UNIVERSITY OF Southampton
Institute of Sound and Vibration Research

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Optimisation of Dynamic Vibration Absorbers to Minimise Kinetic Energy and Maximise Internal Power Dissipation

M. Zilletti, S.J. Elliott and E. Rustighi

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INSTITUTE OF SOUND AND VIBRATION RESEARCH

SIGNAL PROCESSING AND CONTROL GROUP

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Authorised for issue by Professor R. Allen, Group Chairman

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Abstract

The tuning of a dynamic vibration absorber is considered such that either the kinetic energy of the host structure is minimised or the power dissipation within the absorber is maximised. If the host structure is approximated as a lightly damped, single degree of freedom, system, simple expressions are obtained for the optimal ratio of the absorber natural frequency to the host natural frequency and optimal damping ratio of the absorber. These optimal values are shown to be the same whether the kinetic energy of the host structure is minimised or if the power dissipation of the absorber is maximised.

1. Introduction

Dynamic vibration absorbers are single degree of freedom systems that are attached to a host structure to control its motion [1]. Such devices were originally patented in 1911 [2]. They are widely used to control the vibration of civil [3-5], marine [6] and aerospace [7-9] structures and can operate in different ways depending on the application. The first way of operating such a device aims to suppress the vibration only at a particular forcing frequency, in which case the devices natural frequency is tuned to this excitation frequency. The damping of the device should also be as low as possible in this case, so that it presents the greatest impedance to the host structure at the operating frequency. The device is then often known as a "vibration neutraliser", and considerable ingenuity has been put into tuning the natural frequency of the device to track variations in the excitation frequency [4, 7, 10].

Alternatively the device can be used to attenuate the vibration due to a particular mode of the structure over a range of frequencies, when it is sometimes referred to as a "tuned mass damper". The optimum tuning of the natural frequency and damping ratio of the device then become less obvious and depend on exactly how the optimisation criterion is defined. The selected mode of the host structure is generally modelled as a single degree of freedom system for this optimisation, often without any inherent damping.

A survey of tuning criterion for dynamic vibration absorbers when used as tuned mass dampers has been presented by Asami [11], and some of the results from this paper are presented in Table 1. The original optimisation criterion used by Omondroyd and Den Hartog 1928 [12] was that the magnitude of the displacement was equal at the two peaks in the coupled displacement response after the device has been attached. This is also known as mini-max or H_∞ optimisation. Another optimisation criterion would be to minimise the mean square displacement of the host structure when excited by a random force of uniform power spectral density, as first proposed by Crandall and Mark in 1963 [13] and also now known as H_2 optimisation. A third possibility is to adjust the natural frequency and damping of the device such that the poles of the overall system have the greatest negative value, so that the transient response decays as quickly as possible. Asami *et al.* [11] attribute this result in Table 1 to Yamaguchi in 1988 [14], although the same criterion was also considered by Miller and Crawley in

1985 [15]. Krenk in 2005 [16] proposed a further method to tune the parameters of a DVA. He tuned the frequency ratio of the two decoupled oscillators using the same criterion proposed by Omondroyd and den Hartog [12] and proposed a new criterion for the optimal damping ratio. The damping ratio was chosen by simultaneously minimising the displacement of the main mass and the relative displacement of the two masses calculated at the natural frequency of the system when the damper was blocked. He also demonstrate that for the frequency tuning proposed by Omondroyd and den Hartog [12], the complex locus of the natural frequencies has a bifurcation point corresponding to the maximum damping of the two modes. For lower damping ratio the two modes have the same modal damping. Warburton in 1982 [17] proposed the minimisation of the frequency averaged kinetic energy of the host structure as a tuning criterion.

In this paper we consider a further criterion on which to optimise a dynamic vibration absorber based on the maximisation of the power dissipated by the absorber. It is found for a damped host structure that the maximisation of the power absorbed by the damper corresponds to the minimisation of the kinetic energy of the host structure.

The natural frequency of the tuned mass absorber is generally not too difficult to design since the stiffness can be specified. It may be more difficult to tune the damping ratio, however, particularly if the damping mechanism is level dependant. It may be possible to measure the power dissipated in this damper, however, in which case the results below demonstrate that maximising this power dissipation will lead to the same tuning result as minimising the kinetic energy of a lightly damped host structure.

2. Analysis

The dynamic vibration absorber (DVA) is a passive vibration control device which is attached to a vibrating host structure often called primary structure. A single mode of the structure is often modelled as a single degree of freedom primary system, which is shown with the DVA in Figure 1 where m_1 and m_2 are the masses k_1 and k_2 are the stiffness values and c_1 and c_2 the mechanical damping values of the primary system and the DVA respectively. The primary system is subjected to a random excitation f_p, which is assumed to have a flat power spectral density and u_1 and u_2 are the velocities of mass m_1 and m_2.

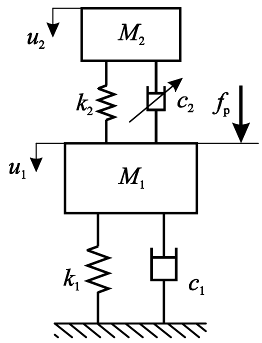

Figure 1: Scheme of the SDOF system with the DVA

The steady state response of the system can be expressed in terms of the five dimensionless parameters defined by:

$\mu = m_2/m_1$: mass ratio
$\nu = \omega_2/\omega_1$: natural frequency ratio
$\lambda = \omega/\omega_1$: forced frequency ratio (1)
$\zeta_1 = c_1/(2m_1\omega_1)$: primary damping
$\zeta_2 = c_2/(2m_2\omega_2)$: secondary damping

where

$\omega_1 = \sqrt{k_1/m_1}$: natural frequency of the host / primary system
(2)
$\omega_2 = \sqrt{k_2/m_2}$: natural frequency of the DVA

Many way of tuning the natural frequencies of the two systems and optimal damping of the DVA have been proposed with the scope of reducing the vibration of the primary system. A summary of various type of optimisations when the primary system is undamped ($\zeta_1 = 0$) are summarised in Table 1.

Table 1: optimisation criteria of the dynamic vibration absorber on a lightly damped SDOF system

	Optimisation criterion	Performance index	Objective	Proposed by:	Optimal parameters
1	H_∞ Optimisation	$A_{1max} = \left\|\frac{x_1}{x_{st}}\right\|_{max}$	Minimise the maximum displacement of the primary mass	Ormondroyd & Den Hrtong 1928 [12]	$\zeta_{\mathrm{opt}} = \sqrt{\frac{3\mu}{8(1+\mu)}}$ $\nu_{\mathrm{opt}} = \frac{1}{1+\mu}$
2	H_2 Optimisation of the mean squared displacement	$I_1 = \frac{E[x_1^2]}{S_f\omega_1/k_1^2}$	Minimise the total displacement of the primary mass over all frequency	Iwata 1982 [18], Warburton 1982 [17]	$\zeta_{\mathrm{opt}} = \sqrt{\frac{\mu(4+3\mu)}{8(1+\mu)(2+\mu)}}$ $\nu_{\mathrm{opt}} = \frac{1}{1+\mu}\sqrt{\frac{2+\mu}{2}}$
3	Stability Maximisation	$\Lambda = -\max_i(\mathrm{Re}[s_i])$	Minimise the transient vibration of the system	Yamaguchi 1988 [14], Millers *et al.* 1985 [15]	$\zeta_{\mathrm{opt}} = \sqrt{\frac{\mu}{1+\mu}}$ $\nu_{\mathrm{opt}} = \frac{1}{1+\mu}$
4	H_∞ Minimisation of relative displacement	$A_{1max} = \left\|\frac{x_1}{x_{st}}\right\|_{max}$ $A_{2max} = \left\|\frac{x_1 - x_2}{x_{st}}\right\|_{max}$	Minimisation of displacement of the main mass and relative displacement	Krenk 2005 [16]	$\zeta_{\mathrm{opt}} = \sqrt{\frac{\mu}{2(1+\mu)}}$ $\nu_{\mathrm{opt}} = \frac{1}{1+\mu}$
5	H_2 Minimisation of kinetic energy	$I_1 = \frac{E[\dot{x}_1^2]}{2\pi S_f\omega_1/k_1}$	Minimise the total kinetic energy of the primary mass over all frequencies	Warburton 1982 [17]	$\zeta_{\mathrm{opt}} = \frac{\sqrt{\mu}}{2}$ $\nu_{\mathrm{opt}} = \frac{1}{\sqrt{1+\mu}}$
6	H_2 Maximisation of the absorbed power	$I_p = \frac{c_2 E[\|\dot{x}_1 - \dot{x}_2\|^2]}{2\pi S_f\omega_1/k_1}$	Minimise the total kinetic energy of the primary mass over all frequencies	This study	$\zeta_{\mathrm{opt}} = \frac{\sqrt{\mu}}{2}$ $\nu_{\mathrm{opt}} = \frac{1}{\sqrt{1+\mu}}$

