Non-orientable surface-plus-one-relation groups

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October 16, 2008

To the memory of Karl Gruenberg

Abstract

Recently Dicks-Linnell determined the L^2-Betti numbers of the orientable surface-plus-one-relation groups, and their arguments involved some results that were obtained topologically by Hempel and Howie. Using algebraic arguments, we now extend all these results of Hempel and Howie to a larger class of two-relator groups, and we then apply the extended results to determine the L^2-Betti numbers of the non-orientable surface-plus-one-relation groups.

2000 Mathematics Subject Classification. Primary: 20F05; Secondary: 16S34, 20J05.
Key words. L^2-Betti number, surface-plus-one-relation group

1 Notation

In this section, we collect together the conventions and basic notation we shall use.

Let G be a (discrete) multiplicative group, fixed throughout the article.

For two subsets A, B of a set X, the complement of $A \cap B$ in A will be denoted by $A - B$ (and not by $A \setminus B$ since we let $G\backslash Y$ denote the set of G-orbits of a left G-set Y).

By an *ordering*, $<$, of a set, we shall mean a binary relation which *totally* orders the set.

A *sequence* is a *set* endowed with a specified listing of its elements, usually represented as a vector in which the coordinates are the elements of (the underlying set of) the sequence. For two sequences A, B, their concatenation will be denoted $A \vee B$.

We use $\mathbb{R} \cup \{-\infty, \infty\}$ with the usual conventions, such as $\frac{1}{\infty} = 0$.

We will find it useful to have a notation for intervals in $\mathbb{Z}$ that is different from the notation for intervals in $\mathbb{R}$.

Let i, $j \in \mathbb{Z}$.

We write

$$[i{\uparrow}j] \coloneqq \begin{cases} (i, i+1, \ldots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leqslant j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$$

Also, $[i{\uparrow}\infty[\ \coloneqq (i, i+1, i+2, \ldots)$ and $[i{\uparrow}\infty] \coloneqq [i{\uparrow}\infty[\ \vee \{\infty\}$. We define $[j{\downarrow}i]$ to be the reverse of the sequence $[i{\uparrow}j]$, that is, $(j, j-1, \ldots, i+1, i)$.

We shall use sequence notation to define families of indexed symbols. Let v be a symbol. For each $k \in \mathbb{Z}$, we let v_k denote the ordered pair (v, k). We let

$$v_{[i\uparrow j]} \coloneqq \begin{cases} (v_i, v_{i+1}, \cdots, v_{j-1}, v_j) & \text{if } i \leqslant j, \\ () & \text{if } i > j. \end{cases}$$

Also, $v_{[i\uparrow\infty[} \coloneqq (v_i, v_{i+1}, v_{i+2}, \ldots)$. We define $v_{[j\downarrow i]}$ to be the reverse of the sequence $v_{[i\uparrow j]}$.

Now suppose that $v_{[i\uparrow j]}$ is a sequence *in* the group G, that is, there is specified a map of sets $v_{[i\uparrow j]} \to G$. We treat the elements of $v_{[i\uparrow j]}$ as elements of G, possibly with repetitions, and we define

$$\Pi v_{[i\uparrow j]} \coloneqq \begin{cases} v_i v_{i+1} \cdots v_{j-1} v_j \in G & \text{if } i \leqslant j, \\ 1 \in G & \text{if } i > j. \end{cases}$$

$$\Pi v_{[j\downarrow i]} \coloneqq \begin{cases} v_j v_{j-1} \cdots v_{i+1} v_i \in G & \text{if } j \geqslant i, \\ 1 \in G & \text{if } j < i. \end{cases}$$

For elements a, b of G, we write $\overline{a} \coloneqq a^{-1}$, ${}^{a}b \coloneqq ab\overline{a}$, and $[a, b] \coloneqq ab\overline{a}\overline{b}$.

For any subsets R and X of G, we let $\langle\, R\,\rangle$ denote the subgroup of G generated by R, we write ${}^{X}R \coloneqq \{{}^{x}r \mid r \in R, x \in X\}$, and $G/\langle\!\langle R \rangle\!\rangle \coloneqq G/\langle\, {}^{G}R\,\rangle$. If $R = \{r\}$, we write simply $\langle\, r\,\rangle$, ${}^{X}r$ and $G/\langle\!\langle r \rangle\!\rangle$, respectively.

For each set X, the *vague cardinal* of X, denoted $|X|$, is the element of $[0{\uparrow}\infty]$ defined as follows. If X is a finite set, then $|X|$ is defined to be the cardinal of X, an element of $[0{\uparrow}\infty[$. If X is an infinite set, then $|X|$ is defined to be ∞.

The *rank* of G, denoted $\operatorname{rank}(G)$, is the smallest element of the subset of $[0{\uparrow}\infty]$ which consists of vague cardinals of generating sets of G.

Mappings of left modules will usually be written on the right of their arguments.

2 Background and summary of results

In outline, the article has the following structure. More detailed definitions can be found in the appropriate sections.

Let S be a surface group, that is, the fundamental group of a closed surface. Here, there exists some $k \in [0{\uparrow}\infty[$ and a presentation $S = \langle\, x_{[1\uparrow k]} \mid w\,\rangle$ where either k is even and $w = \prod_{i \in [1\uparrow \frac{k}{2}]} [x_{2i-1}, x_{2i}]$ (the orientable case), or $k \geqslant 1$ and $w = \prod x_{[1\uparrow k]}^2$ (the non-orientable case). Let $r \in \langle\, x_{[1\uparrow k]} \mid \quad \rangle$ and let $G \coloneqq \langle\, x_{[1\uparrow k]} \mid w, r\,\rangle$; we say that G is a *surface-plus-one-relation group*. A simple example is $\langle\, a, b, c \mid a^2b^2c^2, abc\,\rangle \simeq \langle\, a, b \mid [a, b]\,\rangle$. Under the name 'one-relator surface groups', the orientable surface-plus-one-relation groups were introduced and studied by Hempel [13], and further investigated by Howie [17], with the aim of carrying some of the known theory of one-relator groups over to these two-relator groups.

The purpose of this article is to study G algebraically, and generalize the work of Hempel and Howie to include the non-orientable case. If $k \leqslant 2$, then G is virtually abelian of rank at most two, and we consider such groups to be well understood. Thus we assume that $k \geqslant 3$, and here the closed surface is said to be *hyperbolic*. Recall the following.

2.1 Lemma. *A free generating sequence of* $\langle\, a, b, c \mid \quad \rangle$ *is given by* $(x, y, z) := (\overline{b}\,\overline{a}\,\overline{c}\overline{b}, ab, cab)$, *and here* ${}^{abcab}([x, y]z^2) = (abcab)(\overline{b}\,\overline{a}\,\overline{c}\overline{b})(ab)(bcab)(\overline{b}\overline{a})(cab)(cab)(\overline{b}\,\overline{a}\,\overline{c}\overline{b}\overline{a}) = a^2b^2c^2$. □

Thus, on setting $d = k - 2 \in [1{\uparrow}\infty[$, we can change the generating sequence from $x_{[1\uparrow k]}$ to $(x, y) \vee z_{[1\uparrow d]}$ and arrange for w to take the form $[x, y]u$, with $u \in \langle\, z_{[1\uparrow d]}\,\rangle$.

This leads us to consider the class of two-relator groups described as follows. Let $d \in [0{\uparrow}\infty[$, let $F := \langle\, (x, y) \vee z_{[1\uparrow d]} \mid \quad \rangle$, let u and r be elements of F, suppose that $u \in \langle\, z_{[1\uparrow d]}\,\rangle$, let $S := \langle\, (x, y) \vee z_{[1\uparrow d]} \mid [x, y]u\,\rangle$ and let $G := \langle\, (x, y) \vee z_{[1\uparrow d]} \mid [x, y]u,\ r\,\rangle$. Notice that our surface group has been changed to a form which includes many new groups, while we lose three closed surfaces, namely the sphere, the projective plane, and the Klein bottle.

In Section 3, we introduce a rewriting procedure for S.

In Section 4, we recall the concepts of potency and 'being residually a finite p-group for each prime p', and we show that S enjoys these properties, and we discuss connections with early work of Karl Gruenberg. The potency of S is used later in Section 6 to show that G is virtually torsion free.

In Section 5, we shall see that by changing (x, y) to a different free generating sequence of $\langle\, x, y \mid \quad \rangle$ without changing $[x, y]$, and then by carefully changing r without changing the conjugacy class of the image of r in S, we may assume that we have a presentation in which either $r = x^m$ for some $m \in [0{\uparrow}\infty[$, or $d \geqslant 1$ and r is what we shall call a 'Hempel relator for the presentation $\langle\, (x, y) \vee z_{[1\uparrow d]} \parallel [x, y]u\,\rangle$'. Notice that we use a double bar to distinguish a presentation from the group being presented.

In the case where $r = x^m$, we shall see that G is virtually one-relator, and we consider such groups to be well understood.

The main part of the article then examines the case where r is a Hempel relator; here, we can generalize the results that Hempel and Howie obtained for hyperbolic orientable surface-plus-one-relation groups.

In Section 6, for r a Hempel relator, we perform what Howie calls 'Hempel's trick' and express G as an HNN-extension of a one-relator group over an isomorphism between two free subgroups. We deduce that if r generates a *maximal* cyclic subgroup of F, then G is locally indicable. The proof uses a deep result of Howie which he proved by topological methods; we present a shorter proof, based on Bass-Serre theory, in an appendix. Let m denote the index of $\langle\, r\,\rangle$ in a maximal cyclic subgroup of F; we show that

G is (((free) $\rtimes$ (cyclic of order m)) by (locally indicable)).

In Section 7, for r a Hempel relator, we construct an exact sequence which gives a two-dimensional $\underline{\mathrm{E}}\,G$.

In Section 8, we recall the concept of VFL and show that G enjoys this property, and we calculate Euler characteristics.

In Section 9, we apply all the foregoing Hempel-Howie-type results to calculate the L^2-Betti numbers of the surface-plus-one-relation groups; for the orientable case this was done in [10, Theorem 5.1], in essentially the same way, using the original Hempel-Howie results.

In the appendix, as mentioned, we use Bass-Serre theory to simplify proofs of some important results of Howie on local indicability.

3 Rewriting in $S = \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u\,\rangle$

We shall use the following at various points in the article.

3.1 Notation. Let $d \in [0\uparrow\infty[$, let x, y and z be symbols, let $F := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid \quad\rangle$, and let u and r be elements of F. Suppose that $u \in \langle\, z_{[1\uparrow d]}\,\rangle$. Let $S := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u\,\rangle$, and let $G := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u,\, r\,\rangle$. We shall denote the natural map $F \to S$ by $w \mapsto w \bmod [x,y]u$.

The two-relator group G is a one-relator quotient of S, with relator $r \bmod [x,y]u$.

Let $N(F) := \langle\, \mathbb{Z} \times ((x) \vee z_{[1\uparrow d]}) \mid \quad\rangle$. For each $i \in \mathbb{Z}$, we shall denote the natural map

$$\langle\, (x) \vee z_{[1\uparrow d]} \mid \quad\rangle \quad\simeq\quad \langle\, \{i\} \times ((x) \vee z_{[1\uparrow d]}) \mid \quad\rangle \quad\leqslant\quad \langle\, \mathbb{Z} \times ((x) \vee z_{[1\uparrow d]}) \mid \quad\rangle$$

by $w \mapsto {}^{i}w$. Let y denote the automorphism of $N(F)$ determined by the shifting bijection

$$({}^{i}x) \;\vee\; {}^{i}z_{[1\uparrow d]} \quad\to\quad ({}^{i+1}x) \;\vee\; {}^{i+1}z_{[1\uparrow d]}$$

for all $i \in \mathbb{Z}$. Hence $C_\infty := \langle\, y \mid \quad\rangle$ acts on $N(F)$. If we form the semidirect product $N(F) \rtimes C_\infty$, we get the presentation $\langle\, (x,y) \vee z_{[1\uparrow d]} \mid \quad\rangle$. Thus we may identify $N(F) \rtimes C_\infty$ with F, and $N(F)$ then becomes the normal subgroup of F generated by $(x) \vee z_{[1\uparrow d]}$, that is, the set of elements whose y-exponent sum, with respect to $(x,y) \vee z_{[1\uparrow d]}$, is zero. Notice that, for each $(i,w) \in \mathbb{Z} \times \langle\, (x) \;\vee\; z_{[1\uparrow d]} \mid \quad\rangle$, we have identified ${}^{i}w = {}^{y^i}w$. Bearing in mind that $\mathbb{Z}$ is an abbreviation for $y^{\mathbb{Z}} = C_\infty$, we shall write $({}^{\mathbb{Z}}x)$ for $\mathbb{Z} \times (x)$, and ${}^{\mathbb{Z}}z_{[1\uparrow d]}$ for $\mathbb{Z} \times z_{[1\uparrow d]}$.

Let $N(S) := \langle\, ({}^{\mathbb{Z}}x) \;\vee\; {}^{\mathbb{Z}}z_{[1\uparrow d]} \mid ({}^{i+1}\overline{x}\cdot{}^{i}u\cdot{}^{i}x \mid i \in \mathbb{Z})\,\rangle$. The shifting action of $C_\infty = \langle\, y \mid \quad\rangle$ on $N(F)$ induces a C_∞-action on $N(S)$, and we find that we can identify $N(S) \rtimes C_\infty$ with S. Thus $N(S)$ is the image of $N(F)$ in S.

For each $j \in \mathbb{Z}$, let y act on $\langle\, ({}^{j}x) \;\;\vee\;\; {}^{\mathbb{Z}}z_{[1\uparrow d]} \mid \;\;\rangle$ by ${}^{j}x \mapsto {}^{j}u \cdot {}^{j}x$, and, for each ${}^{i}z_{*} \in {}^{\mathbb{Z}}z_{[1\uparrow d]}$, ${}^{i}z_{*} \mapsto {}^{i+1}z_{*}$. We find that we can make the identification

$$S = \langle\, \langle\, ({}^{j}x) \;\;\vee\;\; {}^{\mathbb{Z}}z_{[1\uparrow d]} \mid \;\;\rangle \rtimes C_{\infty}.$$

Thus $\langle\, ({}^{j}x) \;\;\vee\;\; {}^{\mathbb{Z}}z_{[1\uparrow d]} \mid \;\;\rangle$ can be viewed as a free factor of $N(F) \leqslant F$ which maps bijectively to $N(S)$ in S. By varying j, we get a family of embeddings of $N(S)$ in $N(F) \leqslant F$. □

4 Potency of $S = \langle\, (x, y) \vee z_{[1\uparrow d]} \mid [x, y]u \,\rangle$

In this section, we prove a useful fact about S and recall some related history.

4.1 Definition. A group S is said to be *potent* if, for each $s \in S - \{1\}$ and each $m \in [2\uparrow\infty[$, there exists a homomorphism from S to some finite group which sends s to some element of order exactly m; in this event, we also say that S is a potent group. □

The following fact, whose proof invokes two results of Allenby [1], will be applied in Section 6 to show that certain two-relator groups are virtually torsion free.

4.2 Lemma. *For $u \in \langle\, z_{[1\uparrow d]} \mid \;\;\rangle$, the group $S = \langle\, (x, y) \vee z_{[1\uparrow d]} \mid [x, y]u \,\rangle$ is potent.*

Proof. In the case where $u \neq 1$, $S = \langle\, x, y \mid \;\;\rangle \underset{[y,x]=u}{*} \langle\, z_{[1\uparrow d]} \mid \;\;\rangle$, and then S is potent because the free product of two free groups amalgamating a non-trivial cyclic group is potent [1, Section 4]. (We can also use the latter result as a perverse reference for the fact that free groups are potent.) Now we may assume that $u = 1$, and, hence, S is the free product of a rank-two, free-abelian group and a free group. Observe that free-abelian groups are potent. Recall that the free product of two potent groups is potent [1, Theorem 2.4]. (We can also use the latter result as a less perverse reference for the fact that free groups are potent.) The result now follows. □

We dedicate the remainder of this section to setting the foregoing results of Allenby into an historical context, with particular emphasis on connections with the early research of Karl Gruenberg. This digression will allow us to explain how some of the techniques involved can be used to prove the potency of S directly.

