Realising fusion systems

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Abstract

We show that every fusion system on a p-group S is equal to the fusion system associated to a discrete group G with the property that every p-subgroup of G is conjugate to a subgroup of S.

1 Introduction

Let p be a prime number. By a p-group we shall mean a finite group whose order is a power of p. A *fusion system* on a p-group S is a category $\mathcal{F}$ whose objects are the subgroups of S, and whose morphisms are injective group homomorphisms, subject to certain axioms. The notion of a fusion system is intended to axiomatize the p-local structure of a discrete group $G \geq S$ in which every p-subgroup is conjugate to a subgroup of S. Every such G gives rise to a fusion system $\mathcal{F}_S(G)$ on S, and we say that G realises $\mathcal{F}$ if $\mathcal{F}_S(G) = \mathcal{F}$.

The notion of a *saturated fusion system* is intended to axiomatize the p-local structure of a finite group in which S is a Sylow p-subgroup. It is known that there are saturated fusion systems $\mathcal{F}$ which are not realised by any finite group G, although showing that this is the case is very delicate. In the case when $p = 2$, the only known examples are certain systems discovered by Solomon [4, 10, 16].

In contrast, we show that every fusion system on any p-group S is realised by some discrete group $G \geq S$ in which every maximal p-subgroup is conjugate to S. The groups G that are used in our proofs are constructed

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as graphs of finite groups. In particular each of our groups G contains a free subgroup of finite index. In an appendix we give a brief account of those parts of the theory of graphs of groups that we use.

While preparing this paper, we learned that Robinson has proved a similar, but not identical result [13]. Since [13] was already submitted when we started to write this paper, we have taken it upon ourselves to compare and contrast the two results. Robinson's construction realises a large class of fusion systems, including all saturated fusion systems, but does not realise all fusion systems. The groups that Robinson constructs are iterated free products with amalgamation, whereas the groups that we construct are iterated HNN extensions. In both cases the groups may be viewed as graphs of finite groups.

We state and outline the proof of a version of Robinson's theorem, along the lines of the proof of our main result. We also give examples of fusion systems that cannot be realised by Robinson's method, we give examples of non-saturated fusion systems that are realised by Robinson's method, and we prove an analogue of Cayley's theorem for fusion systems.

The work in this paper grew from the authors' participation in the Banff conference 'Homotopy theory and group actions' and from a VIGRE reading seminar at The Ohio State University which studied the Aschbacher-Chermak approach to the Solomon fusion systems [3]. The authors thank A. Chermak and G. R. Robinson for showing them early versions of [3] and [13].

2 Definitions and results

Let p be a prime, and let G be a discrete group. The p-Frobenius category $\Phi_p(G)$ of the group G is a category whose objects are the p-subgroups of G. If P and Q are p-subgroups of G, or equivalently objects of $\Phi_p(G)$, the morphisms from P to Q are the group homomorphisms $f : P \to Q$ that are equal to conjugation by some element of G. Thus $f : P \to Q$ is in $\Phi_p(G)$ if and only if there exists $g \in G$ with $f(u) = g^{-1}ug$ for all $u \in P$. (Note that the element g is *not* part of the morphism. If $g' = zg$ for some element z in the centralizer of P, then g and g' define the same morphism.)

Now suppose that S is a p-subgroup of G, and let $\mathcal{F}_S(G)$ denote the full subcategory of $\Phi_p(G)$ with objects the subgroups of S. Such categories as these are examples of *fusion systems on S*. According to Puig, a fusion system or 'Frobenius system' on S is a category $\mathcal{F}$. The objects of $\mathcal{F}$ are the subgroups of S, and the morphisms from P to Q form a subset of the set $\mathrm{Inj}(P, Q)$ of injective group homomorphisms from P to Q. These are subject to the following axioms:

1. For any $s \in S$, and any $P, Q \leq S$ with $s^{-1}Ps \leq Q$, the morphism $\phi : P \to Q$ defined by $\phi : u \mapsto s^{-1}us$ is in $\mathcal{F}$;

2. If $f : P \to Q$ is in $\mathcal{F}$, with $R = f(P) \leq Q$, then so are $f : P \to R$ and $f^{-1} : R \to P$.

It is easily checked that these axioms are satisfied in the case when $\mathcal{F} = \mathcal{F}_S(G)$ as defined above. Note that the first axiom could be rewritten as the statement $\mathcal{F}_S(S) \subseteq \mathcal{F}$.

Now consider the special case in which S is a p-subgroup of G that is maximal, and further suppose that every p-subgroup of G is conjugate to a subgroup of S. In this case, every object of $\Phi_p(G)$ is isomorphic within the category $\Phi_p(G)$ to a subgroup of S. It follows that the full subcategory $\mathcal{F}_S(G)$ is equivalent to $\Phi_p(G)$. This was one of the main motivating examples for Puig's definition.

We say that the pair (G, S) realises the fusion system $\mathcal{F}$ if S is a p-subgroup of G, $\mathcal{F}$ is a fusion system on S, and $\mathcal{F}_S(G) = \mathcal{F}$. If G is a group in which every p-subgroup is conjugate to a subgroup of some p-subgroup S, we say that G realises $\mathcal{F}$ if the pair (G, S) realises $\mathcal{F}$.

Remark 1 Another source of fusion systems on a p-group S is the Brauer category of a p-block b [2, 11]. Here H is a finite group, S is the defect group of the p-block b, and the morphisms in the category are those conjugations by elements of H that preserve some extra structure associated to b. In the case when b is the principal block, S is the Sylow p-subgroup of H and this fusion system is just $\mathcal{F}_S(H)$. One corollary of our Theorem 2 is that every such fusion system is realised by some group G in which every p-subgroup is conjugate to a subgroup of S.

Let $\mathcal{F}$ be a fusion system on S and let $\mathcal{F}'$ be a fusion system on S'. A morphism $\alpha : \mathcal{F} \to \mathcal{F}'$ consists of a group homomorphism $\alpha_0 : S \to S'$ and a functor α from $\mathcal{F}$ to $\mathcal{F}'$ such that

1. for all $P \leq S$, $\alpha_0(P) = \alpha(P)$;

2. for each $P, Q \leq S$ and each $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, $\alpha(\phi) \circ \alpha_0 = \alpha_0 \circ \phi$.

With this notion of morphism, the class of all fusion systems on p-groups becomes a category. If S is a p-subgroup of G and S' is a p-subgroup of H, then any group homomorphism $f : G \to H$ with the property that $f(S) \leq S'$ gives rise to a morphism of fusion systems $f_* : \mathcal{F}_S(G) \to \mathcal{F}_{S'}(H)$.

