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THE REVERSING OF INTERFACES IN SLOW DIFFUSION PROCESSES WITH STRONG ABSORPTION*

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Abstract. This paper considers a family of one-dimensional nonlinear diffusion equations with absorption. In particular, the solutions that have interfaces that change their direction of propagation are examined. Although this phenomenon of reversing interfaces has been seen numerically, and some special exact solutions have been obtained, there was previously no analytical insight into how this occurs in the general case. The approach taken here is to seek self-similar solutions local to the interface and local to the reversing time. The analysis is split into two parts, one for the solution prior to the reversing time and the other for the solution after the reversing time. In each case the governing PDE is reduced to an ODE by introducing a self-similar coordinate system. These ODEs do not readily admit any nontrivial exact solutions and so the asymptotic behavior of solutions is studied. By doing this the adjustable parameters, or degrees of freedom, which may be used in a numerical shooting scheme are determined. A numerical algorithm is then proposed to furnish solutions to the ODEs and hence the PDE in the limit of interest. As examples of physical problems in which a PDE of this type may be used as a model the authors study the spreading of a viscous film under gravity and subject to evaporation, the dispersion of a population, and a nonlinear heat conduction problem. The numerical algorithm is demonstrated using these examples. Results are also given on the possible existence of self-similar solutions and types of reversing behavior that can be exhibited by PDEs in the family of interest.

Key words. nonlinear diffusion, slow diffusion, strong absorption, porous medium equation, interface, reversing, self-similarity

AMS subject classifications. 35G20, 35R35, 35C06, 34A34, 35B40, 35A20

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1. Introduction.

This study is concerned with properties of solutions to a one-dimensional slow diffusion equation with strong absorption

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^m \frac{\partial h}{\partial x}\right) - h^{1-q}, \tag{1.1}$$

with $m > 0$, $q > 0$, and $m - q > 0$. Here h is the concentration of some species, x and t denote space and time, respectively, and the exponents m and q are constants. All variables in (1.1) are dimensionless. Boundary conditions and initial conditions to be imposed on (1.1) are discussed below; see (1.2)–(1.4). In this context the term "slow diffusion" refers to (1.1) with $m > 0$ so that the interfaces of compactly supported solutions have a finite propogation speed [10]. The term "strong absorption" refers to (1.1) with $q > 0$, as introduced in [6]. The family of equations shown in (1.1) is widely used as a model for many physical situations including the spreading of viscous gravity currents (typically $m = 3$ and $q = 0, 1$) [1], fluid flow in porous media (typically $m = 1$ and $q = 0, 1$) [2], population modeling (typically $m = 1$ and $q > 0$) [9], and nonlinear heat conduction (typically $m > 1$ and $q > 0$) [10]. In all cases the first term on the right-hand side (RHS) of (1.1) represents diffusion, while the second

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term on the RHS represents absorption or consumption of h according to a power-law. In section 1.1 the derivation of the model for the case of a slowly spreading viscous film under gravity and subject to evaporation is outlined.

In many problems of practical interest, bounded, continuous, nonnegative solutions to (1.1) that have initial data with compact support are sought in infinitely extended regions. In analyzing such solutions it is necessary to consider the points of singular behavior where $h \to 0$. One common approach is to identify such points as interfaces and to only solve the governing PDE on regions between interfaces where h is nonzero. The motions of the interfaces are then determined as part of the solution to the problem by insisting that h is zero at the interfaces and that the interfaces move in such a way that the flux of h through them is zero. (See [6] for a derivation of condition (1.3).) These conditions along with the specification of an initial configuration of h complete the problem definition. Using the notation $s(t)$ to denote the location of the left interface at any given time, the boundary and initial conditions may be written as

$$h = 0 \quad \text{at} \quad x = s(t), \tag{1.2}$$

$$\frac{ds}{dt} = -h^{m-1}\frac{\partial h}{\partial x} + \left(q\frac{\partial}{\partial x}\left(h^q\right)\right)^{-1} \quad \text{at} \quad x = s(t), \tag{1.3}$$

and initial data as

$$h(x, 0) = h_0(x) \tag{1.4}$$

determining the initial position of the interface, $s(0)$. Note that two conditions analogous to (1.2) and (1.3) must also be imposed to determine the motion of the right interface of the solution.