The equation of motion of the system shown in Figure 1 can be written in the matrix form as:

$$\mathbf{M}\ddot{\boldsymbol{x}}(t) + \mathbf{C}\dot{\boldsymbol{x}}(t) + \mathbf{K}\boldsymbol{x}(t) = \boldsymbol{f}(t) \tag{3}$$

Where **M** is the mass matrix, **K** is the stiffness matrix and **C** is the damping matrix given by:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \tag{4}$$

$\boldsymbol{x}(t) = [x_1(t) \quad x_2(t)]^T$ is the column vector containing the displacements of the two masses x_1 and x_2 and $\boldsymbol{f}(t) = [f_{\mathrm{p}}(t) \quad 0]^T$ is the column vector of primary excitation.

Assuming the excitation to be harmonic for the time being and expressing the force and the steady-state response in exponential form, equation (3) becomes:

$$\mathbf{S}(\mathrm{j}\omega)\mathbf{x}(\mathrm{j}\omega) = \mathbf{f}(\mathrm{j}\omega) \tag{5}$$

where

$$\mathbf{S}(\mathrm{j}\omega) = -\omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{C} + \mathbf{K} \tag{6}$$

is the dynamic stiffness matrix. The solution of equation (5) can be obtained as:

$$\mathbf{x}(\mathrm{j}\omega) = \mathbf{S}^{-1}(\mathrm{j}\omega)\mathbf{f}(\mathrm{j}\omega) \tag{7}$$

Integrating equation (7) to obtain the velocities yields:

$$\mathbf{v}(\mathrm{j}\omega) = \mathbf{Y}(\mathrm{j}\omega)\mathbf{f}(\mathrm{j}\omega) \tag{8}$$

where $\mathbf{v}(j\omega) = \mathrm{j}\omega\mathbf{x}(\mathrm{j}\omega)$ and $\mathbf{Y}(\mathrm{j}\omega) = \mathrm{j}\omega\mathbf{S}^{-1}(\mathrm{j}\omega)$ is the mobility matrix. Using the expression of **M**, **K** and **C** of equation (4) the velocity per unit input force of the two masses is given by:

$$Y_{11}(j\omega) = \frac{\mathrm{u}_1}{\mathrm{f_p}} = \\ = \frac{jk_2\omega - \mathrm{c}_2\omega^2 - im_2\omega^3}{k_1k_2 + j\mathrm{c}_2k_1\omega + i\mathrm{c}_1k_2\omega - \mathrm{c}_1\mathrm{c}_2\omega^2 - k_2m_1\omega^2 - k_1m_2\omega^2 - k_2m_2\omega^2 - j\mathrm{c}_2m_1\omega^3 - j\mathrm{c}_1m_2\omega^3 - i\mathrm{c}_2m_2\omega^3 + m_1m_2\omega^4} \tag{9}$$

$$Y_{12}(j\omega) = \frac{\mathrm{u}_2}{\mathrm{f_p}} = \\ = \frac{jk_2\omega - \mathrm{c}_2\omega^2}{k_1k_2 + j\mathrm{c}_2k_1\omega + i\mathrm{c}_1k_2\omega - \mathrm{c}_1\mathrm{c}_2\omega^2 - k_2m_1\omega^2 - k_1m_2\omega^2 - k_2m_2\omega^2 - j\mathrm{c}_2m_1\omega^3 - j\mathrm{c}_1m_2\omega^3 - i\mathrm{c}_2m_2\omega^3 + m_1m_2\omega^4} \tag{10}$$

The five dimensionless coefficients defined in equations (1) and (2) can be written as:

$$\begin{gathered} \mu = m_1^{-1}m_2 \\ \nu = m_1^{1/2}k_1^{-1/2}m_2^{-1/2}k_2^{1/2} \\ \lambda = \omega m_1^{1/2}k_1^{-1/2} \\ \zeta_1 = 2^{-1}c_1m_1^{-1/2}k_1^{-1/2} \\ \zeta_2 = 2^{-1}c_2m_2^{-1/2}k_2^{-1/2} \end{gathered} \tag{11}$$

and thus a generic dimensionless term can be written as:

$$a\nu^b\lambda^c\mu^d\zeta_2^e\zeta_1^f = 2^{-e-f}m_1^{b/2+c/2-d-f/2}k_1^{-b/2-c/2-f/2}m_2^{-b/2+d-c/2}k_2^{b/2-e/2}c_2^ec_1^f\omega^c \tag{12}$$

Each of the coefficient in equations (9) and (10) can be expressed in non dimensional form by setting each of them equal to equation (12) and solving for the parameters *a*, *b*, *c*, *d*, *e*, *f*, so that equation (9) and (10) can be written as:

$$\Gamma = \sqrt{k_1 m_1}\mathrm{Y}_{11}(j\lambda) = \frac{B_0 + (j\lambda)B_1 + (j\lambda)^2 B_2 + (j\lambda)^3 B_3}{A_0 + (j\lambda)A_1 + (j\lambda)^2 A_2 + (j\lambda)^3 A_3 + (j\lambda)^4 A_4} \tag{13}$$

$$\Theta = \sqrt{k_1 m_1}\mathrm{Y}_{12}(j\lambda) = \frac{C_0 + (j\lambda)C_1 + (j\lambda)^2 C_2 + (j\lambda)^3 C_3}{A_0 + (j\lambda)A_1 + (j\lambda)^2 A_2 + (j\lambda)^3 A_3 + (j\lambda)^4 A_4} \tag{14}$$

where

$A_0 = \mu\nu^2$	$B_0 = 0$	$C_0 = 0$
$A_1 = 2\zeta_2\mu\nu + 2\zeta_1\mu\nu^2$	$B_1 = \mu\nu^2$	$C_1 = \mu\nu^2$
$A_2 = \mu\nu^2 + \mu + \mu^2\nu^2 + 4\zeta_2\zeta_2\mu\nu$	$B_2 = 2\zeta_2\mu\nu$	$C_2 = 2\zeta_2\mu\nu$
$A_3 = 2\zeta_2\mu\nu + 2\zeta_2\mu^2\nu + 2\zeta_1\mu$	$B_3 = \mu$	$C_3 = 0$
$A_4 = \mu$		

3. Tuning the dynamic vibration absorber

3.1. Definition of the performance criteria

If the aim of the DVA is to minimise the integral of the kinetic energy of the primary mass calculated over the frequency-band $\pm\infty$, the performance index to be minimised can be defined by:

$$I_\mathrm{k} = \frac{m_1 E[|\mathrm{u}_1|^2]}{2\pi S_\mathrm{f}\omega_1/k_1} \tag{15}$$

where $E[\ ]$ denotes the expectation value. The performance index I_k represents the ratio of the kinetic energy of the primary system to the excitation force with a uniform spectrum density $S_\mathrm{f}(\omega)$. The unit of $S_\mathrm{f}(\omega)$ is N^2s/rad. The constant $2\pi\omega_1/k_1$ is introduced to ensure that the performance index is dimensionless. The mean squared value of the velocity of the primary mass can be written as:

$$E[|\mathrm{u}_1|^2] = \frac{S_\mathrm{f}\omega_1}{m_1 k_1}\int_{-\infty}^{+\infty} |\Gamma|^2 \,\mathrm{d}\lambda \tag{16}$$