Define the category of pairs to have objects the pairs (G, S), where G is a group and S is a p-subgroup, where the morphisms from the pair (G, S) to the pair (H, S') are the group homomorphisms $f : G \to H$ such that $f(S) \leq S'$. With these definitions, there is a realisation functor from the category of pairs to the category of fusion systems, which takes the object (G, S) to the fusion system $\mathcal{F}_S(G)$ on S, and takes the homomorphism f to the morphism f_*.

There is a fusion system $\mathcal{F}_S^{\max}$ on S, in which the set of morphisms from P to Q consists of all injective group homomorphisms from P to Q. Any fusion system on S is a subcategory of $\mathcal{F}_S^{\max}$, and the intersection of a family of fusion systems on S is itself a fusion system. If $\Phi = \{\phi_1, \ldots, \phi_r\}$ is a collection of morphisms in $\mathcal{F}_S^{\max}$, where $\phi_i : P_i \to Q_i$, the fusion system generated by Φ is defined to be the smallest fusion system that contains each ϕ_i.

Theorem 2 *Suppose that $\mathcal{F}$ is the fusion system on S generated by $\Phi = \{\phi_1, \ldots, \phi_r\}$. Let T be a free group with free generators $t_1, \ldots, t_r$, and define G as the quotient of the free product $S * T$ by the relations $t_i^{-1} u t_i = \phi_i(u)$ for all i and for all $u \in P_i$. Then S embeds as a subgroup of G, every p-subgroup of G is conjugate to a subgroup of S, and $\mathcal{F}_S(G) = \mathcal{F}$. Moreover, every finite subgroup of G is conjugate to a subgroup of S, and G has a free normal subgroup of index dividing $|S|!$.*

As was pointed out to us by the referee, the group constructed in Theorem 2 enjoys a universal property.

Corollary 3 *Suppose that H is a group containing S as a subgroup, and that the fusion system $\mathcal{F}$ as in the statement of Theorem 2 is realised by the pair (H, S). Let $h_1, \ldots, h_r$ be any elements of H such that conjugation by h_i induces the morphism $\phi_i : P_i \to Q_i$. For G as defined in the statement of Theorem 2 there is a unique group homomorphism $f : G \to H$ such that $f(s) = s$ for all $s \in S$ and such that $f(t_i) = h_i$. Furthermore $f_* : \mathcal{F}_S(G) \to \mathcal{F}_S(H)$ is an isomorphism.*

Corollary 4 *The category of fusion systems is a retract of the category of pairs as defined above. In other words, there is a functor from the category of fusion systems to the category of pairs which is a pre-inverse to the realisation functor.*

If $f : S' \to S$ is an injective group homomorphism between p-groups, and $\mathcal{F}'$ is a fusion system on S', then there is a functor $f_!$ from $\mathcal{F}'$ to $\mathcal{F}_S^{\max}$, which sends $P' \leq S'$ to $f(P')$ and $\phi' : P' \to Q'$ to

$$f \circ \phi' \circ f^{-1} : f(P') \to f(Q').$$

Theorem 5 *Suppose that $\mathcal{F}$ is the fusion system on S generated by the images $(f_i)_!(\mathcal{F}_{S_i}(G_i))$ for injective group homomorphisms $f_i : S_i' \to S$ for $1 \leq i \leq r$, where G_i is a finite group with S_i' as a Sylow p-subgroup. Define G as the quotient of the free product $S * G_1 * \cdots * G_r$ by the relations $s = f_i(s)$ for all i and for all $s \in S_i'$. Then S embeds as a subgroup of G, every p-subgroup of G is conjugate to a subgroup of S, and $\mathcal{F}_S(G) = \mathcal{F}$. Moreover, every finite subgroup of G is conjugate to a subgroup of one of the G_i, or to a subgroup of S, and G has a free normal subgroup of index dividing $N!$, where N is the least common multiple of $|S|$ and the $|G_i|$.*

Remark 6 The above theorem can be obtained from theorem 1 of [13] by induction. The main result of [13] is theorem 2, which is similar to the above statement except that extra conditions are put on the G_i. These extra conditions allow Robinson to improve the bound on the index of a free normal subgroup, and to deduce some information about the finite quotient by such a subgroup. Another slight difference is that Robinson describes his group as a free product with amalgamation $G_1 * \cdots * G_r$, where G_1 has S as a Sylow p-subgroup. The groups that arise in this way are the same groups as those that arise from our statement, since if S is a subgroup of G_1, then $S *_S G_1 = G_1$.

Theorem 7 *Let Σ denote the group of all permutations of the elements of a p-group S, and identify S with a subgroup of Σ via the Cayley embedding. Every fusion system on S is equal to a subcategory of the Frobenius category $\Phi_p(\Sigma)$ of Σ.*

3 Saturated fusion systems

In this section we present the definition of a saturated fusion system, due to Puig [12], although we shall describe an equivalent definition due to Broto, Levi and Oliver [6]. There are two additional axioms as well as the axioms for a fusion system. These axioms necessitate some preliminary definitions.

As usual, if G is a group and H is a subgroup of G, we write $C_G(H)$ for the centralizer of H in G and $N_G(H)$ for the normalizer of H in G.

Suppose that $\mathcal{F}$ is a fusion system on S. Say that $P \leq S$ is fully $\mathcal{F}$-centralized if

$$|C_S(P)| \geq |C_S(P')|$$

for every P' which is isomorphic to P as an object of $\mathcal{F}$. Suppose that $\mathcal{F} = \mathcal{F}_S(G)$ for some discrete group G in which every p-subgroup is conjugate to a subgroup of S. In this case, if P is fully $\mathcal{F}$-centralized, one sees that $C_S(P)$ is a p-subgroup of $C_G(P)$ of maximal order.

Similarly, say that P is fully $\mathcal{F}$-normalized if

$$|N_S(P)| \geq |N_S(P')|$$

for every P' which is isomorphic to P as an object of $\mathcal{F}$. If $\mathcal{F} = \mathcal{F}_S(G)$ as above and P is fully $\mathcal{F}$-normalized, one sees that $N_S(P)$ is a p-subgroup of $N_G(P)$ of maximal order.