4.3 Definition. Let us say that a group S is $\mathrm{R}\mathfrak{F}p\forall p$ if, for each prime p, S is residually a finite p-group, that is, S embeds in a direct product of finite p-groups; in this event, we also say that S is an $\mathrm{R}\mathfrak{F}p\forall p$-group. □

The history begins with free groups.

In 1935, Magnus obtained the following important result.

(H1) Every free group is residually torsion-free nilpotent.

Let us recall the method that Magnus used.

Let R be an associative ring, let X be a set and let $R\langle\langle X\rangle\rangle$ denote the ring of formal power-series in the set X of non-commuting variables. Each $f \in R\langle\langle X\rangle\rangle$ has a unique expression as a formal sum $\sum f_{[0\uparrow\infty[}$ where, for each $m \in [0\uparrow\infty[$, f_m is homogeneous of degree m. For each $n \in [0\uparrow\infty[$, let I_n denote

$$I_n(R, X) := \{f \in R\langle\langle X\rangle\rangle \mid f_m = 0 \text{ for all } m \in [0\uparrow n-1]\},$$

a closed ideal in $R\langle\langle X\rangle\rangle$. For each $n \in [1\uparrow\infty[$, let U_n denote $U_n(R, X) := 1 + I_n$, a subgroup of the group of units of $R\langle\langle X\rangle\rangle$. If $m \in [1\uparrow n]$, then the natural map from U_m to the group of units of $R\langle\langle X\rangle\rangle/I_{n+1}$ has kernel U_{n+1} and image which can be denoted $1 + (I_m/I_{n+1})$. It follows that U_{n+1} is a normal subgroup of U_1 and that U_n/U_{n+1} is a free R-module which is central in U_1/U_{n+1}.

Consider now the case where $R = \mathbb{Z}$. Here, U_n/U_{n+1} is a free-abelian group, and, hence, $U_1(\mathbb{Z}, X)$ is residually torsion-free nilpotent; see [22, IVa]. Also, by considering leading coefficients in I_1 for reduced expressions, Magnus showed that $1 + X$ freely generates a free subgroup of $U_1(\mathbb{Z}, X)$; see [22, I]. This completes the outline of Magnus' proof of (H1).

Gruenberg [12, p. 29] remarks that A. Mal'cev in 1949, M. Hall in 1950, and Takehasi in 1951, independently found the following result.

(H2) Every free group is $\mathbf{r}\mathfrak{F}p\forall p$.

Let us recall how (H2) is related to power series. Let p be a prime number and let $R = \mathbb{Z}_p$, the integers modulo p. In the case where X is finite, $U_1(\mathbb{Z}_p, X)$ is residually a finite p-group, since the U_1/U_{n+1} are finite p-groups. In the case where X is infinite, we can retract $R\langle\langle X\rangle\rangle$ onto $R\langle\langle Y\rangle\rangle$ for any subset Y of X, and it follows that $U_1(\mathbb{Z}_p, X)$ is again residually a finite p-group. Now, as Luis Paris pointed out to us, a leading-coefficient argument again shows that $1 + X$ freely generates a free subgroup of $U_1(\mathbb{Z}_p, X)$, and this proves (H2).

Since every non-trivial finite p-group has a central, hence normal, subgroup of order p, it is not difficult to see the following result.

(H3) Every $\mathbf{r}\mathfrak{F}p\forall p$-group is potent.

The result (H3) was presented implicitly by Fischer-Karrass-Solitar in 1972; see [11, Proof of Theorem 2]. It was presented explicitly by Kim-McCarron in 1993; see [18, Lemma 2.2]. The facts (H2) and (H3) together prove the following result.

(H4) Every free group is potent.

The result (H4) was presented explicitly by Stebe in 1971 with a proof, attributed to Passman, based on (H1); see [24, Lemma 1]. Independently, it was presented implicitly by Fischer-Karrass-Solitar in 1972, with a proof based on (H2) and (H3); see [11, Proof of Theorem 2].

4.4 Remark. Let $d \in [1\uparrow\infty[$, let $u \in \langle\, z_{[1\uparrow d]} \mid \quad \rangle$, and let $S := \langle\, (x, y) \vee z_{[1\uparrow d]} \mid [x, y]u \,\rangle$.

We sketch an argument that shows how power series can be used to prove that S is $\mathbf{r}\mathfrak{F}p\forall p$.

Let p be a prime number.

As in Notation 3.1, $S = N(S) \rtimes C_\infty$ where $C_\infty = \langle\, y \mid \quad \rangle$ and $N(S) = \langle\, (x) \ \vee\ {}^{\mathbb{Z}}z_{[1\uparrow d]} \mid\ \rangle$. Here y acts on $N(S)$ by $x \mapsto {}^{0}u{\cdot}x$, ${}^{i}z_* \mapsto {}^{i+1}z_*$. Let b be a symbol and choose an identification of the countably infinite set $(x) \vee {}^{\mathbb{Z}}z_{[1\uparrow d]}$ with the subset $1 + b_{[0\uparrow\infty[}$ of $\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle$. Hence, by the p-analogue of Magnus' result, we get an embedding of $N(S)$ in the group of units of $\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle$. The action of y on $N(S)$ extends uniquely to a continuous automorphism of $\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle$, and we can form the skew-group-ring, or skew Laurent-polynomial ring, $(\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle)[C_\infty]$. It is then clear that S embeds in the group of units of $(\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle)[C_\infty]$.

Consider any $n \in [1\uparrow\infty[$.

Let $q \coloneqq p^n$ and let $C_{q^2} \coloneqq \langle y \mid y^{q^2}\rangle$.

We have the closed, y-invariant ideal $I_{n+1} = I_{n+1}(\mathbb{Z}_p, b_{[0\uparrow\infty[})$ of $\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle$, and we can form the skew-group-ring $(\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle/I_{n+1})[C_\infty]$. It is not difficult to check that the image of $(N(S))^q$ is trivial.

Let J_{n+1} denote the (closed, y-invariant) ideal of $\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle$ generated by

$$I_{n+1} \cup \{{}^{i}z_* - {}^{i+q}z_* \mid {}^{i}z_* \in {}^{\mathbb{Z}}z_{[1\uparrow d]}\}.$$

Again, we can form the skew-group-ring $(\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle/J_{n+1})[C_\infty]$. It is not difficult to check that the y^q-action fixes the image ${}^{\mathbb{Z}_q}z_{[1\uparrow d]}$ of ${}^{\mathbb{Z}}z_{[1\uparrow d]}$, and that the y^{q^2}-action fixes the image of x. Thus we can form the skew-group-ring $(\mathbb{Z}_p\langle\langle b_{[0\uparrow\infty[}\rangle\rangle/J_{n+1})[C_{q^2}]$ and find that the image of S is a finite p-group. By varying n, we see that S is residually a finite p-group. By varying p, we see that S is $\mathbf{R}\mathfrak{F}p\forall p$. By (H3), S is potent. □

In 1957, Gruenberg proved the following two results; see [12, Theorem 2.1(i) and Part (iii) of the Corollary on p.44].

(H5) Every residually torsion-free nilpotent group is $\mathbf{R}\mathfrak{F}p\forall p$.

(H6) The free product of any family of $\mathbf{R}\mathfrak{F}p\forall p$-groups is $\mathbf{R}\mathfrak{F}p\forall p$.

Each of these sheds important light on (H2).

In 1981, Allenby obtained the following result; see [1, Theorem 2.4].

(H7) The free product of any family of potent groups is potent.

This sheds light on (H4).

We next consider a class of groups which contains S when $u \neq 1$.

Let A and B be free groups, let $a \in A - \{1\}$, and let $b \in B - \{1\}$. The amalgamated free product $A \underset{a=b}{*} B$ is called a *cyclically-pinched one-relator group.*

In 1968, G. Baumslag used power series to obtain the following result; see [3, Theorem 1].

(H8) If a is not a proper power, and B is cyclic, then $A \underset{a=b}{*} B$ is residually torsion-free nilpotent.

In particular, by (H5), $A \underset{a=b}{*} B$ is then $\mathbf{R}\mathfrak{F}p\forall p$. In 1998, Kim and Tang used this to obtain the following result; see [19, Corollary 3.6].

(H9) $A \underset{a=b}{*} B$ is $\mathrm{R}\mathfrak{F}p\forall p$ if and only if a or b is not a proper power.

This gives the earliest proof we know of that S is $\mathrm{R}\mathfrak{F}p\forall p$, since the case where $u \neq 1$ follows from (H9) with $A = \langle\, x, y \mid \quad \rangle$, $B = \langle\, z_{[1\uparrow d]} \mid \quad \rangle$, $a = [y, x]$, $b = u$, since $[y, x]$ is not a proper power, while the case where $u = 1$ follows from (H6) and the fact that free-abelian groups are $\mathrm{R}\mathfrak{F}p\forall p$.

In 1981, Allenby obtained the following result; see [1, Section 4].

(H10) $A \underset{a=b}{*} B$ is potent.

We have seen that (H10) and (H7) imply that S is potent.

Finally, let us mention fundamental groups of closed surfaces.

In 1968, Chandler [6] obtained the following result.

(H11) For every closed surface except the projective plane and the Klein bottle, the fundamental group is residually torsion-free nilpotent and, hence, by (H5), is $\mathrm{R}\mathfrak{F}p\forall p$.

The following is a consequence of (H10).

(H12) For every closed surface except the projective plane, the fundamental group is potent.

5 Hempel relators

In this section, we introduce Hempel relators and show how to find them.

5.1 Definitions. Let F be a group. Let $r \in F$. If $r \in F - \{1\}$ and $\langle\, r \,\rangle$ is contained in a unique maximal infinite, cyclic subgroup C of F, then there exist a unique $m \in [1\uparrow\infty[$, and a unique $s \in F$, such that $m = [C{:}\langle\, g \,\rangle]$, $\langle\, s \,\rangle = C$ and $s^m = r$; we then define $\sqrt[F]{r} := s$ and $\log_F r := m$, and say that s is *the root* of r in F. For $r = 1$, we define $\sqrt[F]{1} := 1$ and $\log_F 1 = \infty$. In all other cases we say that r *does not have a root* in F. We say that F *has roots* if every element of F has a root in F. □

5.2 Lemma. *With* Notation 3.1*, there is a free generating sequence* (x', y') *of* $\langle\, x, y \mid \quad \rangle$ *such that* $[x', y'] = [x, y]$*, and the* y'*-exponent sum of* r *with respect to* $(x', y') \vee z_{[1\uparrow d]}$ *is zero, and the* x'*-exponent sum of* r *with respect to* $(x', y') \vee z_{[1\uparrow d]}$ *is nonnegative.*

Moreover, S has roots and any element of the free subgroup $N(S)$ has the same root in both S and $N(S)$.

Proof. Let a and b respectively denote the x- and y-exponent sums of r with respect to $(x, y) \vee z_{[1\uparrow d]}$. We want $a \geqslant 0$ and $b = 0$.

Replacing x with $xy^{\pm 1}$ fixes $[x, y]$ and changes (a, b) to $(a, b \pm a)$. Replacing y with $yx^{\pm 1}$ fixes $[x, y]$ and changes (a, b) to $(a \pm b, b)$. If $ab \neq 0$, then one or more of these four operations reduces $|a|+|b|$. By repeating such operations, we can arrange that $ab=0$. If $a=0$, then we can alter $(0,b)$ to (b,b) and then to $(b,0)$; thus we may assume that $b=0$. Finally, if $a<0$, we can successively alter $(a,0)$ to (a,a), $(0,a)$, $(-a,a)$, and $(-a,0)$. Thus we may also assume that $a \geqslant 0$.

We next show that $r \bmod [x,y]u$ has a root in S. We may assume that $r \bmod [x,y]u$ lies in $S-\{1\}$. Let (x',y') be a free generating sequence of $\langle\, x,y \mid \quad \rangle$ such that $[x',y']=[x,y]$, and the y'-exponent sum of r with respect to $(x',y') \vee z_{[1\uparrow d]}$ is zero. To simplify notation, we forget the original (x,y), use (x',y') as the new generating sequence, and name it (x,y). Thus we may assume that $r \bmod [x,y]u$ lies in $N(S)$, a free subgroup of S. Hence $r \bmod [x,y]u$ has a root in $N(S)$. Any cyclic subgroup of S containing $r \bmod [x,y]u$ maps to a finite, hence trivial, subgroup in $S/N(S) = C_\infty = \langle\, y \mid \quad \rangle$. Hence, any cyclic subgroup of S containing $r \bmod [x,y]u$ lies in $N(S)$. Thus the root of $r \bmod [x,y]u$ in the free group $N(S)$ is a root in S. □

5.3 Definition. With Notation 3.1, let $X_1 := ({}^{1}x) \ \vee \ {}^{[0\uparrow\infty[}z_{[1\uparrow d]}$. We say that r is a *Hempel relator* for the presentation $\langle\, (x,y) \vee z_{[1\uparrow d]} \parallel [x,y]u \,\rangle$ if the following hold.

(R1) $r \in \langle\, X_1 \mid \quad \rangle \leqslant N(F) \leqslant F$,

(R2) In $\langle\, X_1 \mid \quad \rangle$, r is not conjugate to any element of $\langle\, {}^{0}\overline{u} \cdot {}^{1}x \,\rangle$.

(R3) With respect to X_1, r is cyclically reduced.

(R4) With respect to X_1, r *involves* some element of ${}^{0}z_{[1\uparrow d]}$, that is, r does not lie in the free factor $\langle\, ({}^{1}x) \ \vee \ {}^{[1\uparrow\infty[}z_{[1\uparrow d]} \mid \quad \rangle$. □

We now want to show that for our purposes we can assume that r is a Hempel relator.

5.4 Lemma. *With* Notation 3.1*, there exists an element w of F, an element v of $\langle\, {}^{F}([x,y]u) \,\rangle$, and an automorphism α of $\langle\, x,y \mid \quad \rangle$ which fixes $[x,y]$, such that $r' := (v)({}^{w\alpha}r)$ is either a non-negative power of x or a Hempel relator for $\langle\, (x,y) \vee z_{[1\uparrow d]} \parallel [x,y]u \,\rangle$.*

Here, $G \simeq \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u,\ r' \,\rangle$ and $\log_F r' = \log_S (r \bmod [x,y]u)$.

Proof. We shall successively alter r and, to avoid extra notation, we shall use the same symbol r to denote the altered element of F at each stage.

We first alter r by automorphisms of $\langle\, x,y \mid \quad \rangle$ which fix $[x,y]$. By Lemma 5.2, we may assume that the y-exponent sum of r with respect to $(x,y) \vee z_{[1\uparrow d]}$ is zero, and the x-exponent sum of r with respect to $(x,y) \vee z_{[1\uparrow d]}$ is nonnegative. Hence $r \bmod [x,y]u$ lies in the free subgroup $N(S)$.

For each $j \in \mathbb{Z}$, we can view $({}^{j}x) \vee {}^{\mathbb{Z}}z_{[1\uparrow d]}$ as a free generating sequence of $N(S)$ and express $r \bmod [x,y]u$ as a word therein and lift the word back to a new element of $N(F)$; this multiplies r by an element of $\langle\, {}^{F}([x,y]u) \,\rangle$. We then reduce the expression cyclically, which corresponds to conjugating r in F.

If $r = {}^{j}x^m$ for some $m \in \mathbb{Z}$, then $m \geqslant 0$. By replacing r with ${}^{y^{-j}}r$, we may assume that $r = x^m$, and the desired conclusions hold.

Thus, we may assume that, for each $j \in \mathbb{Z}$, the cyclically reduced expression for r in $({}^{j}x) \vee {}^{\mathbb{Z}}z_{[1\uparrow d]}$ involves some element of ${}^{\mathbb{Z}}z_{[1\uparrow d]}$, and, hence, there is a unique smallest non-empty interval $[\mu_j\uparrow\nu_j]$ in $\mathbb{Z}$ such that the cyclically reduced expression for r in $({}^{j}x)\vee{}^{\mathbb{Z}}z_{[1\uparrow d]}$ is a word in $({}^{j}x) \vee {}^{[\mu_j\uparrow\nu_j]}z_{[1\uparrow d]}$.

