^*\theta_M)(u) = \theta_M(\hat{\phi}(u))\circ D\hat{\phi}(u) = \kappa_{\hat{\phi}(u)}^{-1}\circ D\pi_M(\hat{\phi}(u))\circ D\hat{\phi}(u)$. But $\kappa_{\hat{\phi}(u)} = D\phi(\pi_M(u))\circ\kappa_u$ and $\pi_M\circ\hat{\phi} = \phi\circ\pi_M$, therefore $(\hat{\phi}^*\theta_M)(u) = \kappa_u^{-1}\circ D\phi(\pi_M(u))^{-1}\circ D\phi(\pi_M(u))\circ D\pi_M(u) = \kappa_u^{-1}\circ D\pi_M(u) = \theta_M(u)$ $\forall u \in GL(M)$. Hence $\ell(\text{Diff}(M)) \leqslant \text{stab}(\theta_M)$. Conversely, suppose $\Psi^*\theta_M = \theta_M$ for some $\Psi \in \text{Aut } GL(M)$. Then $\forall u \in GL(M)$, $\theta_M(\Psi(u))\circ D\Psi(u) = \theta_M(u)$, i.e. $\kappa_{\Psi(u)}^{-1}\circ D\pi_M(\Psi(u))\circ D\Psi(u) = \kappa_u^{-1}\circ D\pi_M(u)$. However, $\pi_M\circ\Psi = b(\Psi)\circ\pi_M$, and, letting $\Psi_o = (\ell\circ b)(\Psi^{-1})\circ\Psi$, we have $\kappa_{\Psi(u)} = \kappa_{((\ell\circ b)(\Psi))(\Psi_o(u))} = D\overline{\Psi}(\pi_M(u))\circ\kappa_{\Psi_o(u)}$. Hence $\kappa_{\Psi_o(u)}^{-1}\circ D\pi_M(u) = \kappa_u^{-1}\circ D\pi_M(u)$. Now $D\pi_M(u)$ is surjective and so $\kappa_{\Psi_o(u)}^{-1} = \kappa_u^{-1}$ as linear isomorphisms of $T_{\pi_M(u)}M$ onto $\mathbb{R}^n$. Hence $\Psi_o(u) = u$, $\forall u \in GL(M)$, and hence $\Psi = \ell(b(\Psi)) \in \ell(\text{Diff}(M))$. This concludes the proof that a frame bundle automorphism Ψ leaves the canonical 1-form invariant if and only if Ψ is the lift of some diffeomorphism of M. In particular, there exists no non-trivial gauge transformation fixing θ_M.

The canonical 1-form θ_M restricts to any sub-bundle of $GL(M)$.

Certain subbundles play an important rôle in geometry and physics and we have used these in the main body of the thesis. An important idea is the following:

Definition (6.1)28: Let M be an n-manifold and G a subgroup of $GL(n,\mathbb{R})$. A G-structure on M is a reduction of the frame bundle $GL(n,\mathbb{R}) \hookrightarrow GL(M) \xrightarrow{\pi_M} M$ to the group G, i.e. a G-structure is a principle subbundle $P \to M$ of $GL(M)$ with structure group G.

The result stated after definition (5.1)5 may be used to determine whether or not a G-structure exists for given $G \leqslant GL(n,\mathbb{R})$: G-structures on M are in bijective correspondence with $\Gamma(GL(M)_G)$, where $GL(M)_G = GL(M) \times_{GL(n,\mathbb{R})} GL(n,\mathbb{R})/G$ is the bundle associated with the frame bundle via the action of $GL(n,\mathbb{R})$ on the quotient space $GL(n,\mathbb{R}) / G$. Obstruction theory then gives necessary topological conditions on M for the existence of sections of $GL(M)_G$ and hence of G-structures on M.

G-structures are important in geometry because many structures on a manifold M are examples of G-structures. We give examples of subgroups G of $GL(n,\mathbb{R})$ which lead to structures used above: For $G = GL^+(n,\mathbb{R})$ we have an orientation structure on M. Such a structure exists if and only if M is orientable and then a choice of section of the $\mathbb{Z}_2$-bundle $GL(M)_{GL^+(n,\mathbb{R})}$ gives a particular orientation on M. For an oriented manifold M, we will use the oriented frame bundle $GL^+(n,\mathbb{R}) \hookrightarrow GL^+(M) \to M$ as in Chapter One (Here, $GL^+(M)$ is the sub-bundle of $GL(M)$ corresponding to the particular $GL^+(n,\mathbb{R})$- structure, i.e. orientation, chosen - obviously there are only two possible $GL^+(n,\mathbb{R})$ - structures on a given orientable manifold M). If M is oriented, then we often refer to reductions of $GL^+(M)$ to the subgroup $G \leqslant GL^+(n,\mathbb{R})$

also as G-structures on M (see, for example, section 1.1).

A second important example arises when $G = SL(n,\mathbb{R})$. Then the set of G-structures is parameterized by $\Gamma(GL(M)_{SL(n,\mathbb{R})})$ which is in bijective correspondence with ${}^{*}\Omega^{n}(M)$, the space of nowhere vanishing n-forms on M. Thus an $SL(n,\mathbb{R})$-structure on a (orientable) manifold M is nothing but a choice of volume element.

Further examples arise in the case $G = CO(p,q) = SO(p,q) \times \mathbb{R}^{+}$ (or $G = O(p,q) \times \mathbb{R}^{+}$) when we have a conformal structure on M (see sections 1.5 and 6.2) and in the case $G = SO(p,q)$ (or $O(p,q)$) when we have a (pseudo-)Riemannian structure (i.e. a metric) on M. When p (or q) $= 0$ such structures always exist, but for $p,q \neq 0$ there are topological obstructions to the existence of sections of the bundles $GL(M)_{CO(p,q)}$ and $GL(M)_{SO(p,q)}$ - for example a Lorentzian (conformal) structure ($p = 1$, $q = n-1$) exists if and only if M has vanishing Euler-invariant or is non-compact. Other important examples are given by $G = Sp(n)$ (symplectic structure) and, for $n = 2m$, $G = GL(m,\mathbb{C})$ (almost complex structure). We refer the reader to Kobayashi [K 6] for more details on these and other examples of G-structures.

The automorphism group of a G-structure $G \hookrightarrow P \longrightarrow M$ is the stabilizer of P in Aut GL(M) and is given by $\mathrm{Aut}(P) = \mathrm{stab}(k) \ltimes \mathrm{Gau}(P)$, where $\mathrm{stab}(k) \leq \mathrm{Diff}(M)$ is the symmetry group of the tensor k on M corresponding to the G-structure P (e.g. for $G = O(p,q)$, $k = g$, a metric on M, $P = O(M,g)$, the bundle of g-orthonormal frames and stab(k) is the isometry group of $g = k$).

We now turn to connections in the frame bundle:

Definition (6.1)29: A connection in the frame bundle $GL(M)$ is called a linear connection on M. Let $\mathrm{Conn}(M) = \mathrm{Conn}\, GL(M)$ denote the affine space of linear connections on M. For $\omega \in \mathrm{Conn}(M)$, the torsion form Θ^{ω} of ω is the $\rho^{(1,0)}$-tensorial 2-form given by $\Theta^{\omega} = d^{\omega}\theta_M$. The torsion map is the map

$\Theta : \mathrm{Conn}(M) \longrightarrow \Omega^2_{\rho^{(1,0)}}(GL(M), \mathbb{R}^n)$; $\omega \longmapsto \Theta^{\omega}$. The torsion tensor of ω is $T^{\omega} = 2\tau(\Theta^{\omega}) \in \Omega^2(TM)$.

Using the formula 6.1.12, we obtain the structural equation $\Theta = d\theta_M + \omega(\theta_M)$ together with the Bianchi identity $d^{\omega}\Theta^{\omega} = \Omega^{\omega}(\theta_M)$. As we have remarked, the vector bundles associated to $GL(M)$ via tensor products of $\rho^{(1,0)}$, $\rho^{(0,1)}$ are the natural tensor bundles over M. Any connection ω in $GL(M)$ defines a covariant derivative operator acting on sections of these tensor bundles (i.e. on tensor fields) as in 6.1.8 and 6.1.9. Using τ we may pullback the structural equations and the Bianchi identities (for both curvature and torsion) to give the standard covariant differential relations between tensor fields on M.

For completeness, we conclude this appendix with a remark on the interaction between the space of metrics on the one hand and the space of linear connections on the other. The fundamental theorem of Riemannian geometry may be stated as follows: Given any $g \in \mathrm{Met}(M)$ (metrics of a given signature), there exists a unique linear connection $\omega \equiv \omega_g$ which is metric (i.e. $d^{\omega}\hat{g} = 0$, where $\hat{g} : GL(M) \longrightarrow S(n,\mathbb{R})$ is the equivariant map corresponding to g) and torsion free (i.e. $d^{\omega}\theta_M = 0$). This unique connection, naturally associated with each metric g is, of course, the Levi-Cività connection of g. We therefore have a natural map

$$\mathrm{LC}: \mathrm{Met}(M) \longrightarrow \mathrm{Conn}(M); \quad g \longmapsto \omega_g \qquad 6.1.4,$$

$\forall g \in \mathrm{Met}(M)$. Since the Levi-Civitā connection ω_g is, in particular, metric, i.e. $d^{\omega}\hat{g} = 0$, it restricts to a connection (also called ω_g) in the bundle of g-orthonormal frames; in other words, the Levi-Civitā connection is the image (see (6.1)21) of a connection under the embedding of $O(M,g)$ (as a sub-bundle) into $GL(M)$.

It is well known that LC is equivariant with respect to the actions of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$ (by pullback) and on $\mathrm{Conn}(M)$ (by natural lift to Aut $GL(M)$ and then by pullback as described above), i.e.

$$LC \circ \phi^* = \hat{\phi}^* \circ LC \qquad 6.1.15,$$

$\forall \phi \in \mathrm{Diff}(M)$.

The natural geometric maps may be constructed using the Levi-Civitā map 6.1.14 together with the curvature map defined in (6.1)20. For example, we have $\mathrm{Riem} = \tau \circ \Omega \circ LC$: $\mathrm{Met}(M) \longrightarrow \mathrm{End}(\Omega^2(M)) \subseteq \Omega^2(\mathrm{AdGL}(M)) \cong \mathrm{End}(\Omega^2(M))$ which associates with each metric its Riemann curvature tensor field. Traces of $\mathrm{Riem}(g)$ give the standard Ricci map $\mathrm{Ric}: \mathrm{Met}(M) \longrightarrow S_2(M)$, and the scalar curvature map $\mathrm{Scal}: \mathrm{Met}(M) \longrightarrow C(M)$. Using 6.1.15 together with the behaviour of the maps Ω, τ and trace under the action of the diffeomorphism group, it follows that Riem, Ric and Scal are equivariant with respect to the actions of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$ and on $\Omega^2(\mathrm{AdGL}(M))$, $S_2(M)$ and $C(M)$ respectively, i.e. we have $F \circ \phi^* = \phi^* \circ F$ for F = Riem, Ric, Scal, and $\forall \phi \in \mathrm{Diff}(M)$.

