The ℓ^2-cohomology of hyperplane complements

M. W. Davis *   T. Januszkiewicz†   I. J. Leary ‡

March 12, 2007

Abstract

We compute the ℓ^2-Betti numbers of the complement of a finite collection of affine hyperplanes in $\mathbb{C}^n$. At most one of the ℓ^2-Betti numbers is non-zero.

AMS classification numbers. Primary: 52B30 Secondary: 32S22, 52C35, 57N65, 58J22.

Keywords: hyperplane arrangements, L^2-cohomology.

1 Introduction

Suppose X is a finite CW complex with universal cover $\widetilde{X}$. For each $p \geq 0$, one can associate to X a Hilbert space, $\mathcal{H}^p(\widetilde{X})$, the p-dimensional "reduced ℓ^2-cohomology," cf. [3]. Each $\mathcal{H}^p(\widetilde{X})$ is a unitary $\pi_1(X)$-module. Using the $\pi_1(X)$-action, one can attach a nonnegative real number called "von Neumann dimension" to such a Hilbert space. The "dimension" of $\mathcal{H}^p(\widetilde{X})$ is called the p^{th} *ℓ^2-Betti number of X*.

Here we are interested in the case where X is the complement of a finite number of affine hyperplanes in $\mathbb{C}^n$. (Technically, in order to be in compliance with the first paragraph, we should replace the complement by a homotopy equivalent finite CW complex. However, to keep from pointlessly complicating the notation, we shall ignore this technicality.) Let $\mathcal{A}$ be the finite collection of hyperplanes, $\Sigma(\mathcal{A})$ their union and $M(\mathcal{A}) := \mathbb{C}^n - \Sigma(\mathcal{A})$.

*The first author was partially supported by NSF grant DMS 0405825.

†The second author was partially supported by NSF grant DMS 0405825.

‡The third author was partially supported by NSF grant DMS 0505471.

The *rank* of $\mathcal{A}$ is the maximum codimension l of any nonempty intersection of hyperplanes in $\mathcal{A}$. It turns out that the ordinary (reduced) homology of $\Sigma(\mathcal{A})$ vanishes except in dimension $l-1$ (cf. Proposition 2.1). Let $\beta(\mathcal{A})$ denote the rank of $\overline{H}_{l-1}(\Sigma(\mathcal{A}))$. Our main result, proved as Theorem 6.2, is the following.

Theorem A. *Suppose $\mathcal{A}$ is an affine hyperplane arrangement of rank l. Only the l^{th} ℓ^2-Betti number of $M(\mathcal{A})$ can be nonzero and it is equal to $\beta(\mathcal{A})$.*

This is reminiscent of a well-known result about the cohomology of $M(\mathcal{A})$ with coefficients in a generic flat line bundle ("generic" is defined in Section 5). This result is proved as Theorem 5.3. We state it below.

Theorem B. *Suppose that L is a generic flat line bundle over $M(\mathcal{A})$. Then $H^*(M(\mathcal{A}); L)$ vanishes except in dimension l and* $\dim_{\mathbb{C}} H^l(M(\mathcal{A}); L) = \beta(\mathcal{A})$.

Both theorems have similar proofs. In the case of Theorem A the basic fact is that the ℓ^2-Betti numbers of S^1 vanish. (In other words, if the universal cover $\mathbb{R}$ of S^1 is given its usual cell structure, then $\mathcal{H}^*(\mathbb{R}) = 0$.) Similarly, for Theorem B, if L is a flat line bundle over S^1 corresponding to an element $\lambda \in \mathbb{C}^*$, with $\lambda \neq 1$, then $H^*(S^1; L) = 0$. By the Künneth Formula, there are similar vanishing results for any central arrangement. To prove the general results, one considers an open cover of $M(\mathcal{A})$ by "small" open neighborhoods each homeomorphic to the complement of a central arrangement. The E_1-page of the resulting Mayer-Vietoris spectral sequence is nonzero only along the bottom row, where it can be identified with the simplicial cochains with constant coefficients on a pair $(N(\mathcal{U}), N(\mathcal{V}))$, which is homotopy equivalent to $(\mathbb{C}^n, \Sigma)$. It follows that the E_2-page can be nonzero only in position $(l, 0)$. (Actually, in the case of Theorem A, technical modifications must be made to the above argument. Instead of reduced ℓ^2-cohomology one takes local coefficients in the von Neumann algebra associated to the fundamental group and the vanishing results only hold modulo modules which don't contribute to the ℓ^2-Betti numbers.)