Properties of (1.1) have been extensively examined by previous authors. See [13] and the bibliography therein for results on existence and uniqueness of solutions. The behavior of solutions depends critically on the values of m and q. One result which is particularly relevant to this study states that when $m \leq 0$, solutions with initial conditions that have compact support have interfaces that move at an infinite velocity. By contrast, if $m > 0$, the interfaces propagate with a finite velocity [10]. Another important result states that for $q > 0$ receding interfaces can exist [6]. In this context a receding interface refers to circumstances where the interface moves in a way that decreases the size of the region of compact support of the solution. For the case $m > 0$ and $0 < q < 1$ it has been shown that $h(x, t) \equiv 0$ for all x after some finite time, so that the problem displays so-called finite extinction time [5], [12]. The asymptotic behavior of the solution near this extinction time was studied in [8], where it was shown that if $m - q \geq 0$ the behavior is completely dominated by absorption. However, if $m - q < 0$, there are regions near the edge of the support where diffusion becomes important. In [15] a generalized version of (1.1) was studied where the growth and connectedness of the support of solutions as $t \to +\infty$ could be determined. It was shown that the compact support can either remain bounded or become unbounded as $t \to +\infty$, depending on the parameters m and q. Predictions for the rate of growth of the compact support were also derived.

Other authors have obtained exact solutions for several special choices of m and q. Essentially, solutions can be found in the case $m + q = 0$, since the effects of diffusion and absorption are matched in the sense that both are proportional to h^{m+1}. Similarly, in the case $q = 0$ the absorption and time derivative terms are matched in the sense that both are proportional to h. Another special case in which an exact solution can be found occurs when $m - q + 1 = 0$ [14]. In [11] separation of variables and self-similar reductions were used to reduce (1.1) to various ODEs. For two of these ODEs first integrals and exact solutions are obtained. In [19] a special exact solution to a generalization of (1.1) in more than one dimension was found by means of a self-similar reduction to the governing equation. In [20] an exact solution was found for the special case $m = 1$ and $q = 0$ by transforming the spatial coordinates. In the same study another exact solution was found for the case $m = 1$ and $q = 1$ by means of separation of variables. In [18] a numerical scheme is developed to find approximate solutions to (1.1). The numerical solutions are compared to one of the aforementioned exact solutions and a good agreement is shown between the two.

Other relevant studies include [3] and [16], in which the problem of waiting time was considered. This refers to the phenomenon exhibited by the purely diffusive version of (1.1) (i.e., without the term h^{1-q} in the PDE) with $m > 0$ where interfaces can remain stationary for some finite time and then begin to move. Using a self-similar reduction to the governing equation, local solutions are found which describe the change in the behavior of the interface.

The authors' original interest in equations of this type came about when modeling the spreading of a viscous film over a horizontal plate under the action of gravity and subject to evaporation. An outline of the derivation of the model is given in section 1.1 and gives rise to (1.1) with $m = 3$ and $q = 1$. Initial analysis of this model was concentrated on looking for approximate traveling wave solutions local to the interface. The notation $s(t)$ was used to denote the location of the left interface at any given time. By transforming to a coordinate system local to this interface by writing $x = s(t) + \xi$ the following PDE in the neighborhood of the interface was derived:

$$\frac{\partial h}{\partial t} - \frac{ds}{dt}\frac{\partial h}{\partial \xi} = \frac{\partial}{\partial \xi}\left(h^3 \frac{\partial h}{\partial \xi}\right) - 1. \tag{1.5}$$

From (1.5) it is relatively straightforward to see that a balance between the second term on the left-hand side (LHS) and the first term on the RHS of (1.5) gives rise to a local advancing traveling wave solution of the form

$$h \sim \left(-3\frac{ds}{dt}\right)^{1/3} \xi^{1/3}, \quad \text{i.e.,} \quad h \sim \left(-3\frac{ds}{dt}\right)^{1/3} (x - s(t))^{1/3} \quad \text{as} \quad x \to s(t)^+. \tag{1.6}$$

Furthermore, a balance between the second term on the LHS and the second term on the RHS of (1.5) gives rise to a receding local traveling wave solution of the form

$$h \sim \left(\frac{ds}{dt}\right)^{-1} \xi, \quad \text{i.e.,} \quad h \sim \left(\frac{ds}{dt}\right)^{-1} (x - s(t)) \quad \text{as} \quad x \to s(t)^+. \tag{1.7}$$