When considering metrics, an important subspace of $\mathrm{Conn}(M)$ is obviously $\mathrm{Metric}(M) = \{\omega \in \mathrm{Conn}(M): \exists\ g \in \mathrm{Met}(M)$ with $d^{\omega}\hat{g} = 0\}$, the space of metric connections on M (so $\omega \in \mathrm{Metric}(M)$ if and only if the holonomy group of ω is a subgroup of $O(p,q)$). Then $\mathrm{Metric}(M) \cong LC(M) \times \Omega^2(TM)$, where $LC(M) \equiv LC(\mathrm{Met}(M))$ is the space of all Levi-Cività connections on M (note that the torsion map $T \equiv T|\mathrm{Metric}(M)\!:\!\mathrm{Metric}\,(M) \dashrightarrow \Omega^2(TM)$ is surjective). For more details concerning the map LC (for example, the question of the injectivity of its projection $\overline{LC}$: $\mathrm{Met}(M)_{/\mathbb{R}^+} \longrightarrow \mathrm{Conn}(M)$, see Schmidt [S5] and Hall [H1].

This concludes our trip through the definitions and main results in the theory of bundles and connections. As we have already remarked, the inter-relationship between metrics, frames and natural groups is explored further in Chapter Four.

6.2 Conformal Structure

The aim of this appendix is to present the main definitions and results in the theory of conformal structures on manifolds. Conformal structures are important in physics, especially in general relativity, and in geometry, and they appear in several sections of this thesis, particularly in section 1.5 and in Chapters Three and Four.

Conformal ideas arise in many areas of physics, for example in quantum field theory, but here we limit ourselves to some remarks concerning gravity and geometry. In differential geometry, a conformal structure may be regarded as a naturally defined subspace of the space of metrics (or geometries) on a given manifold, and a conformal change of metric may be regarded as the simplest non-homothetic deformation of the metric. Conformal structure also has deep interaction with complex geometry, especially in Riemann surface theory and in algebraic geometry.

In general relativity (and in Lorentzian geometry in general), conformal ideas are intimately linked with causal and null structures and thence with the theory of radiation. Appendix 6.3 gives the relevant definitions of conformally compactified spacetime and of null infinity which are necessary for discussions in Chapter Three. The interplay between null and conformal ideas is very physical since a conformal strucure is determined by its null vectors. Indeed, the signature of the metrics in the conformal equivalence class is determined by the topology of the space of null vectors at any event in spacetime. Also, conformal transformations are the most general ones preserving causality. Another area in which conformal ideas enter into general relativity is in the spin-boost-conformal formalism of Geroch, Held and Penrose [G 7] which we discuss in Chapters Two and Three and below in this section. Indeed, much of Penrose's spinorial and twistorial work has a conformal framework (see both volumes of Penrose and Rindler [P 11], [P 12] and references therein).

In the first part of this section notation and ideas are established by giving the relevant definitions and results. For ease of exposition we state definitions for positive definite metrics on oriented manifolds, but the extension to non-positive definite metrics and to non-oriented and non-orientable manifolds is clear. Most results are true for all manifolds of finite dimension, but we emphasize when a compactness or signature condition is required. The second part of this section lists useful formulae concerning conformal deformation. Some of these are standard and these are listed for convenience, but the others give the transformation of GHP and other spinor quantities under a general complex conformal rescaling - the latter have not appeared in the literature in this form before (but see Penrose and Rindler [P 11] and Ludwig [L 13] for related results). For standard definitions and results see Kobayashi [K 6] and Obata [O 1].

Our first definition will cover the standard concepts:

Definition (6.2)1: Let M be a manifold. Metrics $g_1, g_2 \in Met(M)$ are said to be (pointwise) conformal if there exists $f \in C^+(M)$ such that $g_1 = fg_2$. A conformal structure C on M is an equivalence class of pointwise conformal metrics. Let $Con(M) = Met(M)/C^+(M)$ denote the space of all conformal structures on M. Given $g \in Met(M)$, denote by C_g the conformal structure containing

g. A conformal manifold is a pair (M,C) where $C \in \mathrm{Con}(M)$. Metrics $g_1, g_2 \in \mathrm{Met}(M)$ are said to be conformally equivalent if there exists $f \in C^+(M)$, $\phi \in \mathrm{Diff}(M)$ such that $g_1 = f\phi^* g_2$. Given conformal manifolds (M,C), (M',C'), a map $\phi \in C(M,M')$ is said to be conformal if $\phi^* C' \subseteq C$. Conformal manifolds (M,C), (M',C') are said to be (conformally) equivalent if there exists a diffeomorphism ϕ of M onto M' which is conformal. Riemannian manifolds (M,g), (M',g') are said to be conformally equivalent if the corresponding conformal manifolds (M,C_g), $(M',C_{g'})$ are equivalent. A diffeomorphism $\phi \in \mathrm{Diff}(M)$ is said to be a conformeomorphism of the conformal manifold (M,C) if ϕ is conformal. The conformal group $\mathrm{Conf}(M,C)$ of the conformal manifold (M,C) is the group of all conformeomorphisms of (M,C). The Riemannian manifold (M,g) is said to be conformally flat if, for each $x \in M$, there exists a neighbourhood U of x and $f \in C^+(U)$ such that $(U, fg|U)$ is flat.

Some of the above concepts are brought together if we consider the action of the conformorphism group $\mathrm{Conf}(M) = \mathrm{Diff}(M) \ltimes C^+(M)$ (semi-direct product of $\mathrm{Diff}(M)$ and $C^+(M)$ where $\mathrm{Diff}(M)$ acts on $C^+(M)$ by pullback - see Chapter Four and Fischer and Marsden [F 5] for more details). $\mathrm{Conf}(M)$ acts on $\mathrm{Met}(M)$ in the usual manner; $((\phi,f),g) \mapsto f\phi_* g$, $\forall(\phi,f) \in \mathrm{Conf}(M)$, $g \in \mathrm{Met}(M)$, where $\phi_* = (\phi^{-1})^*$. Let $g \in \mathrm{Met}(M)$, then the stabilizer of g under $\mathrm{Conf}(M)$ is just the conformal group $\mathrm{Conf}(M,g) \equiv \mathrm{Conf}(M,C_g)$ of g. The orbit of g under $\mathrm{Conf}(M)$ is the space of metrics conformally equivalent to g. Note that the orbit of g under $C^+(M)$ is just the space of metrics pointwise conformal to g, i.e. it is the conformal structure C_g containing g (and, of course, the orbit of g under Diff(M) is the geometry on M containing g, i.e. the space of metrics isometric to g). In low dimensions we have special results. For example, $C^+(S^1)$ acts transitively on $\mathrm{Met}(S^1)$ (since $\dim S^1 = 1$) and $\mathrm{Conf}(S^2)$ acts transitively on $\mathrm{Met}(S^2)$ (by the uniformization theorem for Riemann surfaces - see Wolf [W 10]). In other words, $\mathrm{Met}(S^1)$ and $\mathrm{Met}(S^2)$ can be realized as homogeneous spaces; $\mathrm{Met}(S^1) \cong C^+(S^1)$ (since $\mathrm{stab}_{C^+(S^1)}(\mathrm{can}) = 1$) and $\mathrm{Met}(S^2) \cong \mathrm{Diff}(S^2) \ltimes C^+(S^2)/SO^\uparrow(1,3)$ (since $\mathrm{stab}_{\mathrm{Conf}(S^2)}(\mathrm{can}) \equiv \mathrm{Conf}(S^2,\mathrm{Can}) \cong SO^\uparrow(1,3)$ - see section 1.5).

For $\dim M \geq 3$ and M compact, Fischer and Marsden prove a slice theorem for the action of Conf(M) on Met(M). This is analogous to the Ebin-Palais slice theorem for the action of Diff(M) on Met(M) (see Ebin [E 4]).

Let $T^*\mathrm{Met}(M)$ denote the L^2-cotangent bundle of Met(M). This is equipped with the canonical (weak) symplectic form $\omega = -d\eta$, where η is the canonical 1-form. Fischer and Marsden show that the induced action of Conf(M) on $(T^*\mathrm{Met}(M),\omega)$ is symplectic and may be reduced using the Marsden-Weinstein technique (see [M 4]). For $\dim M = 3$ the reduced phase space for the action of Conf(M) on $T^*\mathrm{Met}(M)$ is a representation of the space of true gravitational degrees of freedom in the initial value problem of general relativity - another indication of the importance of conformal structure in gravity theory.

Given $g \in \mathrm{Met}(M)$, $\dim M \geq 3$, the conformal group, Conf(M,g) of g is a finite dimensional Lie group with Lie algebra given by $\mathrm{LConf}(M,g) = \{X \in \mathrm{Vect}(M)\colon L_X\, g = hg \text{ for some } h \in C(M)\}$. A vector field X on M is an element of LConf(M,g) if and only if X generates local 1-parameter groups of conformeomorphisms of (M,g), and such an element is called a conformal Killing vector field of (M,g). The Lie algebra of the conformorphism group of any manifold M is given by $LConf(M) = Vect(M) \oplus C(M)$ (where $\oplus$ denotes semi-direct sum of Lie algebras). For positive definite metrics on compact n-manifolds, $Conf(M,g)$ is generically compact - indeed, the only compact manifold M for which $Conf(M,g)$ can be non-compact is the sphere S^n with g in any of the conformal classes defined by a metric of constant (sectional) curvature. For example, (see section 1.5), $Conf(S^2,can) = SO^{\uparrow}(1,3)$, which is non-compact.

We now briefly consider other geometrical ways of studying conformal structures. In definition (6.2)1, a conformal structure C is a subspace of $Met(M)$ and $Con(M)$, the space of conformal structures, is a quotient of $Met(M)$ by the group $C^+(M)$. We may also regard a conformal structure C as a reduction of the frame bundle $GL^+(M)$ to the conformal group $CO(n) = \{a \in GL^+(n,\mathbb{R}) : a^T a = \lambda 1\!\!1_n$, some $\lambda \in \mathbb{R}^+\} \cong SO(n) \times \mathbb{R}^+$, i.e. C may be regarded as a $CO(n)$-structure on M (see definition (6.1)28) with corresponding sub-bundle

$$CO(n) \hookrightarrow CO(M,C) \xrightarrow{\pi_M} M \qquad 6.2.1,$$

of $GL^+(M)$.

Definition (6.2)2: Let $C \in Con(M)$ correspond to the principal $CO(n)$-bundle $CO(M,C)$. Then $CO(M,C)$ is called the conformal frame bundle of the conformal manifold (M,C).