Substituting equation (16) in equation (15) yields:

$$I_\mathrm{k} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\Gamma|^2 \,\mathrm{d}\lambda \tag{17}$$

Thus, substituting equation (13) in (17) yields:

$$I_\mathrm{k} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\frac{B_0 + (j\lambda)B_1 + (j\lambda)^2 B_2 + (j\lambda)^3 B_3}{A_0 + (j\lambda)A_1 + (j\lambda)^2 A_2 + (j\lambda)^3 A_3 + (j\lambda)^4 A_4}\right|^2 \mathrm{d}\lambda \tag{18}$$

Equation (18) can be integrated using the formula in reference [19] leading to:

$$I_{\mathrm{k}} = \frac{\zeta_1(4\zeta_2^2(\nu^3+\nu)+\mu\nu^3)+\zeta_2(4\zeta_2^2\nu^2+(\mu+1)\nu^4-2\nu^2+1)+4\zeta_2\zeta_1^2\nu^2}{\begin{array}{c}4(\zeta_1^2(4\zeta_2^2((\mu+1)\nu^3+\nu)+\mu\nu^3)+\zeta_2\zeta_1(4\zeta_2^2(\mu+1)\nu^2+(\mu+1)^2\nu^4-2\nu^2+1)\\ +\zeta_2^2\mu\nu+4\zeta_2\zeta_1^3\nu^2)\end{array}} \tag{19}$$

On the other end the power absorbed by the DVA is the power dissipated by the damper c_2 and so the absorbed power can be written as:

$$P_{abs}(\omega) = \frac{1}{2}\mathrm{Re}\{\mathrm{f}_\mathrm{d}^*(\mathrm{j}\omega)[\mathrm{u}_1(\mathrm{j}\omega) - \mathrm{u}_2(\mathrm{j}\omega)]\} \tag{20}$$

Where * denotes complex conjugate and the force $\mathrm{f_d}$ is the force produced by the damper given by:

$$\mathrm{f_d}(\mathrm{j}\omega) = c_2(\mathrm{u}_1(\mathrm{j}\omega) - \mathrm{u}_2(\mathrm{j}\omega)) \tag{21}$$

Substituting equation (21) in (20) the absorbed power becomes:

$$P_{abs}(\omega) = \frac{1}{2}c_2|\mathrm{u}_1(\mathrm{j}\omega) - \mathrm{u}_2(\mathrm{j}\omega)|^2 \tag{22}$$

In this case the non-dimensional performance index is defined by:

$$I_\mathrm{p} = \frac{c_2 E[|\mathrm{u}_1 - \mathrm{u}_2|^2]}{2S_\mathrm{f}\omega_1/k_1} \tag{23}$$

which represents the ratio of power absorbed by the DVA to that generated by excitation force with a spectrum density S_f acting on a damper of value k_1/ω_1. The mean squared value of the relative velocity times the mechanical damping c_2 can be expressed as follow:

$$c_2 E[(u_1 - u_2)^2] = \frac{S_\mathrm{f}\omega_1}{k_1} 2\zeta_2\mu\nu \int_{-\infty}^{+\infty} |\Gamma - \Theta|^2 d\lambda \tag{24}$$

Thus the performance index becomes:

$$I_\mathrm{p} = \zeta_2\mu\nu \int_{-\infty}^{+\infty} \left| \frac{D_0 + (j\lambda)D_1 + (j\lambda)^2 D_2 + (j\lambda)^3 D_3}{A_0 + (j\lambda)A_1 + (j\lambda)^2 A_2 + (j\lambda)^3 A_3 + (j\lambda)^4 A_4} \right|^2 d\lambda \tag{25}$$

where

$$D_0 = A_0 - B_0 = 0$$
$$D_1 = A_1 - B_1 = 0$$
$$D_2 = A_2 - B_2 = 0$$
$$D_3 = A_3 - B_3 = \mu$$

The integral over the frequency band between $\pm\infty$ of equation (25) can be calculated using the expression given in reference [19], leading to:

$$I_p = \frac{\pi(\zeta_2\mu\nu(\zeta_2 + 4\zeta_1\zeta_2^2\nu + 4\zeta_1^2\zeta_2^2\nu^2 + \zeta_1(1+\mu)\nu^3))}{\begin{array}{c}2(\zeta_1^2(4\zeta_2^2((\mu+1)\nu^3+\nu)+\mu\nu^3)+\zeta_2\zeta_1(4\zeta_2^2(\mu+1)\nu^2+(\mu+1)^2\nu^4-2\nu^2+1)\\ +\zeta_2^2\mu\nu+4\zeta_2\zeta_1^3\nu^2)\end{array}} \tag{26}$$

Although the denominators are the same in equations (19) and (26), the dependence of their numerators on ζ_2 and ν is clearly different.

3.2. Minimisation of the total kinetic energy and maximisation of the power absorbed

In order to minimise the total kinetic energy of the primary mass m_1, the following conditions have to be satisfied:

$$\begin{cases} \dfrac{\partial I_\mathrm{k}}{\partial \zeta_2} = 0 \\ \dfrac{\partial I_\mathrm{k}}{\partial \nu} = 0 \end{cases} \tag{27}$$

while to maximise the total power absorbed by the DVA the following conditions have to be satisfy:

$$\begin{cases} \dfrac{\partial I_\mathrm{p}}{\partial \zeta_2} = 0 \\ \dfrac{\partial I_\mathrm{p}}{\partial \nu} = 0 \end{cases} \tag{28}$$

Differentiating the performance index I_k expressed in equation (19) with respect to ζ_2 and ν, and setting these equal to zero, yields a pair of simultaneous equations:

$$\begin{aligned} -\mu\nu[(\zeta_1^2\mu(\mu+1)\nu^6 + \zeta_2^2(4\zeta_1^2\nu^2(4\zeta_1^2\nu^2 + (2\mu-3)\nu^2 + 2) + (\mu+1)\nu^4 - 2\nu^2 + 1) \\ + 2\zeta_1\zeta_2\mu\nu^3(4\zeta_1^2\nu^2 + 1) + 4(4\zeta_1^2 - 1)\zeta_2^4\nu^2 + 8\zeta_1\zeta_2^3\nu(4\zeta_1^2\nu^2 - 2\nu^2 \\ + 1))] = 0 \end{aligned} \tag{29a}$$

$$\begin{aligned} -\zeta_2\mu[(\zeta_1^2\mu(\mu+1)\nu^6 + \zeta_2^2(4\zeta_1^2\nu^2(4\zeta_1^2\nu^2 + (2\mu+1)\nu^2 + 2) - 3(\mu+1)\nu^4 + 2\nu^2 + 1) \\ + 2\zeta_1\zeta_2\nu^3(4\zeta_1^2(\mu+1)\nu^2 - 2(\mu+1)\nu^2 + \mu + 2) + 4(4\zeta_1^2 - 1)\zeta_2^4\nu^2 \\ + 8\zeta_1\zeta_2^3\nu(4\zeta_1^2\nu^2 - \nu^2 + 1))] = 0 \end{aligned} \tag{29b}$$