In addition to this local analysis, numerical experiments were carried out that indicated, for certain initial conditions, numerical solutions exhibit a behavior in which the interface of the solution would change its direction of propagation. Throughout this study such a phenomenon is referred to as the reversing of an interface. Although the reversing behavior of solutions to (1.1) has been observed numerically and some of the aforementioned exact solutions also exhibit this behavior, there appears to be no analytical explanation of how this occurs in the general case [17], [18]. In other words, there appears to be no analytical explanation of how the advancing wave (1.6) gives way to the receding wave (1.7). With this in mind, an approach is taken in this study similar to that used in [16], namely, to seek solutions to (1.1) local to the interface and local to the reversing time. To find such solutions a self-similar reduction will be made to (1.1) in the parameter regime $m > 0$, $q > 0$ and $m - q > 0$. The first of these restrictions ensures that the interfaces propagate with a finite velocity [10], while the second allows the existence of receding interfaces [6]. It will become apparent that the third means that an advancing interface moves due to a balance between the time derivative and diffusion. Conversely, a receding interface moves due to a balance between the time derivative and absorption. Note that an interface may reverse even if the third restriction is not satisfied—however, it appears that if $m - q \leq 0$ the solution local to the interface and local to the reversing time will behave differently from the case studied here.

For algebraic clarity the origins of time and space are chosen so that the reversing time is $t = 0$ when the position of the interface is $s(0) = 0$. Additionally we assume, without loss of generality, that h is nonzero in $x > 0$ and h is zero for all $x \leq 0$. Throughout this study it is assumed that $s(t)$ and its first derivative are continuous. While this seems to be a physically sensible assumption, there appears to be no rigorous proofs on this behavior. To find a suitable similarity reduction the authors, refer to [7], which, by consideration of classical point symmetries of (1.1), lists possible reductions for equations of this type. Included in this list is a reduction of the form

$$h = (\pm t)^{\alpha} H(\phi) \quad \text{with} \quad \phi = x(\pm t)^{\beta} \quad \text{and} \quad s(t) = \Lambda(\pm t)^{-\beta}, \tag{1.8}$$

where Λ is an arbitrary constant and α and β are constants that are fixed by the exponents m and q. Later in this work it will be seen that $\alpha = 1/q$ and $\beta = -(m+q)/2q$. Though in [7] other reductions to (1.1) are derived, these are unsuitable to describe the reversing behavior since they necessarily give a discontinuity in the velocity of the interface as t passes through zero. It is noted, however, that some of these other reductions do give a sensible behavior of the solution away from $t = 0$. The form of (1.8) means that the position of the interface moves in proportion to $(\pm t)^{-\beta}$ with $\beta < -1$ (since $m > 0$, $q > 0$ and $m - q > 0$) and hence the velocity of the interface changes smoothly as t passes through zero.

The analysis is split into two sections. In section 2 the solution for $t < 0$ (with an advancing interface) is studied. Using a self-similar reduction of the form (1.8) to (1.1) an ODE is derived for the dependent self-similar variable H as a function of ϕ. The asymptotic behavior of solutions to this ODE as $H \to 0^+$ and $\phi \to +\infty$ are examined by assuming a power-law type expansion in each limit. In section 3 a similar analysis is carried out for the solution with $t > 0$ (and a receding interface). In order to close the problem continuity of h across $t = 0$ is enforced so that h is continuous as the interface changes its direction of propagation. This is equivalent to insisting that the solutions both prior to and after the reversing time have the same asymptotic behavior as $\phi \to +\infty$. In section 4 a numerical scheme which makes use of the determined asymptotic behaviors is proposed to determine solutions local to $x = 0$ and $t = 0$. (See [21] for another example of applying this technique.) In sections 4.1, 4.2, and 4.3 this algorithm is demonstrated using several physically motivated examples. Section 4.4 discusses the practicalities of finding solutions for other pairs of values of m and q. Finally, in section 5 the results and conclusions are discussed. Before proceeding to analyze (1.1) a physical application that leads to a reversing interface of the type outlined above is discussed.