The conformal frame bundle consists of all frames comprising pairwise orthogonal tangent vectors of equal length (relative to any $g \in C$), i.e. $CO(M,C) = \{u \in GL^+(M)$: $\exists\, g \in C$ with

$u \in SO(M,g)\} = \bigcup_{g \in C} SO(M,g)$. The canonical 1-form θ_M on $GL^+(M)$ restricts to a 1-form (also called θ_M) on $CO(M,C)$, i.e. $\theta_M \in \Omega^1_{\rho(1,0)}(CO(M,C),\mathbb{R}^n)$. See section 1.5 for a discussion of connections and torsion in the conformal frame bundle.

We know that reductions of $GL^+(M)$ to the group $CO(n)$ are in bijective correspondence with the sections of the bundle associated to $GL^+(M)$ via the action of $GL^+(n,\mathbb{R})$ on the homogeneous space $GL^+(n,\mathbb{R})/CO(n)$, i.e. $Con(M) \cong \Gamma(GL^+(M)_{CO(n)})$, where $GL^+(M)_{CO(n)} = GL^+(M) \times_{GL^+(n,\mathbb{R})} (GL^+(n,\mathbb{R})/CO(n))$ (see section 6.1 - note that since we are developing the ideas for oriented manifolds we use the oriented frame bundle $GL^+(M)$. The theory for non-oriented manifolds is identical, but with $GL^+(M)$ replaced by $GL(M)$ and $SO(n)$ replaced by $O(n)$).

There is also a principal $\mathbb{R}^+$-bundle associated with every conformal structure: Let $C \in Con(M)$ and let $\mathbb{R}^+(M,C) = \{g(x): g \in C, x \in M\}$. We have the free $\mathbb{R}^+$-action given by $(g(x),r) \mapsto rg(x)$ and the projection $\pi^+: \mathbb{R}^+(M,C) \to M$; $g(x) \mapsto x$. The space $\mathbb{R}^+(M,C)$ is then the total space of a principal $\mathbb{R}^+$-bundle over M. Note that this bundle is trivializable since each metric in the conformal structure gives a global section. $\mathbb{R}^+(M,C)$ may be regarded as a line sub-bundle of $\odot^2 T^*M$, the bundle of symmetric covariant tensors of rank two.

Definition (6.2)3: Let $C \in Con(M)$. The conformal line bundle of the conformal manifold (M,C) is the principal $\mathbb{R}^+$-fibration:

$$\mathbb{R}^+ \hookrightarrow \mathbb{R}^+(M,C) \xrightarrow{\pi^+} M \qquad 6.2.2.$$

In fact the bundle π^+ characterizes the conformal structure C since $C = \Gamma(\pi^+)$. Note that there exists a natural principal bundle homomorphism f_C: $CO(M,C) \to \mathbb{R}^+(M,C)$ given by $u \mapsto g(\pi_M(u))$, where $g \in C$ is the unique metric such that $u \in SO(M,g)$. Conformal connections then give rise to connections in $\mathbb{R}^+(M,C)$ (see definition (6.1)21).

Given any action of $CO(n)$ on a manifold F, we may form the bundle $CO(M,C) \times_{CO(n)} F$ associated with the conformal frame bundle. For example, see the discussion in section 1.5 where the action is a representation of $CO(n)$ on a vector space. A natural family of line bundles associated with $CO(M,C)$ may be defined as follows: for $w \in \mathbb{R}$, define $\hat{w} \in \mathrm{Hom}(CO(n),\mathbb{R}^+)$ by $\hat{w}(a,r) = (\det(ar))^{w/n}$ $= r^w$, $\forall (a,r) \in CO(n)$. This defines a representation of $CO(n)$ on $\mathbb{R}$ and we let $\mathbb{R}_w \equiv \mathbb{R}_w(M,C)$ denote the associated line bundle.

<u>Definition (6.2)4</u>: Sections of $\mathbb{R}_w$ are called <u>functions of conformal weight w</u>. (Cf. definition (1.5)3).

Each metric $g \in C$ induces a trivialization $j_{g,w}$: $\mathbb{R}_w \longrightarrow \mathbb{R}_o$ $= M \times \mathbb{R}$; $[(u,t)] \mapsto (\pi_M(u),\ r^{-\frac{1}{2}w}t)$, where $u \in SO(M,rg), \forall [(u,t)] \in \mathbb{R}_w$. Deforming the metric g within C gives rise to trivializations $j_{fg,w}$, where $j_{fg,w}|(\mathbb{R}_w)_x = f(x)^{-\frac{1}{2}w} j_{g,w}$. We have corresponding linear maps $k_{g,w}$: $\Gamma(\mathbb{R}_w) \longrightarrow \Gamma(\mathbb{R}_o) = C(M)$ and these may be used to construct conformally invariant differential operators. For example, we may define the <u>conformal Laplacian</u> Δ_C^w associated with each $w \in \mathbb{R}$ and $C \in \mathrm{Con}(M)$. This is given by Δ_C^w $= k^{-1}_{g,w+2} \circ Y_g \circ k_{g,w}$, where $Y_g = \Delta_g + \frac{(n-2)}{4(n-1)} \mathrm{Scal}(g)$: $C(M) \to C(M)$, for any $g \in C$, so that $\Delta_C^w : \Gamma(\mathbb{R}_w) \longrightarrow \Gamma(\mathbb{R}_{w+2})$. Note that the definition of Δ_C^w does not depend on the choice of representative metric $g \in C$ because of the transformation properties of $k_{g,w}$, $\mathrm{Scal}(g)$ and $\Delta_g \equiv \delta_g \circ d$ under conformal deformation of g (see equations 6.2.7 and 6.2.11 for the latter two). See Parker and Rosenberg [P 3], Branson [B24] and Ørsted [Ø 4] for more details on conformally invariant differential operators.

For convenience we now list useful formulae concerning the conformal transformation properties of quantities of interest in geometry and in general relativity. The first list is standard and concerns geometric maps F on the space of metrics $\mathrm{Met}(M)$. Our notation is as follows: $g \in \mathrm{Met}(M)$, $f \in C^+(M)$, $\hat{g} = fg$, $F \equiv F(g)$ and $\hat{F} \equiv F(\hat{g})$ for any map F with domain $\mathrm{Met}(M)$. We also make use of the symmetrized tensor product $\odot$ and the Kulkarni-Nomizu product , $\oslash : S_2(M) \times S_2(M) \longrightarrow \Gamma(\otimes^4 T^* M)$ defined by $(h \oslash k)(U,V,W,X) = h(U,W)k(V,X) + h(V,X)k(U,W) - h(U,X)k(V,W) - h(V,W)k(U,X)$, $\forall U,V,W,X \in \mathrm{Vect}(M)$, $h,k \in S_2(M)$. We also let $\phi = \frac{1}{2} d \log f \in \Omega^1(M)$ as this considerably simplifies the formulae:

For $LC: \mathrm{Met}(M) \longrightarrow \mathrm{Conn}(M)$, we express the transformation in terms of the associated covariant derivative acting on $\mathrm{Vect}(M)$. Let $X,Y \in \mathrm{Vect}(M)$, then:

$$\hat{\nabla}_X Y = \nabla_X Y + \phi(X)Y + \phi(Y)X - g(X,Y)\phi^{\#} \qquad 6.2.3.$$

Instead of using $\mathrm{Riem}: \mathrm{Met}(M) \longrightarrow \Omega^2(\mathrm{AdGL}(M)) \subseteq \Gamma((\otimes^3 T^* M) \otimes TM)$, we use the totally covariant form $\underline{\mathrm{Riem}}: \mathrm{Met}(M) \longrightarrow \Gamma(\otimes^4 T^* M)$, and we have

$$\underline{\hat{\mathrm{Riem}}} = f(\underline{\mathrm{Riem}} - g \oslash (\nabla\phi - \phi\odot\phi + \tfrac{1}{2}|\phi|^2 g)) \qquad 6.2.4.$$

The conformally invariant part of the Riemann curvature tensor is the Weyl tensor, so that $\mathrm{Weyl}: \mathrm{Met}(M) \longrightarrow \Gamma((\otimes^3 T^* M) \otimes TM)$ is invariant under the action of $C^+(M)$:

$$\hat{\mathrm{Weyl}} = \mathrm{Weyl} \qquad 6.2.5.$$

Curvature information not contained in the conformally invariant Weyl tensor is stored in the Ricci tensor and, for Ric: Met(M) $\rightarrow S_2(M)$, we have:

$$\hat{\mathrm{Ric}} = \mathrm{Ric} - (n-2)(\nabla\phi - \phi\otimes\phi) + (\delta\phi - (n-2)|\phi|^2)g \qquad 6.2.6.$$

Next, we have the scalar curvature function, Scal: Met(M) $\rightarrow$ C(M), and, by taking the $\hat{g}$-trace of equation 6.2.6, we obtain:

$$\hat{\mathrm{Scal}} = f^{-1}(\mathrm{Scal} + 2(n-1)\delta\phi - (n-2)(n-1)|\phi|^2) \qquad 6.2.7.$$

The canonical measure, vol: Met(M) $\rightarrow {}^*\Omega^n(M)$, transforms as follows:

$$\hat{\mathrm{vol}} = f^{\frac{n}{2}}\,\mathrm{vol} \qquad 6.2.8,$$

and the associated Hodge star operator: Met(M) $\rightarrow$ End(Ω(M)) restricted to $\Omega^k(M)$ is:

$$\hat{*} = f^{\frac{n}{2}-k} * \qquad 6.2.9.$$

Using 6.2.9 we obtain the codifferential restricted to $\Omega^k(M)$

$$\hat{\delta} = f^{-1}(\delta - (n-2k)\iota_{\phi^\#}) \qquad 6.2.10,$$

and thence the Laplace-Beltrami operator, $\Delta = d \circ \delta + \delta \circ d$, restricted to $\Omega^k(M)$:

$$\hat{\Delta} = f^{-1}(\Delta - (n-2k)d \circ \iota_{\phi^\#} - (n-2k-2)\iota_{\phi^\#} \circ d$$
$$+ 2(n-2k)\phi_\Lambda\iota_{\phi^\#} - 2\phi_\Lambda\delta) \qquad 6.2.11.$$

In conformal geometry the important metric dependent quantities are those which only depend on the conformal class of the metric, the most important of which is the Weyl tensor (6.2.5). Indeed, for $n \geq 4$, the Weyl tensor is the sole obstruction to conformal flatness (see definition (6.2)1) and we have that the Riemannian manifold (M,g) is conformally flat if and only if $\mathrm{Weyl}(g) = 0$ (see Eisenhart [E18]). For $n = 3$, the Weyl tensor vanishes identically (so the curvature is determined by the Ricci tensor alone) and the obstruction to conformal flatness is known as the <u>Weyl-Schouten tensor field</u> - this is an $O(p,q)$-invariant component of the covariant derivative of the Riemann tensor field (see Schouten [S29]). Again in dimension two, the Weyl tensor vanishes identically (and the curvature is determined by the scalar curvature alone), and in this case there is no obstruction to conformal flatness:- Any two dimensional Riemannian manifold (M, g) is conformally flat (See Kobayashi and Nomizu [K7]).