In [2] the first and third authors proved a similar result for the ℓ^2-cohomology of the universal cover of the Salvetti complex associated to an arbitrary Artin group (as well as a formula for the cohomology of the Salvetti complex with generic, 1-dimensional local coefficients). This can be interpreted as a computation of the ℓ^2-cohomology of universal covers of hyperplane complements associated to infinite reflection groups. Although the main argument in [2] uses an explicit description of the chain complex of the Salvetti complex, an alternative argument, similar to the one outlined above, is given in [2, Section 10].

Our thanks to the referee for finding some mistakes in the first version of this paper.

2 Hyperplane arrangements

A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes in $\mathbb{C}^n$. A *subspace* of $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$. Denote by $L(\mathcal{A})$ the poset of subspaces, partially ordered by inclusion, and let $\overline{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbb{C}^n\}$. An arrangement is *central* if $L(\mathcal{A})$ has a minimum element. Given $G \in L(\mathcal{A})$, its *rank*, $\text{rk}(G)$, is the codimension of G in $\mathbb{C}^n$. The minimal elements of $L(\mathcal{A})$ are a family of parallel subspaces and they all have the same rank. The *rank* of an arrangement $\mathcal{A}$ is the rank of a minimal element in $L(\mathcal{A})$. $\mathcal{A}$ is *essential* if $\text{rk}(\mathcal{A}) = n$.

The *singular set* $\Sigma(\mathcal{A})$ of the arrangement is the union of hyperplanes in $\mathcal{A}$ (so that $\Sigma(\mathcal{A})$ is a subset of $\mathbb{C}^n$). The complement of $\Sigma(\mathcal{A})$ in $\mathbb{C}^n$ is denoted $M(\mathcal{A})$. When there is no ambiguity we will drop the "$\mathcal{A}$" from our notation and write L, Σ or M instead of $L(\mathcal{A})$, $\Sigma(\mathcal{A})$ or $M(\mathcal{A})$.

Proposition 2.1. *Σ is homotopy equivalent to a wedge of $(l-1)$-spheres, where $l = \text{rk}(\mathcal{A})$. (So, if $\mathcal{A}$ is essential, the spheres are $(n-1)$-dimensional.)*

Proof. The proof follows from the usual "deletion-restriction" argument and induction. If the rank l is 1, then Σ is the disjoint union of a finite family of parallel hyperplanes. Hence, Σ is homotopy equivalent to a finite set of points, i.e., to a wedge of 0-spheres. Similarly, when $l = 2$, it is easy to see that Σ is homotopy equivalent to a connected graph; hence, a wedge of 1-spheres. So, assume by induction that $l > 2$. Choose a hyperplane $H \in \mathcal{A}$, let $\mathcal{A}' = \mathcal{A} - \{H\}$ and let $\mathcal{A}''$ be the restriction of $\mathcal{A}$ to H (i.e., $\mathcal{A}'' := \{H' \cap H \mid H' \in \mathcal{A}'\}$). Put $\Sigma' = \Sigma(\mathcal{A}')$, $\Sigma'' = \Sigma(\mathcal{A}'')$, $l' = \text{rk}(\mathcal{A}')$ and $l'' = \text{rk}(\mathcal{A}'')$. We can also assume by induction on $\text{Card}(\mathcal{A})$ that Σ' and Σ'' are homotopy equivalent to wedges of spheres. If $l' < n$ and H is transverse to the minimal elements of $L(\mathcal{A}')$, then $l'' = l$, the arrangement splits as a product, $\Sigma = \Sigma'' \times \mathbb{C}$, and we are done by induction. In all other cases $l' = l$ and $l'' = l - 1$. We have $\Sigma = \Sigma' \cup H$ and $\Sigma' \cap H = \Sigma''$. H is simply connected and since $l > 2$, Σ' is simply connected and Σ'' is connected. By van Kampen's Theorem, Σ is simply connected. Consider the exact sequence of the pair (Σ, Σ'):

$$\to H_*(\Sigma') \to H_*(\Sigma) \to H_*(\Sigma, \Sigma') \to .$$

There is an excision isomorphism, $H_*(\Sigma, \Sigma') \cong H_*(H, \Sigma'')$. Since H is contractible it follows that $H_*(H, \Sigma'') \cong \overline{H}_{*-1}(\Sigma'')$. By induction, $\overline{H}_*(\Sigma')$ is concentrated in dimension $l-1$ and $\overline{H}_*(\Sigma'')$ in dimension $l-2$. So, $\overline{H}_*(\Sigma)$ is also concentrated in dimension $l-1$. It follows that Σ is homotopy equivalent to a wedge of $l-1$ spheres. □