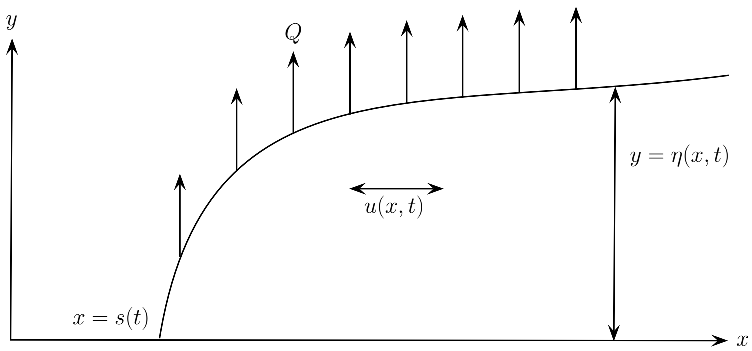

FIG. 1.1. *Definition diagram for the slow spreading of a viscous film over a plate with evaporation.*

1.1. The slow spreading of a viscous film with evaporation. Consider the slow spreading of a Newtonian viscous film along a fixed horizontal plate under the action of gravity and subject to evaporation. The definition diagram for the flow is shown in Figure 1.1. By assuming that the Reynolds number is sufficiently small that inertial effects can be neglected the equations of momentum and continuity are

$$\nabla p = \mu \nabla^2 \mathbf{u} - \rho \mathbf{g}, \quad \nabla \cdot \mathbf{u} = 0. \tag{1.9}$$

Here p is the pressure, μ is the dynamic viscosity of the fluid, $\mathbf{u} = (u, v)$ is the velocity vector, ρ is the density of the fluid, and $\mathbf{g}$ is the acceleration due to gravity. The no-slip boundary condition between the film and the plate is

$$\mathbf{u} = 0 \quad \text{on} \quad y = 0. \tag{1.10}$$

It is assumed that the capillary number, $Ca = \mu V \gamma^{-1}$ (where V is the velocity scale of the flow and γ is the interfacial tension), is sufficiently large so that surface tension effects are negligible. The free surface of the film is modeled as stress free and assumed to satisfy a modified kinematic condition which takes into account the loss of fluid due to a constant evaporation normal to the free surface. Hence

$$\mathbf{t}'\mathbf{T}\mathbf{n} = \mathbf{n}'\mathbf{T}\mathbf{n} = 0 \quad \text{and} \quad v - \frac{Q}{\rho} = u\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} \quad \text{on} \quad y = \eta(x, t). \tag{1.11}$$

Here a prime denotes transpose, $y = \eta(x, t)$ is the free surface of the film, $\mathbf{t}$ and $\mathbf{n}$ are vectors tangential and normal to the surface $y = \eta(x, t)$, Q is the evaporation rate per unit area, and $\mathbf{T}$ is the stress tensor (defined in the usual way).

Assuming that the film is slender motivates the nondimensionalization $x = L\bar{x}$, $y = \epsilon L\bar{y}$ and $\eta = \epsilon L\bar{\eta}$. Here L is a typical length of the film and $\epsilon \ll 1$ is the ratio of typical depth to typical length. An as yet undetermined horizontal velocity scale for the flow is introduced by writing $u = u_0\bar{u}$. In order to conserve mass the vertical velocity scale is $v = \epsilon u_0 \bar{v}$. Pressure and time are scaled the natural way by writing $p = \rho g \epsilon L \bar{p}$ and $t = L u_0^{-1} \bar{t}$. So that the pressure in the film is hydrostatic (to leading order in ϵ) this sets the undetermined velocity scale $u_0 = \rho g \epsilon^3 L^2 \mu^{-1}$. To leading order in ϵ the system of equations (1.9), (1.10), and (1.11) may be written as

$$\frac{\partial \bar{p}}{\partial \bar{y}} = -1, \quad \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} = \frac{\partial \bar{p}}{\partial \bar{x}}, \quad \text{and} \quad \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{1.12}$$

$$\bar{u} = \bar{v} = 0 \quad \text{on} \quad \bar{y} = 0, \tag{1.13}$$

$$\frac{\partial \bar{u}}{\partial \bar{y}} = 0, \quad \bar{p} = 0, \quad \text{and} \quad \bar{v} - \frac{Q}{\epsilon u_0 \rho} = \bar{u}\frac{\partial \bar{\eta}}{\partial \bar{x}} + \frac{\partial \bar{\eta}}{\partial \bar{t}} \quad \text{on} \quad \bar{y} = \bar{\eta}(\bar{x}, \bar{t}). \tag{1.14}$$

The system of equations (1.12), (1.13), and (1.14) can be solved to obtain $\bar{u}$ and $\bar{v}$ in terms of $\bar{y}$ and $\bar{\eta}$. Substitution of these solutions into the third equation in (1.14) yields the following equation for $\bar{\eta}$:

$$\frac{\partial \bar{\eta}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left(\bar{\eta}^3 \frac{\partial \bar{\eta}}{\partial \bar{x}} \right) - \frac{Q}{\epsilon u_0 \rho}. \tag{1.15}$$