Other conformal invariants (i.e. functions on $\mathrm{Met}(M)$ which project to $\mathrm{Con}(M) \equiv \mathrm{Met}(M)/C^+(M))$ may be obtained from equations 6.2.3 - 6.2.11. We have already mentioned the <u>Yamabe operator</u> $Y;\ g \longrightarrow \Delta_g + \frac{(n-2)}{4(n-1)}\mathrm{Scal}(g):\ C(M) \longrightarrow C(M)$ above and another example is the Hodge star operator acting on $\Omega^k(M)$ where M is of dimension $2k$. The latter example is of great importance in the study of self-duality (see Atiyah et al. [A30]).

In general relativity, the interaction between conformal and null structures is very useful. For instance, let (M,g) be a pseudo-Riemannian manifold (i.e. g is not definite) and let $c\colon I \longrightarrow M$ be a null geodesic on M. Then the curve $\alpha^* c$ is a null geodesic for the conformally related pseudo-Riemannian manifold (M, fg) so long as $\alpha\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$\alpha'' + (\nabla_{\dot{c}} \log(c^*f))\alpha' = 0$. In other words, the null geodesics of (M,g) are precisely the null geodesics of (M,fg) (up to a change of parameterization). See Friedrich and Schmidt [F 11] for a discussion of geodesics, conformal structures and general relativity.

We may also study the change in quantities associated with embeddings under a conformal deformation of metric. For example. suppose $k \in \mathrm{Emb}(M,N)$ and $g \in \mathrm{Met}(N)$. Let K, $\hat{K}$ be the second fundamental forms (see section 2.1) of the Riemannian embeddings $k\colon (M,k^*g) \longrightarrow (N,g)$, $k\colon (M,(k^*f)k^*g) \longrightarrow (N,fg)$ respectively. Then, using the definition of second fundamental form, it is straightforward to show that K, $\hat{K}$ are related as follows: $\hat{K}(X,Y) = K(X,Y) - (k^*g)(X,Y)(df)^{\perp}$, $\forall X,Y \in \mathrm{Vect}(M)$, where $(df)^{\perp}$ denotes the normal component of the gradient of f, $(df)^{\#}$.

We refer the reader to Besse [B 10], Kobayashi [K 6] and Weber and Goldberg [W 3] for more general details concerning conformal structures, but now we turn to the specific question of four dimensional Lorentzian geometry. In particular we now discuss the conformal transformation of spinor quantities in the GHP formalism which we considered in section 2.3. The GHP formalism provides a framework for performing calculations involving the components of a g-spin connection on a spacetime (M,g). If the metric g is deformed conformally then the corresponding spin connection (see section 1.3) also changes which in turn induces a transformation of the GHP spin coefficients. A knowledge of how these quantities transform is obviously essential when performing GHP calculations in a conformally rescaled spacetime. We present here a list of the transformation properties of the basic GHP spin connection coefficients and of various other useful spinor quantities.

We consider a general complex rescaling so that our formulae are more general than those in Penrose and Rindler [P 11]. For a treatment of conformal transformation of the (less geometrical) NP quantities, see Ludwig [L 13]. Note, however, that Ludwig discusses complex conformal rescalings within the context of general $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$ transformations (see also section 2.3).

We use the standard GHP notation (see [G 7]) as introduced in section 2.3 and we consider a general complex rescaling of ε given by some fixed $f \in C(M, \mathbb{C}^*)$. We refer the reader to section 1.8 for details on how such a rescaling influences the geometry of spacetime. The possible importance of allowing complex rescalings of the symplectic form ε rather than just real rescalings has been noted by Penrose [P 9]. Penrose remarks that if we wish to maintain the conformal invariance of the massless free field equations and of the twistor equation, then any complex conformal deformation must be accompanied by a change in the torsion of the connection. In particular, if the connection is initially torsion-free, then torsion must appear. For speculation on the possible physical significance of the introduction of torsion in this manner, see the Penrose paper cited.

Our formulae are considerably simplified if we introduce the map $\alpha\colon \mathbb{R}^2 \times C(M,\mathbb{C}^*) \to C(M,\mathbb{C}^*)$; $((p,q),f) \mapsto f^p\, \bar{f}^q$. This map has the following useful properties:- $\alpha((p,q),\cdot)\colon C(M,\mathbb{C}^*) \to C(M,\mathbb{C}^*)$ is a homomorphism, indeed an automorphism for $p^2 \neq q^2$ ($C(M,\mathbb{C}^*)$ with pointwise multiplication as group structure) and $\alpha(\cdot,f)\colon \mathbb{R}^2 \to C(M,\mathbb{C}^*)$ is also a homomorphism. In addition, for any derivation D on $C(M,\mathbb{C}^*)$, we have $D(\alpha((p, q),f))$ $= (pD\log f + qD\log\bar{f})\alpha((p, q),f)$. Finally, we have

$\alpha((p,q),f) = \alpha((q,p),\bar{f})$. For convenience we write $f(p,q) \equiv \alpha((p,q),f)$, $\forall(p,q) \in \mathbb{R}^2$, $f \in C(M,\mathbb{C}^*)$. Note that $|f|^2 \equiv f(1,1)$ and $2i \arg f = \log f(1,-1)$, $\forall f \in C(M,\mathbb{C}^*)$.

Let $f \in C(M,\mathbb{C}^*)$ and $\hat{\varepsilon}_{AB} = f(1,0)\varepsilon_{AB} \equiv f\,\varepsilon_{AB}$. Then, necessarily, we must have $\hat{\varepsilon}_{A'B'} = f(0,1)\varepsilon_{A'B'}$, $\hat{\varepsilon}^{AB} = f(-1,0)\varepsilon^{AB}$, $\hat{\varepsilon}^{A'B'} = f(0,-1)\varepsilon^{A'B'}$, $\hat{g}_{ab} = f(1,1)g_{ab} \equiv |f|^2 g_{ab}$ and $\hat{g}^{ab} = f(-1,-1)g^{ab}$. Note that in the current notation the (real) pointwise conformal rescaling of the spacetime metric is given by $|f|^2$ rather than by $f(\in C^+(M))$ as elsewhere in this section.

We now consider a general complex rescaling of the spin frame $\tilde{u} = \{o_A, \iota_A\}$ given by $\hat{o}^A = f(p,q)o^A$ for some fixed $(p,q) \in \mathbb{R}^2$. Then, since $\varepsilon^{AB} = o^A\iota^B - o^B\iota^A$ and $o_A\iota^A = 1$, we must have $\hat{\iota}^A = f(-1-p, -q)\iota^A$, $\hat{o}_A = f(p+1,q)o_A$ and $\hat{\iota}_A = f(-p,-q)\iota_A$ (together with the complex conjugate transformations $\hat{\bar{o}}^{A'} = f(q,p)\bar{o}^{A'}$, etc.). We have the GHP prime operation given by $(o^A)' = i\iota^A$, $(\iota^A)' = io^A$ and this induces the map $(p,q) \mapsto (-1-p,-q)$ of $\mathbb{R}^2$ onto itself. Suppose ω is a quantity of well defined GHP type, then $\hat{\omega} = f(ap+bq+c,\ bp+aq+d)\omega$ where $\{a,b\}$ is the type of ω.

The spin frame $\tilde{u}$ projects to the null tetrad $u = \eta(\tilde{u}) = \{l,n,m,\bar{m}\}$ in the usual manner and this transforms as follows: $\hat{l}^a = f(p+q,p+q)l^a$, $\hat{l}_a = f(p+q+1,\ p+q+1)l_a$, $\hat{n}^a = f(-1-p-q,\ -1-p-q)n^a$, $\hat{n}_a = f(-p-q,\ -p-q)n_a$, $\hat{m}^a = f(p-q,\ -1-p+q)m^a$, $\hat{m}_a = f(p-q+1,\ -p+q)m_a$, $\hat{\bar{m}}^a = f(-1-p+q,\ p-q)\bar{m}^a$ and $\hat{\bar{m}}_a = (-p+q,\ p-q+1)\bar{m}_a$, using the usual null tetrad inner products:- $l\cdot n = 1 = -\,m\cdot\bar{m}$ with all other inner products vanishing.

We also use formula 6.2.3, $\hat{\nabla}_X Y = \nabla_X Y + \phi(X)Y + \phi(Y)X - g(X,Y)\phi^{\#}$, where $\phi = \frac{1}{2}d\log|f|^2$, and $\nabla_X f(p,q) = f(p,q)(p\nabla_X \log f + q\nabla_X \log\bar{f})$, $\forall X,Y \in \mathrm{Vect}(M)$. Note that, as usual, we use the notation $l,n,m,\bar{m}$

to mean local sections of the frame bundle as well as for a null tetrad at a particular point. Let us first present a couple of examples of how the calculations are performed, and then we give the full list of transformation formulae.

First consider the spin coefficient $\kappa = \langle m^b, \nabla_l l\rangle$. Under the rescalings introduced, we have $\hat{\kappa} = \langle \hat{m}^b, \hat{\nabla}_{\hat{l}}\hat{l}\rangle$ since both the metric and the spin frame undergo rescaling. Hence $\hat{\kappa} = \langle \hat{m}^b, \nabla_{\hat{l}}\hat{l} + 2\phi(\hat{l})\hat{l}\rangle$
$= \langle f(p-q+1, -p+q)m^b,\ \ f(p+q,p+q)\nabla_l(f(p+q,p+q)l)\rangle$
$= f(2p+1,2q)\langle m^b, \nabla_l(f(p+q,p+q))l\rangle + f(2p+1,2q)f(p+q,p+q)\langle m^b, \nabla_l l\rangle$
$= f(3p+q+1,p+3q)\kappa$. Similarly, $\tau = \langle m^b, \nabla_n l\rangle$ and so we have
$\hat{\tau} = \langle \hat{m}^b, \hat{\nabla}_{\hat{n}}\hat{l}\rangle = \langle \hat{m}^b, \nabla_{\hat{n}}\hat{l} + \phi(\hat{n})\hat{l} + \phi(\hat{l})\hat{n} - g(\hat{l},\hat{n})\phi^{\#}\rangle$
$= \langle f(p-q+1,-p+q)m^b,\ \ f(-1-p-q,-1-p-q)\nabla_n(f(p+q,p+q)l) - f(-1,-1)\phi^{\#}\rangle$
(since $g(\hat{l},\hat{n}) = f(-1,-1)\hat{g}(\hat{l},\hat{n}) = f(-1,-1)$)
$= f(-2q,-2p-1)\langle m^b, \nabla_n(f(p+q,p+q)l\rangle - f(p-q,-p+q-1)\langle m^b, \phi^{\#}\rangle$
$= f(p-q,-p+q-1)(\langle m^b, \nabla_n l\rangle - \langle m^b, \phi^{\#}\rangle)$.
Now $\langle m^b, \phi^{\#}\rangle = \langle \phi, m\rangle = \langle \tfrac{1}{2}d\log|f|^2, m\rangle = m(\log|f|) = \eth\log|f|$, and hence
$\hat{\tau} = f(p-q,-p+q-1)(\tau - \eth\log|f|)$.