3 Certain covers and their nerves

Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ is a cover of some space X (where I is some index set). Given a subset $\sigma \subset I$, put $U_\sigma := \bigcap_{i\in\sigma} U_i$. Recall that the *nerve* of $\mathcal{U}$ is the simplicial complex $N(\mathcal{U})$, defined as follows. Its vertex set is I and a finite, nonempty subset $\sigma \subset I$ spans a simplex of $N(\mathcal{U})$ if and only if U_σ is nonempty.

We shall need to use the following well-known lemma several times in the sequel, see [4, Cor. 4G.3 and Ex. 4G(4)]

Lemma 3.1. *Let $\mathcal{U}$ be a cover of a paracompact space X and suppose that either (a) each U_i is open or (b) X is a CW complex and each U_i is a subcomplex. Further suppose that for each simplex σ of $N(\mathcal{U})$, U_σ is contractible. Then X and $N(\mathcal{U})$ are homotopy equivalent.*

Suppose $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{C}^n$. An open convex subset U in $\mathbb{C}^n$ is *small* (with respect to $\mathcal{A}$) if the following two conditions hold:

(i) $\{G \in \overline{L}(\mathcal{A}) \mid G \cap U \neq \emptyset\}$ has a unique minimum element $\mathrm{Min}(U)$.

(ii) A hyperplane $H \in \mathcal{A}$ has nonempty intersection with U if and only if $\mathrm{Min}(U) \subset H$.

The intersection of two small convex open sets is also small; hence, the same is true for any finite intersection of such sets.

Now let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of $\mathbb{C}^n$ by small convex sets. (Such covers clearly exist.) Put

$$\mathcal{U}_{\text{sing}} := \{U \in \mathcal{U} \mid U \cap \Sigma \neq \emptyset\}.$$

Lemma 3.2. *$N(\mathcal{U})$ is a contractible simplicial complex and $N(\mathcal{U}_{\mathrm{sing}})$ is a subcomplex homotopy equivalent to Σ. Moreover, $H_*(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$ is concentrated in dimension l, where $l = \operatorname{rk} \mathcal{A}$.*

Proof. $\mathcal{U}_{\mathrm{sing}}$ is an open cover of a neighborhood of Σ which deformation retracts onto Σ. For each simplex σ of $N(\mathcal{U})$, U_σ is contractible (in fact, it is a small convex open set). By Lemma 3.1, $N(\mathcal{U})$ is homotopy equivalent to $\mathbb{C}^n$ and $N(\mathcal{U}_{\mathrm{sing}})$ is homotopy equivalent to Σ. The last sentence of the lemma follows from Proposition 2.1. □

Remark 3.3. Lemma 3.1 can also be used to show that the geometric realization of L is homotop[y equivalent to Σ.

Definition 3.4. $\beta(\mathcal{A})$ is the rank of $H_l(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$.

Equivalently, $\beta(\mathcal{A})$ is the rank of $H_l(\mathbb{C}^n, \Sigma(\mathcal{A}))$ (or of $\overline{H}_{l-1}(\Sigma(\mathcal{A}))$. Also, it is not difficult to see that $(-1)^l\beta(\mathcal{A}) = \chi(\mathbb{C}^n, \Sigma) = 1 - \chi(\Sigma) = \chi(M)$, where $\chi(\,)$ denotes the Euler characteristic.

Remark 3.5. Suppose $\mathcal{A}_{\mathbb{R}}$ is an arrangement of real hyperplanes in $\mathbb{R}^n$ and $\Sigma_{\mathbb{R}} \subset \mathbb{R}^n$ is the singular set. Then $\mathbb{R}^n - \Sigma_{\mathbb{R}}$ is a union of open convex sets called *chambers* and $\beta(\mathcal{A}_{\mathbb{R}})$ is the number of bounded chambers. If $\mathcal{A}$ is the complexification of $\mathcal{A}_{\mathbb{R}}$, then $\Sigma(\mathcal{A}) \sim \Sigma(\mathcal{A}_{\mathbb{R}})$. Hence, $\beta(\mathcal{A}) = \beta(\mathcal{A}_{\mathbb{R}})$.