We claim that for some $k \in \mathbb{Z}$, $\mu_{k+1} = k$.

To pass from $({}^{j+1}x) \vee {}^{\mathbb{Z}}z_{[1\uparrow d]}$ to $({}^{j}x) \vee {}^{\mathbb{Z}}z_{[1\uparrow d]}$, we replace ${}^{j+1}x$ with ${}^{j}u \cdot {}^{j}x$ and reduce cyclically. In passing from $[\mu_{j+1}\uparrow\nu_{j+1}]$ to $[\mu_j\uparrow\nu_j]$, we may add j, we may delete some values, and we then take the convex hull in $\mathbb{Z}$. By repeating this change sufficiently often, we find that, for $j << 0$, $j \leqslant \mu_j$. Among all $j \in \mathbb{Z}$ such that $j \leqslant \mu_j$, let us choose one that minimizes $\nu_j - \mu_j \in [0\uparrow\infty[$.

To pass from $({}^{j}x) \vee {}^{\mathbb{Z}}z_{[1\uparrow d]}$ to $({}^{j+1}x) \vee {}^{\mathbb{Z}}z_{[1\uparrow d]}$, we replace ${}^{j}x$ with ${}^{j}\overline{u} \cdot {}^{j+1}x$ and reduce cyclically. In passing from $[\mu_j\uparrow\nu_j] \subseteq [j\uparrow\infty[$ to $[\mu_{j+1}\uparrow\nu_{j+1}]$, we may add j, we may delete some values, and we then take the convex hull in $\mathbb{Z}$. Hence, $[\mu_{j+1}\uparrow\nu_{j+1}] \subseteq [j\uparrow\nu_j]$; that is, $\mu_{j+1} \geqslant j$ and $\nu_{j+1} \leqslant \nu_j$.

If $\mu_{j+1} = j$, we take $k = j$.

To prove the claim, it remains to consider the case where $\mu_{j+1} \geqslant j+1$. By the minimality assumption, $\mu_{j+1} \leqslant \mu_j$. Hence, $j+1 \leqslant \mu_{j+1} \leqslant \mu_j$. Since $j+1 \leqslant \mu_j$, when we replace ${}^{j}x$ with ${}^{j}\overline{u} \cdot {}^{j+1}x$ the occurrences of ${}^{j}\overline{u}$ survive unaffected by (cyclic) reduction. Since $j+1 \leqslant \mu_{j+1}$, we see that ${}^{j}\overline{u} = 1$. Hence $u = 1$ and, hence, μ_j does not depend on j. We take $k = \mu_0$, and then $\mu_{k+1} = \mu_0 = k$.

In all cases, then, we have some $k \in \mathbb{Z}$ such that $\mu_{k+1} = k$. By replacing r with ${}^{y^{-k}}r$, we may arrange that $k = 0$. Now r has become a Hempel relator. Notice that r is not conjugate to a power of ${}^{0}\overline{u} \cdot {}^{1}x$ in $\langle\, ({}^{1}x) \vee {}^{[0\uparrow\infty[}z_{[1\uparrow d]} \mid \quad \rangle$ because $r \bmod [x,y]u$ is not conjugate to a power of ${}^{0}x$ in $\langle\, ({}^{1}x) \vee {}^{\mathbb{Z}}z_{[1\uparrow d]} \mid \quad \rangle$. $\square$

In Examples 8.3, we shall see that, in the case where $r \in \langle\, x \,\rangle$, G is virtually one-relator; we consider such groups to be well understood.

6 HNN decomposition, local indicability and torsion

In this section, we shall extend three types of results of Hempel and Howie, namely the HNN decomposition, the local indicability, and the analysis of torsion.

6.1 Notation. We shall use a left-right twisting of the notation in [8, Examples I.3.5(v)], and write $G_v \underset{G_e}{*} t_e$ to denote an HNN extension, where it is understood that G_v is a group, G_e is a subgroup of G_v, there is specified some injective homomorphism $\overline{t}_e \colon G_e \to G_v$, $g \mapsto {}^{\overline{t}_e}g$, and the associated HNN extension is

$$G_v \underset{G_e}{*} t_e := (G_v * \langle\, t_e \mid \quad \rangle)/\langle\!\langle\, \{\, \overline{t}_e{\cdot}\overline{g}{\cdot}t_e{\cdot}^{\overline{t}_e}g \mid g \in G_e \}\, \rangle\!\rangle.$$

If $G = G_v \underset{G_e}{*} t_e := G_v$, then the Bass-Serre G-tree has vertex set G/G_v and edge set G/G_e, with gG_e joining gG_v to gt_eG_v, for each $g \in G$. $\square$

In the case where r is a Hempel relator we shall now see that we get an HNN extension of a one-relator group over a free group.

6.2 Notation. With Notation 3.1, suppose that r is a Hempel relator for the presentation $\langle\, (x, y) \vee z_{[1\uparrow d]} \parallel [x, y]u \,\rangle$; thus, for $X_1 := (\,{}^1x) \;\vee\; {}^{[0\uparrow\infty[}z_{[1\uparrow d]}$, the following hold.

(R1) $r \in \langle\, X_1 \mid \quad \rangle \leqslant N(F) \leqslant F$.

(R2) In $\langle\, X_1 \mid \quad \rangle$, r is not conjugate to any element of $\langle\, x \,\rangle$, where $x := {}^0\overline{u} \cdot {}^1x$. In particular, $r \neq 1$ and, hence, $d \neq 0$.

(R3) With respect to X_1, r is cyclically reduced.

(R4) With respect to X_1, r involves some element of ${}^0z_{[1\uparrow d]}$.

(R5) With respect to X_1, r involves some element of ${}^\nu z_{[1\uparrow d]}$, where $\nu = \nu(r)$ denotes the least element of $[0\uparrow\infty[$ such that $r \in \langle\, (\,{}^1x) \;\vee\; {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \mid \quad \rangle$.

Since ${}^1x = {}^0u \cdot x$, we can identify $\langle\, (x) \;\vee\; {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \mid \quad \rangle = \langle\, (\,{}^1x) \;\vee\; {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \mid \quad \rangle$, and thus view r as an element of a free group with two specified free generating sets. With respect to $(x) \;\vee\; {}^{[0\uparrow\nu]}z_{[1\uparrow d]}$, r involves some element of ${}^\nu z_{[1\uparrow d]}$, even if $\nu = 0$, by (R5) and (R2). With respect to $(\,{}^1x) \;\vee\; {}^{[0\uparrow\nu]}z_{[1\uparrow d]}$, r involves some element of ${}^0z_{[1\uparrow d]}$, by (R4). We define

$$\begin{aligned} G_{[0\uparrow\nu]} &:= \langle\, (x) \;\vee\; {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \mid r \,\rangle = \langle\, (\,{}^1x) \;\vee\; {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \mid r \,\rangle, \\ G_{[0\uparrow(\nu-1)]} &:= \langle\, (x) \;\vee\; {}^{[0\uparrow(\nu-1)]}z_{[1\uparrow d]} \mid \quad \rangle, \\ G_{[1\uparrow\nu]} &:= \langle\, (\,{}^1x) \;\vee\; {}^{[1\uparrow\nu]}z_{[1\uparrow d]} \mid \quad \rangle. \end{aligned}$$

By Magnus' Freiheitssatz, which appears as Corollary A.3.2 in the appendix, the natural maps from $G_{[0\uparrow(\nu-1)]}$ and $G_{[1\uparrow\nu]}$ to $G_{[0\uparrow\nu]}$ are injective.

We have an isomorphism $y\colon G_{[0\uparrow(\nu-1)]} \to G_{[1\uparrow\nu]}$ given by the natural bijection on the specified free generating sets. We can then form the HNN extension $G_{[0\uparrow\nu]} \underset{G_{[1\uparrow\nu]}}{*} y$. On simplifying the presentation we recover the presentation of G and thus obtain the HNN decomposition $G = G_{[0\uparrow\nu]} \underset{G_{[1\uparrow\nu]}}{*} y$. $\square$

In the case where G is a hyperbolic orientable surface-plus-one-relation group, this HNN decomposition was obtained topologically by Howie [17, Proposition 2.1.2(c)] who attributed it to an argument implicit in [13, Proof of Theorem 2.2] which in turn is attributed to Howie.

6.3 Definition. A group is said to be *locally indicable* if each finitely generated subgroup *either* is trivial *or* has some infinite, cyclic quotient. □

6.4 Theorem. *Let $d \in [0{\uparrow}\infty[$, let $F := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid \quad \rangle$, and let u and r be elements of F. Suppose that $u \in \langle\, z_{[1\uparrow d]} \,\rangle$, and that r is a Hempel relator for $\langle\, (x,y) \vee z_{[1\uparrow d]} \parallel [x,y]u \,\rangle$. Let $G := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u, r \,\rangle$. If $\sqrt[F]{r} = r$, then G is locally indicable.*

Proof. We use Notation 6.2.

In G, define, for each $i \in \mathbb{Z}$,

$$\begin{aligned} G_{[i\uparrow(i+\nu-1)]} &:= {}^{y^i}(G_{[0,(\nu-1)]}) = \langle\, (\,{}^{i}x) \;\vee\; {}^{[i\uparrow(i+\nu-1)]}z_{[1\uparrow d]} \mid \quad \rangle, \\ G_{[i\uparrow(i+\nu)]} &:= {}^{y^i}(G_{[0,\nu]}) = \langle\, (\,{}^{i}x) \;\vee\; {}^{[i\uparrow(i+\nu)]}z_{[1\uparrow d]} \mid \; {}^{i}r \,\rangle. \end{aligned}$$

We can then write

$$G_{[i\uparrow(i+\nu)]} = (G_{[i\uparrow(i+\nu-1)]} * \langle\, {}^{i+\nu}z_{[1\uparrow d]} \mid \quad \rangle)/\langle\!\langle {}^{i}r \rangle\!\rangle. \tag{1}$$

Since $\sqrt[F]{r} = r$, ${}^{i}r$ is not a proper power in $G_{[i\uparrow(i+\nu-1)]} * \langle\, {}^{i+\nu}z_{[1\uparrow d]} \mid \quad \rangle$.

By (R2), (R3) and (R5), ${}^{0}r \in G_{[0\uparrow(\nu-1)]} * \langle\, {}^{\nu}z_{[1\uparrow d]} \mid \quad \rangle$ is not conjugate to any element of $G_{[0\uparrow(\nu-1)]}$. Hence ${}^{i}r \in G_{[i\uparrow(i+\nu-1)]} * \langle\, {}^{i+\nu}z_{[1\uparrow d]} \mid \quad \rangle$ is not conjugate to any element of $G_{[i\uparrow(i+\nu-1)]}$.

By using $i+\nu+1$, we can form the free product with amalgamation

$$\begin{aligned} G_{[i\uparrow(i+\nu+1)]} &:= G_{[i\uparrow(i+\nu)]} \underset{\langle\, (\,{}^{i}u \cdot {}^{i}x = {}^{i+1}x) \;\vee\; {}^{[(i+1)\uparrow(i+\nu)]}z_{[1\uparrow d]} \mid \quad \rangle}{*} G_{[(i+1)\uparrow(i+\nu+1)]} \\ &= G_{[i\uparrow(i+\nu)]} \underset{G_{[(i+1)\uparrow(i+\nu)]}}{*} G_{[(i+1)\uparrow(i+\nu+1)]}. \end{aligned}$$

By varying i, we get a bi-infinite chain of free products with amalgamation

$$N(G) := \quad \cdots\; G_{[(i-1)\uparrow(i+\nu-1)]} \underset{G_{[i\uparrow(i+\nu-1)]}}{*} G_{[i\uparrow(i+\nu)]} \underset{G_{[(i+1)\uparrow(i+\nu)]}}{*} G_{[(i+1)\uparrow(i+\nu+1)]} \;\cdots.$$

Here $C_\infty = \langle\, y \mid \quad \rangle$ acts by shifting and we find that $N(G) \rtimes C_\infty = G$. For any finite, non-empty interval $[j{\uparrow}i]$ in $\mathbb{Z}$ let us define

$$G_{[j\uparrow(i+\nu)]} := \quad G_{[j\uparrow(j+\nu)]} \underset{G_{[(j+1)\uparrow(j+\nu)]}}{*} \cdots \underset{G_{[i\uparrow(i+\nu-1)]}}{*} G_{[i\uparrow(i+\nu)]},$$

a subgroup of G. By using (1), we see that

$$G_{[j\uparrow(i+\nu)]} = (G_{[j\uparrow(i+\nu-1)]} * \langle\, {}^{i+\nu}z_{[1\uparrow d]} \mid \quad \rangle)/\langle\!\langle {}^{i}r \rangle\!\rangle.$$

Now ${}^{i}r \in G_{[i\uparrow(i+\nu-1)]} * \langle\, {}^{i+\nu}z_{[1\uparrow d]} \mid \quad \rangle \subseteq G_{[j\uparrow(i+\nu-1)]} * \langle\, {}^{i+\nu}z_{[1\uparrow d]} \mid \quad \rangle$, and ${}^{i}r$ has the same cyclically reduced expression in both free products. In particular, ${}^{i}r$ is not a proper power and is not conjugate to any element of $G_{[j\uparrow(i+\nu-1)]}$. By a result of Howie given as Corollary A.3.6 in the appendix, the subgroup $G_{[j\uparrow\infty[} := \bigcup\limits_{i\in[j\uparrow\infty[} G_{[j\uparrow(i+\nu)]}$ is then locally indicable. It follows that $\bigcup\limits_{j\in[0\downarrow(-\infty)[} G_{[j\uparrow\infty[}$ is locally indicable, that is, $N(G)$ is locally indicable. Hence, $N(G)\rtimes C_\infty$ is locally indicable, that is, G is locally indicable. This completes the proof. $\square$

In the case where G is a hyperbolic orientable surface-plus-one-relation group, the above result was given by Hempel [13, Theorem 2.2] with its proof attributed to Howie.

We now discuss torsion.

6.5 Theorem. *Let $d \in [0\uparrow\infty[$, let $F := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid \quad \rangle$, and let u and r be elements of F. Suppose that $u \in \langle\, z_{[1\uparrow d]} \,\rangle$ and that r is a Hempel relator for $\langle\, (x,y) \vee z_{[1\uparrow d]} \parallel [x,y]u \,\rangle$. Let $G := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u, r \,\rangle$.*

Let $m := \log_F r$ and let $C_m := \langle\, \sqrt[F]{r} \,\rangle / \langle\!\langle r \rangle\!\rangle$. Then the following hold.

(i). *C_m can be identified with the subgroup of G generated by the image of $\sqrt[F]{r}$.*

(ii). *$G/\langle\, {}^{G}C_m \,\rangle = \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u,\ \sqrt[F]{r} \,\rangle$ which is locally indicable.*

(iii). *There exists a subset X of G such that $\langle\, {}^{G}C_m \,\rangle = \ast({}^{X}C_m) := \mathop{\ast}\limits_{g\in X}({}^{g}C_m)$. Let K denote the kernel of the homomorphism $\ast({}^{X}C_m) \to C_m$ which acts as conjugation by $\overline{g}$ on ${}^{g}C_m$, for each $g \in X$. Then K is a free group, and $\langle\, {}^{G}C_m \,\rangle = K \rtimes C_m$.*

(iv). *Each torsion subgroup of G lies in some conjugate of C_m.*

(v). *Every torsion-free subgroup of G is locally indicable.*

(vi). *G has some torsion-free finite-index subgroup.*

Proof. We again use Notation 6.2.

Notice that $d \neq 0$ and $r \neq 1$, by (R2).

Notice that $\sqrt[F]{r}$ too is a Hempel relator for $\langle\, (x,y)\vee z_{[1\uparrow d]} \parallel [x,y]u \,\rangle$, and that $\nu(\sqrt[F]{r})=\nu(r)$.

(i). By results of Magnus, we can view C_m as a subgroup of the one-relator group $G_{[0\uparrow\nu]}$. By results of Higman-Neumann-Neumann, we can view $G_{[0\uparrow\nu]}$ as a subgroup of G. Thus (i) holds.

(ii) follows easily from Theorem 6.4.