The full list is as follows:

$$\hat{\kappa} = f(3p+q+1,\ p+3q)\kappa \qquad 6.2.12,$$

$$\hat{\sigma} = f(3p-q+1,\ -p+3q-1)\sigma \qquad 6.2.13,$$

$$\hat{\rho} = f(p+q,p+q)(\rho - \text{Þ}\ \log|f|) \qquad 6.2.14,$$

$$\hat{\tau} = f(p-q,-p+q-1)(\tau - \eth\ \log|f|) \qquad 6.2.15,$$

$$\hat{\beta} = f(p-q,-p+q-1)(\beta + (p+\tfrac{3}{4})\eth\ \log f + (q+\tfrac{1}{4})\ \eth\ \log\bar{f}) \qquad 6.2.16,$$

$$\hat{\varepsilon} = f(p+q,p+q)(\varepsilon + (p+\tfrac{3}{4})\ \text{Þ}\ \log f + (q+\tfrac{1}{4})\ \bar{\text{Þ}}\ \log\bar{f}) \qquad 6.2.17,$$

$$\hat{\kappa}' = f(-3p-q-2,\ -p-3q-1)\kappa' \qquad 6.2.18,$$

$$\hat{\sigma}' = f(-3p+q-2,\ p-3q)\sigma' \qquad 6.2.19,$$

$$\hat{\rho}' = f(-p-q-1,\ -p-q-1)(\rho' - \text{þ}'\log|f|) \qquad 6.2.20,$$

$$\hat{\tau}' = f(-p+q-1,\ p-q)(\tau' - \eth'\log|f|) \qquad 6.2.21,$$

$$\hat{\beta}' = f(-p+q-1,p-q)(\beta' + (-p-\tfrac{1}{4})\,\eth'\log f + (-q+\tfrac{1}{4})\,\eth'\log \bar{f}) \qquad 6.2.22,$$

$$\hat{\varepsilon}' = f(-p-q-1,-p-q-1)(\varepsilon'+(-p-\tfrac{1}{4})\text{þ}'\log f + (-q+\tfrac{1}{4})\,\text{þ}'\log\bar{f}) \qquad 6.2.23.$$

Note that $\hat{(\omega)}' = \hat{(\omega')}$ and similarly $\bar{\hat{\omega}} = \hat{\bar{\omega}}$, so that the transformation properties of $\bar{\kappa},\ldots,\bar{\kappa}',\ldots$ may also be written down (using the fact that $\overline{f(p,q)} = f(q,p)$).

The spin structure s_g which we have chosen gives us the principal $SL(2,\mathbb{C})$-bundle $\widetilde{SO}(M,g)$ over spacetime as in section 1.7. This bundle may be regarded as the complex symplectic frame bundle corresponding to the symplectic vector bundle $(S(s_g),\varepsilon)$. When we consider complex conformal rescalings $\varepsilon \mapsto f\varepsilon$ of the symplectic form ε, it is appropriate to consider the corresponding complex conformal symplectic structure with structure group $SL(2,\mathbb{C}) \times \mathbb{C}^*$ (see section 1.8 for more discussion of this matter and Kobayashi [K 6] for general remarks concerning conformal symplectic structures). Corresponding to the representations $(A,z) \mapsto z^r\,\bar{z}^s$ of $SL(2,\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}$, $(r,s) \in \mathbb{R}^2$, we have the two parameter family $\mathbb{C}_{(r,s)}$ of complex line bundles over M. These are analogous to the bundles $\mathbb{C} \hookrightarrow E(s,w) \to S^2$ of section 1.5 and also to the real line bundles $\mathbb{R} \hookrightarrow \mathbb{R}_w \to M$ introduced above in this section. Now consider a spinor field $\lambda \in \Gamma(S(s_g) \otimes_{\mathbb{C}} \mathbb{C}_{(-r,-s)})$ which, when regarded as an equivariant map on the total space, transforms as

$\hat{\lambda}^A = f(r,s)\lambda^A$ with components $\hat{\lambda}_o = f(r+p+1,\ s+q)\lambda_o$, $\hat{\lambda}_1 = f(r-p,\ s-q)\lambda_1$ with respect to the spin frame $\{\hat{o}_A, \hat{\iota}_A\}$. We now introduce some differential operators which generalize the spin coefficients of GHP. These operators arise from quantities defined by J. Vickers (unpublished).

Definition (6.2)5: Let λ be as above and define differential operators K, S, R and T by:

$$K\lambda = o^A \text{þ}\, \lambda_A \equiv \text{þ}\, \lambda_o + \kappa\lambda_1 \qquad 6.2.24,$$

$$S\lambda = o^A \eth\, \lambda_A \equiv \eth\, \lambda_o + \sigma\lambda_1 \qquad 6.2.25,$$

$$R\lambda = o^A \eth'\lambda_A \equiv \eth'\lambda_o + \rho\lambda_1 \qquad 6.2.26,$$

$$T\lambda = o^A \text{þ}'\lambda_A \equiv \text{þ}'\lambda_o + \tau\lambda_1 \qquad 6.2.27.$$

Since these operators depend on the spin frame, we may regard them as operators on the space of equivariant functions on the total space of the conformal symplectic bundle. The combinations 6.2.24 - 6.2.27 arise naturally in many calculations, for example in Chapter Three of this thesis. We now consider the conformal transformation properties of these operators. For fixed λ, let us write $K \equiv K\lambda$, etc. The calculations are analogous to those for the GHP spin coefficients and the latter are obtained as a special case when we put $\lambda^A = o^A$ and $(r,s) = (p,q)$:

$$\hat{K} = f(r+2p+q+1, s+p+2q)(K+((r+\tfrac{1}{4})\ \text{þ} \log f + (s-\tfrac{1}{4})\ \text{þ} \log\bar{f})\lambda_o) \qquad 6.2.28,$$

$$\hat{S} = f(r+2p-q+1, s-p+2q-1)(S+((r+\tfrac{1}{4})\ \eth \log f + (s-\tfrac{1}{4})\ \eth \log\bar{f})\lambda_o) \qquad 6.2.29,$$

$$\hat{R} = f(r+q, s+p)(R + ((r+\tfrac{3}{4})\ \eth'\log f + (s+\tfrac{1}{4})\ \eth'\log\bar{f})\lambda_o - (\text{þ} \log|f|)\lambda_1) \qquad 6.2.30,$$

$$T = f(r-q,s-p-1)(T + ((r+\tfrac{3}{4})\ \text{þ}'\log f + (s+\tfrac{1}{4})\ \text{þ}'\log\bar{f})\lambda_o - (\eth\log|f|)\lambda_1) \qquad 6.2.31.$$

Note that the types of K, S, R, T are {2,1}, {2,-1}, {0,1}, {0,-1} respectively. Useful quantities of type {0,0} may be defined as follows:

$$P = \bar{\lambda}_{o'}(T - iR') \qquad 6.2.32,$$

$$Q = \lambda_o(\bar{T} - i\bar{R}') \qquad 6.2.33,$$

with transformation properties given by:

$$\hat{P} = f(r+s.r+s)(P + ((r+\tfrac{1}{4})\ \text{þ}'\log f + (s-\tfrac{1}{4})\ \text{þ}'\log\bar{f}|\lambda_o|^2$$
$$+ ((r+\tfrac{1}{4})\eth\log f + (s-\tfrac{1}{4})\eth\log\bar{f})\bar{\lambda}_{o'}\ \lambda_1) \qquad 6.2.34,$$

$$Q = f(r+s,r+s)(Q + ((s+\tfrac{3}{4})\ \text{þ}'\log f + (r+\tfrac{5}{4})\ \text{þ}'\log\bar{f})|\lambda_o|^2$$
$$- ((s+\tfrac{3}{4})\eth'\log f + (4+\tfrac{5}{4})\eth'\log\bar{f})\lambda_o\bar{\lambda}_{1'}) \qquad 6.2.35.$$

Certain physical quantities may be written down in terms of P, Q. For example, the <u>Ludvigsen-Vickers quasi-local momentum integrand</u> is given by

$$J = 4\phi_{AB}\, o^A\, \iota^B = -J' \qquad 6.2.36,$$

where

$$2\phi_{AB} = \lambda_{(A}\nabla_{B)}{}^{C'}\bar{\lambda}_{C'} - \lambda_{C'}\nabla^{C'}{}_{(A}\lambda_{B)} \qquad 6.2.37.$$

A long but straightforward calculation now shows that J may be expressed as $J = \lambda_o(\bar{T} - i\bar{R}') - \bar{\lambda}_{o'}(T - iR') + \lambda_1(\bar{R} - i\bar{T}') + \bar{\lambda}_{1'}(R - iT') = (Q + P') - (P + Q') = A - A'$, where $A = Q + P'$.

The change in the Ludvigsen-Vickers integrand induced by a complex conformal deformation may be calculated using 6.2.34 and 6.2.35, and we obtain:

$$\hat{J} = f(r+s,r+s)(J + (-(r-s-\tfrac{1}{2})\,\text{Þ}'\log f + (r-s+\tfrac{3}{2})\,\text{Þ}'\log\bar{f})|\lambda_o|^2$$

$$+ ((r-s-\tfrac{1}{2})\,\text{Þ}\log f - (r-s+\tfrac{3}{2})\,\text{Þ}\log\bar{f})|\lambda_1|^2$$

$$+ (-(r-s-\tfrac{1}{2})\eth\log f + (r-s+\tfrac{3}{2})\eth\log\bar{f})\bar{\lambda}_{o'}\lambda_1$$

$$+ ((r-s-\tfrac{1}{2})\eth'\log f - (r-s+\tfrac{3}{2})\eth'\log\bar{f})\lambda_o\bar{\lambda}_{1'}) \qquad 6.2.38.$$

To actually calculate quasi-local momentum, we only need the real part of J and this transforms as follows:

$$\widehat{\mathrm{Re}(J)} = f(r+s,r+s)(\mathrm{Re}(J) + 2(|\lambda_o|^2\,\text{Þ}' - |\lambda_1|^2\,\text{Þ})\log|f|$$

$$+ i(2r-2s+1)(\lambda_o\bar{\lambda}_{1'}\,\eth' - \bar{\lambda}_{o'}\lambda_1\,\eth)\arg f) \qquad 6.2.39.$$

Differential operators of physical importance may also be expressed in terms of the quantities defined above and their conformal transformation properties deduced. For example, let $S_{A'AB} = \nabla_{A'(A}\lambda_{B)}$ and $A_{A'AB} = \nabla_{A'[A}\lambda_{B]}$. Then:

$$S_{A'AB} = -iK'o_A\bar{o}_{A'}o_B - K\iota_A\bar{\iota}_{A'}\iota_B + iS'o_A\bar{\iota}_{A'}o_B + S\iota_A\bar{o}_{A'}\iota_B$$

$$-\tfrac{1}{2}(T-iR')(o_A\bar{o}_{A'}\iota_B + \iota_A\bar{o}_{A'}o_B) + \tfrac{1}{2}(R-iT')(o_A\bar{\iota}_{A'}\iota_B + \iota_A\bar{\iota}_{A'}o_B)$$

$$6.2.40,$$

and

$$A_{A'AB} = \tfrac{1}{2}(T + iR')(\iota_A \bar{o}_{A'} o_B - o_A \bar{o}_{A'} \iota_B)$$

$$+ \tfrac{1}{2}(R + iT')(o_A \bar{\iota}_{A'} \iota_B - \iota_A \bar{\iota}_{A'} o_B) \qquad 6.2.41.$$