For any small convex open set U, put

$$\widehat{U} := U - \Sigma(\mathcal{A}) = U \cap M(\mathcal{A}).$$

Since U is convex, $(U, U \cap \Sigma(\mathcal{A}))$ is homeomorphic to $(\mathbb{C}^n, \Sigma(\mathcal{A}_G))$, where $G = \operatorname{Min}(U)$ and $\mathcal{A}_G$ is the central subarrangement defined by

$$\mathcal{A}_G := \{H \in \mathcal{A} \mid G \subset H\}.$$

(G might be $\mathbb{C}^n$, in which case $\mathcal{A}_G = \emptyset$.) Hence, $\widehat{U}$ is homeomorphic to $M(\mathcal{A}_G)$, the complement of a central subarrangement.

The next lemma is well-known.

Lemma 3.6. *Suppose U is a small open convex set. Then $\pi_1(\widehat{U})$ is a retract of $\pi_1(M(\mathcal{A}))$.*

Proof. The composition of the two inclusions, $\widehat{U} \hookrightarrow M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_G)$ is a homotopy equivalence (where $G = \operatorname{Min}(U) \in L(\mathcal{A})$). □

By intersecting the elements of $\mathcal{U}$ with M $(= \mathbb{C}^n - \Sigma)$ we get an induced cover $\widehat{\mathcal{U}}$ of M. An element of $\widehat{\mathcal{U}}$ is a deleted small convex open set $\widehat{U}$ for some $U \in \mathcal{U}$. Similarly, by intersecting $\mathcal{U}_{\text{sing}}$ with M we get an induced cover $\widehat{\mathcal{U}}_{\text{sing}}$ of a deleted neighborhood of Σ. The key observation is the following.

Observation 3.7. *$N(\widehat{\mathcal{U}}) = N(\mathcal{U})$ and $N(\widehat{\mathcal{U}}_{\text{sing}}) = N(\mathcal{U}_{\text{sing}})$.*

4 The Mayer-Vietoris spectral sequence

Let X be a space, $\pi = \pi_1(X)$ and $r : \widetilde{X} \to X$ the universal cover. Given a left π-module A, define

$$C^*(X; A) := \mathrm{Hom}_\pi(C_*(\widetilde{X}), A),$$

the cochains with *local coefficients in A*. Taking cohomology gives $H^*(X; A)$.

Let $\mathcal{U}$ be an open cover of X and $N = N(\mathcal{U})$ its nerve. Let $N^{(p)}$ denote the set of p-simplices in N. There is an induced cover $\widetilde{\mathcal{U}} := \{r^{-1}(U)\}_{U \in \mathcal{U}}$ with the same nerve. Suppose that for each simplex σ of N, U_σ is connected and that $\pi_1(U_\sigma) \to \pi_1(X)$ is injective. (This means that $r^{-1}(U_\sigma)$ is a disjoint union of copies of the universal cover $\widetilde{U}_\sigma$.) There is a Mayer-Vietoris double complex

$$C_{p,q} = \bigoplus_{\sigma \in N^{(p)}} C_q(r^{-1}(U_\sigma))$$

(cf. [1, §VII.4]) and a corresponding double cochain complex with local coefficients:

$$C^{p,q}(A) := \mathrm{Hom}_\pi(C_{p,q}; A).$$

The cohomology of the total complex is $H^*(X; A)$. Now suppose that for each simplex σ of N, U_σ is connected and that $\pi_1(U_\sigma) \to \pi_1(X)$ is injective. (This means that $r^{-1}(U_\sigma)$ is a disjoint union of copies of the universal cover $\widetilde{U}_\sigma$.) We get a spectral sequence with E_1-page

$$E_1^{p,q} = \bigoplus_{\sigma \in N^{(p)}} H^q(U_\sigma; A). \tag{1}$$

Here $H^q(U_\sigma; A)$ means the cohomology of $\mathrm{Hom}_\pi(C_*(r^{-1}(U_\sigma)), A)$ or equivalently, of $\mathrm{Hom}_{\pi_1(U_\sigma)}(C_*(\widetilde{U}_\sigma); A)$. The E_2-page has the form $E^{p,q} = H^p(N; \mathfrak{H}^q)$, where $\mathfrak{H}^q$ means the functor $\sigma \to H^q(U_\sigma; A)$. The spectral sequence converges to $H^*(X; A)$.