Following the same procedure, the partial derivates of the performance index I_p expressed in equation (26) are given by:

$$\begin{aligned} \pi\zeta_1\mu\nu[\zeta_1^2\mu(\mu+1)\nu^6 + \zeta_2^2(4\zeta_1^2\nu^2(4\zeta_1^2\nu^2 + (2\mu-3)\nu^2 + 2) + (\mu+1)\nu^4 - 2\nu^2 + 1) \\ + 2\zeta_1\zeta_2\mu\nu^3(4\zeta_1^2\nu^2 + 1) + 4(4\zeta_1^2 - 1)\zeta_2^4\nu^2 \\ + 8\zeta_1\zeta_2^3\nu(4\zeta_1^2\nu^2 - 2\nu^2 + 1)] = 0 \end{aligned} \tag{30a}$$

$$\begin{aligned} \zeta_1\mu\nu[(\zeta_1^2\mu(\mu+1)\nu^6 + \zeta_2^2(4\zeta_1^2\nu^2(4\zeta_1^2\nu^2 + (2\mu+1)\nu^2 + 2) - 3(\mu+1)\nu^4 + 2\nu^2 + 1) \\ + 2\zeta_1\zeta_2\nu^3(4\zeta_1^2(\mu+1)\nu^2 - 2(\mu+1)\nu^2 + \mu + 2) + 4(4\zeta_1^2 - 1)\zeta_2^4\nu^2 \\ + 8\zeta_1\zeta_2^3\nu(4\zeta_1^2\nu^2 - \nu^2 + 1))] = 0 \end{aligned} \tag{30b}$$

Simultaneous equation (29) and (30) are both satisfied for $\zeta_2 = 0$ and $\nu = 0$ corresponding to maximising I_k and I_p. The other solutions can be found setting to zero the terms in squared brackets. If $\zeta_1 \neq 0$ the term in square brackets in equation (29a) is equal to the term in square brackets in equation (30a) and the term in square brackets in equation (29b) is equal to the term in square brackets in equation (30b) which means that minimum of the total kinetic energy and the maximum of the total power absorbed correspond.

If ζ_1 is equal zero the primary system is undamped. Equations (29a) and (29b) for $\partial I_k/\partial\zeta_2$ and $\partial I_k/\partial\nu$ then reduce to:

$$\begin{aligned} -1 + (2 + 4\zeta_2^2)\nu^2 - (1 + \mu)\nu^4 = 0 \\ -1 + (-2 + 4\zeta_2^2)\nu^2 + 3(1 + \mu)\nu^4 = 0 \end{aligned} \tag{31}$$

Solving the two equations simultaneously the two positive real optimal values of $\zeta_{2\mathrm{opt}}$ and ν_{opt} are obtained as:

$$\begin{cases} \zeta_{2\mathrm{opt}} = \dfrac{\sqrt{\mu}}{2} \\ \nu_{\mathrm{opt}} = \dfrac{1}{\sqrt{1+\mu}} \end{cases} \tag{32}$$

In this case the performance index I_{p}, however, becomes equal to $\pi/2$ if ζ_1 is exactly zero. The absorbed power is then independent on ζ_2 and ν, as can be seen from equation (30a) and equation (30b), since they both are proportional to ζ_1.

4. Comparison of tuning strategies

Provided that ζ_1 has a very small value thus singular condition will not occur, these optimum values of ζ_2 and ν will be the same for maximising power absorption as ζ_1 tends to zero. Figure 2 shows the performance index I_{p} as function of ζ_2 when ν is equal ν_{opt} (top plot) and I_{p} as function of ν when ζ_2 is equal $\zeta_{2\mathrm{opt}}$ (bottom plot) for different values of the primary damping ration ζ_1. The plot shows that when ζ_1 is equal zero the absorbed power is constant. As ζ_1 is increased the absorbed power has a maximum.

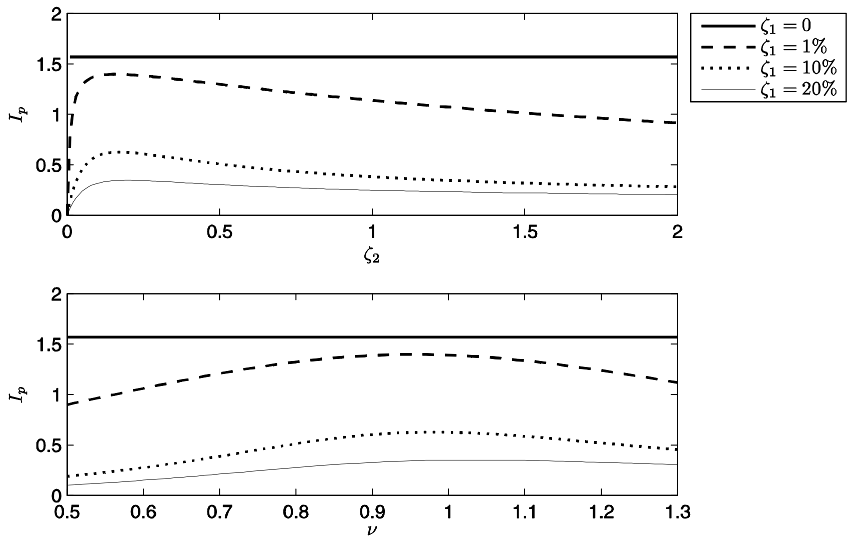

Figure 2: I_p as function of ζ_2 when $\nu = \nu_{opt}$ (top plot) and I_p as function of ν when $\zeta_2 = \zeta_{2opt}$ (bottom plot) for $\mu = 0.1$.

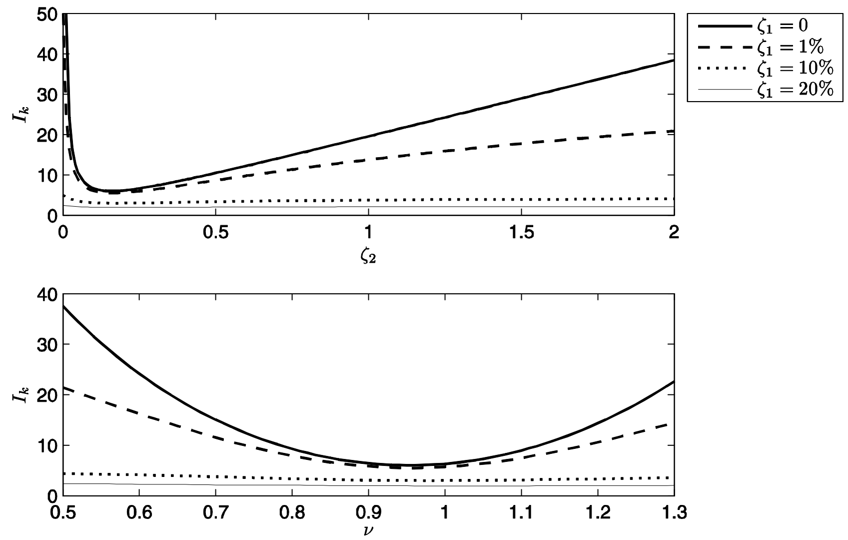

Figure 3: I_k as function of ζ_2 when $\nu = \nu_{opt}$ (top plot) and I_k as function of ν when $\zeta_2 = \zeta_{2opt}$ (bottom plot) for $\mu = 0.1$.

Figure 3 shows the performance index I_k as function of ζ_2 when ν is equal ν_{opt} (top plot) and I_k as function of ν when ζ_2 is equal ζ_{2opt} (bottom plot) for different values of the primary damping ration ζ_1. The plot shows that I_k is minimised for a single value of ζ_2 and ν. As ζ_1 is increased the gradient of I_k around the minimum decreases.

Figure 4(a) and (b) show the PSD of the velocity and displacement respectively of the primary mass in dimensionless form for five different strategies of tuning the DVA. In Figure 4(a) the area under the curve is minimised when the minimisation of kinetic energy is implemented. Figure 4(b) show that the H_∞ optimisation set the two peaks at the minimum magnitude and the area under the curve is minimised when the H_2

optimisation is implemented. The minimisation stability optimisation is not designed to minimise the steady state response but only the transient response.