(iii). It is straightforward to see the following.

$$\begin{aligned} G/\langle\, {}^{G}C_m \,\rangle &= \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u,\ \sqrt[F]{r} \,\rangle \\ &= \langle\, (x) \vee {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \mid \sqrt[F]{r} \,\rangle \mathop{\ast}\limits_{\langle\, ({}^{1}x)\vee {}^{[1\uparrow\nu]}z_{[1\uparrow d]} \mid \quad \rangle} y \\ &= (G_{[0\uparrow\nu]}/\langle\, {}^{G_{[0\uparrow\nu]}}C_m \,\rangle) \mathop{\ast}\limits_{G_{[1\uparrow\nu]}} y. \end{aligned}$$

If we consider $\langle\,{}^{G}C_m\,\rangle$ acting on the Bass-Serre tree for the HNN decomposition of G, we now see that $\langle\,{}^{G}C_m\,\rangle$ acts freely on the edge set, and that the quotient graph is the Bass-Serre tree for the above HNN decomposition of $G/\langle\,{}^{G}C_m\,\rangle$. Hence $\langle\,{}^{G}C_m\,\rangle$ is a free product of conjugates of $\langle\,{}^{G_{[0\uparrow\nu]}}C_m\,\rangle$, with one conjugate for each vertex of the latter tree. By [11, Theorem 1], $\langle\,{}^{G_{[0\uparrow\nu]}}C_m\,\rangle$ in turn is a free product of conjugates of C_m. Hence, $\langle\,{}^{G}C_m\,\rangle$ is a free product of conjugates of C_m. Hence we have the desired homomorphism $\langle\,{}^{G}C_m\,\rangle \to C_m$, and its kernel K. Clearly $\langle\,{}^{G}C_m\,\rangle = K \rtimes C_m$. If we consider K acting on the Bass-Serre tree associated with the graph-of-groups decomposition of $\langle\,{}^{G}C_m\,\rangle$ as a free product of copies of C_m, we see that K acts freely on the vertex set, and, hence, K is a free group.

(iv). Suppose that H is some torsion subgroup of G. The image of H in the torsion-free quotient $G/\langle\,{}^{G}C_m\,\rangle$ is then trivial, that is, $H \leqslant \langle\,{}^{G}C_m\,\rangle = K \rtimes C_m$. Since $H \cap K$ is necessarily trivial, H embeds in C_m and, in particular, H is finite. Also, H lies in $\langle\,{}^{G}C_m\,\rangle = \ast({}^{X}C_m)$. By Bass-Serre theory, or the Kurosh subgroup theorem, H lies in some conjugate of C_m.

(v). Suppose that H is some torsion-free subgroup of G. Then, by Bass-Serre theory, or the Kurosh subgroup theorem, $H \cap (\ast({}^{X}C_m)) = H \cap \langle\,{}^{G}C_m\,\rangle$ is free, and, hence, locally indicable. Now $H/(H \cap \langle\,{}^{G}C_m\,\rangle)$ embeds in the locally indicable group $G/\langle\,{}^{G}C_m\,\rangle$, and hence is locally indicable. It follows that H is locally indicable.

(vi). We imitate the proof of [11, Theorem 2]. By Lemma 4.2, or Remark 4.4, there exists some finite group Φ and some homomorphism $\alpha\colon \langle\,(x,y) \vee z_{[1\uparrow d]} \mid [x,y]u\,\rangle \to \Phi$ such that $\alpha(\sqrt[F]{r}\,)$ has order exactly m. This induces a homomorphism

$$\beta\colon \langle\,(x,y) \vee z_{[1\uparrow d]} \mid [x,y]u, r\,\rangle \to \Phi$$

which is injective on C_m. Let N denote the kernel of $\beta\colon G \to \Phi$. If H is some torsion subgroup of N, then, by (iv), $H \subseteq {}^{g}C_m \cap N$ for some $g \in G$. Since β is injective on ${}^{g}C_m$ and vanishes on N, we see that ${}^{g}C_m \cap N = \{1\}$. Thus N is torsion free and of finite index in G. □

In the case where G is a hyperbolic orientable surface-plus-one-relation group, Howie [17] proved several of these results.

7 Exact sequences

In this section, we construct an exact sequence that shows the existence of a two-dimensional $\underline{\mathrm{E}}\,G$ when we have a Hempel presentation.

7.1 Remark. We shall find ourselves considering diagrams of abelian groups of the forms that appear in Fig 7.1.1, where maps are written on the right of their arguments. If the left-hand diagram is commutative and its two columns and long row are exact, then the right-hand diagram is an exact sequence; this implication follows from the theory of total complexes of double complexes, and is easy to check by chasing diagrams. □

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & 0 \arrow[d] & D \arrow[d, "f"] & & \\
 & A \arrow[r, "c"] \arrow[d, "a"'] & E \arrow[d, "g"] & & \\
 & B \arrow[r, "d"] \arrow[d, "b"'] & F \arrow[d, "h"] & & \\
0 \arrow[r] & C \arrow[r, "e"] \arrow[d] & G \arrow[r, "i"] \arrow[d] & H \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
0 \arrow[d] \\
A \oplus D \arrow[d, "\left(\begin{smallmatrix} -a & c \\ 0 & f \end{smallmatrix}\right)"] \\
B \oplus E \arrow[d, "\left(\begin{smallmatrix} d \\ g \end{smallmatrix}\right)"] \\
F \arrow[d, "hi"] \\
H \arrow[d] \\
0
\end{tikzcd}
$$

Figure 7.1.1: A double complex and its total complex

7.2 Notation. Let W be a set and let G be a group.

We identify $\mathbb{Z}[G\times W] = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}W$, and, hence, for each $(p, w) \in \mathbb{Z}G \times W$, we view $p \otimes w$ as an element of $\mathbb{Z}[G\times W]$.

We also identify $\mathbb{Z}[G\times W]$ with the direct sum of a family of copies of $\mathbb{Z}G$ indexed by W, denoted $(\mathbb{Z}G)^W$.

For any subgroup C of G, we consider C as acting trivially on W and write $\mathbb{Z}[G \times_C W] = \mathbb{Z}G \otimes_{\mathbb{Z}C} \mathbb{Z}W$.

For any map of sets $\alpha\colon W \to G$, $w \mapsto \alpha(w)$, the map of sets

$$W \to (\mathbb{Z}[G\times W]) \rtimes G = \begin{pmatrix} 1 & 0 \\ \mathbb{Z}[G\times W] & G \end{pmatrix}, \quad w \mapsto (1 \otimes w, \alpha(w)) = \begin{pmatrix} 1 & 0 \\ 1 \otimes w & \alpha(w) \end{pmatrix},$$

induces a unique group homomorphism $\langle\, W \mid \quad \rangle \to (\mathbb{Z}[G\times W]) \rtimes G$, denoted $r \mapsto (\frac{\partial r}{\partial W}, \alpha(r))$. We call $\frac{\partial r}{\partial W}$ the *total Fox derivative* of r (with respect to W, relative to α). If the map α is understood, then, for each $w \in W$, we write $\frac{\partial r}{\partial w} \otimes w$ for the summand of $\frac{\partial r}{\partial W}$ corresponding to w. $\square$

7.3 Theorem. *Let $d \in [0{\uparrow}\infty[$, let $F := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid \quad \rangle$, and let u and r be elements of F. Suppose that $u \in \langle\, z_{[1\uparrow d]} \,\rangle$ and that r is a Hempel relator for $\langle\, (x,y) \vee z_{[1\uparrow d]} \parallel [x,y]u \,\rangle$. Let $G := \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u, r \,\rangle$.*

Let C denote the subgroup of G generated by the image of $\sqrt[F]{r}$. Let $(x,y,z_{[1\uparrow d]})$ denote $(x,y) \vee z_{[1\uparrow d]}$, and let $G \times ([x,y]u, {}_C r)$ denote $(G \times \{[x,y]u\}) \cup (G \times_C \{r\})$. Then the sequence of left $\mathbb{Z}G$-modules given by

$$0 \to \mathbb{Z}[G \times ([x,y]u, {}_C r)] \xrightarrow{\begin{pmatrix} 1\otimes[x,y]u \\ \mapsto \frac{\partial[x,y]u}{\partial(x,y,z_{[1\uparrow d]})} \\ 1\otimes_C r \\ \mapsto \frac{\partial r}{\partial(x,y,z_{[1\uparrow d]})} \end{pmatrix}} \mathbb{Z}[G \times (x,y,z_{[1\uparrow d]})] \xrightarrow{\begin{pmatrix} 1\otimes w \\ \mapsto (w-1)1 \end{pmatrix}} \mathbb{Z}[G/1] \xrightarrow{(1\mapsto G)} \mathbb{Z}[G/G] \to 0$$

is exact.

7.4 Remark. The exact sequence of left $\mathbb{Z}G$-modules in Theorem 7.3, which has the form

$$0 \to \mathbb{Z}G \oplus \mathbb{Z}[G/C] \to \mathbb{Z}G^{d+2} \to \mathbb{Z}G \to \mathbb{Z} \to 0,$$

is the augmented cellular chain complex of the 'adjusted' universal cover of the Cayley complex of the presentation

$$\langle\, (x,y) \vee z_{[1\uparrow d]} \parallel [x,y]u, r \,\rangle,$$

where 'adjusted' means the following. In the universal cover, at each zero-cell $g \in G$, there is attached a two-cell $g\mathbf{r}$ whose boundary reads representatives of ${}^g r$ in F off the one-skeleton. Let c denote the image of $\sqrt[F]{r}$ in G. Then $c^m = 1$ and the boundaries of $gc\mathbf{r}$ and $g\mathbf{r}$ read the same elements of F; if $m \geqslant 2$, this creates a two-sphere in the universal cover. By 'adjusting' the universal cover we mean identifying $g\mathbf{r} = gc\mathbf{r}$ for each $g \in G$. To make the G-action cellular, we should subdivide $\mathbf{r}$ into m regions permuted by c with a single point fixed by c. This would allow us to divide out by the action of G to recover an 'adjusted' Cayley complex, a two-dimensional $\underline{\mathrm{E}}\,G$, which has a two-cell $\mathbf{r}'$, whose boundary reads $\sqrt[F]{r}$, with an interior $\frac{1}{m}$th-point, or cone-point of angle $\frac{2\pi}{m}$, which means that a path travelling m times around the boundary of $\mathbf{r}'$ is null-homotopic. $\square$

Proof of Theorem 7.3. In Notation 6.2, let $(x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]})$ denote $(x) \vee {}^{[0\uparrow\nu]}z_{[1\uparrow d]}$, let $(x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]})$ denote $(x) \vee {}^{[1\uparrow\nu]}z_{[1\uparrow d]}$, and let

$$\langle\, x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \,\rangle := G_{[0\uparrow\nu]} := \langle\, (x) \vee {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \mid r \,\rangle,$$

$$\langle\, {}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]} \,\rangle := G_{[1\uparrow\nu]} := \langle\, (x) \vee {}^{[1\uparrow\nu]}z_{[1\uparrow d]} \mid \quad \rangle.$$

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & 0 \arrow[d] & \mathbb{Z}[G/C] \arrow[d, "\begin{pmatrix} C \mapsto \\ \frac{\partial r(x,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})}{\partial (x,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})} \end{pmatrix}"] & & \\
 & \mathbb{Z}[G \times ({}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]})] \arrow[r, "\begin{pmatrix} 1\otimes {}^{1}x \mapsto \\ (y - {}^{0}u)\otimes x \\ -\frac{\partial\, {}^{0}u}{\partial\, {}^{0}z_{[1\uparrow d]}} \\ \\ 1\otimes {}^{i}z_{*} \mapsto \\ y\otimes {}^{i-1}z_{*} \\ -1\otimes {}^{i}z_{*} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1\otimes w \mapsto \\ (w-1)\otimes y \end{pmatrix}"] & \mathbb{Z}[G \times (x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]})] \arrow[d, "\begin{pmatrix} 1\otimes w \mapsto \\ (w-1)1 \end{pmatrix}"] & & \\
 & \mathbb{Z}[G \times (y)] \arrow[r, "\begin{pmatrix} 1\otimes y \mapsto \\ (y-1)1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1\otimes y \mapsto \\ \langle\, {}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]}\,\rangle \end{pmatrix}"] & \mathbb{Z}[G/1] \arrow[d, "\begin{pmatrix} 1 \mapsto \\ \langle\, x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]}\,\rangle \end{pmatrix}"] & & \\
0 \arrow[r] & \mathbb{Z}[G/\langle\, {}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]}\,\rangle] \arrow[r, "\begin{pmatrix} \langle\, {}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]}\,\rangle \mapsto \\ (y-1)\langle\, x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]}\,\rangle \end{pmatrix}"] \arrow[d] & \mathbb{Z}[G/\langle\, x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]}\,\rangle] \arrow[r] \arrow[d] & \mathbb{Z}[G/G] \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

Figure 7.4.1: A commuting diagram.

Recall that $y\colon \langle\, x, {}^{[0\uparrow(\nu-1)]}z_{[1\uparrow d]}\,\rangle \to \langle\, {}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]}\,\rangle$ acts by $x \mapsto {}^{1}x$, ${}^{i}z_{*} \mapsto {}^{i+1}z_{*}$, and that $G = \langle\, x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]}\,\rangle \underset{\langle\, {}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]}\,\rangle}{*} y$. All the notation involved in Fig. 7.4.1 has now been explained.

We now make five observations about Fig. 7.4.1.

(i). The long row is the exact augmented cellular chain complex of the Bass-Serre tree corresponding to the HNN graph-of-groups decomposition of G with one vertex and one edge. See, for example, [8, Examples I.3.5(v) and Theorem I.6.6].

(ii). The left column is the exact sequence obtained by applying the exact functor

$\mathbb{Z}G \otimes_{\mathbb{Z}[\langle\, {}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]}\,\rangle]} (\quad)$ to the exact augmented cellular chain complex of the Cayley tree of the free group $\langle\, {}^{1}x, {}^{[1\uparrow\nu]}z_{[1\uparrow d]}\,\rangle$, or, equivalently, the Bass-Serre tree corresponding to the graph-of-groups decomposition with one vertex and $1+\nu d$ edges. See, for example, [8, Examples I.3.5(i) and Theorem I.6.6].

(iii). The right column is the exact sequence obtained by applying the exact functor $\mathbb{Z}G\otimes_{\mathbb{Z}[\langle\, x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]}\,\rangle]}(\quad)$ to Lyndon's exact sequence for the one-relator group $\langle\, x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \mid r\,\rangle$; see, for example, [7, $(*)$ on p. 167].

(iv). It is clear that the lower square commutes.

(v). To see that the upper square commutes, we note that along the upper route in the upper square,

$$\begin{aligned} 1\otimes {}^{1}x \quad &\mapsto \quad (y-{}^{0}u)\otimes x \;-\sum_{w\in\, {}^{0}z_{[1\uparrow d]}} \frac{\partial\, {}^{0}u}{\partial w}\otimes w \\ &\mapsto \quad (y-{}^{0}u)(x-1)-({}^{0}u-1) = yx-{}^{0}ux-y+1 \\ &\qquad = {}^{1}x\cdot y-{}^{1}x-y+1 = ({}^{1}x-1)(y-1), \end{aligned}$$

and

$$\begin{aligned} 1\otimes {}^{i}z_{*} \quad &\mapsto \quad y\otimes {}^{i-1}z_{*} - 1\otimes {}^{i}z_{*} \\ &\mapsto \quad y({}^{i-1}z_{*}-1)-({}^{i}z_{*}-1) = y\cdot {}^{i-1}z_{*}-y-{}^{i}z_{*}+1 \\ &\qquad = {}^{i}z_{*}\cdot y - y - {}^{i}z_{*}+1 = ({}^{i}z_{*}-1)(y-1). \end{aligned}$$

It is now clear that the upper square commutes.

With these five observations in mind, we can apply Remark 7.1 to Fig. 7.4.1, and what we get is the exact sequence which appears as the left column of Fig. 7.4.2. After some adjustments, it becomes the right column of Fig. 7.4.2, which can be viewed as the augmented cellular chain complex of the adjusted universal cover of the Cayley complex of the presentation

$$\langle\, (x,y) \;\vee\; {}^{[0\uparrow\nu]}z_{[1\uparrow d]} \parallel [x,y]u,\; (y\cdot {}^{i-1}z_{*}\cdot \overline{y}\cdot {}^{i}\overline{z}_{*} \mid z_{*}^{(i)} \in {}^{[1\uparrow\nu]}z_{[1\uparrow d]}),\; r(x, {}^{[0\uparrow\nu]}z_{[1\uparrow d]})\,\rangle.$$

If $\nu=0$, we have the desired exact sequence.