Now suppose that $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G, and that $P \leq S$ is fully $\mathcal{F}$-normalized. In this case, $N_S(P)$ must be a Sylow p-subgroup of the finite group $N_G(P)$. Moreover, $C_G(P) \cap N_S(P) = C_S(P)$ must be a Sylow p-subgroup of $C_G(P)$, and $\mathrm{Aut}_S(P) = N_S(P)/C_S(P)$ must be a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P) = N_G(P)/C_G(P)$. This gives the first of two extra axioms for a saturated fusion system:

3. If P is fully $\mathcal{F}$-normalized, then P is also fully $\mathcal{F}$-centralized, and $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

Next, suppose that $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G and that $f : P \to Q \leq S$ is an isomorphism in $\mathcal{F}$ such that Q is fully $\mathcal{F}$-centralized. This implies that $C_S(Q)$ is a Sylow p-subgroup of $C_G(Q)$. Pick an element $h \in G$ so that f is equal to conjugation by h, i.e., so that $f(u) = c_h(u) = h^{-1}uh$ for all $u \in P$. The image $c_h(C_S(P))$ is a p-subgroup of $C_G(c_h(P)) = C_G(Q)$, and so there exists $h' \in C_G(Q)$ so that $c_{h'} \circ c_h(C_S(P)) \leq C_S(Q)$. Since $c_{h'}$

acts as the identity on Q, if we define $k = hh'$, we see that c_k extends f and $c_k(C_S(P)) \leq C_S(Q)$.

The map c_k clearly extends to a map from $N_f = N_S(P) \cap c_k^{-1}(N_S(Q))$ to $N_S(Q)$. But since $C_S(P)$ is a subgroup of $c_k^{-1}(N_S(Q))$, we may rewrite this as

$$\begin{aligned} N_f &= \{g \in N_S(P) \colon c_k \circ c_g \circ c_k^{-1} \in \mathrm{Aut}_S(Q)\} \\ &= \{g \in N_S(P) \colon f \circ c_g \circ f^{-1} \in \mathrm{Aut}_S(Q)\}, \end{aligned}$$

which does not depend on choice of k. This leads to the second extra axiom:

4. If $f : P \to Q$ is an isomorphism in $\mathcal{F}$ and Q is fully $\mathcal{F}$-centralized, then f extends in $\mathcal{F}$ to a map from N_f to $N_S(Q)$, where

$$N_f = \{g \in N_S(P) \colon f \circ c_g \circ f^{-1} \in \mathrm{Aut}_S(Q)\}.$$

Remark 8 It has been shown [8] that the axioms for a saturated fusion system can be simplified to:

3′. $\mathrm{Aut}_S(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$.

4′. If $f : P \to Q$ is an isomorphism in $\mathcal{F}$ and Q is fully $\mathcal{F}$-normalized, then f extends in $\mathcal{F}$ to a map from N_f to $N_S(Q)$, where N_f is as defined in axiom 4.

Remark 9 In the case when S is abelian, axioms 3 and 4 simplify. In this case, every subgroup of S is fully $\mathcal{F}$-centralized and fully $\mathcal{F}$-normalized for any fusion system $\mathcal{F}$, and for any $f \in \mathcal{F}$, $N_f = S$. Hence a fusion system $\mathcal{F}$ on an abelian p-subgroup S is saturated if and only if $\mathrm{Aut}_{\mathcal{F}}(S)$ is a p'-group and every morphism $f : P \to S$ in $\mathcal{F}$ extends to an automorphism of S.

Remark 10 As mentioned in the introduction, there are saturated fusion systems which are not realised by any finite group. One source of saturated fusion systems is the fusion systems associated to p-blocks of finite groups [2, 11]. The question of whether every such fusion system can be realised by a finite group is a long-standing open problem.

4 Examples

Let E be an elementary abelian p-group of rank at least three, i.e., a direct product of at least three copies of the cyclic group of order p. Let $A = \mathrm{Aut}(E)$ be the full group of automorphisms of E, which is of course isomorphic to a general linear group over the field of p elements. Let B be a subgroup of A of order a power of p, and let C be a non-trivial subgroup of A of order coprime to p. Note that A is generated by its subgroups of order coprime to p.

Each of A, B and C may be viewed as a collection of morphisms in the fusion system $\mathcal{F}_E^{\max}$. For $X = A$, B or C, let $\mathcal{F}_E(X)$ denote the fusion system generated by all the morphisms in X.

Example 11 The fusion system $\mathcal{F}_E(C)$ is saturated, and is equal to the fusion system $\mathcal{F}_E(G)$, where G is the semi-direct product $G = E \rtimes C$.

Example 12 The fusion system $\mathcal{F}_E(A)$ is not saturated, since in $\mathcal{F}_E(A)$ the automorphism group of the object E does not have $E/Z(E)$ as a Sylow p-subgroup. However, $\mathcal{F}_E(A)$ can be realised by the procedure of Theorem 5. Let $C_1, \ldots, C_r$ be p'-subgroups of A that together generate A. If we put $G_i = E \rtimes C_i$ with f_i the identity map of E, then the fusion system generated by all of the $(f_i)_!(\mathcal{F}_E(G_i))$ is equal to $\mathcal{F}_E(A)$.

Example 13 The fusion system $\mathcal{F}_E(B)$ cannot be realised by the procedure used in Theorem 5. For suppose that $G_1, \ldots, G_r$ are finite groups with Sylow p-subgroups $E_1, \ldots, E_r$, each of which is isomorphic to a subgroup of E, and suppose that $\mathcal{F}_E(B)$ is generated by the fusion systems $(f_i)_!\mathcal{F}_{E_i}(G_i)$. Those G_i for which $f_i : E_i \to E$ is not an isomorphism do not contribute any morphisms to $\mathrm{Aut}_{\mathcal{F}}(E)$. If $f_i : E_i \to E$ is an isomorphism, then either $\mathrm{Aut}_{G_i}(E_i)$ contains non-identity elements of p' order, implying that $\mathcal{F} \neq \mathcal{F}_E(B)$, or E_i is central in G_i and G_i does not contribute any morphisms to $\mathrm{Aut}_{\mathcal{F}}(E)$.

Next we consider some examples of fusion systems $\mathcal{F}$ on an abelian p-group E in which $\mathrm{Aut}_{\mathcal{F}}(E)$ is a p'-group, but for which some isomorphisms between proper subgroups of E do not extend to elements of $\mathrm{Aut}_{\mathcal{F}}(E)$.

Example 14 Let F and F' be distinct order p subgroups of E, and let $\phi : F \to F'$ be an isomorphism. Let $\mathcal{F}_E(\phi)$ be the fusion system generated by ϕ. Every morphism in $\mathcal{F}_E(\phi)$ is equal to either an inclusion map or the composite of either ϕ or ϕ^{-1} with an inclusion map. In particular, in $\mathcal{F}_E(\phi)$, the automorphism group of each object $E' \leq E$ is trivial. The fusion system $\mathcal{F}_E(\phi)$ cannot be realised by the procedure of Theorem 5, as will be explained below.

In view of Remark 9, $\mathcal{F}_E(\phi)$ is not a saturated fusion system, since the morphism $\phi : F \to F'$ does not extend to an automorphism in $\mathcal{F}_E(\phi)$ of the group E.