In the next two sections we will apply this spectral sequence to the case where X is $M(\mathcal{A})$ and the open cover is $\widehat{\mathcal{U}}$ from the previous section. By Lemma 3.6, $\pi_1(\widehat{U}_\sigma) \to \pi_1(M(\mathcal{A})$ is injective so we get a spectral sequence with E_1-page given by (1). Moreover, the π-module A will be such that for any simplex σ in $N(\widehat{\mathcal{U}}_{\text{sing}})$, $H^q(\widehat{U}_\sigma; A) = 0$ for all q (even for $q = 0$) while for a simplex σ of $N(\widehat{\mathcal{U}})$ which is not in $N(\widehat{\mathcal{U}}_{\text{sing}})$, $H^q(U_\sigma; A) = 0$ for all $q > 0$ and is constant (i.e., independent of σ) for $q = 0$. Thus $E_1^{p,q}$ will vanish for $q > 0$ and $E_1^{*,0}$ can be identified with the cochain complex $C^*(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ with constant coefficients.

5 Generic coefficients

Here we will deal with 1-dimensional local coefficient systems. We begin by considering such local coefficients on S^1. Let α be a generator of the infinite cyclic group $\pi_1(S^1)$. Suppose k is a field of characteristic 0 and $\lambda \in k^*$. Let A_λ be the $k[\pi_1(S^1)]$-module which is a 1-dimensional k-vector space on which α acts by multiplication by λ.

Lemma 5.1. *If $\lambda \neq 1$, then $H^*(S^1; A_\lambda)$ vanishes identically.*

Proof. If S^1 has its usual CW structure with one 0-cell and one 1-cell, then in the chain complex for its universal cover both C_0 and C_1 are identified with the group ring $k[\pi_1(S^1)]$ and the boundary map with multiplication by $1 - t$, where t is the generator of $\pi_1(S^1)$. Hence, the coboundary map $C^0(S^1; A_\lambda) \to C^1(S^1; A_\lambda)$ is multiplication by $1 - \lambda$. □

Next, consider $M(\mathcal{A})$. Its fundamental group π is generated by loops a_H for $H \in \mathcal{A}$, where the loop a_H goes once around the hyperplane H in the "positive" direction. Let α_H denote the image of a_H in $H_1(M(\mathcal{A}))$. Then $H_1(M(\mathcal{A}))$ is free abelian with basis $\{\alpha_H\}_{H\in\mathcal{A}}$. So, a homomorphism $H_1(M(\mathcal{A})) \to k^*$ is determined by an $\mathcal{A}$-tuple $\Lambda \in (k^*)^{\mathcal{A}}$, where $\Lambda = (\lambda_H)_{H\in\mathcal{A}}$ corresponds to the homomorphism sending α_H to λ_H. Let $\psi_\Lambda : \pi \to k^*$ be the composition of this homomorphism with the abelianization map $\pi \to H_1(M(\mathcal{A}))$. The resulting local coefficient system on $M(\mathcal{A})$ is denoted A_Λ. The next lemma follows from Lemma 5.1.

Lemma 5.2. *Suppose $\mathcal{A}$ is a nonempty central arrangement and Λ is such that $\prod_{H\in\mathcal{A}} \lambda_H \neq 1$. Then $H^q(M(\mathcal{A}))$ vanishes for all q.*

Proof. Without loss of generality we can suppose the elements of $\mathcal{A}$ are linear hyperplanes. The Hopf bundle $M(\mathcal{A}) \to M(\mathcal{A})/S^1$ is trivial (cf. [6, Prop. 5.1, p. 158]); so, $M(\mathcal{A}) \cong B \times S^1$, where $B = M(\mathcal{A})/S^1$. Let $i : S^1 \to M(\mathcal{A})$ be inclusion of the fiber. The induced map on $H_1(\ )$ sends α to $\sum \alpha_H$. Thus, if we pull back A_Λ to S^1, we get A_λ, where $\lambda = \prod_{H \in \mathcal{A}} \lambda_H$. The condition on Λ is $\lambda \neq 1$, which by Lemma 5.1 implies that $H^*(S^1; A_\lambda)$ vanishes identically. By the Künneth Formula $H^*(M(\mathcal{A}); A_\Lambda)$ also vanishes identically. □

Returning to the case where $\mathcal{A}$ is a general arrangement, for each simplex σ in $N(\widehat{\mathcal{U}})$, let $\mathcal{A}_\sigma := \mathcal{A}_{\mathrm{Min}(U_\sigma)}$ be the corresponding central arrangement (so that $\widehat{U}_\sigma \cong M(\mathcal{A}_\sigma)$). Given $\Lambda \in (k^*)^{\mathcal{A}}$, put

$$\lambda_\sigma := \prod_{H \in \mathcal{A}_\sigma} \lambda_H.$$

Call Λ *generic* if $\lambda_\sigma \neq 1$ for all $\sigma \in N(\mathcal{U}_{\mathrm{sing}})$.