This nonlinear PDE for the free surface of the film $\bar{\eta}(\bar{x}, \bar{t})$ contains one nondimensional parameter, $Q\epsilon^{-1}u_0^{-1}\rho^{-1}$, that gives a measure of the ratio of the spreading rate to the evaporation rate. Equation (1.15) can be reduced to an equation of type (1.1) for $\bar{\eta}(x^*, t^*)$ with $m = 3$ and $q = 1$ by writing

$$\bar{t} = \frac{\epsilon u_0 \rho}{Q} t^* \quad \text{and} \quad \bar{x} = \sqrt{\frac{\epsilon u_0 \rho}{Q}} x^*. \tag{1.16}$$

2. Prior to the reversing time. In this section analysis is concentrated on solutions to (1.1) when $t < 0$ and the interface is advancing. As discussed in section 1 a similarity reduction of the form

$$h = (-t)^{1/q} H(\phi), \quad \text{where} \quad \phi = x(-t)^{-(m+q)/2q} \quad \text{and} \quad s(t) = A(-t)^{(m+q)/2q}, \tag{2.1}$$

is employed. Here A is an undetermined constant. Using (2.1), (1.1) and its corresponding local boundary condition reduce to

$$-\frac{1}{q} H + \frac{m+q}{2q} \phi H' = (H^m H')' - H^{1-q} \quad \text{with} \quad H(A) = 0. \tag{2.2}$$

Equation (2.2) readily admits one nontrivial exact solution

$$H = \left(\frac{(m+q)^2}{2(2+m-q)} \right)^{1/(m+q)} \phi^{2/(m+q)}. \tag{2.3}$$

However, (2.3) has $H(0) = 0$ and as such cannot lead to a solution to (1.1) that exhibits reversing behavior. Hence, other solutions to (2.2) must be sought. To this end the asymptotic behavior of solutions near $\phi = A$ and as $\phi \to +\infty$ are examined. As $\phi \to A^+$ an asymptotic solution that has the form of a power-law is sought. It is found that

$$H \sim \left(\frac{m+q}{2q} Am \right)^{1/m} (\phi - A)^{1/m} \quad \text{as} \quad \phi \to A^+. \tag{2.4}$$

The prefactor of the asymptotic behavior (2.4) depends on the values of both m and q, whereas the exponent depends on the value of m only. Physically this may be understood by noting that interfaces of solutions to (1.1) advance due to diffusion. Since the solution prior to the reversing time has an advancing interface it may not come as a complete surprise that the value of m largely determines the behavior of the solution close to the interface.

In the far field a solution that has the form of a power-law is also sought. It is found that

$$H \sim N\phi^{2/(m+q)} \quad \text{as} \quad \phi \to +\infty. \tag{2.5}$$

Here N is an undetermined constant. A natural question to ask is whether there are solutions to (2.2) that have the behavior (2.4) near $\phi = A$ and reach the behavior (2.5) in the far field. In addition one may also wish to know whether there is an infinite family, countably many or a unique solution with these behaviors. One ad hoc approach to answering this question (which will subsequently be shown to be a correct approach in this case) is to implement a numerical shooting scheme to integrate (2.2) from near $\phi = A$ using (2.4) toward the far field. The value of A can then be adjusted as a shooting parameter with the aim of reaching a solution with behavior (2.5) in the far field. Alternatively one could integrate (2.2) numerically from the far field using (2.5) and adjust the value of N as a shooting parameter with the aim of reaching a solution with behavior (2.4) near $\phi = A$. The results of such numerical experiments are shown in Figures 4.1 and 4.2. From these results one might conclude that there is only one solution with the aforementioned behaviors; however, this is rather naive. In order to support this claim the authors carry out a more thorough analysis near the point $\phi = A$ and as $\phi \to +\infty$. The approach will be to linearize about a particular solution $\bar{H}$ by writing

$$H = \bar{H} + H_1 \tag{2.6}$$

and requiring that $H_1 \ll \bar{H}$. Using (2.2) and (2.6) the following second order linear homogeneous ODE for H_1 is derived for H_1:

$$-\frac{1}{q} H_1 + \frac{m+q}{2q}\,\phi H_1' = \left(\bar{H}^m H_1' + m H_1 \bar{H}^{m-1}\bar{H}'\right)' - (1-q)\bar{H}^{-q} H_1. \tag{2.7}$$

By examining the possible asymptotic solutions for H_1 the number of adjustable parameters, or degrees of freedom, when implementing a shooting scheme will be determined.