If $\nu\geqslant 1$, then we shall delete ${}^{[1\uparrow\nu]}z_{[1\uparrow d]}$ from the set of generators and delete

$$(y\cdot {}^{i-1}z_{*}\cdot \overline{y}\cdot {}^{i}\overline{z}_{*} \mid z_{*}^{(i)} \in {}^{[1\uparrow\nu]}z_{[1\uparrow d]})$$

from the set of relators, and understand, henceforth, that ${}^{i}z_{*}$ denotes ${}^{y^{i}}z_{*}$ in the new generators. We consider G as being unaltered, but we alter the exact sequence by dividing out by the exact subcomplex

$$0\to \mathbb{Z}[G\times({}^{[1\uparrow\nu]}z_{[1\uparrow d]})])] \xrightarrow{\begin{pmatrix} 1\otimes {}^{i}z_{*}\mapsto \\ \frac{\partial(y\cdot {}^{i-1}z_{*}\cdot\overline{y}\cdot {}^{i}\overline{z}_{*})}{\partial(x,y,\, {}^{[0\uparrow\nu]}z_{[1\uparrow d]})}\end{pmatrix}} \mathbb{Z}[G\times(\tfrac{\partial(y\cdot {}^{i-1}z_{*}\cdot\overline{y}\cdot {}^{i}\overline{z}_{*})}{\partial(x,y,\, {}^{[0\uparrow\nu]}z_{[1\uparrow d]})} \mid {}^{i}z_{*}\in {}^{[1\uparrow\nu]}z_{[1\uparrow d]})] \to 0\to 0\to 0.$$

$$
\begin{tikzcd}[column sep=huge]
0 \arrow[d] & 0 \arrow[d] \\
\mathbb{Z}[G\times({}^{1}x,{}^{[1\uparrow\nu]}z_{[1\uparrow d]})]\oplus\mathbb{Z}[G/C] \arrow[r, "{\begin{pmatrix} 1\otimes w \mapsto 1\otimes w \\ C \mapsto 1\otimes_C r \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} 1\otimes{}^{1}x & \mapsto & \begin{matrix}(1-{}^{1}x)\otimes y\\ +(y-{}^{0}u)\otimes x\\ -\frac{\partial\,{}^{0}u}{\partial\,{}^{0}z_{[1\uparrow d]}}\end{matrix} \\ 1\otimes{}^{i}z_{*} & \mapsto & \begin{matrix}(1-{}^{i}z_{*})\otimes y\\ +y\otimes{}^{i-1}z_{*}\\ -1\otimes{}^{i}z_{*}\end{matrix} \\ C & \mapsto & \frac{\partial(r(x,{}^{[0\uparrow\nu]}z_{[1\uparrow d]}))}{\partial(x,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})} \end{pmatrix}}"'] \arrow[dr, phantom, "="] & \mathbb{Z}[G\times({}^{1}x,{}^{[1\uparrow\nu]}z_{[1\uparrow d]},{}_{C}r)] \arrow[d, "{\begin{pmatrix} -[x,y]\otimes{}^{1}x & \mapsto & \frac{\partial([x,y]\,{}^{0}u)}{\partial(x,y,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})} \\ 1\otimes{}^{i}z_{*} & \mapsto & \frac{\partial(y\cdot{}^{i-1}z_{*}\cdot\overline{y}\cdot{}^{i}\overline{z}_{*})}{\partial(x,y,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})} \\ 1\otimes_C r & \mapsto & \frac{\partial(r(x,{}^{[0\uparrow\nu]}z_{[1\uparrow d]}))}{\partial(x,y,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})} \end{pmatrix}}"] \\
\mathbb{Z}[G\times(y)]\oplus\mathbb{Z}[G\times(x,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})] \arrow[r, equal] \arrow[d, "(1\otimes w\mapsto(w-1)1)"] \arrow[dr, phantom, "="] & \mathbb{Z}[G\times(x,y,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})] \arrow[d, "(1\otimes w\mapsto(w-1)1)"'] \\
\mathbb{Z}[G/1] \arrow[r, equal] \arrow[d, "(1\mapsto G)"] & \mathbb{Z}[G/1] \arrow[d, "(1\mapsto G)"'] \\
\mathbb{Z}[G/G] \arrow[r, equal] \arrow[d] & \mathbb{Z}[G/G] \arrow[d] \\
0 & 0
\end{tikzcd}
$$

Figure 7.4.2: An exact sequence rewritten.

Thus, for each $i\in[1\uparrow d]$, in the quotient of $\mathbb{Z}[G\times({}^{1}x,{}^{[1\uparrow\nu]}z_{[1\uparrow d]},{}_{C}r)]$, we are identifying $1\otimes{}^{i}z_{*}$ with 0, while, in the quotient of $\mathbb{Z}[G\times(x,y,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})]$, we are identifying $1\otimes{}^{i}z_{*}$ with $(1-{}^{i}z_{*})\otimes y+y\otimes{}^{i-1}z_{*}$, which, by induction, is identified with $\frac{\partial(y^{i}\cdot z_{*}\cdot\overline{y}^{i})}{\partial(y,z_{*})}$. It follows that, in the quotient of $\mathbb{Z}[G\times(x,y,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})]$, $\frac{\partial(r(x,{}^{[0\uparrow\nu]}z_{[1\uparrow d]}))}{\partial(x,y,{}^{[0\uparrow\nu]}z_{[1\uparrow d]})}$ becomes identified with $\frac{\partial(r(x,y,z_{[1\uparrow d]}))}{\partial(x,y,z_{[1\uparrow d]})}$, and the exact quotient sequence is the augmented cellular chain complex of the adjusted universal cover of our original Cayley complex. $\square$

In the case where G is a hyperbolic orientable surface-plus-one-relation group, this result was obtained by Howie [17, Corollary 3.6].

8 VFL and Euler characteristics

In this section we calculate the Euler characteristics of the surface-plus-one-relation groups.

8.1 Definitions. Consider any resolution of $\mathbb{Z}$ by projective, left $\mathbb{Z}G$-modules

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{2}$$

The *length* of the projective $\mathbb{Z}G$-resolution (2) is the supremum, in $[0{\uparrow}\infty]$, of the non-empty set $\{n \in [0{\uparrow}\infty] \mid P_n \neq 0\}$.

Let $\mathcal{P}$ denote the unaugmented complex $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$, and view $\mathbb{Q}$ as a $\mathbb{Q}$-$\mathbb{Z}G$-bimodule. For each $n \in [0{\uparrow}\infty[$, $\mathrm{H}_n(G;\mathbb{Q}) := \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Q},\mathbb{Z}) := \mathrm{H}_n(\mathbb{Q} \otimes_{\mathbb{Z}G} \mathcal{P})$; for the purposes of this article, it will be convenient to understand that $\mathrm{H}_n(G;-)$ applies to *right* $\mathbb{Z}G$-modules. The *nth Betti number of* G is $b_n(G) := \dim_{\mathbb{Q}} \mathrm{H}_n(G;\mathbb{Q}) \in [0{\uparrow}\infty]$. The value of the Betti numbers does not depend on the choice of projective $\mathbb{Z}G$-resolution (2).

The *cohomological dimension of* G, denoted $\mathrm{cd}\, G$, is the smallest element of the subset of $[0{\uparrow}\infty]$ which consists of lengths of projective $\mathbb{Z}G$-resolutions of $\mathbb{Z}$. The *virtual cohomological dimension of* G, denoted $\mathrm{vcd}\, G$, is the smallest element of the subset of $[0{\uparrow}\infty]$ which consists of cohomological dimensions of finite-index subgroups of G. The *cohomological dimension of* G *with respect to an associative ring* Q, denoted $\mathrm{cd}_Q\, G$, is the smallest element of the subset of $[0{\uparrow}\infty]$ which consists of lengths of projective QG-resolutions of Q.

If there exists a resolution (2) of finite length such that all the P_n are finitely generated, free left $\mathbb{Z}G$-modules, then we say that G is *of type* FL and we define the *Euler characteristic of* G to be

$$\chi(G) := \sum_{n\in[0{\uparrow}\infty]} (-1)^n b_n(G).$$

If G has some finite-index subgroup H of type FL, then we say that G is *of type VFL* and, if G is not of type FL, we define the *Euler characteristic of* G to be

$$\chi(G) := \tfrac{1}{[G:H]}\chi(H);$$

this value, which is sometimes called the 'virtual Euler characteristic', does not depend on the choice of H. □

8.2 Remark. If G has some finitely generated, one-relator, finite-index subgroup H, say $H = \langle\, X \mid r\,\rangle$, then G is of type VFL and

$$-\chi(G) = \tfrac{-1}{[G:H]}\chi(H) = \tfrac{1}{[G:H]}\big(|X| - 1 - \tfrac{1}{\log_{\langle\, X \mid \ \rangle}(r)}\big).$$

See, for example, [7, (∗) on p. 167]. □

We now discuss the case where $r = x^m$.

8.3 Examples. Let $d,\, m \in [0{\uparrow}\infty[$, let $F \coloneqq \langle\, (x,y) \vee z_{[1\uparrow d]} \mid \quad \rangle$, and let u and r be elements of F. Suppose that $u \in \langle\, z_{[1\uparrow d]}\,\rangle - \{1\}$ and that $r = x^m$. Let $G = \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u, r\,\rangle$.

In $[1{\uparrow}\infty]$, let $m_F \coloneqq \log_F x^m \in \{m,\infty\}$, let m' denote the supremum of the orders of the finite subgroups of G, and let $m'' \coloneqq \max(m_F, m')$.

It is straightforward to verify the following assertions. It then follows that G is virtually one-relator, and hence of type VFL.

(i). If $m = 0$, then $G \;=\; \langle\, (x,y) \vee z_{[1\uparrow d]} \mid [x,y]u\,\rangle$.

Here, $m_F = \infty$, $m' = 1$, $m'' = \infty$, and $-\chi(G) = d = d - \frac{1}{m''} \geqslant 0$.

(ii). If $m = 1$, then $G \;=\; \langle\, (y) \vee z_{[1\uparrow d]} \mid u\,\rangle$.

Here, $m_F = 1$, $m' = \log_F u$, $m'' = \log_F u$, and $-\chi(G) = d - \frac{1}{\log_F u} = d - \frac{1}{m''} \geqslant 0$.

(iii). If $m \geqslant 2$, then $G \;=\; \langle\, (y) \vee {}^{\mathbb{Z}_m}z_{[1\uparrow d]} \mid \prod\limits_{i\in[(m-1)\downarrow 0]} {}^{i}u\,\rangle \;\rtimes\; \langle\, x \mid x^m\,\rangle$, with the x-action defined by ${}^{x}y = {}^{0}\overline{u}\cdot y$ and ${}^{x}({}^{i}z_*) = {}^{i+1}z_*$.

Here, $m_F = m' = m'' = m$, and $-\chi(G) = d - \frac{1}{m} = d - \frac{1}{m''} \geqslant 0$. $\square$

It is now convenient to go back to writing $(x,y) \vee z_{[1\uparrow d]}$ in the form $x_{[1\uparrow k]}$ with $k = d+2$.

8.4 Corollary. *Let $k \in [3{\uparrow}\infty[$, let $F \coloneqq \langle\, x_{[1\uparrow k]} \mid \quad \rangle$, and let w and r be elements of F. Suppose that $w \in [x_1,x_2]\langle\, x_{[3\uparrow k]}\,\rangle$ and that r is a Hempel relator for $\langle\, x_{[1\uparrow k]} \;\|\; w\,\rangle$. Let $G \coloneqq \langle\, x_{[1\uparrow k]} \mid w, r\,\rangle$.*

Let $m \coloneqq \log_F r$. Then $\operatorname{vcd} G \leqslant 2$, G is of type VFL and $-\chi(G) = k - 2 - \frac{1}{m} \geqslant 0$.

Let C denote the subgroup of G generated by the image of $\sqrt[F]{r}$, let Q be any associative ring such that $mQ = Q$, let $R \coloneqq QG$, and let $e \coloneqq \frac{1}{m}\sum\limits_{c\in C} c \in R$. Then the sequence of left R-modules

$$0 \to R \oplus Re \xrightarrow{\begin{pmatrix} \frac{\partial w}{\partial x_1} & \cdots & \frac{\partial w}{\partial x_k} \\ \frac{\partial r}{\partial x_1} & \cdots & \frac{\partial r}{\partial x_k} \end{pmatrix}} R^k \xrightarrow{\begin{pmatrix} x_1 - 1 \\ \vdots \\ x_k - 1 \end{pmatrix}} R \xrightarrow{G\to\{1\}} Q \to 0$$

is exact, and $\operatorname{cd}_Q G \leqslant 2$.

Proof. By Theorem 6.5(vi), there exists some torsion-free finite-index subgroup H of G. Clearly, H acts freely on G on the left and the number of orbits is $[G\colon H] = |H\backslash G|$. Since H meets each conjugate of C trivially, C acts freely on $H\backslash G$ on the right, H acts freely on G/C on the left, and the number of orbits in each case is $|H\backslash G/C| = \frac{|H\backslash G|}{|C|} = \frac{[G:H]}{m}$.

By Theorem 7.3, with $d = k-2$, we see that $\mathrm{cd}\, H \leqslant 2$, H is of type FL, and $\chi(H) = [G{:}H] - (d+2)[G{:}H] + ([G{:}H] + \frac{[G:H]}{m})$. Hence $\mathrm{vcd}\, G \leqslant 2$, G is of type VFL, and $\chi(G) = 1 - (d+2) + (1 + \frac{1}{m})$. Since $(QC)e \simeq Q$ as left QC-modules, we see that

$$Q[G/C] = QG \otimes_{QC} Q \simeq QG \otimes_{QC} (QC)e \simeq Re.$$

It is now clear that the results hold. □

We can now describe the Euler characteristics of the surface-plus-one-relation groups.

8.5 Remarks. Let $k \in [0{\uparrow}\infty[$, let $F := \langle\, x_{[1\uparrow k]} \mid \quad \rangle$, and let w and r be elements of F. Suppose either that k is even and $w = \prod_{i\in[1\uparrow \frac{k}{2}]} [x_{2i-1}, x_{2i}]$, or that $k \geqslant 1$ and $w = \prod x^2_{[1\uparrow k]}$. Let $S := \langle\, x_{[1\uparrow k]} \mid w \,\rangle$ and let $G := \langle\, x_{[1\uparrow k]} \mid w, r \,\rangle$.

In $[1{\uparrow}\infty]$, let $m := \log_S r \bmod w$, let m' denote the supremum of the orders of the finite subgroups of G, and let $m'' := \max(m, m')$.

Case 1. $k \leqslant 2$.

Here, G is virtually abelian of rank at most two. In particular, G is virtually one-relator, and it follows from Remark 8.2 that G is of type VFL and $\chi(G) = \frac{1}{|G|} \geqslant 0$.

Case 2. $k \geqslant 3$ and, for some $m \in [0{\uparrow}\infty[$, $r \bmod w$ is conjugate in S to the mth power of some free generator of the subgroup $\langle\, x_1x_2,\, x_2x_3 \mid \quad \rangle$ of S.

Here, by using Lemma 2.1 together with Examples 8.3, we find that G is virtually one-relator, that G is of type VFL, and that $-\chi(G) = k - 2 - \frac{1}{m''} \geqslant 0$. There are two possibilities for m''.

Case 2a. $(k, m) = (3, 1)$.

That is, $r \bmod w$ is conjugate in S to some free generator of the rank-two, free subgroup $\langle\, x_1x_2,\, x_2x_3,\, x_3x_1 \mid (x_1x_2)(x_2x_3)(x_3x_1) \,\rangle$ of S. Here, $G \simeq C_\infty * C_2$, $-\chi(G) = \frac{1}{2} \geqslant 0$, $m = 1$, $m' = 2$, and $m'' = 2$.