Now suppose that $\mathcal{F}$ is a fusion system on E generated by the images $(f_i)_!\mathcal{F}_{E_i}(G_i)$ of some fusion systems for finite groups. If $\phi : F \to F'$ is a morphism in $\mathcal{F}$, then there exists i so that $F, F' \leq f_i(E_i)$ and $\phi \in (f_i)_!\mathcal{F}_{E_i}(G_i)$. But then (by the same argument as used above) there is a morphism $\tilde{\phi} : f_i(E_i) \to f_i(E_i)$ extending $\phi : F \to F'$. Thus $\mathcal{F}$ cannot be equal to the fusion system $\mathcal{F}_E(\phi)$, since this fusion system contains no such $\tilde{\phi}$.

Example 15 Let F be a proper subgroup of E, and suppose that D is a non-trivial p'-group of automorphisms of F. Let $F \rtimes D$ denote the semi-direct product of F and D, let G be the free product with amalgamation $G = E *_F (F \rtimes D)$, and let $\mathcal{F}$ be the fusion system $\mathcal{F}_E(G)$. From this definition one sees that $\mathcal{F}$ can be obtained by the procedure of Theorem 5. On the other hand, since $\mathrm{Aut}_{\mathcal{F}}(E)$ is trivial, one sees that the non-trivial automorphisms of F do not extend to automorphisms of E, and hence $\mathcal{F}$ is not saturated.

As remarked earlier, Robinson does not consider all fusion systems that can be built by the procedure of Theorem 5, but only those fusion systems that he calls Alperin fusion systems [13]. With the notation of Theorem 5 (and bearing in mind Remark 6), a fusion system is Alperin if the following conditions hold:

1. Inside each G_i there is a subgroup E_i which is the largest normal p-subgroup of G_i, and the centralizer of this subgroup is as small as possible, in the sense that $C_{G_i}(E_i) = Z(E_i)$;

2. The quotient G_i/E_i is isomorphic to $\mathrm{Out}_{\mathcal{F}}(E_i) := \mathrm{Aut}_{\mathcal{F}}(E_i)/\mathrm{Aut}_{E_i}(E_i)$;

3. Inside S, the image of the subgroup S'_i (the Sylow p-subgroup of G_i which is to be identified with a subgroup of S) is equal to the normalizer of the image of E_i, i.e., $f_i(S'_i) = N_S(f_i(E_i))$.

In terms of this definition, the content of Alperin's fusion theorem with some later embellishments [1, 7] is that the fusion system for any finite group is Alperin. Robinson remarks [13] that work of Broto, Castellana, Grodal, Levi and Oliver implies that every saturated fusion system is Alperin [5]. It is easy to see that a fusion system on an abelian p-group is Alperin if and only if it is saturated. We finish this section by giving an example of a fusion system that is Alperin but not saturated.

Example 16 Let p be an odd prime, let $A = (C_p)^3$, and let B be a subgroup of $\mathrm{Aut}(A)$ of order p such that A is indecomposable as a B-module. (Equivalently, the action of a generator for B on A should be a single Jordan block.) Let S be the semi-direct product $S = A \rtimes B$. The centre Z of S has order p. Let $E = Z \times B \leq S$, a subgroup isomorphic to $C_p \times C_p$. It is readily seen that $C_S(E) = E$ and that $P = N_S(E)$ is isomorphic to a semi-direct product $(C_p)^2 \rtimes C_p$, the unique non-abelian group of order p^3 and exponent p. Let G_1 be the semi-direct product $G_1 = E \rtimes \mathrm{Aut}(E)$. Since the Sylow p-subgroups of $\mathrm{Aut}(E)$ are cyclic of order p, there is an isomorphism between P and a Sylow p-subgroup of G_1 that extends the inclusion of E.

By construction, the fusion system $\mathcal{F}$ for the free product with amalgamation $S *_P G_1$ is Alperin in the sense of Robinson [13], but this fusion system is not saturated. For example, there are non-identity self-maps of Z inside $\mathcal{F}$, and if $\mathcal{F}$ were saturated, any self-map of Z inside $\mathcal{F}$ would extend to a self-map of S. But in $\mathcal{F}$, S has only inner automorphisms, and these restrict to Z as the identity.

5 Proofs

Proof. (of Theorem 7.) As in the statement, let Σ be the group of all permutations of S, and identify S with a subgroup of Σ. Let P and Q be subgroups of $S \leq \Sigma$, and let $\phi : P \to Q$ be any injective group homomorphism. It suffices to show that there is some $\sigma \in \Sigma$ such that for all $u \in P$, $\sigma^{-1} u \sigma = \phi(u)$. Let Ω denote the group S viewed as a set with a left S-action. There are two ways to view Ω as a set with a left P-action, via $P \leq S$ and via $\phi : P \to Q \leq S$. Denote these two P-sets by Ω and ${}^{\phi}\Omega$ respectively. Each of Ω and ${}^{\phi}\Omega$ is isomorphic as a P-set to the disjoint union of $|S : P|$ copies of P. In particular, there is an isomorphism of P-sets $\sigma : {}^{\phi}\Omega \to \Omega$. Viewing σ as an element of Σ, one has that $\sigma\phi(u)\omega = u\sigma\omega$ for all $u \in P$ and $\omega \in \Omega$.

Hence $\sigma^{-1}u\sigma = \phi(u)$ for all u as required. $\square$

Remark 17 A version of Theorem 7 appeared in [9], although fusion systems were not mentioned there.

Before proving Theorem 2 we give a result concerning extending group homomorphisms, and two corollaries, one of which will be used in the proof.

Lemma 18 *Let S and G be as in the statement of Theorem 2, let $j : S \to G$ be the natural map from S to G, let H be a group and let $f : S \to H$ be a group homomorphism. There is a group homomorphism $\tilde{f} : G \to H$ with $f = \tilde{f} \circ j$ if and only if for each i, the homomorphisms $f : P_i \to H$ and $f \circ \phi_i : P_i \to H$ differ by an inner automorphism of H.*

Proof. Given a homomorphism $\tilde{f}$ as in the statement, one has that for each i and for each $u \in P_i$, $f\phi_i(u) = h_i^{-1}f(u)h_i$, where $h_i = \tilde{f}(t_i)$. For the converse, suppose that there exists, for each i, an element h_i satisfying the equation $f\phi_i(u) = h_i^{-1}f(u)h_i$ for all $u \in P_i$. In this case one may define $\tilde{f}$ on the generators of G by $\tilde{f}(s) = f(s)$ for all $s \in S$ and $\tilde{f}(t_i) = h_i$. $\square$

Corollary 19 *With notation as in the statement of Theorem 2, there is a homomorphism from G to Σ, the group of all permutations of the set S, extending the Cayley representation of S.*

Proof. The argument used in the proof of Theorem 7 shows that the conditions of Lemma 18 hold. $\square$

Remark 20 Corollary 19 gives an alternative way to prove Corollary 26, at least in the special case of a rose-shaped graph.