First consider the asymptotic behavior (2.4) near $\phi = A$. Linearizing about this,

$$H \sim \left(\frac{m+q}{2q} Am\right)^{1/m} (\phi - A)^{1/m} + H_1 \quad \text{as} \quad \phi \to A^+, \tag{2.8}$$

and substituting into (2.2) leads to a second order linear homogeneous ODE for H_1 with the following possible asymptotic behaviors:

$$H_1 \sim M_1 (\phi - A)^{(1/m)-1} \quad \text{or} \quad H_1 \sim M_2 \quad \text{as} \quad \phi \to A^+. \tag{2.9}$$

Here M_1 and M_2 are undetermined constants. The first behavior in (2.9) corresponds to changing A in (2.4) by a small amount. This can be seen by performing a Taylor expansion on (2.4) for small adjustments to A. Hence without loss of generality M_1 can be set equal to zero. The second behavior in (2.9) is necessarily larger than (2.4) in the limit that $\phi \to A^+$ unless M_2 is zero. Hence the solution with $M_2 = 0$ is required. Therefore there is an adjustable parameter (degree of freedom) only when shooting from near $\phi = A$, namely, the value of A.

An analogous analysis in the far field is performed by linearizing about the asymptotic behavior (2.5) by writing

$$H \sim N\phi^{2/(m+q)} + H_1 \quad \text{as} \quad \phi \to +\infty. \tag{2.10}$$

As before this leads to a second order linear homogeneous ODE for H_1 with the following possible asymptotic behaviors:

$$H_1 \sim N_1 \phi^{2/(m+q)} \quad \text{or} \quad H_1 \sim N_2 \exp\left(\frac{(m+q)^2}{4q^2} N^{-m} \phi^{2q/(m+q)}\right) \quad \text{as} \quad \phi \to +\infty. \tag{2.11}$$

Here N_1 and N_2 are undetermined constants. The first behavior in (2.11) is the same as (2.5); hence, without loss of generality N_1 is set equal to zero. For sufficiently large ϕ the second behavior in (2.11) becomes larger than (2.5) unless N_2 is zero. Hence the solution with $N_2 = 0$ is required. It is noted that numerical evidence suggests that solutions that do deviate from (2.5) (with N_2 nonzero) have $H = 0$ at some large but finite ϕ. Hence, such solutions are not viable in this problem. Therefore there is only one adjustable parameter (degree of freedom) when shooting from the far field, namely, the value of N.

Having determined the adjustable parameters near $\phi = A$ and as $\phi \to +\infty$ the shooting problem from near $\phi = A$ is reconsidered. Picking a particular value of A one might expect that for some large but finite ϕ

$$H \sim N\phi^{2/(m+q)} + N_2 \exp\left(\frac{(m+q)^2}{4q^2} N^{-m} \phi^{2q/(m+q)}\right). \tag{2.12}$$

Hence as ϕ continues to grow the solution will deviate from the behavior (2.5) unless $N_2 = 0$. Examples of such solutions can be seen in Figure 4.1. Thus it is anticipated that there are at most countably many values of A that correspond to the far field behavior (2.5).

Alternatively, consider the shooting problem from the far field. Picking a particular value of N one might expect that for some small but nonzero H

$$H \sim \left(\frac{m+q}{2q} Am\right)^{1/m} (\phi - A)^{1/m} + M_2. \tag{2.13}$$

Hence the solution deviates from the behavior (2.4) unless $M_2 = 0$. Examples of such solutions can be seen in Figure 4.2. Therefore it is expected that there are at most countably many values of N that correspond to the behavior (2.4) near $\phi = A$.