Using 6.2.28 - 6.2.31, we may show, for instance, that $\hat{S}_{A'AB} = f(r+1,s)S_{A'AB}$ + (terms which vanish for $r = s = 0$ and f $\mathbb{R}$-valued) which is just the usual conformal invariance of the twistor equation $S_{A'AB} = 0$. By contracting 6.2.41, we obtain the operator appearing in the neutrino equation, namely

$$A^{A'A}{}_{A} \equiv \nabla^{AA'}\lambda_A = (T + iR')\bar{o}^{A'} - (R + iT')\bar{\iota}^{A'} \qquad 6.2.42.$$

An equation used in Chapter Three in the Ludvigsen-Vickers quasi-local momentum definition is the null limit of the Sen-Witten equation. The corresponding Sen-Witten operator is given by $\overline{W}_{A'} = \nabla_{AA'}\lambda^A - t_{AA'}\nabla_t\lambda^A$, where t is the timelike future directed normal to the spacelike hypersurface $\Sigma \hookrightarrow M$. It is easily shown that

$$\overline{W}_{A'} = ((\tfrac{1}{2}t^2 - 1)T + (t\cdot n)^2K - iR')\bar{o}_{A'}$$

$$- i((\tfrac{1}{2}t^2 - 1)T' + (t\cdot l)^2K' + iR)\bar{\iota}_{A'} \qquad 6.2.43,$$

and, assuming $\hat{t} = f(-\tfrac{1}{2},-\tfrac{1}{2})t$, so that $\hat{t}^2 = t^2$, we find

$$\hat{\overline{W}}_{A'} = f(r,s)(\overline{W}_{A'} + x_{A'} + (x_{A'})')$$

where

$$x_{A'} = (((r+\tfrac{3}{4})\, \text{þ}'\log f + (s+\tfrac{1}{4})\, \text{þ}'\log\bar{f})\lambda_o - (\eth\log|f|)\lambda_1)(\tfrac{1}{2}t^2 - 1)$$

$$+ ((r+\tfrac{1}{4})\, \text{þ}\log f + (s-\tfrac{1}{4})\, \text{þ}\log\bar{f})\lambda_o(t\cdot n)^2$$

$$+ ((r+\tfrac{3}{4})\eth\log f + (s+\tfrac{1}{4})\eth\log\bar{f})\lambda_1 - (\text{þ}'\log|f|)\lambda_o \qquad 6.2.44.$$

We remark that if we are interested in a small number of quantities, it is often possible to choose the parameters p,q,r,s in a particular manner so as to simplify as many transformation formulae as possible. For example, the Hawking gauge (Hawking [H 4]) is often utilized. This is given by $\kappa = \varepsilon = \rho - \bar{\rho}' = \rho' - \bar{\rho}' = 0$, $\tau = -\bar{\tau}' = \beta - \bar{\beta}'$. Suppose we let $p+q+1 = 0$ and let f be $\mathbb{R}$-valued. Then, using 6.2.12 - 6.2.23, we see $\hat{\kappa} = f^{-3}\varepsilon$, $\hat{\varepsilon} = f^{-2}\varepsilon$, $(\rho - \bar{\rho})^{\wedge} = f^{-2}(\rho - \bar{\rho})$, $(\rho' - \bar{\rho}')^{\wedge} = \rho' - \bar{\rho}'$, $(\tau + \bar{\tau}')^{\wedge} = f^{-1}((\tau + \bar{\tau}') - 2\eth\log f)$, $(\tau - \beta + \bar{\beta}')^{\wedge} = f^{-1}(\tau - \beta + \bar{\beta}')$, $(\bar{\tau}' + \beta - \bar{\beta}')^{\wedge} = f^{-1}((\bar{\tau}' + \beta - \bar{\beta}') - 2\eth\log f)$. Thus, the choice $p+q+1 = 0$, f $\mathbb{R}$-valued makes the trasnformation of the Hawking gauge particularly simple. If we actually wish to leave invariant this gauge, then we must also have $\eth f = 0$.

A particular application of the conformal transformation formula presented in this section is in the study of asymptotically simple spacetimes when a conformal compactification is made. For example, we could study quasi-local quantities in the compactified spacetime. In the next section, we briefly discuss asymptotic simplicity, which may be regarded as an asymptotic boundary condition on spacetime in a conformal setting.

6.3 Asymptotic Simplicity

For a physical theory to be useful, we must be able to discuss a certain class of solutions which represent isolated systems. In general relativity, these are bounded systems such as stars and black holes whose corresponding spacetime is asymptotically flat, i.e. the metric approaches that of Minkowski spacetime at large distances from the source. Examples are the Schwarzschild, Reissner-Nordström and Kerr-Newman spacetimes and these solutions have asymptotically flat regions whose conformal structure is similar to that of Minkowski spacetime. A useful definition of asymptotic flatness may be abstracted from these examples (Penrose [P7]) and we discuss such a definition in this section. For physical reasons indicated in Chapter Three, we restrict our attention to spacetimes whose metric approaches flatness along <u>null directions</u>. For more details, see Beem and Ehrlich [B 5], Geroch [G 5], Hawking and Ellis [H 5] and Penrose and Rindler [P 12]. These references also discuss the case of asymptotic flatness at spacelike infinity.

Reasons for studying isolated systems, in particular asymptotically flat spacetimes in general relativity, include the following:- physical attributes such as mass, momentum, angular momentum and other "charges" may often be assigned to such systems so as to describe the system using only a small number of parameters. Other ideas fitting into the conceptual framework of isolated systems are multipoles and radiation, both of which have important physical significance.

From a mathematical viewpoint, the imposition of boundary conditions such as asymptotic flatness leads to a simpler structure at infinity. This structure is <u>universal</u> in that it does not depend on the particular spacetime and it provides an arena in which to study the true degrees of freedom of the gravitational fidld. Indeed a well defined symmetry group, the BMS group, naturally occurs and this may be used as the basis of definitions of asymptotic kinematical quantities. In particular, spinor methods have proved very useful when combined with the imposition of the conformal boundary condition of asymptotic flatness. Indeed, it is fair to say that many of the remaining problems in general relativity would stand a better chance of being solved if the combination of spinor, embedding and conformal techniques at infinity could be propagated into the interior of spacetime in a consistent way. This idea of propagation from infinity lies at the heart of the Ludvigsen-Vickers definition of quasi-local momentum discussed in Chapter Three of this thesis. Whether or not more of the structure available asymptotically may be extended into spacetime itself remains to be seen. A major problem, besides practical details such as obstructions to propagation by caustics, etc., is the desire for some notion of universality when defining symmetry groups and kinematical quantities.

We turn now to the definition of the useful notions of asymptotic flatness. We use the term spacetime as introduced in section 1.7, and the following definition is essentially that given in the references cited above.

<u>Definition (6.3)1</u>: An <u>asymptote</u> of a spacetime (M,g) is a quadruple $(\hat{M},\hat{g},f,\phi)$ where $(\hat{M},\hat{g})$ is a spacetime with boundary $\partial\hat{M}$, $f \in C(\hat{M})$ and $\phi \in \mathrm{Emb}(M,\hat{M})$ such that ϕ is a diffeomorphism

of M onto $\hat{M} - \partial\hat{M}$, $(\phi^{*}f)^{2}g = \phi^{*}\hat{g}$, $f|\partial\hat{M} = 0$ and $df|\partial\hat{M} \neq 0$. A spacetime (M,g) is said to be asymptotically simple and empty if it admits an asymptote $(\hat{M},\hat{g},f,\phi)$ such that $\partial\hat{M} \subseteq \hat{M}$ - support $(\phi_{*}(Ric(g)))$, $(\hat{M},\hat{g})$ is strongly causal (i.e., for all $x \in \hat{M}$ and for all neighbourhoods U of x, there exists a neighbourhood V of x, $V \subseteq U$, such that every non-spacelike curve in $(\hat{M},\hat{g})$ intersects V at most once), and every inextendible null geodesic γ of (M,g) is such that $\phi \circ \gamma$ admits both future and past endpoints on $\partial\hat{M}$.

Asymptotically simple and empty spacetimes include Minkowski spacetime $(\mathbb{R}^4,\eta)$ together with isolated systems which do not undergo gravitational collapse. However they do not include important solutions such as Schwarzschild, Reissner-Nordström or Kerr-Newman, because there exist null geodesics in these spacetimes which do not get out to infinity. We modify the definition slightly to include such spaces:

A spacetime (M,g) is said to be weakly asymptotically simple and empty if there exists an asymptotically simple and empty spacetime (M',g') (with asymptote $(\hat{M}',\hat{g}',f',\phi')$) and a neighbourhood U' of $\partial\hat{M}'$ in $\hat{M}'$ such that $((\phi')^{-1}(U'),g')$ is isometric to (U,g) for some open set of M.

Examples of weakly asymptotically simple and empty spacetimes include those mentioned above and, in general, such spacetimes possess a whole family of asymptotically flat regions. When discussing weakly asymptotically simple and empty spacetimes we consider only one of these regions.

If a (weakly) asymptotically simple and empty spacetime (M,g) is a solution of Einstein's equations with vanishing cosmological constant, we say that (M,g) is asymptotically flat. The basic underlying philosophy in the study of asymptotically flat spacetimes is that, as far as physics and geometry are concerned, the boundary $\partial\hat{M}$ is to be treated as being at finite distance from the sources of the conformally compactified spacetime $(\hat{M},\hat{g})$. In particular, the conformal factor, the rescaled metric, curvature tensor, matter fields are amooth on $\partial\hat{M}$. The powerful local techniques of spinor geometry may be utilized on $(\hat{M},\hat{g})$ with implications for the asymptotic structure of the physical spacetime (M,g).