Corollary 21 *With notation as in the statement of Theorem 2, a complex representation of S with character χ extends to a complex representation of G if and only if for each i and for each $u \in P_i$, $\chi(u) = \chi(\phi_i(u))$.*

Remark 22 Of course, a representation of S will extend to G in many different ways if it extends at all.

Proof. (of Theorem 2.) As in Appendix 6.2, one sees that the group G presented in the statement is the fundamental group of a graph of groups with one vertex group, S, and one edge group P_i for each ϕ_i, $1 \leq i \leq r$.

From Corollary 26 it follows that S is a subgroup of G. From Corollary 30, it follows that any finite subgroup of G, and in particular any p-subgroup of G, is conjugate to a subgroup of S. By Theorem 28, there is a cellular action of G on a tree T, with one orbit of vertices and r orbits of edges. By suitable choice of orbit representatives, we may choose a vertex v whose stabilizer is S, and edges $e_1, \ldots, e_r$ so that the stabilizer of e_i is P_i, and so that the initial vertex of e_i is v while the final vertex is $t_i{\cdot}v$.

Since every p-subgroup of G is conjugate to a subgroup of S, there is a fusion system $\mathcal{F}_S(G)$ associated to G. By construction $\mathcal{F}_S(G)$ contains each ϕ_i, which corresponds to conjugation by t_i.

Conversely, suppose that $g \in G$ has the property that $g^{-1}Pg \leq Q$ for some subgroups P, Q of S. It suffices to show that conjugation by g, as a map from P to Q, is equal to a composite of (restrictions of) the maps ϕ_j and their inverses with conjugation maps by elements of S.

Consider the action of P on the tree T. By hypothesis, the action of P fixes both the vertex v and the vertex $g{\cdot}v$. Since T is a tree, P must fix all the vertices and edges on the unique shortest path from v to $g{\cdot}v$. Let this path have length n. Define $g_0 = 1_G$, $g_n = g$, and for $1 \leq i \leq n-1$, choose $g_i \in G$ so that $g_0{\cdot}v, g_1{\cdot}v, \ldots, g_n{\cdot}v$ is the shortest path in T from v to $g{\cdot}v$. For each i, P is contained in the stabilizer of the vertex $g_i{\cdot}v$, and so $P \leq g_iSg_i^{-1}$, or equivalently $g_i^{-1}Pg_i \leq S$.

The edge joining $g_i{\cdot}v$ and $g_{i+1}{\cdot}v$ is an edge of the form $g_i{\cdot}e_j$ or $g_{i+1}{\cdot}e_j$ for some j depending on i. Consider the two cases separately, first supposing that the edge is of the form $g_i{\cdot}e_j$. In this case it follows that $P \leq g_iP_ig_i^{-1}$, since P stabilizes the edge $g_i{\cdot}e_j$. Also one sees that $g_{i+1}{\cdot}v = g_it_j{\cdot}v$, and hence $g_{i+1}^{-1}g_it_j \in S$. Hence conjugation by $g_i^{-1}g_{i+1}$, viewed as a map from $g_i^{-1}Pg_i$ to $g_{i+1}^{-1}Pg_{i+1}$ is equal to the composite of the map ϕ_j (restricted to $g_i^{-1}Pg_i \leq P_i$) followed by conjugation by an element of S.

The other case is similar. Here it follows that $P \leq g_{i+1}P_{i+1}g_{i+1}^{-1}$, and one has that $g_i{\cdot}v = g_{i+1}t_j{\cdot}v$, from which $g_i^{-1}g_{i+1}t_j = s \in S$. In this case conjugation by $g_i^{-1}g_{i+1}$, as a map from $g_i^{-1}Pg_i$ to $g_{i+1}^{-1}Pg_{i+1}$, is equal to the composite map given by conjugation by s followed by the map ϕ_j^{-1} (restricted to $s^{-1}g_i^{-1}Pg_is \leq \phi_j(P_{i+1})$).

Thus conjugation by $g = g_n$ as a map from P to Q can be expressed as a composite of maps inside the fusion system generated by the ϕ_i, and so $\mathcal{F}_S(G)$ is equal to this fusion system.

It remains to show that the group G contains a free normal subgroup of index at most $|S|!$. Let Σ denote the symmetric group on the set S. By Corollary 19, there is a homomorphism $G \to \Sigma$ which extends the natural injection $S \to \Sigma$. By Corollary 31, the kernel of this homomorphism is a free normal subgroup of G, and its index is a factor of $|\Sigma| = |S|!$. □

Proof. (of Corollary 3.) Define a function f on the union of S and the generators of T by $f(s) = s$ and $f(t_i) = h_i$. This extends uniquely to a group homomorphism f from G to H by an argument similar to that used in the proof of Lemma 18. Since the pairs (G, S) and (H, S) both realise the same fusion system, it is immediate that f_* is an isomorphism of fusion systems. □

Proof. (of Corollary 4.) For $\mathcal{F}$ a fusion system on a p-group S, let $\Phi(\mathcal{F})$ be the (finite) set of all morphisms in $\mathcal{F}$, and define $G_m(\mathcal{F})$ to be the group constructed as in Theorem 2 using the set $\Phi(\mathcal{F})$ as the chosen generators for $\mathcal{F}$. Any morphism of fusion systems $\alpha : \mathcal{F} \to \mathcal{F}'$ will give rise to a function from $\Phi(\mathcal{F})$ to $\Phi(\mathcal{F}')$ and hence a group homomorphism from $G_m(\mathcal{F})$ to $G_m(\mathcal{F}')$. Hence the map sending $\mathcal{F}$ to the pair $(G_m(\mathcal{F}), S)$ is a functor from fusion systems to pairs. It is easily checked that the fusion system on S realised by $G_m(\mathcal{F})$ is equal to $\mathcal{F}$, which shows that this functor is a pre-inverse to the realisation functor. □

Proof. (of Theorem 5—sketch.) In this case, the group G is the fundamental group of a star-shaped graph of groups, with one central vertex labelled S and r outer vertices labelled $G_1, \ldots, G_r$. The edge from G_i to S is labelled by the group S'_i. By Theorem 28, there is a cellular action of G on a tree T, with $r + 1$ orbit of vertices and r orbits of edges. We may choose orbit representatives $v_0, v_1, \ldots, v_r$ of vertices and $e_1, \ldots, e_r$ of edges so that the stabilizer of v_0 is S, and for $1 \leq i \leq r$, the stabilizer of v_i is G_i (resp. of e_i is S'_i). Moreover, we may assume that e_i has initial vertex v_i and terminal vertex v_0.