Case 2b. $(k, m) \neq (3, 1)$.

Here, $-\chi(G) = k - 2 - \frac{1}{m} \geqslant 0$. If $r \bmod w \neq 1$, then $m = m' = m'' < \infty$. If $r \bmod w = 1$, then $m = \infty$, $m' = 1$, and $m'' = \infty$.

Case 3. $k \geqslant 3$ and $r \bmod w$ is not conjugate in S to any power of any free generator of the subgroup $\langle\, x_1x_2,\, x_2x_3 \mid \quad \rangle$ of S.

By using Lemmas 2.1 and 5.4, we find that there exists some presentation for G as in Corollary 8.4.

Hence, G is of type VFL, $-\chi(G) = k - 2 - \frac{1}{m} \geqslant 0$, and $m = m' = m'' < \infty$. □

We summarize the preceding, but do not record that one can use $m'' = m$ except for one case, or $m'' = m'$ except for one case.

8.6 Corollary. *Let $k \in [0{\uparrow}\infty[$, let $F := \langle\, x_{[1\uparrow k]} \mid \quad \rangle$, and let w and r be elements of F. Suppose either that k is even and $w = \prod_{i \in [1\uparrow \frac{k}{2}]}[x_{2i-1}, x_{2i}]$, or that $k \geqslant 1$ and $w = \prod x^2_{[1\uparrow k]}$. Let $S := \langle\, x_{[1\uparrow k]} \mid w\,\rangle$ and let $G := \langle\, x_{[1\uparrow k]} \mid w, r\,\rangle$. In $[1{\uparrow}\infty]$, let $m := \log_S r \bmod w$, let m' denote the supremum of the orders of the finite subgroups of G, and let $m'' := \max(m, m')$. Then*

$$\chi(G) = \begin{cases} \frac{1}{|G|} \geqslant 0 & \text{if } k \leqslant 2, \\ -k+2+\frac{1}{m''} \leqslant 0 & \text{if } k \geqslant 3. \end{cases} \qquad \square$$

9 L^2-Betti numbers of surface-plus-one-relation groups

Let us begin with a brief algebraic review of Atiyah's theory of L^2-Betti numbers of groups.

9.1 Review. Let $\mathbb{C}[[G]]$ denote the set of all functions from G to $\mathbb{C}$ expressed as formal sums, that is, a function $x\colon G \to \mathbb{C}$, $g \mapsto x_g$, will be written as $\sum\limits_{g \in G} x_g g$. Then $\mathbb{C}[[G]]$ has a natural $\mathbb{C}G$-bimodule structure, and contains a copy of $\mathbb{C}G$ as $\mathbb{C}G$-sub-bimodule. For each $x \in \mathbb{C}[[G]]$, let $\|x\| := \sqrt{\sum\limits_{g \in G} |x_g|^2} \in [0, \infty]$, and let $\operatorname{tr}(x) := x_{1_G} \in \mathbb{C}$.

Let $\ell^2(G) := \{x \in \mathbb{C}[[G]] : \|x\| < \infty\}$. For x, $y \in \ell^2(G)$, $g \in G$, and S a finite subset of G, it follows from the Cauchy-Schwarz inequality that $\sum\limits_{h \in S} \left| x_h y_{\overline{h}g} \right| \leqslant \|x\| \cdot \|y\|$; hence, there exists a well-defined limit $\sum\limits_{h \in G} (x_h y_{\overline{h}g}) \in \mathbb{C}$; hence, there exists a well-defined element $x{\cdot}y := \sum\limits_{g \in G} ((\sum\limits_{h \in G} x_h y_{\overline{h}g}) g) \in \mathbb{C}[[G]]$, called the *external product* of x and y.

The *group von Neumann algebra of G* is defined as the additive abelian group

$$\mathcal{N}(G) := \{p \in \ell^2(G) \mid p{\cdot}\ell^2(G) \subseteq \ell^2(G)\}$$

endowed with the ring structure induced by the external product; it can be shown that this agrees with the definition of the group von Neumann algebra given in [21, Section 1.1], and that $\mathcal{N}(G) = \{p \in \ell^2(G) \mid \ell^2(G){\cdot}p \subseteq \ell^2(G)\}$. We then have a chain of $\mathbb{C}G$-bimodules $\mathbb{C}G \subseteq \mathcal{N}(G) \subseteq \ell^2(G) \subseteq \mathbb{C}[[G]]$, and $\mathbb{C}G$ is a subring of $\mathcal{N}(G)$, and $\ell^2(G)$ is an $\mathcal{N}(G)$-bimodule containing $\mathcal{N}(G)$ as $\mathcal{N}(G)$-sub-bimodule.

It can be shown that the elements of $\mathcal{N}(G)$ which act faithfully on the left $\mathcal{N}(G)$-module $\ell^2(G)$ are precisely the two-sided non-zerodivisors in $\mathcal{N}(G)$, and that these form a left and right Ore subset of $\mathcal{N}(G)$; see [21, Theorem 8.22(1)]. The *ring of unbounded operators affiliated to* $\mathcal{N}(G)$, denoted $\mathcal{U}(G)$, is defined as the left, and the right, Ore localization of

$\mathcal{N}(G)$ at the set of its two-sided non-zerodivisors; see [21, Section 8.1]. For example, it is then clear that

(3) if g is an element of G of infinite order, then $g-1$ is invertible in $\mathcal{U}(G)$.

It can be shown that $\mathcal{U}(G)$ is a von Neumann regular ring in which one-sided inverses are two-sided inverses, and, hence, one-sided zerodivisors are two-sided zerodivisors; see [21, Section 8.2].

It can be shown that there exists a continuous, additive von Neumann dimension that assigns to every left $\mathcal{U}(G)$-module M a value $\dim_{\mathcal{U}(G)} M \in [0,\infty]$; see Definition 8.28 and Theorem 8.29 of [21]. For example, if e is an idempotent element of $\mathcal{N}(G)$, then $\dim_{\mathcal{U}(G)} \mathcal{U}(G)e = \mathrm{tr}(e)$; see Theorem 8.29 and Sections 6.1-2 of [21].

For each $n \in [0{\uparrow}\infty[$, the *nth L^2-Betti number of G* is defined as

$$b_n^{(2)}(G) \coloneqq \dim_{\mathcal{U}(G)}\ \mathrm{H}_n(G;\mathcal{U}(G)) \quad \in \quad [0,\infty],$$

where $\mathcal{U}(G)$ is to be viewed as a $\mathcal{U}(G)$-$\mathbb{Z}G$-bimodule; see Definition 6.50, Lemma 6.51 and Theorem 8.29 of [21].

It is easy to show that if G is finite, then, for each $n \in [0{\uparrow}\infty[$,

$$b_n^{(2)}(G) = \begin{cases} \chi(G) = \frac{1}{|G|} & \text{if } n = 0, \\ 0 & \text{if } n \in [1{\uparrow}\infty[\,. \end{cases}$$

By [21, Theorem 6.54(8b)],

$$b_0^{(2)}(G) = \tfrac{1}{|G|}. \tag{4}$$

By [21, Theorem 1.9(8)], if H is a finite-index subgroup of G, then

$$b_n^{(2)}(H) = [G:H]\,b_n^{(2)}(G). \tag{5}$$

In general, there is little relation between the nth L^2-Betti number and the nth (ordinary) Betti number, $b_n(G) = \dim_{\mathbb{Q}} \mathrm{H}_n(G;\mathbb{Q}) \in [0{\uparrow}\infty]$. However, by [21, Remark 6.81], if G is of type VFL, we can define and calculate the *L^2-Euler characteristic*

$$\chi^{(2)}(G) \coloneqq \sum_{n\in[0\uparrow\infty[} (-1)^n b_n^{(2)}(G) = \sum_{n\in[0\uparrow\infty[} (-1)^n b_n(G) \eqqcolon \chi(G). \tag{6}$$ □

Recall that, for any finitely generated, virtually one-relator group G, a formula for $\chi(G)$ was given in Remark 8.2.

9.2 Lemma. *If G is a finitely generated, virtually one-relator group, then G is of type VFL and, for each $n \in [0{\uparrow}\infty[$,*

$$b_n^{(2)}(G) = \begin{cases} \max\{\chi(G), 0\} = \frac{1}{|G|} & \text{if } n = 0, \\ \max\{-\chi(G), 0\} & \text{if } n = 1, \\ 0 & \text{if } n \in [2{\uparrow}\infty[\,. \end{cases}$$

Proof. If G is itself a finitely generated, one-relator group, the conclusions hold, by [10, Theorem 4.2]. By Review 9.1(5), this then extends to overgroups of finite index on dividing by the index. □

For Hempel presentations we have the following information.

9.3 Lemma. *Let $k \in [3{\uparrow}\infty[$, let $F := \langle\, x_{[1\uparrow k]} \mid \quad \rangle$, and let w and r be elements of F. Suppose that $w \in [x_1, x_2]\langle\, x_{[3\uparrow k]}\,\rangle$ and that r is a Hempel relator for $\langle\, x_{[1\uparrow k]} \parallel w\,\rangle$. Let $G := \langle\, x_{[1\uparrow k]} \mid w, r\,\rangle$. Let $m := \log_F r$, let C_m denote the subgroup of G generated by the image of $\sqrt[F]{r}$, and let $e := \frac{1}{m}\sum\limits_{c \in C_m} c \in \mathbb{C}G$.*

(i). *For all $q \in \mathcal{U}(G)$ and all $a \in \mathbb{C}G$, if $qea = 0$ then either $qe = 0$ or $ea = 0$.*

(ii). *The homology of*

$$0 \to \mathcal{U}(G) \oplus \mathcal{U}(G)\, e \xrightarrow{\begin{pmatrix} \frac{\partial w}{\partial x_1} & \cdots & \frac{\partial w}{\partial x_k} \\ \frac{\partial r}{\partial x_1} & \cdots & \frac{\partial r}{\partial x_k} \end{pmatrix}} \mathcal{U}(G)^k \xrightarrow{\begin{pmatrix} x_1 - 1 \\ \vdots \\ x_k - 1 \end{pmatrix}} \mathcal{U}(G) \to 0,$$

is $\mathrm{H}_*(G; \mathcal{U}(G))$, and, for each $n \in (0) \vee [3{\uparrow}\infty[$, $b_n^{(2)}(G) = 0$.

Proof. (i). Recall that [10, Theorem 3.1(iii)], which depends on results in [5] and [20], asserts that if G is ((free $\rtimes C_m$) by (locally indicable)), then (i) holds. By Theorem 6.5(ii),(iii), we now see that (i) holds.

(ii). By using Corollary 8.4 and Review 9.1(4), it is not difficult to see that (ii) holds. □

Let us now consider the special case of hyperbolic surface-plus-one-relation groups.

9.4 Lemma. *Let $k \in [3{\uparrow}\infty[$, let $F := \langle\, x_{[1\uparrow k]} \mid \quad \rangle$, and let w and r be elements of F. Suppose either that $w = [x_1, x_2] \prod x^2_{[3\uparrow k]}$ or that k is even and $w = \prod_{j \in [1\uparrow \frac{k}{2}]} [x_{2j-1}, x_{2j}]$. Suppose that r is a Hempel relator for $\langle\, x_{[1\uparrow k]} \parallel w\,\rangle$. Let $G := \langle\, x_{[1\uparrow k]} \mid w, r\,\rangle$. Then $b_2^{(2)}(G) = 0$.*

Proof. Let $m := \log_F r$, let C denote the subgroup of G generated by the image of $\sqrt[F]{r}$, and let $e := \frac{1}{m}\sum_{c\in C} c \in \mathbb{C}G$. If the map

$$\mathcal{U}(G) \oplus \mathcal{U}(G)\, e \xrightarrow{\begin{pmatrix} \frac{\partial w}{\partial x_1} & \cdots & \frac{\partial w}{\partial x_k} \\ \frac{\partial r}{\partial x_1} & \cdots & \frac{\partial r}{\partial x_k}\end{pmatrix}} \mathcal{U}(G)^k$$

is injective, then $b^{(2)}(G) = 0$, by Lemma 9.3(ii). It remains to consider the case where there exists some $(p, qe) \neq (0,0)$ in the kernel of this map, that is, $p, q \in \mathcal{U}(G)$ and

$$\text{for each } j \in [1{\uparrow}k], \quad p\tfrac{\partial w}{\partial x_j} + qe\tfrac{\partial r}{\partial x_j} = 0 \text{ in } \mathcal{U}(G). \tag{7}$$

Let G act by right multiplication on the set of all subsets of $\mathcal{U}(G)$, let $V := p + qe\mathbb{C}G$, a subset of $\mathcal{U}(G)$, and let $G_V := \{g \in G \mid Vg = V\} = \{g \in G \mid p(g-1) \in qe\mathbb{C}G\}$, a subgroup of G. The subset V of $\mathcal{U}(G)$ is then closed under the right G_V-action on $\mathcal{U}(G)$. By Lemma 9.3(i), the surjective map $e\mathbb{C}G \to qe\mathbb{C}G$, $ea \mapsto qea$, is either injective or zero. In either event, $qe\mathbb{C}G$ is a projective right $\mathbb{C}G$-module, and hence a projective right $\mathbb{C}G_V$-module. By the left-right dual of [9, Corollary 5.6], there exists a right G_V-tree T with finite edge stabilizers and vertex set the right G_V-set $p + qe\mathbb{C}G$.

We claim that $0 \notin p + qe\mathbb{C}G$. Since $(p, qe) \neq (0,0)$, we may assume that $qe \neq 0$ for this argument. Consider any $a \in \mathbb{C}G$. By Corollary 8.4, with $Q = \mathbb{C}$, the map

$$\mathbb{C}G \oplus \mathbb{C}Ge \xrightarrow{\begin{pmatrix} \frac{\partial w}{\partial x_1} & \cdots & \frac{\partial w}{\partial x_k} \\ \frac{\partial r}{\partial x_1} & \cdots & \frac{\partial r}{\partial x_k}\end{pmatrix}} \mathbb{C}G^k$$

is injective, and, in particular, $(-ea, e)$ does not lie in the kernel, since $e \neq 0$. Hence there exists some $j \in [1{\uparrow}k]$, such that $0 \neq -ea\frac{\partial w}{\partial x_j} + e\frac{\partial r}{\partial x_j} = e(-a\frac{\partial w}{\partial x_j} + \frac{\partial r}{\partial x_j})$. By Lemma 9.3(i),

$$0 \neq qe(-a\tfrac{\partial w}{\partial x_j} + \tfrac{\partial r}{\partial x_j}) = (-qea)\tfrac{\partial w}{\partial x_j} + qe\tfrac{\partial r}{\partial x_j}.$$

It is now clear from (7) that $p \neq -qea$. This proves that $0 \notin p + qe\mathbb{C}G$, as claimed.

By using Review 9.1(3), we now find that each vertex stabilizer for T is torsion. Hence, by Theorem 6.5(iv), each vertex stabilizer for T has order at most m. It follows that G_V is virtually free; see, for example, [8, Theorem IV.1.6,(b)⇒(c)].

Let

$$G_{\pm V} := \{g \in G \mid \text{ either } p(g-1) \in qe\mathbb{C}G \text{ or } p(g+1) \in qe\mathbb{C}G\} = \{g \in G \mid Vg = \pm V\},$$

a subgroup of G. We shall see that $G_{\pm V} = G$.

By (7), we see that, for each $j \in [1{\uparrow}k]$, $p\,\frac{\partial w}{\partial x_j} \in qe\mathbb{C}G$, and, hence,

$$p\,(1 - x_1x_2\overline{x}_1) = p\,\tfrac{\partial w}{\partial x_1} \in qe\mathbb{C}G, \tag{8}$$

$$p\,(x_1 - [x_1, x_2]) = p\,\tfrac{\partial w}{\partial x_2} \in qe\mathbb{C}G, \tag{9}$$

$$\text{for all } j \in [3{\uparrow}k], \quad p\,\tfrac{\partial w}{\partial x_j} \in qe\mathbb{C}G. \tag{10}$$

By (8), ${}^{x_1}x_2 \in G_V$. Right multiplying (9) by $\overline{x}_1$, we see that $p\,(1 - {}^{x_1x_2}\overline{x}_1) \in qe\mathbb{C}G$. Thus ${}^{x_1x_2}x_2$ and ${}^{x_1x_2}x_1$ lie in G_V. Hence, their product x_1x_2 lies in G_V, and then x_1 and x_2 lie in G_V. In particular, $V[x_1, x_2] = V$.