Now let (M,g) be an asymptotically flat spacetime with asymptote $(\hat{M},\hat{g},\hat{f},\phi)$. The major consequences of the definition are as follows:-

The boundary $\partial\hat{M}$ has two components, denoted $\mathscr{I}^-$ (past null infinity) and $\mathscr{I}^+$ (future null infinity), each of which is a non-shearing null hypersurface in $(\hat{M},\hat{g})$. Null geodesics have past endpoints on $\mathscr{I}^-$ and future endpoints on $\mathscr{I}^+$. The spacetime (M,g) is necessarily globally hyperbolic (so, in particular, M is topologically $\Sigma \times \mathbb{R}$ where $\Sigma \hookrightarrow M$ is a Cauchy surface for (M,g)), and each of $\mathscr{I}^\pm$ are topologically $S^2 \times \mathbb{R}$. The field $n = (-d\hat{f})^{\#} \in \mathrm{Vect}(\hat{M})$, when restricted to $\mathscr{I}^\pm$, is a null normal and its integral curves are the null generators of $\mathscr{I}^\pm$. Any two cross sections of $\mathscr{I}^\pm \cong S^2 \times \mathbb{R} \longrightarrow S^2$ are mapped conformally to one another by the flow of n. The Weyl tensor of $\hat{g}$ vanishes identically on $\mathscr{I}^\pm$ and this leads to the peeling off theorem - along a null geodesic in a neighbourhood of $\mathscr{I}^\pm$, the various spinor components of the Weyl tensor vary as different powers of an affine parameter. The peeling off theorem relates the outgoing (for $\mathscr{I}^+$) null direction given by the geodesic to the algebraic type and asymptotic behaviour of the conformally invariant part of the gravitational field.

We now make some brief remarks concerning asymptotes. Suppose (M,g) is asymptotically flat. A natural question to ask is how many asymptotes does (M,g) admit? To answer this it is necessary to consider equivalence and extension of asymptotes: Two asymptotes $(\hat{M}_1,\hat{g}_1,f_1,\phi_1)$, $(\hat{M}_2,\hat{g}_2,f_2,\phi_2)$ of (M,g) are said to be <u>equivalent</u> if there exists $h \in C^+(\hat{M})$, $\psi \in \mathrm{Diff}(\hat{M}_1,\hat{M}_2)$ such that $\hat{g}_1 = h^2(\psi^*\hat{g}_2)$ and $f_1 = h(\psi^* f_2)$. The asymptote $(\hat{M},\hat{g},f,\phi)$ is said to be an <u>extension</u> of $(\hat{M} - W, \hat{g}|(\hat{M} - W), f|(\hat{M} - W),\phi)$ for any closed subset W of $\partial\hat{M}$. We then say an asymptote is <u>maximal</u> if it admits no non-trivial extension. It may then be shown that for any asymptotically flat spacetime (M,g), there exists a maximal asymptote, unique up to equivalence (Actually, we must restrict to <u>regular asymptotes</u> - see Geroch [G 5] for more details). Thus, given any asymptotically flat spacetime we may use the maximal regular asymptote. This will be defined only up to a conformeomorphism and we should ensure that all physically meaningful quantities are invariant under such transformations.

Let us now focus on, say, $\mathscr{I}^+ \subseteq \partial\hat{M}$ where $(\hat{M},\hat{g},f,\phi)$ is, up to equivalence, the maximal regular asymptote of the asymptotically flat spacetime (M,g). We regard $\mathscr{I}^+$ as the space $S^2 \times \mathbb{R}$ equipped with various fields induced by the embedding ϕ of spacetime into $\hat{M}$. The two most important fields are the degenerate metric $q = j^*\hat{g}$ (where $j: \mathscr{I}^+ \to \hat{M}$ is inclusion) of signature $(o - -)$ and the vector field $n = (-df)^{\hat{\#}}$ which, when restricted to $\mathscr{I}^+$, is a null normal to $\mathscr{I}^+$. Suppose we choose an asymptote $(\hat{M},h^2\hat{g},hf,\phi)$ equivalent to the original one (where, without loss

of generality, we have taken the compactified spacetime to be $\hat{M}$, rather than some other differomorph). Then q becomes $h^2 q$ and n becomes $h^{-1} n$ (since $d(hf) = h\,df$ on $\mathscr{I}^+$ and $\hat{g}^{-1}$ becomes $h^{-2}\hat{g}^{-1}$). Thus the field $S = q \otimes n \otimes n \in \Gamma((\odot^2 T^* \mathscr{I}^+) \otimes (\odot^2 T \mathscr{I}^+))$ is independent of the choice of representative of the equivalence class of asymptotes. Given a signature $(0 - -)$ conformal structure C on $\mathscr{I}^+ \cong S^2 \times \mathbb{R}$, it is easily shown (see Schmidt et al. [S 6]) that a type $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ tensor field, symmetric in both pairs of slots, is characterized by the two properties
(i) $\forall \alpha \in \Omega^1(\mathscr{I}^+)$, $S(\cdot, \cdot, \alpha, \alpha) \in C \cup \{0\}$ and (ii) any contraction of S itself vanishes.

<u>Definition (6.2)2</u>: Let $(\mathscr{I}^+, C)$ be as just described. The tensor field $S \in \Gamma((\odot^2 T^* \mathscr{I}^+) \otimes (\odot^2 T \mathscr{I}^+))$ uniquely defined by properties (i), (ii) is called the <u>strong conformal</u> geometry on $(\mathscr{I}^+, C)$ (Penrose and Rindler [P 12]).

The triple $(\mathscr{I}^+, C, S)$ is unique up to diffeomorphism and we regard the strong conformal geometry as representing the universal first order structure on $\mathscr{I}^+$. The asymptotic symmetry group is the automorphism group of this first order structure:-

<u>Definition (6.3)3</u>: Let $(\mathscr{I}^+, C, S)$ be as above. The <u>Bondi-Metzner-Sachs (BMS) group</u> is defined by $BMS = \{\phi \in \mathrm{Diff}(\mathscr{I}^+) : \phi^* S = S\}$.

The BMS group is unique up to isomorphism, i.e. it does not depend on the choice of degenerate conformal structure C. This group arose originally (Bondi et al. [B 15], Sachs [S 2]) as the group of coordinate transformations preserving Bondi et al's form of a future asymptotically flat metric, but it is more geometrical to view the BMS group as the automorphism group of the universal structure $(\mathcal{J}^+, C, S)$.

Using definition (6.3)3, the isomorphism class of BMS may be deduced. Here we just state the relevant results. For a comprehensive treatment of representation theoretical and topological aspects of the BMS group, see McCarthy [M 1].

The BMS group, like the Poincaré group, possesses a semi-direct product structure, so we first recall the important facts concerning such structures. See, for example, Jacobson, p. 79, for the basic definition. Suppose that we are given groups H and K together with an action θ of H on K by automorphisms, i.e. $\theta \in \mathrm{Hom}(H, \mathrm{Aut}(K))$. Consider the set $H \times K$. Then, using θ, there are two natural group structures on $H \times K$. The first is given by $(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 \theta_{h_1}(k_2))$ and the second is given by $(h_1, k_1)(h_2, k_2) = (h_1 h_2, \theta_{h_2^{-1}}(k_1) k_2)$, $\forall (h_1, k_1), (h_2, k_2) \in H \times K$. It is straightforward to check that these are indeed group structures on $H \times K$. Let G, G' denote $H \times K$ equipped with the first, second group structure respectively. Then the map $\psi: G \rightarrow G'$; $(h,k) \mapsto (h, \theta_{h^{-1}}(k))$ is easily seen to be an isomorphism of groups. The isomorphism class containing G (and hence G') is called the <u>semi-direct product</u> of H and K with respect to θ and we denote it $H \ltimes_\theta K$ (or $H \ltimes K$ if θ is understood). Depending on the situation, we pick the concrete representative G (or G') and write $H \ltimes_\theta K = G$ (or G').

The semi-direct product often arises as the splitting of a short exact sequence of groups $1 \rightarrow K \xhookrightarrow{\iota} G \xrightarrow{\lambda} H \rightarrow 1$ where ι is inclusion of $K \leqslant G$ and λ is an epimorphism. Given a splitting $\beta \in \mathrm{Hom}(H, G)$ (so that $\lambda \circ \beta = \mathrm{id}_H$) we may define $\theta \in \mathrm{Hom}(H, \mathrm{Aut}(K))$ by $\theta_h(k) = \beta(h) k \beta(h^{-1})$, $\forall h \in H$, $k \in K$. Then

G is (isomorphic to) the semi-direct product $H \ltimes_\theta K$ - define $\xi: H \ltimes_\theta K \longrightarrow G$ by $\xi(h,k) = k\beta(h)$, then ξ is an isomorphism of groups with inverse given by $\xi^{-1}(g) = (\lambda(g),\ g((\beta\circ\lambda)(g^{-1})))$. An example of this phenomenon is the splitting of sequence 6.1.2 (or 6.1.3) in the case of the frame bundle $GL(M)$. This gives rise to the semi-direct product structure $\mathrm{Diff}(M) \ltimes \mathrm{GauGL}(M)$ for $\mathrm{Aut}\ GL(M)$ (see definition (6.1)27 and equation 6.1.13).

We now return to the BMS group. Recall (see sections 1.5 and 1.7) that the restricted Lorentz group $SO^\uparrow(1,3)$ is isomorphic to $\mathrm{Conf}(S^2, \mathrm{Can})$, the conformal group of the two sphere equipped with its standard conformal structure (1.5.33). In equation 1.5.20, we gave the conformal action of $SL(2,\mathbb{C})$, the double cover of $SO^\uparrow(1,3)$, on S^2 - as we remarked, this action projects to $SO^\uparrow(1,3)$ and is the action realizing the isomorphism $SO^\uparrow(1,3) \cong \mathrm{Conf}(S^2, \mathrm{Can})$. Let us therefore write $\phi \in \mathrm{Hom}(SO^\uparrow(1,3), \mathrm{Diff}(S^2))$ for this action, so that $\phi_a^*\,\mathrm{can} = K_a^2\,\mathrm{can}$, $\forall a \in SO^\uparrow(1,3)$, where $K_a \equiv K_A$ (any $A \in \Lambda^{-1}(a)$) (see equation 1.5.36). Note that the conformal factor $K_a^2 \in C^+(S^2)$ is given by $K_a^2 = \frac{1}{2}\mathrm{trace}(\phi_a^*\,\mathrm{can})$ (trace with respect to $\mathrm{can} \in \mathrm{Met}(S^2)$) so that, $\forall a_1, a_2 \in SO^\uparrow(1,3)$, $K^2_{a_1a_2} = \frac{1}{2}\mathrm{trace}(\phi^*_{a_1a_2}\mathrm{can})$ $= \frac{1}{2}\mathrm{trace}(\phi^*_{a_2}\phi^*_{a_1}\,\mathrm{can}) = \frac{1}{2}\mathrm{trace}(\phi^*_{a_2}(K^2_{a_1}\,\mathrm{can})) = \frac{1}{2}(\phi^*_{a_2}K^2_{a_1})\mathrm{trace}(\phi^*_{a_2}\,\mathrm{can})$ $= (\phi^*_{a_2}K^2_{a_1})K^2_{a_2}$.