In this case, one sees that any finite subgroup of G is conjugate to either a subgroup of S or to a subgroup of G_i for some i. Since S'_i is a Sylow p-subgroup of G_i, it follows that any p-subgroup of G is conjugate to a subgroup of S as required.

As in the previous proof, it is clear that the fusion system $\mathcal{F}_S(G)$ contains the image of each $\mathcal{F}_{S'_i}(G_i)$, but an argument is needed to show that these images generate $\mathcal{F}_S(G)$. Given $g \in G$ and $P, Q \leq S$ so that $g^{-1}Pg \leq Q$, one argues that the action of P fixes the vertices v_0 and $g{\cdot}v_0$ in the tree T, and hence fixes the shortest path (necessarily of even length, say $2n$) that joins these vertices.

Let $g_0 = 1_G$, $g_{2n} = g$, and pick group elements so that the vertices on the shortest path from v_0 to $g{\cdot}v_0$ are:

$$g_0{\cdot}v_0\ ,\ g_1{\cdot}v_{j(1)}\ ,\ g_2{\cdot}v_0\ ,\ g_3{\cdot}v_{j(2)}\ ,\ \cdots\ ,\ g_{2n-1}{\cdot}v_{j(n)}\ ,\ g_{2n}{\cdot}v_0$$

for some function $j : \{1, \ldots, n\} \to \{1, \ldots, r\}$. If i is even, then $g_i^{-1}Pg_i \leq S$, and if i is odd then $g_i^{-1}Pg_i \leq G_{j((i+1)/2)}$. Since P stabilizes each edge, one sees that $P \leq g_i^{-1}S_kg_i$, where S_k denotes the image of S'_k inside S, and $k = j((i+1)/2)$ if i is odd and $k = j(i/2)$ if i is even. In particular, each $g_i^{-1}Pg_i$ is a subgroup of S.

One may show that in the case when i is odd, $g_i^{-1}g_{i+1} \in G_{j((i+1)/2)}$ and that in the case when i is even, $g_i^{-1}g_{i+1} \in S$. Thus the map from $g_i^{-1}Pg_i$ to $g_{i+1}^{-1}Pg_{i+1}$ given by conjugation by $g_i^{-1}g_{i+1}$ is a map inside the fusion system generated by the images of the $\mathcal{F}_{S'_i}(G_i)$, and conjugation by $g = g_{2n}$ as a map from P to $Q \leq S$ is expressed as a composite of maps of the required form.

Finally, if Ω is a finite set so that $|\Omega|$ is divisible by $|S|$ and by each $|G_i|$, one may define free actions of S and each G_i on Ω which give rise to the same (free) action of $S_i = f_i(S'_i)$. This gives rise to a group homomorphism from G to Σ, the symmetric group on Ω, whose kernel is free by Corollary 31.

□

6 Appendix: graphs of groups

In this section we give proofs of those results about graphs of groups that we use. Our treatment of graphs of groups is topological and follows that of Scott and Wall [14]; an alternative, more algebraic, treatment of this subject can be found in Serre's book [15]. There is no a direct correspondence between the two treatments but we give references when appropriate to the closest results following Serre's approach.

For the purposes of this paper, a graph Γ consists of two sets, the vertices V and the directed edges E, together with two functions $\iota, \tau : E \to V$. For $e \in E$, $\iota(e)$ is called the initial vertex of e and $\tau(e)$ is the terminal vertex of e. Multiple edges and loops are allowed in this definition. Γ is connected if the only equivalence relation on V that contains all pairs $(\iota(e), \tau(e))$ is the relation with just one class.

A graph Γ may be viewed as a category, with objects the disjoint union of V and E and two non-identity morphisms with domain e for each $e \in E$, one morphism $e \to \iota(e)$ and one morphism $e \to \tau(e)$.

A graph of groups is a connected graph Γ together with groups G_v, G_e for each vertex and edge, and injective group homomorphisms $f_{e,\iota} : G_e \to G_{\iota(e)}$ and $f_{e,\tau} : G_e \to G_{\tau(e)}$ for each edge e. If Γ is viewed as a category, this is just a functor from Γ to the category of groups and injective group homomorphisms. Without loss of generality, one may assume that each map $f_{e,\iota} : G_e \to G_{\iota(e)}$ is the inclusion of a subgroup.

6.1 The fundamental group of a graph of groups

For a topologist, and arguably for anybody, the easiest way to define the fundamental group of a graph of groups is via the notion of a graph of spaces.

A graph of spaces is a connected graph Γ together with topological spaces X_v, X_e for each vertex and edge, and continuous maps $f_{e,\iota} : X_e \to X_{\iota(e)}$ and $f_{e,\tau} : X_e \to X_{\tau(e)}$. A graph of spaces is just a functor from the category Γ to the category of topological spaces and continuous functions. A graph of based spaces is defined similarly: each X_e and X_v is equipped with a base point, and the maps must preserve base points. Let I denote the closed unit interval $[0, 1]$. The total space of a graph of spaces is the space X made from the disjoint union

$$\coprod_{v \in V} X_v \ \amalg \ \coprod_{e \in E} X_e \times I$$

by identifying $(x, 0) \in X_e \times I$ with $f_{e,\iota}(x) \in X_{\iota(e)}$ and identifying $(x, 1) \in X_e \times I$ with $f_{e,\tau}(x) \in X_{\tau(e)}$. As an example, consider the graph of spaces in which each X_e and X_v is a single point. For this graph of spaces the total space is the usual topological realization of the graph as a 1-dimensional CW-complex. The reader who is familiar with the homotopy colimit construction will note that if one views a graph of spaces as a functor $X_{(-)}$ on the category Γ, then the total space X is naturally homeomorphic to the homotopy colimit of the functor $X_{(-)}$, or in symbols, $X = \mathrm{hocolim}_\Gamma X_{(-)}$.

Given a graph of groups, one may define a graph of connected based spaces by taking classifying spaces as the spaces X_e and X_v:

$$X_e = BG_e = K(G_e, 1) \qquad X_v = BG_v = K(G_v, 1).$$

For the continuous map $f_{e,\iota} : X_e \to X_{\iota(e)}$ (resp. $f_{e,\tau} : X_e \to X_{\tau(e)}$) one may take any continuous map that induces the given map $G_e \to G_{\iota(e)}$ (resp.

$G_e \to G_{\tau(e)}$) on fundamental groups. Define a total space X as the realization of this graph of spaces.