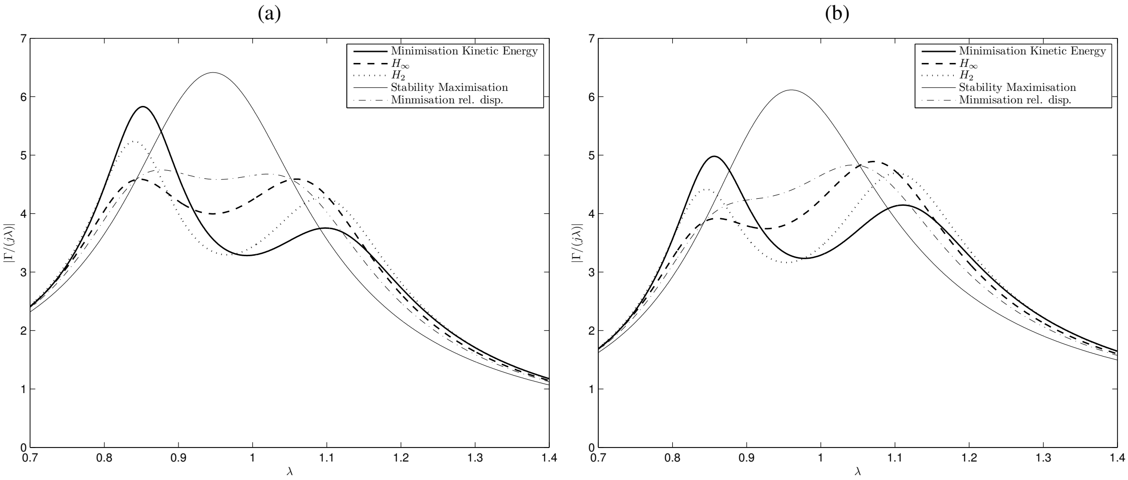

Figure 4: Optimal PSD a) of the dimensionless velocity and b) the displacement of the primary mass in dimensionless form when the four different criteria are implemented ($\zeta_1 = 0$, $\mu = 0.1$)

Figure 5 and Figure 6 show the optimal values of the frequency ratio and the damping ratio as function of the mass ratio for five different tuning strategies. The five tuning strategies give similar optimal values when μ is small. For grater values of μ the optimal conditions diverges. It is interesting to notice that for the minimisation of kinetic energy the optimal damping always increases for increasing values of μ. For all the other strategies the optimal damping ratio converges to a finite value.

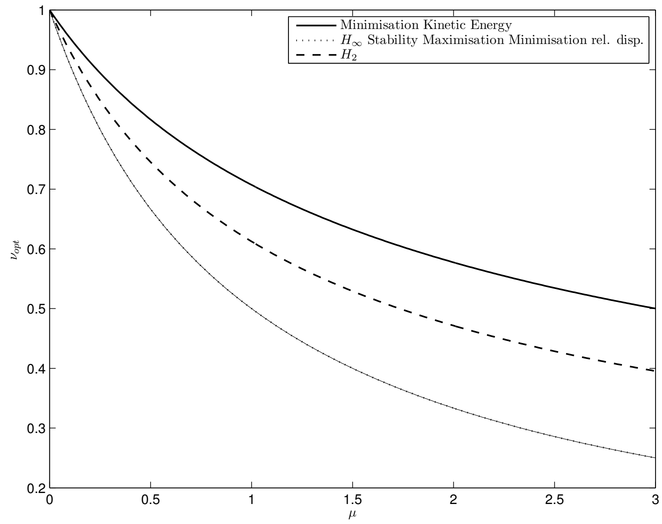

Figure 5: Optimal frequency ratio ν as function of the mass ratio μ for the 5 different tuning strategies

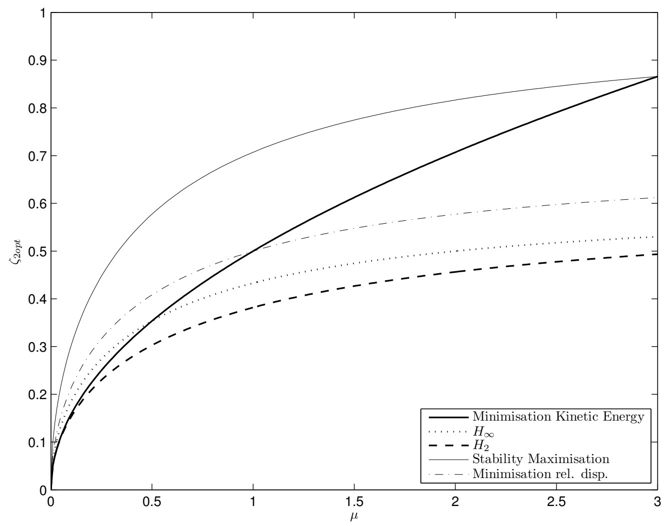

Figure 6: Optimal damping ratio ζ_2 as function of the mass ratio μ for the 5 different tuning strategies

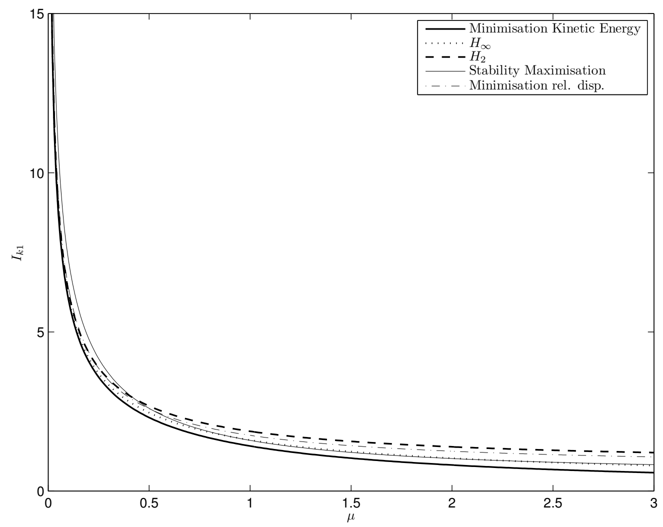

Figure 7: Performance index I_k as function of the mass ratio μ for the 5 different tuning strategies

Figure 7 shows the performance index I_k as function of the mass ratio μ when the optimal values for the different strategies are implemented. The curves in Figure 7 are obtained substituting the optimal value in Table 1 in equation (19). The plot shows that the lowest curve is the one obtained when the DVA is set to minimise the kinetic energy of the primary mass as one would expect.

5. Effect of damping in the host structures

It has not been possible to solve equations (29a) and (30a) when $\zeta_1 \neq 0$ in order to find analytical expression for $\zeta_{2\text{opt}}$ and ν_{opt}.

In this case only an approximate solution of the location of the minimum of the total kinetic energy and thus the maximum of the total absorbed power can be found using the perturbation method. First of all it is assumed that the primary damping ζ_1 is so small that it is regarded as a perturbation. To emphasize that ζ_1 is small a new symbol ε instead of the parameter ζ_1 is introduced:

$$\zeta_1 = \varepsilon \tag{33}$$

Next, the solutions of equation (29a) and (30a) (which it has been shown to be the same if $\zeta_1 \neq 0$) are assumed in the form of a power series of ε:

$$\begin{gathered} \nu = \upsilon_0 + \varepsilon\upsilon_1 + \varepsilon\upsilon_2^2 + \ldots \\ \zeta = \zeta_{20} + \varepsilon\zeta_{21} + \varepsilon\zeta_{22}^2 + \ldots \end{gathered} \tag{34}$$

Finally, equations (34) is substituted into equations (29a), and collect terms of like powers of ε and equate them to zero (starting with the constant terms, the terms containing ε, the terms containing ε^2, and so on) so that the equation is satisfied for all values of ε. As a result, we have a series of equations from which we can determinate the unknown coefficients in equation (34) successively. The zero-order approximation leads to the result where υ_0 and ζ_0 are the optimal values found in equations (32) when $\zeta_1 = 0$. Equating first order terms to zero, yields to:

$$\begin{gathered} a_1\zeta_{21} + a_2\nu_1 + a_3 = 0 \\ b_1\zeta_{21} + b_2\nu_1 + b_3 = 0 \end{gathered} \tag{35}$$

where

$a_1 = 2 + 2\mu$	$b_1 = 2\sqrt{\mu(1+\mu)}$
$a_2 = \sqrt{\mu(1+\mu)} + \mu^{3/2}\sqrt{1+\mu}$	$b_2 = 4 + 5\mu + \mu^2$
$a_3 = -2\mu\sqrt{1+\mu}$	$b_3 = -2\sqrt{\mu} - 2\mu^{3/2}$