Now consider the action Φ of $SO^\uparrow(1,3)$ on $C(S^2)$ given by:-

$$\Phi_a(f) = K_{a^{-1}}\,\phi^*_{a^{-1}}\,f \qquad 6.3.1,$$

$\forall f \in C(S^2)$, $a \in SO^\uparrow(1,3)$. From a representation theoretical viewpoint Φ is the closest relative representation associated with the representation $D^{o}(\frac{1}{2},\frac{1}{2})$ (see Gel'fand et al. [G17] and section 1.7 for a brief discussion of the D^{o} representations). We regard $C(S^2)$ as an abelian group and then the action Φ is by automorphisms. Let $B = SO^{\uparrow}(1,3) \ltimes_{\Phi} C(S^2)$ denote the corresponding semi-direct product, where, to conform with standard practice, we use the "G'-representation", so that the group structure in B is given by $(a_1,f_1)(a_2,f_2) = (a_1a_2, \Phi_{a_2^{-1}}(f_1) + f_2) = (a_1a_2, K_{a_2}\phi^*_{a_2}f_1 + f_2)$, $\forall (a_1,f_1),(a_2,f_2) \in B$.

Now define $\Psi: B \longrightarrow \mathrm{Diff}(S^2 \times \mathbb{R})$ by:-

$$\Psi_{(a,f)}(x,t) = (\phi_a(x), K_a(x)^{-1}(t + f(x))) \qquad 6.3.2,$$

$\forall (x,t) \in S^2 \times \mathbb{R}$, $(a,f) \in B$. Note that $\Psi_{(a,f)}$ is indeed a diffeomorphism. We now show that Ψ is a monomorphism. We have

$$\Psi_{(a_1,f_1)(a_2,f_2)}(x,t) = \Psi_{(a_1 a_2, K_{a_2}\phi^*_{a_2} f_1+f_2)}(x,t)$$

$$= (\phi_{a_1a_2}(x), K_{a_1a_2}(x)^{-1}(t + K_{a_2}(x)f_1(\phi_{a_2}(x)) + f_2(x)))$$

$$= (\phi_{a_1}(\phi_{a_2}(x)), K_{a_1}(\phi_{a_2}(x))^{-1}K_{a_2}(x)^{-1}(t + K_{a_2}(x)f_1(\phi_{a_2}(x)) + f_2(x)))$$

$$= (\phi_{a_1}(\phi_{a_2}(x)), K_{a_1}(\phi_{a_2}(x))^{-1}(K_{a_2}(x)^{-1}(t + f_2(x)) + f_1(\phi_{a_2}(x))))$$

$$= \Psi_{(a_1,f_1)}(\phi_{a_2}(x), K_{a_2}(x)^{-1}(t + f_2(x))) = \Psi_{(a_1,f_1)}(\Psi_{(a_2,f_2)}(x,t))$$

$= (\Psi_{(a_1,f_1)} \circ \Psi_{(a_2,f_2)})(x,t)$, $\forall (x,t) \in S^2 \times \mathbb{R}$. Hence Ψ is a homomorphism. By inspection, Ψ is injective, and so Ψ is an isomorphism of $SO^{\uparrow}(1,3) \times_{\Phi} C(S^2)$ onto $\Psi(B) \leq \mathrm{Diff}(S^2 \times \mathbb{R})$.

It is straightforward to show (see Schmidt et al. [S 6]) that $\Psi(B)$ coincides with BMS, as defined by (6.3)3, so that the BMS group is isomorphic to $SO^{\uparrow}(1,3) \times_{\Phi} C(S^2)$ with action on $\mathcal{J}^+ \cong S^2 \times \mathbb{R}$ given by 6.3.2. From now on we identify BMS with $B = SO^{\uparrow}(1,3) \ltimes_{\Phi} C(S^2)$.

A more geometrical interpretation may be given as follows:- Let $\pi: \mathcal{J}^+ \to S^2$ be the trivial bundle given by projection onto the first factor. We regard the fibre of π as an affine space modelled on the vector space $\mathbb{R}$. Let $A = \Gamma(\mathcal{J}^+)$ denote the space of sections of this bundle, so that A is the cone space of Bramson (see [B21] and section 2.2) and is an affine space modelled on the vector space $C(S^2)$ where the action is given by $\mu;(f,c) \mapsto (f + c);\ x \mapsto f(x) + c(x),\ \forall (f,c) \in C(S^2) \times A$. Note that we identify a section c with the corresponding function given by $x \mapsto (x, c(x)) \in \mathcal{J}^+$, $\forall x \in S^2$. The vector space $C(S^2)$, when regarded as a normal subgroup of BMS, is the space of supertranslations of $\mathcal{J}^+$. Using Φ (6.3.1), $SO^{\uparrow}(1,3)$ may be regarded as a subgroup of $GL(C(S^2))$ ($\equiv$ {Bounded invertible linear operators on $C(S^2)$}).

Note that the action Ψ given by 6.3.2 is by bundle automorphisms of $\pi: \mathcal{J}^+ \to S^2$, and, for each $(a,f) \in$ BMS, $\Psi_{(a,f)}: \mathcal{J}^+ \to \mathcal{J}^+$ covers the diffeomorphism $\phi_a: S^2 \to S^2$. We may therefore define an action $\hat{\Psi}$ of BMS on $A = \Gamma(\mathcal{J}^+)$ by $\hat{\Psi}_{(a,f)}(c) = \Psi_{(a,f)} \circ c \circ \phi_{a^{-1}}$, $\forall c \in A$, $(a,f) \in$ BMS. Using 6.3.2, we see that

$$\hat{\Psi}_{(a,f)} = \Phi_a \circ \mu_f \qquad 6.3.3,$$

where $\mu_f: A \to A$ is the affine action given above, and $\Phi_a: A \to A$ is given by the formula 6.3.1, i.e. $\hat{\Psi}_{(a,f)}(c) = K_{a^{-1}}\, \phi^*_{a^{-1}}\, (f + c)$.

Now recall that given any affine space A modelled on a vector space V via the action μ, we may project any bijection

$q: A \to A$ to a bijection $\bar{q}: V \to V$ given (uniquely) by $\mu_{\bar{q}(v)} = q \circ \mu_v \circ q^{-1}$, $\forall v \in V$. The affine space, $\mathrm{Aff}(A)$, of A is given by $\mathrm{Aff}(A) = \{q \in \mathrm{Bij}(A) : \bar{q} \in GL(V)\}$ and we have the short exact sequence $1 \to V \xhookrightarrow{\mu} \mathrm{Aff}(A) \xrightarrow{-} GL(V) \to 1$. Any subgroup H of $GL(V)$ gives rise to a corresponding sequence $1 \to V \xhookrightarrow{\mu} \mathrm{Aff}_H(A) \xrightarrow{-} H \to 1$.

Consider, for fixed $(a,f) \in \mathrm{BMS}$, the bijection $\hat{\Psi}_{(a,f)}$ (6.3.3) of the affine space A (modelled on $C(S^2)$). Let $\overset{\Delta}{\Psi}_{(a,f)}$ denote the corresponding bijection of $C(S^2)$ so that, $\forall f' \in C(S^2)$,

$$\mu_{\overset{\Delta}{\Psi}_{(a,f)}(f')} = \hat{\Psi}_{(a,f)} \circ \mu_{f'} \circ \hat{\Psi}^{-1}_{(a,f)} = \Phi_a \circ \mu_f \circ \mu_{f'} \circ \mu_{-f} \circ \Phi_{a^{-1}}$$

$= \Phi_a \circ \mu_{f'} \circ \Phi_{a^{-1}} = \mu_{\bar{\Phi}_a(f')}$ so $\overset{\Delta}{\Psi}_{(a,f)} = \Phi_a$ (since, as we have remarked above, Φ acts on both $C(S^2)$ and on A according to the same formula 6.3.1). In other words, the BMS group, regarded as a subgroup of $\mathrm{Bij}(A)$ via the action $\hat{\Psi}$, projects to the group $SO^{\uparrow}(1,3)$, regarded as a subgroup of $GL(C(S^2))$ via the action Φ. We have thus demonstrated that the BMS group is the group of affine transformations of A (cone space) characterized by the short exact sequence

$$1 \to C(S^2) \hookrightarrow \mathrm{BMS} \longrightarrow SO^{\uparrow}(1,3) \to 1 \qquad 6.3.4.$$

The isomorphism $\mathrm{BMS} \cong SO^{\uparrow}(1,3) \times C(S^2)$ may now be regarded as arising from a splitting of 6.3.4. Of course such a splitting is not unique - one must choose an origin $c_o \in A$ (i.e. some <u>cut</u> of $\mathscr{I}^+$) and then a splitting $\beta_o \in \mathrm{Hom}(SO^{\uparrow}(1,3),\mathrm{BMS})$ is defined by $\beta_o(a) = \mu_o \circ \Phi_a \circ \mu_o^{-1}: A \to A$. This <u>origin dependence</u> of $SO^{\uparrow}(1,3)$ as a subgroup of BMS is the root of the problems involved in attempting to define <u>angular momentum</u> in general relativity. The origin dependence also exists in non-relativistic and special relativistic mechanics (where we deal with the <u>Galileo group</u> $SO(3) \ltimes \mathbb{R}^3$ and <u>Poincaré group</u> $SO^{\uparrow}(1,3) \ltimes \mathbb{R}^4$ respectively) but in general relativity there is an additional obstruction to defining angular momentum. Indeed, a choice of origin cannot even be made if gravitational radiation is present since such radiation will supertranslate any initial choice of origin. We discuss this problem further in section 3.2.

The relationship between the BMS group and the Poincaré group may be seen if we consider the translation subgroup of BMS. It can be shown (see, for example, Sachs [S 2]) that there is a unique four dimensional normal subgroup $T \cong \mathbb{R}^4$ of BMS (T is the unique non-trivial subspace of $C(S^2)$ invariant under the $SO^{\uparrow}(1,3)$ action Φ (6.3.1)). T is called the <u>translation</u> subgroup of BMS. The action μ of $C(S^2)$ on A is free and transitive, so restricting μ to $T \triangleleft C(S^2)$ partitions A into an uncountable number of subspaces each of which may be identified with $\mathbb{R}^4$. Thus there are many possible choices of Minkowski subspace of cone space A. A choice of origin $c_o \in A$ picks out a particular copy of Minkowski space in A - we just take the orbit of c_o under T. We may now define an origin dependent Poincaré subgroup of BMS by restricting the action Φ of $SO^{\uparrow}(1,3)$ on $C(S^2)$ to the subspace T.

This completes our discussion of the abstract BMS group. The importance of BMS in physics is that it is the asymptotic symmetry group for a wide class of spacetimes, namely those which are asymptotically flat. It turns out that, in addition to the strong conformal geometry discussed above, future (or past) null infinity is universally equipped with a phase space Γ consisting of equivalence classes of connections (the second order structure). This phase space Γ is an affine space modelled on a Fréchet space and possesses a naturally defined (weak) symplectic structure ω. The BMS group acts symplectomorphically on (Γ,ω) and the corresponding moment maps : $\Gamma \rightarrow LBMS^* = (LSO^{\uparrow}(1,3) \oplus C(S^2))^*$ may be identified with fluxes of supermomentum and of angular momentum. We refer the reader to Ashtekar and Streubel [A23] for a treatment of the symplectic geometry of the BMS action of (Γ,ω).

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