For discrete groups K and H, the space BK is unique up to based homotopy, and homotopy class of based maps from BK to BH are in bijective correspondence with group homomorphisms from K to H. It follows that the homotopy type of the space X defined above depends only on the graph of groups, rather than on the particular choices of classifying spaces and maps between them. The fundamental group G of the graph of groups can now be defined as the fundamental group of X. This describes the fundamental group of the graph of groups up to isomorphism. The inclusion of each X_v in X defines a conjugacy class of homomorphism $G_v \to G$ (which will be shown to be injective, below). For many purposes one wants a more precise description of G, together with a single choice of homomorphism $G_v \to G$. This can be done by choosing a basepoint for the space X, and for each v, a path in X from the basepoint for X to the basepoint for $X_v \subseteq X$.

6.2 Presentations for graphs of groups

We shall only consider presentations for graphs of groups where the underlying graph is either a 'rose' or a 'star'. By a rose we mean a graph with only one vertex, so that every edge has the same initial and terminal vertices. By a star we mean a connected graph with $n+1$ vertices and n edges, for some $n > 0$, with one central vertex, such that all the edges have this vertex as their terminal vertex and so that every other vertex is the initial vertex of exactly one edge.

Suppose one is given a p-group S, subgroups $P_i, Q_i \leq S$, and injective group homomorphisms $\phi_i : P_i \to Q_i$ for $1 \leq i \leq r$, as in the statement of Theorem 2. Use this data to make a rose-shaped graph of groups with r edges. Let S be the vertex group, let P_i be the ith edge group, with the inclusion map $P_i \leq S$ (resp. the composite $\phi_i : P_i \to Q_i \leq S$) as the ith initial (resp. terminal) homomorphism. There is a model for BP_i having just one 0-cell and one 1-cell for each element of P_i. Take a model for BS having just one 0-cell and take this 0-cell as the base point. To make a CW-complex of the homotopy type of the total space of the graph of groups, it suffices to add to BS one 1-cell t_i for each i (with both ends at the unique 0-cell), one 2-cell $D_{i,u}$ for $1 \leq i \leq r$ and for each $u \in P_i$, and higher dimensional cells (which will not affect the fundamental group). The attaching map for the 2-cell $D_{i,u}$ spells out the word $u\, t_i\, \phi_i(u)\, t_i^{-1}$, and so the presentation coming from this

CW-structure is the presentation given in the statement of Theorem 2.

Next suppose that one is given a p-group S, groups G_i for $1 \leq i \leq r$ with Sylow p-subgroups S_i, and injective group homomorphisms $f_i : S_i \to S$, i.e., the data found in the statement of Theorem 5. In this case, define a star of groups with central vertex group P, other vertex groups $G_1, \ldots, G_r$, and edge groups $S_1, \ldots, S_r$. The map of each edge group into its initial vertex group is the inclusion $S_i \to G_i$, and the map of each edge group into its terminal vertex group is $f_i : S_i \to S$. An argument similar to that given in the previous paragraph shows that the fundamental group of this graph of groups has the presentation given in the statement of Theorem 5. Note that here one can make a space homotopy equivalent to the total space of the graph of spaces by starting from the one-point union of BS and the BG_i, without adding any extra 1-cells. This is reflected in the fact that the vertex groups generate the fundamental group of the graph of groups.

6.3 Properties of graphs of groups

Proposition 23 *Let G be the fundamental group of a graph of groups based on a graph Γ. Every subgroup $H \leq G$ is itself the fundamental group of a graph of groups, indexed by a graph Δ equipped with a map $f : \Delta \to \Gamma$ which does not collapse any edges. For each v and $e \in \Delta$, the group H_v (resp. H_e) is a subgroup of $G_{f(v)}$ (resp. $G_{f(e)}$).*

This proposition appears as [14, Theorem 3.7] in the special case when the graph is either an interval or a loop, i.e., the case when the fundamental group of the graph of groups is either a free product with amalgamation or an HNN extension.

Proof. Use the bijection between connected covering spaces of a connected CW-complex (with a choice of base point) and subgroups of its fundamental group. Let X be the total space of the graph of spaces used in the definition of G, so that there is a covering space of X whose fundamental group is H. Any connected covering space of X can be expressed as the total space of a graph of spaces indexed by some Δ as in the statement. This gives an expression for the fundamental group of any connected covering space of X as the fundamental group of a graph of groups as claimed. □

Theorem 24 *Let X be the total space of the graph of spaces used in the definition of the fundamental group G of a graph of groups. The universal* *covering space of X is contractible, and hence X is homotopy equivalent to BG.*

Proof. We shall build a space Y, in such a way that it is clear that Y is contractible, and that Y is a covering space of X.

For v a vertex, define the subspace X'_v of X by

$$X'_v = X_v \cup \bigcup_{\iota(e)=v} X_e \times [0, 0.5) \cup \bigcup_{\tau(e)=v} X_e \times (0.5, 1].$$

Similarly, define for e an edge, $X'_e = X_e \times (0, 1)$. The inclusions $X_v \to X'_v$ and $X_e \cong X_e \times \{0.5\} \to X'_e$ are homotopy equivalences, and it may be useful to think of X'_v as a nice open neighbourhood of X_v in X. Let Y_v, Y'_v, Y_e, and Y'_e be the universal covering spaces of X_v, X'_v, X_e and X'_e respectively. Each Y'_v (resp. Y'_e) is contractible since it is the universal covering space of the classifying space BG_v (resp. BG_e).

The definition of the space X'_v lifts to a description of the space Y'_v. The complement $Y'_v - Y_v$ is identified with a collection of disjoint copies of $Y_e \times (0, 0.5)$, and $Y_e \times (0.5, 1)$, for different edges e. There are copies of $Y_e \times (0, 0.5)$ if and only if $\iota(e) = v$. In this case the copies are in bijective correspondence with the cosets of $f_{e,\iota}(G_e)$ in G_v. Similarly, there are copies of $Y_e \times (0.5, 1)$ for each e with $\tau(e) = v$, and these copies are indexed by cosets of $f_{e,\tau}(G_e)$ in G_v.

By induction, we shall construct a sequence $Y_0 \subseteq Y_1 \subseteq Y_2 \cdots$ of spaces so that: each Y_n is contractible; there is a map $\pi : Y_n \to X$ which is locally a covering map except at some points of X; for any $x \in X$ and any $n \geq 0$, at least one of $\pi : Y_n \to X$ and $\pi : Y_{n+1} \to X$ is locally a covering map at x.

Pick a vertex v of the graph Γ, and define Y_0 to be the space Y'_v. Define a map $\pi : Y'_v \to X$ as the composite of the map $Y'_v \to X'_v$ and the inclusion $X'_v \subseteq X$. As remarked earlier, $Y'_v - Y_v$ consists of lots of subspaces of the form $Y_e \times (0, 0.5)$ for $\iota(e) = v$ and lots of subspaces of the form $Y_e \times (0.5, 1)$ for $\tau(e) = v$. Define Y_1 by attaching to each such subspace a copy of Y'_e. The map $\pi : Y_0 \to X$ extends uniquely to $\pi : Y_1 \to X$ by insisting that on each newly-added Y'_e subspace, π is equal to the composite map $Y'_e \to X'_e \subseteq X$. From the construction of Y_1, it is apparent that Y_1 is contractible.