In equations (35) the values of ν_0 and ζ_{20} have been already substituted. The solution of equitation (35) is given by:

$$\begin{gathered} \nu_1 = \frac{\sqrt{\mu}}{2 + 2\mu} \\ \zeta_{21} = \frac{3\mu}{4\sqrt{1+\mu}} \end{gathered} \tag{36}$$

The first order approximate solution of equations (29a) and (30a) is therefore given by:

$$\begin{gathered} \upsilon'_{\mathrm{opt}} = \frac{1}{\sqrt{1+\mu}} + \zeta_1\frac{\sqrt{\mu}}{2+2\mu} \\ \zeta'_{2\mathrm{opt}} = \frac{\sqrt{\mu}}{2} + \zeta_1\frac{3\mu}{4\sqrt{1+\mu}} \end{gathered} \tag{37}$$

Figure 8 a and b show the performance indexes I_k and I_p as function of the damping ratio ζ_2 and the frequency ratio ν respectively when $\zeta_1 = 0.2$ and $\mu = 0.1$. Figure 8 shows that I_k has a global minimum which corresponds to the global maximum of I_p represented by ○. The symbol × in Figure 8a and b mark the position of the optimum conditions when $\zeta_1 = 0$ while □ mark the first order approximate optimum given by equation (37).

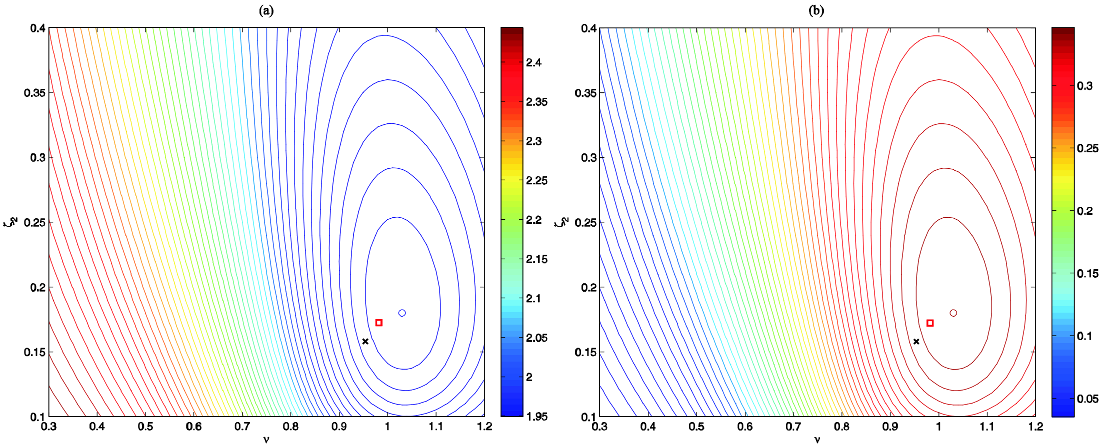

Figure 8: a) I_k and b) I_p when $\zeta_1 = 0.2$ and $\mu = 0.1$. The solutions given by equations (32) are shown as ×, the approximate solutions given in equation (37) are shown as □, and the true minimum and maximum are shown as ○

6. Discussion and Conclusions

It is shown that even if the damping of the host structure is not very light, the ratio of natural frequencies and absorber damping ratio that maximise the power dissipation in the absorber are the same as those that minimise the kinetic energy of the host structure. This may provide a method of self-tuning such a dynamic vibration absorber. If the power dissipation in the absorber could be measured and the disturbance was stationary, a tuning strategy might be used that is similar to that used for feedback controllers by Zilletti *et al.* (2010) [20]. This might be important if the damping mechanism of the absorber or the host structure were level dependant, for example, when subject to stationary disturbances.

One method of measuring the power dissipation within the absorber may be to measure its temperature. If the tuned vibration absorber was implemented with an efficient inertial electromagnetic actuator, most of the mechanical power dissipation could then be arranged to be the electrical power generated in a tuneable shunting impedance. It may be possible to use this power both to tune the absorber, by adjusting this electrical impedance, and also, using energy harvesting techniques, to power the electronic system used to implement the self-tuning.

Appendix A - Wolframe Mathematica programs

In this appendix there are the Wolframe Mathematica scripts to verify equations: (9), (10), (13), (14), (19), (26), (29), (31), (30) and (32)

In[1]:=
```
(* Equations (9) and (10) *)

Clear[S, ω, Y, y11, y12, M, K, Cc]
```

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix};$$

$$\mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix};$$

$$\mathbf{Cc} = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix};$$

```
S = -ω^2 * M + I * ω * Cc + K;

Y = I * ω * ExpandNumerator[ExpandDenominator[Inverse[S]]];

y11 = ExpandDenominator[ExpandNumerator[Y[[1, 1]]]]
```

Out[7]=

$$\frac{-c_2\,\omega^2 + i\,k_2\,\omega - i\,m_2\,\omega^3}{i\,c_2\,k_1\,\omega + i\,c_1\,k_2\,\omega - i\,c_2\,m_1\,\omega^3 - i\,c_1\,m_2\,\omega^3 - i\,c_2\,m_2\,\omega^3 - c_1\,c_2\,\omega^2 - k_2\,m_1\,\omega^2 - k_1\,m_2\,\omega^2 - k_2\,m_2\,\omega^2 + k_1\,k_2 + m_1\,m_2\,\omega^4}$$

In[8]:=
```
y12 = ExpandDenominator[ExpandNumerator[Y[[1, 2]]]]
```

Out[8]=

$$\frac{-c_2\,\omega^2 + i\,k_2\,\omega}{i\,c_2\,k_1\,\omega + i\,c_1\,k_2\,\omega - i\,c_2\,m_1\,\omega^3 - i\,c_1\,m_2\,\omega^3 - i\,c_2\,m_2\,\omega^3 - c_1\,c_2\,\omega^2 - k_2\,m_1\,\omega^2 - k_1\,m_2\,\omega^2 - k_2\,m_2\,\omega^2 + k_1\,k_2 + m_1\,m_2\,\omega^4}$$

In[9]:=
```
ExpandDenominator[ExpandNumerator[Simplify[y11 - y12]]]
```

Out[9]=

$$-\frac{i\,m_2\,\omega^3}{i\,c_2\,k_1\,\omega + i\,c_1\,k_2\,\omega - i\,c_2\,m_1\,\omega^3 - i\,c_1\,m_2\,\omega^3 - i\,c_2\,m_2\,\omega^3 - c_1\,c_2\,\omega^2 - k_2\,m_1\,\omega^2 - k_1\,m_2\,\omega^2 - k_2\,m_2\,\omega^2 + k_1\,k_2 + m_1\,m_2\,\omega^4}$$

In[1]:=
```
(* Equation (13) *)
Clear[μ, ν, λ]
```

$$\begin{aligned}
&\mathbf{A0} = \mu\,\nu^2;\\
&\mathbf{A1} = 2\,\zeta_2\,\mu\,\nu + 2\,\zeta_1\,\mu\,\nu^2;\\
&\mathbf{A2} = \mu\,\nu^2 + \mu + \mu^2\,\nu^2 + 4\,\zeta_1\,\zeta_2\,\mu\,\nu;\\
&\mathbf{A3} = 2\,\zeta_2\,\mu\,\nu + 2\,\zeta_2\,\mu^2\,\nu + 2\,\zeta_1\,\mu;\\
&\mathbf{A4} = \mu;\\
&\mathbf{B0} = 0;\\
&\mathbf{B1} = \mu\,\nu^2;\\
&\mathbf{B2} = 2\,\zeta_2\,\mu\,\nu;\\
&\mathbf{B3} = \mu;
\end{aligned}$$

$$\Gamma = \mathrm{ExpandDenominator}\left[\mathrm{ExpandNumerator}\left[\frac{\mathrm{B0} + (\mathrm{I}\,\lambda)\,\mathrm{B1} + (\mathrm{I}\,\lambda)^2\,\mathrm{B2} + (\mathrm{I}\,\lambda)^3\,\mathrm{B3}}{\mathrm{A0} + (\mathrm{I}\,\lambda)\,\mathrm{A1} + (\mathrm{I}\,\lambda)^2\,\mathrm{A2} + (\mathrm{I}\,\lambda)^3\,\mathrm{A3} + (\mathrm{I}\,\lambda)^4\,\mathrm{A4}}\right]\right]$$