Theorem 5.3. (Compare [7, Thm. 4.6, p. 160]). *Let $\mathcal{A}$ be an affine arrangement of rank l and Λ a generic $\mathcal{A}$-tuple in k^*. Then $H^*(M(\mathcal{A}); A_\Lambda)$ is concentrated in degree l and*

$$\dim_k H^l(M(\mathcal{A}); A_\Lambda) = \beta(\mathcal{A}).$$

Proof. We have an open cover of $\widetilde{M}(\mathcal{A})$, $\{r^{-1}(\widehat{U})\}_{U \in \mathcal{U}}$. By Observation 3.7, its nerve is $N(\mathcal{U})$. By Lemma 5.2 and the last paragraph of Section 4, the E_1-page of the Mayer-Vietoris spectral sequence is concentrated along the bottom row where it can be identified with $C^*(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}); k)$. So, the E_2-page is concentrated on the bottom row and $E_2^{p,0} = H^p(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}); k)$. By Lemma 3.2 these groups are nonzero only for $p = l$ and

$$\dim_k E_2^{l,0} = \dim_k H^l(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}); k) = \beta(\mathcal{A}).$$

□

Remark 5.4. When $k = \mathbb{C}$, a 1-dimensional local coefficient system on X is the same thing as a flat line bundle over X.

6 ℓ^2-cohomology

For a discrete group π, $\ell^2\pi$ denotes the Hilbert space of complex-valued, square integrable functions on π. There are unitary π-actions on $\ell^2\pi$ by either left or right multiplication; hence, $\mathbb{C}\pi$ acts either from the left or right as an algebra of operators. The *associated von Neumann algebra* $\mathcal{N}\pi$ is the commutant of $\mathbb{C}\pi$ (acting from, say, the right on $\ell^2\pi$).

Given a finite CW complex X with fundamental group π, the space of ℓ^2-cochains on its universal cover $\widetilde{X}$ is the same as $C^*(X;\ell^2\pi)$, the cochains with local coefficients in $\ell^2\pi$. The image of the coboundary map need not be closed; hence, $H^*(X;\ell^2\pi)$ need not be a Hilbert space. To remedy this, one defines the *reduced* ℓ^2-cohomology $\mathcal{H}^*(\widetilde{X})$ to be the quotient of the space of cocycles by the closure of the space of coboundaries. We shall also use the notation $\mathcal{H}^*(X;\ell^2\pi)$ for the same space.

The von Neumann algebra admits a trace. Using this, one can attach a "dimension," $\dim_{\mathcal{N}\pi} V$, to any closed, π-stable subspace V of a finite direct sum of copies of $\ell^2\pi$ (it is the trace of orthogonal projection onto V). The nonnegative real number $\dim_{\mathcal{N}\pi}(\mathcal{H}^p(X;\ell^2\pi))$ is the p^{th} *ℓ^2-Betti number* of X.

A technical advance of Lück [5, Ch. 6] is the use local coefficients in $\mathcal{N}\pi$ in place of the previous version of ℓ^2-cohomology. He shows there is a well-defined dimension function on $\mathcal{N}\pi$-modules, $A \to \dim_{\mathcal{N}\pi} A$, which gives the same gives the same answer for ℓ^2-Betti numbers, i.e., for each p one has that $\dim_{\mathcal{N}\pi} H^p(X;\mathcal{N}\pi) = \dim_{\mathcal{N}\pi} \mathcal{H}^p(X;\ell^2\pi)$. Let $\mathcal{T}$ be the class of $\mathcal{N}\pi$-modules of dimension 0. The dimension function is additive with respect to short exact sequences. This allows one to define ℓ^2-Betti numbers for spaces more general than finite complexes. The class $\mathcal{T}$ is a Serre class of $\mathcal{N}\pi$-modules [8], which allows one to compute ℓ^2-Betti numbers by working with spectral sequences modulo $\mathcal{T}$.