We claim that $x_{[3\uparrow k]} \subseteq G_{\pm V}$. Arguing inductively, we suppose that $j \in [3{\uparrow}k]$ and $x_{[1\uparrow(j-1)]} \subseteq G_{\pm V}$. For this step, we shall consider only the non-orientable case, $w = [x_1, x_2] \prod x^2_{[3\uparrow k]}$; the orientable case is similar, and was done in [10, Lemma 5.15]. Let $u = [x_1, x_2] \prod x^2_{[3\uparrow(j-1)]}$. Then $u \in G_V$, that is, $p - pu \in qe\mathbb{C}G$. Now $\frac{\partial w}{\partial x_j} = u(1 + x_j)$, and, by (10), $pu(1 + x_j) \in qe\mathbb{C}G$. Summing these two elements of $qe\mathbb{C}G$, we see that $p + pux_j \in qe\mathbb{C}G$. Thus $ux_j \in G_{\pm V}$, and, hence, $x_j \in G_{\pm V}$.

By induction, the claim is proved.

Hence $G_{\pm V} = G$, and, hence $[G{:}G_V] \leqslant 2$, and, hence, G is virtually free. Thus $\operatorname{vcd} G \leqslant 1$, and, hence, $b_2^{(2)}(G) = 0$. □

Recall that, for any surface-plus-one-relation group G, a formula for $\chi(G)$ was given in Corollary 8.6.

9.5 Theorem. *Let G be a surface-plus-one-relation group. Then G is of type VFL and, for each $n \in [0{\uparrow}\infty[$,*

$$b_n^{(2)}(G) = \begin{cases} \max\{\chi(G), 0\} = \frac{1}{|G|} & \text{if } n = 0, \\ \max\{-\chi(G), 0\} & \text{if } n = 1, \\ 0 & \text{if } n \in [2{\uparrow}\infty[\,. \end{cases}$$

Proof. If G is virtually one-relator, then, by Lemma 9.2, the desired conclusions hold. Thus, we may assume that G is *not* virtually one-relator. It then follows from Remarks 8.5 that we may assume that G has a presentation as in Lemma 9.4 and that G is of type VFL. Then, by Lemma 9.4 and Lemma 9.3(ii), $b_2^{(n)}(G) = 0$ for all $n \in (0) \vee [2{\uparrow}\infty[$. By Review 9.1(6), $\chi(G) = -b_1^{(2)}(G)$, as desired. □

APPENDIX
Howie towers via Bass-Serre Theory

In this appendix, we use Bass-Serre Theory to prove some of Howie's results on local indicability.

We shall use [8] as our reference for Bass-Serre theory.

Throughout, let F be a group.

A.1 Actions on trees

A.1.1 Notation. Let E be a subset of an F-set X.

Two elements of E that lie in the same F-orbit in X are said to be *glued together by* F, and we write $\text{glue}(F, E) \coloneqq \{g \in F \mid gE \cap E \neq \emptyset\}$. □

A.1.2 Definitions. Let $T = (T, VT, ET, \iota, \tau)$ be an F-tree.

(i). Let r be an element of F that fixes no vertex of T.

The smallest $\langle\, r\,\rangle$-subtree of T, denoted $\text{axis}(r)$, has the form of the real line, and r acts on it by shifting it; see, for example, [8, Proposition I.4.11]. We write $E\text{axis}(r) \coloneqq E(\text{axis}(r))$.

The (F, ET)-*support* of r is defined as

$$\text{supp}(r) \coloneqq \{Fe \in F\backslash ET \mid e \in E\text{axis}(r)\}.$$

For all $f \in F$, $\text{axis}({}^f r) = f\,\text{axis}(r)$ and $\text{supp}({}^f r) = \text{supp}(r)$.

For all $n \in [1{\uparrow}\infty[$, $\text{axis}(r^n) = \text{axis}(r)$.

If F acts freely on ET, then r has a root in F.

If $\text{glue}(F, E\text{axis}(r)) = \langle\, r\,\rangle$, then $\sqrt[F]{r} = r$.

(ii). Let $<$ be a (total) ordering of $F\backslash ET$.

A subset R of F is said to be $(F, ET, <)$-*staggered* if each element of R fixes no vertex of T and, for each $(r_1, r_2) \in R \times R$, exactly one of the following three conditions holds:

${}^F r_1 = {}^F r_2$;

$\min(\text{supp}(r_1), <) \;<\; \min(\text{supp}(r_2), <)$ and $\max(\text{supp}(r_1), <) \;<\; \max(\text{supp}(r_2), <)$;

$\min(\text{supp}(r_2), <) \;<\; \min(\text{supp}(r_1), <)$ and $\max(\text{supp}(r_2), <) \;<\; \max(\text{supp}(r_1), <)$.

When either of the latter two conditions holds, we say that r_1 and r_2 have *staggered supports* (with respect to $<$).

(iii). A subset R of F is said to be (F, ET)-*staggerable* if there exists an ordering $<$ of $F\backslash ET$ such that R is $(F, ET, <)$-staggered. □

A.1.3 Example. For a free product $F = A*B$, the *Bass-Serre tree* is the F-graph T with vertex set the disjoint union of F/A and F/B and edge set F, in which each $f \in ET = F$ has initial vertex fA and terminal vertex fB. It can be shown that T is a tree; see, for example, [8, Theorem I.7.6]. Notice that F acts *freely* on the edge set of T.

Let $r \in F$.

Clearly, r fixes some vertex of T if and only if r lies in some conjugate of A or some conjugate of B.

If r fixes no vertex of T, then there exists some $n \in [1{\uparrow}\infty[$, and some sequence $a_{[1\uparrow n]}$ in $A - \{1\}$, and some sequence $b_{[1\uparrow n]}$ in $B - \{1\}$ such that some conjugate of r can be expressed in the form ${}^f r = \prod_{i\in[1\uparrow n]}(a_i b_i)$. The entire conjugacy class ${}^F r$ can be represented by writing $\prod_{i\in[1\uparrow n]}(a_i b_i)$ cyclically.

Here, $\operatorname{supp}(r) = \{F\} = F\backslash ET$. If $<$ denotes the unique ordering of $F\backslash ET$, then $\{r\}$ is $(F, ET, <)$-staggered.

If $r = \prod_{i\in[1\uparrow n]}(a_i b_i)$, then $\langle\, r\,\rangle\backslash \operatorname{axis}(r) = \{\langle\, r\,\rangle w \mid w \text{ is an initial subword of } r\}$ and there is a direct description of $\sqrt[F]{r}$ and $\log_F r$, as follows. We can write $\log_F r = \frac{n}{m}$, where m is smallest element of $[1{\uparrow}n]$ with the property that, for each $i \in [1{\uparrow}(n-m)]$, $a_i = a_{i+m}$ and $b_i = b_{i+m}$. Here, $\sqrt[F]{r} = \prod\limits_{i\in[1\uparrow m]}(a_i b_i)$. □

A.2 Staggerability

A.2.1 Definitions. A group is said to be *indicable* if *either* it is trivial *or* it has some infinite, cyclic quotient. A group is said to be *locally indicable* if every finitely generated subgroup is indicable. □

For any subset R of F in which each element of R has a root in F, we let $\sqrt[F]{R}$ denote the set of roots of elements of R in F. In this section, we let F be a locally indicable group, and let T be an F-tree with trivial edge stabilizers, and let R be an (F, ET)-staggerable subset of F with $\sqrt[F]{R} = R$. We will prove that $F/\langle\, {}^F R\,\rangle$ is locally indicable, and that $\langle\, {}^F R\,\rangle$ acts freely on T, or, equivalently, the natural map $F \twoheadrightarrow F/\langle\, {}^F R\,\rangle$ is injective on the vertex stabilizers. The following result deals with an extreme case.

A.2.2 Lemma. *Let F be a locally indicable group, let T be an F-tree with trivial edge stabilizers, and let R be an (F, ET)-staggerable subset of F such that $\sqrt[F]{R} = R$.*

Suppose that F is finitely generated and that $F\backslash T$ is finite.

Then $F/\langle\, {}^F R\,\rangle$ is indicable; if $F/\langle\, {}^F R\,\rangle$ is trivial then F acts freely on T and, for each $r \in R$, $\operatorname{glue}(F, E\operatorname{axis}(r)) = \langle\, r\,\rangle$.

Proof. We argue by induction on $|F\backslash ET|$. Since F is a finitely generated, locally indicable group, F is indicable, which means that the implications hold when R is empty. Thus we may assume that R is non-empty; in particular, $|F\backslash ET| \geqslant 1$. By induction, we may suppose that the implications hold for all smaller values of $|F\backslash ET|$.

We may assume that $R = {}^F R$, and we may assume that we are given an ordering $<$ of the finite set $F\backslash ET$ such that R is $(F, ET, <)$-staggered. In particular, there exists some $e_{\max} \in \bigcup\limits_{r\in R} E\text{axis}(r)$ such that

$$Fe_{\max} \quad = \quad \max(\{Fe \mid e \in \bigcup_{r\in R} E\text{axis}(r)\},\ <).$$

There then exists some $r_{\max} \in R$ such that $e_{\max} \in E\text{axis}(r_{\max})$, and, by the definition of $(F, ET, <)$-staggered, $Fe_{\max}$ does not meet the axis of any element of $R - {}^F(r_{\max})$. Thus there exists some pair (r, e), for example, $(r_{\max}, e_{\max})$, such that the following hold.

(11) $\quad r \in R$, $e \in E\text{axis}(r)$, and Fe does not meet the axis of any element of $R - {}^F r$.

(12) $\quad \text{glue}(F, E\text{axis}(r)) = \langle\, r\,\rangle$ and/or $(r, e) = (r_{\max}, e_{\max})$.

In the forest $T - Fe$, let T_ι denote the component containing ιe, and let T_τ denote the component containing τe. Let F_ι denote the F-stabilizer of $\{T_\iota\}$, and let F_τ denote the F-stabilizer of $\{T_\tau\}$. Let $R_\iota := R \cap F_\iota$ and $R_\tau := R \cap F_\tau$. By (11), for each $r' \in R - \{{}^F r\}$, $\text{axis}(r')$ lies in $T - Fe$ and hence lies in a component of $T - Fe$. It follows that ${}^F R_\iota \cup {}^F R_\tau \cup {}^F r$ is all of R. Notice that if $F\{T_\iota\} = F\{T_\tau\}$ then ${}^F R_\iota = {}^F R_\tau$.

By applying the Bass-Serre Structure Theorem to the F-tree whose vertices are the components of $T - Fe$, and whose edge set is Fe, with fe joining fT_ι to fT_τ, we see that

$$F = \begin{cases} F_\iota * F_\tau & \text{if } F\{T_\iota\} \neq F\{T_\tau\}, \\ F_\iota * \langle\, f \mid \quad \rangle & \text{if } f \in F \text{ and } fT_\iota = T_\tau. \end{cases} \tag{13}$$

Hence,

$$F/\langle\!\langle R - \{r\} \rangle\!\rangle = \begin{cases} (F_\iota/\langle\!\langle R_\iota \rangle\!\rangle) * (F_\tau/\langle\!\langle R_\tau \rangle\!\rangle) & \text{if } F\{T_\iota\} \neq F\{T_\tau\}, \\ (F_\iota/\langle\!\langle R_\iota \rangle\!\rangle) * \langle\, f \mid \quad \rangle & \text{if } f \in F \text{ and } fT_\iota = T_\tau. \end{cases} \tag{14}$$

Consider the case where both $F_\iota/\langle\!\langle R_\iota \rangle\!\rangle$ and $F_\tau/\langle\!\langle R_\tau \rangle\!\rangle$ have infinite, cyclic quotients. By (14), $F/\langle\!\langle R - \{r\} \rangle\!\rangle$ has a rank-two, free-abelian quotient. On incorporating r, we see that $F/\langle\!\langle R \rangle\!\rangle$ has an infinite, cyclic quotient, and the desired conclusion holds.

Thus, it remains to consider the case where one of $F_\iota/\langle\!\langle R_\iota \rangle\!\rangle$, $F_\tau/\langle\!\langle R_\tau \rangle\!\rangle$ does not have an infinite, cyclic quotient; by replacing e with $\overline{e}$, if necessary, we may assume that $F_\iota/\langle\!\langle R_\iota \rangle\!\rangle$ does not have an infinite, cyclic quotient.

By the induction hypothesis applied to $(F_\iota, T_\iota, R_\iota)$, we see that $F_\iota/\langle\!\langle R_\iota \rangle\!\rangle$ is trivial, that F_ι acts freely on T_ι, and, for each $r_\iota \in R_\iota$, $\text{glue}(F_\iota, E\text{axis}(r_\iota)) = \langle\, r_\iota\,\rangle$, and, hence, $\text{glue}(F, E\text{axis}(r_\iota)) = \langle\, r_\iota\,\rangle$.

By replacing r with $\overline{r}$ if necessary, we may assume the following.

(15) There exists a segment of axis(r) of the form e, p, re.

If $F\{T_\iota\} \neq F\{T_\tau\}$, then, by (15), we have a path $\overline{r}p$ in axis(r) from $\overline{r}\tau e \in \overline{r}T_\tau \neq T_\iota$ to $\iota e \in T_\iota$. Now $\overline{r}p$ necessarily enters T_ι through an edge of the form $g\overline{e}$ where $g \in F_\iota$ and $\overline{e}$ is the inverse of the edge e. Notice that $g \in \mathrm{glue}(F_\iota, E\mathrm{axis}(r)) - \langle\, r\,\rangle$. This proves the following.

(16) If $\mathrm{glue}(F_\iota, E\mathrm{axis}(r)) \subseteq \langle\, r\,\rangle$ then $F\{T_\iota\} = F\{T_\tau\}$.

Consider the case where R_ι is empty. Here, $F_\iota = F_\iota/\langle\!|\, R_\iota\,|\!\rangle = \{1\}$. By (16), $F\{T_\iota\} = F\{T_\tau\}$, and, then, by (13), $F = \langle\, t \mid \quad\rangle$. Now $\{t, \overline{t}\} = \sqrt[F]{F} \supseteq R = \{r\}$. Hence, $F = \langle\, r\,\rangle$, and, hence, $\mathrm{glue}(F, E\mathrm{axis}(r)) = \langle\, r\,\rangle$ This proves the following.

(17) If $\mathrm{glue}(F, E\mathrm{axis}(r)) \neq \langle\, r\,\rangle$ then R_ι is non-empty.

Case 1. $\mathrm{glue}(F, E\mathrm{axis}(r)) = \langle\, r\,\rangle$.

Here, by (16), $F\{T_\iota\} = F\{T_\tau\}$. Hence, $F(VT_\iota) = VT$ and $R = {}^F R_\iota \cup {}^F r$. Let $v \in VT$. We wish to show that $F_v = 1$, and, we may assume that $v \in VT_\iota$. Here $F_v \leqslant F_\iota$, and, since F_ι acts freely on T_ι, $F_v = 1$, as desired. Thus F acts freely on T. In (15), the path p from τe to $r\iota e$ in axis(r) does not meet Fe, and, hence, p stays within T_τ, and, hence $r\iota e \in T_\tau$. Thus, $rT_\iota = T_\tau$, and, by (13), $F = F_\iota * \langle\, r \mid \quad\rangle$. Since $F_\iota/\langle\!|\, R_\iota\,|\!\rangle$ is trivial, we see that $F/\langle\!|\, R\,|\!\rangle$ is trivial, and, hence, $F/\langle\!|\, R\,|\!\rangle$ is indicable. Here all the required conclusions hold.

Case 2. $\mathrm{glue}(F, E\mathrm{axis}(r)) \neq \langle\, r\,\rangle$.