Out[11]=

$$\frac{-2\,\zeta_2\,\lambda^2\,\mu\,\nu - i\,\lambda^3\,\mu + i\,\lambda\,\mu\,\nu^2}{-2\,i\,\zeta_2\,\lambda^3\,\mu^2\,\nu - 2\,i\,\zeta_2\,\lambda^3\,\mu\,\nu - 2\,i\,\zeta_1\,\lambda^3\,\mu - 4\,\zeta_1\,\zeta_2\,\lambda^2\,\mu\,\nu + 2\,i\,\zeta_1\,\lambda\,\mu\,\nu^2 + 2\,i\,\zeta_2\,\lambda\,\mu\,\nu + \lambda^4\,\mu - \lambda^2\,\mu^2\,\nu^2 - \lambda^2\,\mu\,\nu^2 - \lambda^2\,\mu + \mu\,\nu^2}$$

In[12]:=

$$\begin{aligned}
&\nu = \mathrm{k_2}^{1/2}\,\mathrm{m_1}^{1/2}\,\mathrm{k_1}^{-1/2}\,\mathrm{m_2}^{-1/2};\\
&\lambda = \omega\,\mathrm{m_1}^{1/2}\,\mathrm{k_1}^{-1/2};\\
&\mu = \mathrm{m_2}\,\mathrm{m_1}^{-1};\\
&\zeta_1 = 2^{-1}\,\mathrm{c_1}\,2^{-1}\,\mathrm{k_1}^{-1/2}\,\mathrm{m_1}^{-1/2};\\
&\zeta_2 = 2^{-1}\,\mathrm{c_2}\,\mathrm{k_2}^{-1/2}\,\mathrm{m_2}^{-1/2};
\end{aligned}$$

$$\mathrm{y11} = \mathrm{ExpandDenominator}\left[\mathrm{ExpandNumerator}\left[\mathrm{Simplify}\left[\frac{\mathrm{B0} + (\mathrm{I}\,\lambda)\,\mathrm{B1} + (\mathrm{I}\,\lambda)^2\,\mathrm{B2} + (\mathrm{I}\,\lambda)^3\,\mathrm{B3}}{\mathrm{A0} + (\mathrm{I}\,\lambda)\,\mathrm{A1} + (\mathrm{I}\,\lambda)^2\,\mathrm{A2} + (\mathrm{I}\,\lambda)^3\,\mathrm{A3} + (\mathrm{I}\,\lambda)^4\,\mathrm{A4}}\,\frac{1}{\sqrt{\mathrm{k_1}}\,\sqrt{\mathrm{m_1}}}\right]\right]\right]$$

Out[17]=

$$\frac{-2\,c_2\,\omega^2 + 2\,i\,k_2\,\omega - 2\,i\,m_2\,\omega^3}{2\,i\,c_2\,k_1\,\omega + i\,c_1\,k_2\,\omega - 2\,i\,c_2\,m_1\,\omega^3 - i\,c_1\,m_2\,\omega^3 - 2\,i\,c_2\,m_2\,\omega^3 - c_1\,c_2\,\omega^2 - 2\,k_2\,m_1\,\omega^2 - 2\,k_1\,m_2\,\omega^2 - 2\,k_2\,m_2\,\omega^2 + 2\,k_1\,k_2 + 2\,m_1\,m_2\,\omega^4}$$

In[1]:=
```
(* Equation (14) *)
Clear[μ, ν, λ]
A0 = μ ν^2;
A1 = 2 ζ2 μ ν + 2 ζ1 μ ν^2;
A2 = μ ν^2 + μ + μ^2 ν^2 + 4 ζ1 ζ2 μ ν;
A3 = 2 ζ2 μ ν + 2 ζ2 μ^2 ν + 2 ζ1 μ;
A4 = μ;
C0 = 0;
C1 = μ ν^2;
C2 = 2 ζ2 μ ν;
C3 = 0;

θ = ExpandDenominator[ExpandNumerator[(C0 + (I λ) C1 + (I λ)^2 C2 + (I λ)^3 C3)/(A0 + (I λ) A1 + (I λ)^2 A2 + (I λ)^3 A3 + (I λ)^4 A4)]]
```

Out[11]=

$$\frac{-2\zeta_2\lambda^2\mu\nu+i\lambda\mu\nu^2}{-2i\zeta_2\lambda^3\mu^2\nu-2i\zeta_2\lambda^3\mu\nu-2i\zeta_1\lambda^3\mu-4\zeta_1\zeta_2\lambda^2\mu\nu+2i\zeta_1\lambda\mu\nu^2+2i\zeta_2\lambda\mu\nu+\lambda^4\mu-\lambda^2\mu^2\nu^2-\lambda^2\mu\nu^2-\lambda^2\mu+\mu\nu^2}$$

In[12]:=
```
ν = k2^(1/2) m1^(1/2) k1^(-1/2) m2^(-1/2);
λ = ω m1^(1/2) k1^(-1/2);
μ = m2 m1^-1;
ζ1 = 2^-1 c1 2^-1 k1^(-1/2) m1^(-1/2);
ζ2 = 2^-1 c2 k2^(-1/2) m2^(-1/2);

y12 = ExpandDenominator[ExpandNumerator [Simplify[(C0 + (I λ) C1 + (I λ)^2 C2 + (I λ)^3 C3)/(A0 + (I λ) A1 + (I λ)^2 A2 + (I λ)^3 A3 + (I λ)^4 A4) 1/(Sqrt[k1] Sqrt[m1])]]]
```

Out[17]=

$$\frac{-2c_2\omega^2+2ik_2\omega}{2ic_2k_1\omega+ic_1k_2\omega-2ic_2m_1\omega^3-ic_1m_2\omega^3-2ic_2m_2\omega^3-c_1c_2\omega^2-2k_2m_1\omega^2-2k_1m_2\omega^2-2k_2m_2\omega^2+2k_1k_2+2m_1m_2\omega^4}$$

In[1]:=
```
(* Equation (19) *)
A0 = μ ν^2;
A1 = 2 ζ2 μ ν + 2 ζ1 μ ν^2;
A2 = μ ν^2 + μ + μ^2 ν^2 + 4 ζ1 ζ2 μ ν;
A3 = 2 ζ2 μ ν + 2 ζ2 μ^2 ν + 2 ζ1 μ;
A4 = μ;
B0 = 0;
B1 = μ ν^2;
B2 = 2 ζ2 μ ν;
B3 = μ;
Ik = Simplify[(A0 B3^2 (A0 A3 - A1 A2) + A0 A1 A4 (2 B1 B3 - B2^2) - A0 A3 A4 (B1^2 - 2 B0 B2) + A4 B0^2 (A1 A4 - A2 A3))/(A0 A4 (A0 A3^2 + A1^2 A4 - A1 A2 A3))]
```

Out[10]=

$$\frac{\zeta_1(4\zeta_2^2(\nu^3+\nu)+\mu\nu^3)+\zeta_2(4\zeta_2^2\nu^2+(\mu+1)\nu^4-2\nu^2+1)+4\zeta_2\zeta_1^2\nu^2}{2\left(\zeta_1^2(4\zeta_2^2((\mu+1)\nu^3+\nu)+\mu\nu^3)+\zeta_2\zeta_1\left(4\zeta_2^2(\mu+1)\nu^2+(\mu+1)^2\nu^4-2\nu^2+1\right)+\zeta_2^2\mu\nu+4\zeta_2\zeta_1^3\nu^2\right)}$$