Lemma 6.1. *Suppose $\mathcal{A}$ is a nonempty central arrangement. Then, for all $q \geq 0$, $H^q(M(\mathcal{A});\mathcal{N}\pi)$ lies in $\mathcal{T}$. In other words, all ℓ^2-Betti numbers of $M(\mathcal{A})$ are zero.*

Proof. The proof is along the same line as that of Lemma 5.2. It is well-known that the reduced ℓ^2-cohomology of $\mathbb{R}$ vanishes. Since $M(\mathcal{A}) = S^1 \times B$, the result follows from the Künneth Formula for ℓ^2-cohomology in [5, 6.54 (5)]. $\square$

Theorem 6.2. *Suppose $\mathcal{A}$ is an affine hyperplane arrangement. Then*

$$H^*(M(\mathcal{A});\mathcal{N}\pi) \cong H^*(N(\mathcal{U}), N(\mathcal{U}_{\rm sing})) \otimes \mathcal{N}\pi \pmod{\mathcal{T}}$$

Hence, for $l = \mathrm{rk}(\mathcal{A})$, the ℓ^2-Betti numbers of $M(\mathcal{A})$ vanish except in dimension l, where $\dim_{\mathcal{N}\pi} \mathcal{H}^l(\widetilde{M}(\mathcal{A})) = \beta(\mathcal{A})$.

Proof. For each $\sigma \in N(\mathcal{U}_{\rm sing})$, let $\pi_\sigma := \pi_1(U_\sigma)$. By Lemma 6.1,

$$\dim_{\mathcal{N}\pi_\sigma} H^*(M(\mathcal{A}_\sigma);\mathcal{N}\pi_\sigma) = 0.$$

Since the $\mathcal{N}_\pi$-module $H^*(M(\mathcal{A}_\sigma),\mathcal{N}\pi)$ is induced from $H^*(M(\mathcal{A}_\sigma),\mathcal{N}\pi)$,

$$\dim_{\mathcal{N}\pi} H^*(M(\mathcal{A}_\sigma);\mathcal{N}\pi) = \dim_{\mathcal{N}\pi_\sigma} H^*(M(\mathcal{A}_\sigma);\mathcal{N}\pi_\sigma) = 0.$$

As in the proof of Theorem 5.3, it follows that the E_1-page of the spectral sequence consists of modules in $\mathcal{T}$, except that $E_1^{*,0}$ is identified with $C^*(N(\mathcal{U}), N(\mathcal{U}_{\rm sing})) \otimes \mathcal{N}(\pi)$. Similarly, the E_2-page consists of modules in $\mathcal{T}$, except that $E_2^{*,0}$ is identified with $H^*(N(\mathcal{U}), N(\mathcal{U}_{\rm sing})) \otimes \mathcal{N}\pi$. For each subsequent differential, either the source or the target is a module in $\mathcal{T}$, and hence for each i and j one has that $E_\infty^{i,j} \cong E_2^{i,j} \pmod{\mathcal{T}}$. The claim follows since the filtration of $H^*(M(\mathcal{A});\mathcal{N}\pi)$ given by the E_∞-page of the spectral sequence is finite. ☐

References

[1] K. S. Brown, *Cohomology of Groups*, Springer-Verlag, Berlin and New York, 1982.

[2] M. W. Davis and I. J. Leary, *The ℓ^2-cohomology of Artin groups*, J. London Math. Soc. (2) **68** (2003), 493–510.

[3] B. Eckmann, *Introduction to ℓ_2-methods in topology: reduced ℓ_2-homology, harmonic chains, ℓ_2-Betti numbers*, Israel J. Math. **117** (2000), 183–219

[4] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, Cambridge, 2001.

[5] W. Lück, *L^2-invariants and K-theory*, Springer-Verlag, Berlin and New York, 2002.

[6] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, Berlin and New York, 1992.

[7] V. V. Schechtman and A. N. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Inventiones Math. **106** (1991), 139–194.

[8] J.-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. Math. **58** (1953) 258–294.

M. W. Davis `mdavis@math.ohio-state.edu`
T. Januszkiewicz `tjan@math.ohio-state.edu`
I. J. Leary `leary@math.ohio-state.edu`

Department of Mathematics,
The Ohio State University,
231 W. 18th Ave.,
Columbus
Ohio 43210