In constructing Y_1, one attached to Y_0 many spaces of the form Y'_e, by identifying one end of Y'_e with part of Y_0. For each copy of Y'_e that was attached via its initial end, take a copy of $Y'_{\tau(e)}$, and attach this at the other end of Y'_e. Similarly, for each copy of Y'_e that was attached to Y_0 by its terminal end, take a copy of $Y'_{\iota(e)}$ and attach this at the other end of Y'_e. This defines a space Y_2, which is clearly contractible, and the map π extends uniquely to a map $Y_2 \to X$ which agrees with the covering map $Y'_e \to X'_e$ or $Y'_v \to X'_v$ on each such subspace.

Now suppose that n is even, and that Y_n has been constructed from Y_{n-1} by attaching subspaces Y'_v in such a way that the intersection of Y_{n-1} and each new Y'_v is equal to one of the components of $Y'_v - Y_v$. Furthermore, suppose that the map π on each new Y'_v is equal to the map $Y'_v \to X'_v \subseteq X$. Form Y_{n+1} by attaching a copy of Y'_e to each other component of $Y'_v - Y_v$ for each of the copies of Y'_v. Extend the map π as before.

In the case when n is odd, suppose that Y_n has been constructed from Y_{n-1} by attaching subspaces Y'_e in such a way that the intersection of Y_{n-1} and each new Y'_e is equal to one of the two components of $Y'_e - Y_e \times \{0.5\}$. Suppose also that the map π on each of the new Y'_e is equal to the map $Y'_e \to X'_e \subseteq X$. Form Y_{n+1} by attaching a copy of Y'_v to the other component of each $Y'_e - Y_e \times \{0.5\}$, where v is either $\iota(e)$ or $\tau(e)$ depending which component of $Y'_e - Y_e \times \{0.5\}$ was used. Extend the map π in the same way as before.

By construction, each Y_n is contractible, and comes equipped with a map $\pi : Y_n \to X$. If n is even, this map is locally a covering except possibly at points of X contained in the union of the images of the X_v. If n is odd, this map is a covering except possibly at point of X contained in the union of the images of the $X_e \times \{0.5\}$. Now define Y by $Y = \bigcup_n Y_n$. This space Y is contractible, and the map $\pi : Y \to X$ is a covering map, since it is locally a covering map at every point of X. It follows that Y is the universal covering space of X. Since the universal covering space of X has been shown to be contractible, it follows that X is a model for BG. $\square$

Remark 25 The above proof relies on the fact that the edge groups map injectively to the vertex groups. The theorem can be found in [14, Proposition 3.2 (ii)]. There is no direct analogue in the more algebraic treatment of [15]. The closest result to this theorem is arguably Theorem 12 of [15].

Corollary 26 *Each vertex group G_v maps injectively into the fundamental group of a graph of groups.*

Proof. Given a vertex v, construct the universal covering space as in the proof of Theorem 24, with $Y_0 = Y'_v$. The group of all deck transformations of Y is naturally isomorphic to G, the fundamental group of X. Under this isomorphism, the subgroup of those deck transformations that preserve Y_0 is identified with G_v. $\square$

Remark 27 Corollary 26 is [14, Proposition 3.2 (i)]. There is also an algebraic proof that each G_v embeds in G; see for instance [15, Corollary 1 to Theorem 11]. In the case when the graph is a rose, this argument is given in Corollary 19.

6.4 The action on a tree

Say that an action of a group on a tree is cellular if no element of the group exchanges the ends of any edge. The following theorem is implicit in [14, Section 4].

Theorem 28 *Let G be the fundamental group of a graph of groups indexed by the graph Γ. There is a tree T with a cellular G-action and an isomorphism $f : T/G \cong \Gamma$. If $\tilde{x}$ is either a vertex or edge of T, and $x = f(\tilde{x})$ is the image of $G{\cdot}\tilde{x}$ under f, then the stabilizer of $\tilde{x}$ is conjugate to G_x.*

Proof. Let X be the total space of the graph of spaces used in defining G. As remarked earlier, the underlying topological space of the graph Γ can be identified with the total space of the constant graph of 1-point spaces indexed by Γ. The unique map from each X_v and X_e to a point induces a map from X to Γ.

Now let Y be the universal covering space of X, as constructed in the proof of Theorem 24. This Y can be viewed as a graph of spaces over some graph Δ, with vertex spaces copies of the spaces Y_v and edges spaces copies of the spaces Y_e. The group G acts on Y in such a way that the setwise stabilizer of each copy of Y_v is a conjugate of G_v, and similarly the setwise stabilizer of each copy of $Y_e \times (0, 1)$ is a conjugate of G_e. Define T to be the total space of the graph of 1-point spaces over the graph Δ. By construction, T is a graph equipped with a G-action, an equivariant map $\phi : Y \to T$, and an isomorphism $f : T/G \to \Gamma$. To check that T is a tree, let $T_n = \phi(Y_n)$. As in the proof of Theorem 24, one shows inductively that T_n is contractible, and $T = \bigcup_n T_n$. $\square$

Lemma 29 *Any cellular action of a finite group H on a tree T fixes a vertex.*

Proof. Take any vertex $t \in T$, and define a finite subtree T' to be the union of all the shortest paths between elements of the orbit $H \cdot t$. If T' is not itself fixed by H, remove an H-orbit of 'leaves' (i.e., vertices of valency one) from T', and continue this process until a subtree fixed by H is all that remains. □

As a consequence of the previous two results we get a very useful corollary, which is stated as [15, Corollary to Theorem 8] in the special case of an interval of groups. (This is the case when the fundamental group of the graph of groups is a free product with amalgamation.)

Corollary 30 *Every finite subgroup of the fundamental group of a graph of groups is conjugate to a subgroup of a vertex group.*

Proof. Let G be the fundamental group of the graph of groups and let T be the corresponding tree. If H is a finite subgroup of G then H fixes some vertex of T. The stabilizer of each vertex of T is a conjugate of one of the vertex groups G_v. □

The following corollary is stated as [15, Proposition 18] in the special case of an interval of groups.

Corollary 31 *Let H be a subgroup of a graph of groups whose intersection with each conjugate of each vertex group is trivial. Then H is a free group.*

Proof. The hypotheses imply that H acts freely on the tree T, and so the quotient space T/H is a 1-dimensional classifying space for H. □

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