In[11]:=
```
(* Equation (29a) *)
derIkzeta2 = FullSimplify[D[Ik, ζ2]]
```

Out[11]=

$$\frac{\mu\nu(\zeta_1^2(\mu+1)(-\mu)\nu^6-\zeta_2^2(4\zeta_1^2\nu^2(4\zeta_1^2\nu^2+(2\mu-3)\nu^2+2)+(\mu+1)\nu^4-2\nu^2+1)-2\zeta_1\zeta_2\mu\nu^3(4\zeta_1^2\nu^2+1)+4(1-4\zeta_1^2)\zeta_2^4\nu^2-8\zeta_1\zeta_2^3\nu(4\zeta_1^2\nu^2-2\nu^2+1))}{2\left(\zeta_1^2\mu\nu^3+4\zeta_1\zeta_2^3(\mu+1)\nu^2+\zeta_2^2\nu(4\zeta_1^2((\mu+1)\nu^2+1)+\mu)+\zeta_1\zeta_2\left(4\zeta_1^2\nu^2+(\mu+1)^2\nu^4-2\nu^2+1\right)\right)^2}$$

In[12]:=
```
(* Equation (29b) *)
derIkni = FullSimplify[D[Ik, ν]]
```

Out[12]=

$$\frac{\zeta_2\mu(\zeta_1^2(\mu+1)(-\mu)\nu^6-\zeta_2^2(4\zeta_1^2\nu^2(4\zeta_1^2\nu^2+(2\mu+1)\nu^2+2)-3(\mu+1)\nu^4+2\nu^2+1)-2\zeta_1\zeta_2\nu^3(4\zeta_1^2(\mu+1)\nu^2-2(\mu+1)\nu^2+\mu+2)+4(1-4\zeta_1^2)\zeta_2^4\nu^2+8\zeta_1\zeta_2^3\nu(-4\zeta_1^2\nu^2+\nu^2-1))}{2\left(\zeta_1^2\mu\nu^3+4\zeta_1\zeta_2^3(\mu+1)\nu^2+\zeta_2^2\nu(4\zeta_1^2((\mu+1)\nu^2+1)+\mu)+\zeta_1\zeta_2\left(4\zeta_1^2\nu^2+(\mu+1)^2\nu^4-2\nu^2+1\right)\right)^2}$$

In[13]:=
```
(* Equation (31) *)
ζ1 = 0;
Simplify[Numerator[derIkzeta2]]
Simplify[Numerator[derIkni]]
```

Out[14]= $\zeta_2^2\left(4\zeta_2^2\nu^2-(\mu+1)\nu^4+2\nu^2-1\right)$

Out[15]= $\zeta_2^2\left(4\zeta_2^2\nu^2+3(\mu+1)\nu^4-2\nu^2-1\right)$

In[16]:=
```
(* Equation (32) *)
Solve[{4 ζ2^2 ν^2 - (μ + 1) ν^4 + 2 ν^2 - 1 = 0, 4 ζ2^2 ν^2 + 3 (μ + 1) ν^4 - 2 ν^2 - 1 = 0}, {ζ2, ν}]
```

Out[16]=

$$\left\{\left\{\zeta_2\to-\frac{\sqrt{\mu}}{2},\nu\to-\frac{1}{\sqrt{\mu+1}}\right\},\left\{\zeta_2\to-\frac{\sqrt{\mu}}{2},\nu\to\frac{1}{\sqrt{\mu+1}}\right\},\left\{\zeta_2\to\frac{\sqrt{\mu}}{2},\nu\to-\frac{1}{\sqrt{\mu+1}}\right\},\left\{\zeta_2\to\frac{\sqrt{\mu}}{2},\nu\to\frac{1}{\sqrt{\mu+1}}\right\}\right\}$$

In[1]:=
```
(* Equation (26)*)
A0 = μ ν^2;
A1 = 2 ζ2 μ ν + 2 ζ1 μ ν^2;
A2 = μ ν^2 + μ + μ^2 ν^2 + 4 ζ1 ζ2 μ ν;
A3 = 2 ζ2 μ ν + 2 ζ2 μ^2 ν + 2 ζ1 μ;
A4 = μ;
D0 = 0;
D1 = 0;
D2 = 0;
D3 = μ;

integral = Simplify[π (A0 D3^2 (A0 A3 - A1 A2) + A0 A1 A4 (2 D1 D3 - D2 ^ 2) - A0 A3 A4 (D1 ^ 2 - 2 D0 D2) + A4 D0 ^ 2 (A1 A4 - A2 A3)) / (A0 A4 (A0 A3 ^ 2 + A1 ^ 2 A4 - A1 A2 A3))];

Ip = Simplify[ζ2 μ ν integral]
```

Out[11]=

$$\frac{\pi \zeta_2 \mu \nu (\zeta_1 (4 \zeta_2^2 \nu + (\mu+1) \nu^3) + 4 \zeta_2 \zeta_1^2 \nu^2 + \zeta_2)}{2 \left(\zeta_1^2 (4 \zeta_2^2 ((\mu+1) \nu^3 + \nu) + \mu \nu^3) + \zeta_2 \zeta_1 \left(4 \zeta_2^2 (\mu+1) \nu^2 + (\mu+1)^2 \nu^4 - 2 \nu^2 + 1\right) + \zeta_2^2 \mu \nu + 4 \zeta_2 \zeta_1^3 \nu^2\right)}$$

In[12]:=
```
(* Equation (30a)*)
derIpzeta2 = FullSimplify[D[Ip, ζ2]]
```

Out[12]=

$$\frac{\pi \zeta_1 \mu \nu (\zeta_1^2 \mu (\mu+1) \nu^6 + \zeta_2^2 (4 \zeta_1^2 \nu^2 (4 \zeta_1^2 \nu^2 + (2\mu-3) \nu^2 + 2) + (\mu+1) \nu^4 - 2 \nu^2 + 1) + 2 \zeta_1 \zeta_2 \mu \nu^3 (4 \zeta_1^2 \nu^2 + 1) + 4 (4 \zeta_1^2 - 1) \zeta_2^4 \nu^2 + 8 \zeta_1 \zeta_2^3 \nu (4 \zeta_1^2 \nu^2 - 2 \nu^2 + 1))}{2 \left(\zeta_1^2 \mu \nu^3 + 4 \zeta_1 \zeta_2^3 (\mu+1) \nu^2 + \zeta_2^2 \nu (4 \zeta_1^2 ((\mu+1) \nu^2 + 1) + \mu) + \zeta_1 \zeta_2 \left(4 \zeta_1^2 \nu^2 + (\mu+1)^2 \nu^4 - 2 \nu^2 + 1\right)\right)^2}$$

In[13]:=
```
(* Equation (30b)*)
derIpni = FullSimplify[D[Ip, ν]]
```

Out[13]=

$$\frac{\pi \zeta_1 \zeta_2 \mu (\zeta_1^2 \mu (\mu+1) \nu^6 + \zeta_2^2 (4 \zeta_1^2 \nu^2 (4 \zeta_1^2 \nu^2 + (2\mu+1) \nu^2 + 2) - 3 (\mu+1) \nu^4 + 2 \nu^2 + 1) + 2 \zeta_1 \zeta_2 \nu^3 (4 \zeta_1^2 (\mu+1) \nu^2 - 2 (\mu+1) \nu^2 + \mu + 2) + 4 (4 \zeta_1^2 - 1) \zeta_2^4 \nu^2 + 8 \zeta_1 \zeta_2^3 \nu (4 \zeta_1^2 \nu^2 - \nu^2 + 1))}{2 \left(\zeta_1^2 \mu \nu^3 + 4 \zeta_1 \zeta_2^3 (\mu+1) \nu^2 + \zeta_2^2 \nu (4 \zeta_1^2 ((\mu+1) \nu^2 + 1) + \mu) + \zeta_1 \zeta_2 \left(4 \zeta_1^2 \nu^2 + (\mu+1)^2 \nu^4 - 2 \nu^2 + 1\right)\right)^2}$$

In[14]:=
```
ζ1 = 0;
FullSimplify[Ip]
```

Out[15]=

$$\frac{\pi}{2}$$

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