By (17), R_ι is non-empty, and, hence, there exists some $e_\iota \in \bigcup\limits_{r_\iota \in R_\iota} E\mathrm{axis}(r_\iota)$ such that

$$Fe_\iota = \min(\{Fe \mid e \in \bigcup_{r_\iota \in R_\iota} E\mathrm{axis}(r_\iota)\},\ <).$$

There then exists some $r_\iota \in R$ such that $e_\iota \in E\mathrm{axis}(r_\iota)$, and we then know that $\mathrm{glue}(F, E\mathrm{axis}(r_\iota)) = \langle\, r_\iota\,\rangle$. By (12), $(r, e) = (r_{\max}, e_{\max})$. Using the definition of $(F, T, <)$-staggered, one can show that Fe_ι does not meet the axis of any element of $R - {}^F r_\iota$. We then replace (r, e) with (r_ι, e_ι), and, by Case 1, all the required conclusions hold.

This completes the proof. $\square$

We next deal with the case where R is finite.

A.2.3 Theorem. *Let F be a locally indicable group, let T be an F-tree with trivial edge stabilizers, and let R be an (F, ET)-staggerable subset of F such that $\sqrt[F]{R} = R$.*

Suppose that R is finite and that H is a finitely generated subgroup of F such that H contains R, and $H/\langle\, {}^H R\,\rangle$ has no infinite, cyclic quotient.

Then there exists some finitely generated subgroup F' of F such that F' contains H, R is (F', ET)-staggerable, $F'/\langle {}^{F'}\!R \rangle$ is trivial, F' acts freely on T and, for each $r \in R$, $\mathrm{glue}(F', E\mathrm{axis}(r)) = \langle r \rangle$. *Here, $H \leqslant F' = \langle {}^{F'}\!R \rangle \leqslant \langle {}^{F}\!R \rangle$, and, hence, H acts freely on T.*

Proof. Let v be an arbitrary vertex of T. Choose a finite generating set S of H such that S contains $R \cup \{1\}$, and let Y be the smallest subtree of T containing Sv. Then, for each $s \in S$, $sVY \cap VY$ is non-empty since it contains sv; thus

$$S \subseteq \mathrm{glue}(F, VY). \tag{18}$$

If F' is any subgroup of F and T' is any F'-subtree of T, for the purposes of this proof let us say that (F', T') is an *admissible* pair if $F' \supseteq S$, and $T' \supseteq Y$, and R is (F', ET')-staggerable. By hypothesis, (F, T) is admissible. Let $<$ be an ordering of $F \backslash ET$ such that R is $(F, ET, <)$-staggered.

The Type 1 transformation. Suppose that $F \neq \langle S \cup \mathrm{glue}(F, EY) \rangle$ or $T \neq FY$. Define $F' := \langle S \cup \mathrm{glue}(F, EY) \rangle$ and $T' := F'Y$. We shall prove that (F', T') is an admissible pair and $\mathrm{glue}(F, EY) = \mathrm{glue}(F', EY)$.

Clearly

$$\mathrm{glue}(F, EY) = \mathrm{glue}(F', EY). \tag{19}$$

It follows from (18) that

$$S \ \cup \ \mathrm{glue}(F', EY) \quad \subseteq \quad \mathrm{glue}(F', Y).$$

If we consider the set of components of the F'-forest $F'Y$ in T as an F'-set, we see that the component containing Y is fixed by a generating set of F', and hence is fixed by F'. This component must then be all of $F'Y$. Hence T' is connected, and, hence, T' is an F'-tree.

It is straightforward to show that

$$\mathrm{glue}(F, ET') = F'. \tag{20}$$

By (20), the natural map from $F' \backslash ET'$ to $F \backslash ET$ is an embedding; we again denote by $<$ the ordering of $F' \backslash ET'$ induced from $F \backslash ET$. We claim that, with the conjugation action by F on F, $\mathrm{glue}(F, R) \subseteq F'$. Suppose that $f \in F$, that $r \in R$, and that ${}^{f}r \in R$. Then $\mathrm{axis}(r)$ and $\mathrm{axis}({}^{f}r)$ lie in T' and are glued together by f since $\mathrm{axis}(r^{f}) = f\,\mathrm{axis}(r)$. By (20), $f \in F'$, and the claim is proved. It follows that R is $(F', ET', <)$-staggered. Hence (F', T') is admissible. This completes the verification of the Type 1 transformation.

The Type 2 transformation. Suppose that $F = \langle S \cup \mathrm{glue}(F, EY) \rangle$, that $T = FY$, and that $F/\langle {}^{F}\!R \rangle$ has some infinite, cyclic quotient F/N; here, N is a normal subgroup of F such that $N \geq \langle {}^{F}\!R \rangle$. We shall prove that (N, T) is admissible and $\mathrm{glue}(N, EY) \subset \mathrm{glue}(F, EY)$.

Since $H/\langle\, {}^{H}R \,\rangle$ has no infinite, cyclic quotient, it follows that $H \subseteq N$.

Since F/N is cyclic, we can choose $x \in F$ such that xN generates F/N. Then $F = \langle\, x \,\rangle N$. Since F/N is infinite, $\langle\, x \,\rangle \cap N = \{1\}$.

We now give $N\backslash ET$ an ordering. Give $(F\backslash ET) \times \mathbb{Z}$ the lexicographic ordering and choose a left F-transversal E_0 in ET. Then $\langle\, x \,\rangle E_0$ is a left N-transversal in ET, and there is a bijective map

$$(F\backslash ET) \times \mathbb{Z} \quad \to \quad N\backslash ET, \quad (Fe, n) \mapsto Nx^n e, (e, n) \in E_0 \times \mathbb{Z}.$$

Let $<_N$ denote the ordering induced on $N\backslash ET$ by this bijection.

We claim that R is $(N, ET, <_N)$-staggered. To see this, consider any r_1, $r_2 \in R$ such that ${}^{N}r_1 \neq {}^{N}r_2$. If ${}^{F}r_1 = {}^{F}r_2$, then r_1 and r_2 will have equal (F, ET)-supports, while their (N, ET)-supports, viewed in $(F\backslash ET) \times \mathbb{Z}$, will differ by a non-zero shift in the second coordinate, and hence be staggered. If ${}^{F}r_1 \neq {}^{F}r_2$, then r_1 and r_2 will have staggered (F, ET)-supports, and, hence, their (N, ET)-supports, viewed in $(F\backslash ET) \times \mathbb{Z}$, will also be staggered.

Hence (N, T) is admissible.

Since $N \not\supseteq F = \langle\, S \cup \mathrm{glue}(F, EY) \,\rangle$, we see that $N \not\supseteq S \cup \mathrm{glue}(F, EY)$. Since $N \supseteq S$, we see that $N \not\supseteq \mathrm{glue}(F, EY)$, and, hence, $\mathrm{glue}(N, EY) \subset \mathrm{glue}(F, EY)$. This completes the verification of the Type 2 transformation.

Since EY is finite and edge stabilizers are trivial, $\mathrm{glue}(F, EY)$ is finite. Notice that transformations of Type 2 reduce $\mathrm{glue}(F, EY)$, while transformations of Type 1 do not change $\mathrm{glue}(F, EY)$. Notice that we cannot apply two transformations of Type 1 consecutively. Thus, after applying a finite number of transformations of Types 1 and 2, we arrive at a pair (F', T') such that $F' = \langle\, S \cup \mathrm{glue}(F', EY) \,\rangle$, $T' = F'Y$, R is (F', ET')-staggerable, and $F'/\langle\, R^{F'} \,\rangle$ has no infinite, cyclic quotient. By Lemma A.2.2, $F'/\langle\, {}^{F'}R \,\rangle$ is trivial, F' acts freely on T', and, for each $r \in R$, $\mathrm{glue}(F', E\mathrm{axis}(r)) = \langle\, r \,\rangle$. Since F' acts freely on T', it follows that F' acts freely on all of T. Since R is (F', ET')-staggerable, it follows that R is (F', ET)-staggerable, because any ordering on $F'\backslash ET'$ can be extended to some ordering of $F'\backslash ET$, by the axiom of choice. □

The finite descending chain of subgroups implicit in the above argument is the chain of subgroups considered by Howie in his tower arguments.

We now have a general result.

A.2.4 Corollary. *Let F be a locally indicable group, let T be an F-tree with trivial edge stabilizers, and let R be an (F, ET)-staggerable subset of F.*

Then $\langle\, {}^{F}R \,\rangle$ acts freely on T, that is, each vertex stabilizer embeds in $F/\langle\, {}^{F}R \,\rangle$ under the natural map.

If, moreover, $\sqrt[F]{R} = R$, then $F/\langle\, {}^{F}R \,\rangle$ is locally indicable.

Proof. Since $\langle\, {}^{F}\!R\,\rangle \leqslant \langle\, {}^{F}(\sqrt[F]{R})\,\rangle$, and $\sqrt[F]{R}$ is again (F, ET)-staggerable, we may assume that $\sqrt[F]{R} = R$.

We first show that $\langle\, {}^{F}\!R\,\rangle$ acts freely on T. Let R' be an arbitrary finite subset of ${}^{F}\!R$, and let $H = \langle\, R'\,\rangle$. On applying Theorem A.2.3, we see that $\langle\, R'\,\rangle$ acts freely on T. It then follows that all of $\langle\, {}^{F}\!R\,\rangle$ acts freely on T.

We now show that $F/\langle\, {}^{F}\!R\,\rangle$ is locally indicable. Consider an arbitrary finitely generated subgroup of $F/\langle\, {}^{F}\!R\,\rangle$, and express it in the form $(\langle\, S\,\rangle\langle\, {}^{F}\!R\,\rangle)/\langle\, {}^{F}\!R\,\rangle$ where S is a finite subset of F. It remains to show that $(\langle\, S\,\rangle\langle\, {}^{F}\!R\,\rangle)/\langle\, {}^{F}\!R\,\rangle$ is indicable. We may assume that $(\langle\, S\,\rangle\langle\, {}^{F}\!R\,\rangle)/\langle\, {}^{F}\!R\,\rangle$ has no infinite, cyclic quotient, and it remains to show that $\langle\, S\,\rangle \leqslant \langle\, {}^{F}\!R\,\rangle$.

For the purposes of this proof, let $\langle\, S\,\rangle'$ denote the derived subgroup of $\langle\, S\,\rangle$, and let $\langle\, S\,\rangle^{\mathrm{ab}}$ denote the abelianization $\langle\, S\,\rangle/\langle\, S\,\rangle'$. Now

$$(\langle\, S\,\rangle\langle\, {}^{F}\!R\,\rangle)/(\langle\, S\,\rangle'\langle\, {}^{F}\!R\,\rangle) = ((\langle\, S\,\rangle\langle\, {}^{F}\!R\,\rangle)/\langle\, {}^{F}\!R\,\rangle)^{\mathrm{ab}}$$

which is a finite abelian group by supposition. Let d denotes its exponent. Then $S^d \subseteq \langle\, S\,\rangle'\langle\, {}^{F}\!R\,\rangle$ and, hence, there exists some finite subset R_0 of ${}^{F}\!R$ such that S^d lies in the set $\langle\, S\,\rangle'\langle\, R_0\,\rangle$. Then, $(\langle\, S \cup R_0\,\rangle/\langle\!\langle\, R_0\,\rangle\!\rangle)^{\mathrm{ab}}$ is an abelian group of exponent at most d, and, hence, $\langle\, S \cup R_0\,\rangle/\langle\!\langle\, R_0\,\rangle\!\rangle$ has no infinite, cyclic quotient, and, hence, by Theorem A.2.3, $\langle\, S \cup R_0\,\rangle \leqslant \langle\, {}^{F}\!R\,\rangle$, as desired. □

A.3 Consequences

The main application is the following.

A.3.1 Theorem (Howie)**.** *Let A and B be locally indicable groups and let r be an element of $A{*}B$ such that r is not conjugate to any element of $(A \cup B) - \{1\}$. Then the natural maps from A and B to $(A{*}B)/\langle\!\langle\, r\,\rangle\!\rangle$ are injective. If, moreover, $\sqrt[A*B]{r} = r$, then $(A{*}B)/\langle\!\langle\, r\,\rangle\!\rangle$ is locally indicable.*

Proof. Let T be the Bass-Serre $(A{*}B)$-tree as in Example A.1.3; then T is an $(A{*}B)$-tree with trivial edge stabilizers.

If H is some finitely generated subgroup of $A{*}B$, then the Bass-Serre Structure Theorem for the H-action on T, or the Kurosh Subgroup Theorem, shows that H is a free product of a family of finitely generated groups each of which is free, or isomorphic to a subgroup of A, or isomorphic to a subgroup of B. Hence all these free factors are indicable, and hence H is indicable. Thus $A{*}B$ is locally indicable. Thus we may assume that $r \neq 1$; then r is not conjugate to any element of $A \cup B$ and $\{r\}$ is $(A{*}B, ET)$-staggerable.

By Corollary A.2.4, the natural maps from A and B to $(A{*}B)/\langle\!\langle\, r\,\rangle\!\rangle$ are injective, and, if $\sqrt[A*B]{r} = r$, then $(A{*}B)/\langle\!\langle\, r\,\rangle\!\rangle$ is locally indicable. □

A.3.2 Corollary (Magnus' Freiheitssatz)**.** *Let F_1 and F_2 be free groups. If r is an element of F_1*F_2 such that r is not conjugate to any element of $F_1 - \{1\}$, then the natural map from F_1 to $(F_1*F_2)/\langle\!\langle r \rangle\!\rangle$ is injective.* $\square$

A.3.3 Corollary (Brodskiĭ)**.** *Let F be a free group. If $r \in F$ and $\sqrt[F]{r} = r$, then $F/\langle\!\langle r \rangle\!\rangle$ is locally indicable.* $\square$

A.3.4 Remarks. The injectivity result in Theorem A.3.1 is called the *local indicability Freiheitssatz* since it generalizes Magnus' Freiheitssatz, Corollary A.3.2.

The local indicability Freiheitssatz was proved independently by Brodskiĭ [4, Theorem 1], Howie [14, Theorem 4.3], and Short [23]. The proof by Brodskiĭ was algebraic, while the proofs by Howie and Short were topological, with Howie using topological towers and Short using diagrams. B. Baumslag [2] rediscovered Brodskiĭ's algebraic proof.

The local indicability conclusion in Theorem A.3.1 was proved by Howie [15, Theorem 4.2(iii)⇒(i)], by topological-tower methods. The one-relator case, Corollary A.3.3, had been proved earlier by Brodskiĭ [4, Theorems 1 and 2]. Howie [16] later gave a direct proof of Brodskiĭ's result using elementary groupoid methods, and his groupoid proof of the special case led us to the Bass-Serre proof of the general case. $\square$

A.3.5 Corollary. *Let G be a locally indicable group, let F be a free group, and let r be an element of $G*F$ that is not a proper power and that is not conjugate to any element of G. Then $(G*F)/\langle\!\langle r \rangle\!\rangle$ is locally indicable.*

Proof. Consider first the case where r is conjugate to an element of F. By conjugating r, we may assume that $r \in F$. Here, $F/\langle\!\langle r \rangle\!\rangle$ is locally indicable by Corollary A.3.3. Now $(G*F)/\langle\!\langle r \rangle\!\rangle = G*(F/\langle\!\langle r \rangle\!\rangle)$ is locally indicable by the degenerate case of Theorem A.3.1. Thus we may assume that r is not conjugate to any element of $(G \cup F) - \{1\}$. Here, $(G*F)/\langle\!\langle r \rangle\!\rangle$ is locally indicable by Theorem A.3.1. $\square$

It is now straightforward to deduce the following special case of a result of Howie [15, Corollary 4.5] on 'reducible presentations'.

A.3.6 Corollary (Howie)**.** *Let $G_{[0\uparrow\infty[}$ be a family of groups such that $G_0 = 1$ and, for all $n \in [0\uparrow\infty[$, $G_{n+1} = (G_n*\langle\, X_{n+1} \mid \quad \rangle)/\langle\!\langle r_{n+1} \rangle\!\rangle$ where X_{n+1} is a set, and r_{n+1} is an element of $G_n*\langle\, X_{n+1} \mid \quad \rangle$ that is not a proper power and that is not conjugate to any element of G_n. Then $\bigcup G_{[0\uparrow\infty[}$ is locally indicable.* $\square$

Acknowledgments

The research of the first- and second-named authors was jointly funded by the MEC (Spain) and the EFRD (EU) through Project MTM2006-13544.

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