3. After the reversing time. In this section analysis is concentrated on (1.1) when $t > 0$ and the interface is receding. In this case a reduction of the same form as that used in section 2 is employed, taking care that t is now positive:

$$h = t^{1/q} H(\phi) \quad \text{where} \quad \phi = xt^{-(m+q)/2q} \quad \text{and} \quad s(t) = Bt^{(m+q)/2q}. \tag{3.1}$$

Here B is an undetermined constant. Using (3.1), (1.1) and its corresponding local boundary condition reduce to

$$\frac{1}{q} H - \frac{m+q}{2q} \phi H' = (H^m H')' - H^{1-q} \quad \text{with} \quad H(B) = 0. \tag{3.2}$$

This second order nonlinear ODE does not readily admit any exact nontrivial solutions aside from (2.3). Hence, in a manner similar to the previous section, the asymptotic behavior of solutions to (3.2) near $\phi = B$ and as $\phi \to +\infty$ are examined. As $\phi \to B^+$, the behavior is examined by assuming a power-law type expansion. It is found that

$$H \sim \left(\frac{m+q}{2q^2} B\right)^{-1/q} (\phi - B)^{1/q}. \tag{3.3}$$

The prefactor of the asymptotic behavior (2.4) depends on the values of both m and q, whereas the exponent depends on the value of q only. Physically this may be understood by noting that interfaces of solutions to (1.1) recede due to absorption effects. Since the solution after the reversing time has a receding interface it may not be unexpected that the value of q largely determines the behavior of the solution close to the interface.

As $\phi \to +\infty$ the asymptotic behavior is also examined by assuming a power-law type expansion. It is found that

$$H \sim Q\phi^{2/(m+q)}. \tag{3.4}$$

Here Q is an undetermined constant. For reasons that have been discussed in section 2 the authors now linearize about the asymptotic solutions (3.3) and (3.4). Linearizing about (3.3) using

$$H \sim \left(\frac{m+q}{2q^2}B\right)^{-1/q}(\phi - B)^{1/q} + H_1 \tag{3.5}$$

and substituting into (3.2) leads to a second order linear homogeneous ODE for H_1 with the following asymptotic behaviors:

$$\begin{aligned} &H_1 \sim P_1(\phi - B)^{(1/q)-1} \\ \text{or}\quad &H_1 \sim P_2 \exp\left(\left(\frac{B}{2}\right)\frac{m+q}{m-q}\left(\frac{m+q}{2q^2}B\right)^{m/q}(\phi - B)^{-(m-q)/q}\right) \\ &\text{as}\quad \phi \to B^+. \end{aligned} \tag{3.6}$$

The first behavior in (3.6) corresponds to a small change in B in the behavior (3.3). This can be seen by performing a Taylor expansion on (3.3) for small adjustments to B. Hence, without loss of generality P_1 is set equal to zero. The second behavior in (3.6) becomes larger than (3.3) in the limit $\phi \to B^+$ unless P_2 is zero. Hence the solution with $P_2 = 0$ is required. Therefore there is only one adjustable parameter (degree of freedom) when shooting from near $\phi = B$, namely, the value of B.

The solution (3.4) in the far field is linearized about by writing

$$H \sim Q\phi^{2/(m+q)} + H_1. \tag{3.7}$$

Again this leads to a second order linear homogeneous ODE for H_1 with the following asymptotic behavior:

$$H_1 \sim Q_1\phi^{2/(m+q)} \quad \text{or} \quad H_1 \sim Q_2 \exp\left(-\frac{(m+q)^2}{4q^2}Q^{-m}\phi^{2q/(m+q)}\right). \tag{3.8}$$

The first behavior in (3.8) is the same as (3.4) and so without loss of generality Q_1 is set equal to zero. To fix values for Q and Q_2 it is necessary to discuss a condition of continuity which should be imposed on the solution. On physical grounds it must be insisted that the concentration h is continuous as t passes through zero. In terms of the self-similar variables this is equivalent, requiring that solutions to (2.2) and (3.2) have the same behavior as $\phi \to +\infty$ corresponding to $t \to 0^+$ and $t \to 0^-$. It was shown in the previous section that solutions to (2.2) take the form (2.5) in the far field. Therefore, continuity across $t = 0$ requires that the solution to (3.2) has $Q = N$.

Now consider the shooting problem from near $\phi = B$. Picking a value of B one might expect that for some large but finite ϕ

$$H \sim Q\phi^{2/(m+q)} + Q_2 \exp\left(-\frac{(m+q)^2}{4q^2}Q^{-m}\phi^{2q/(m+q)}\right). \tag{3.9}$$

The value of B must be adjusted as a shooting parameter until the behavior (3.9) with $Q = N$ is reached in the limit that $\phi \to +\infty$. By doing this a value for Q_2 is also fixed; however, in the far field limit the second term in (3.9) tends to zero, independent of Q_2. As before one expects there to be at most countably many values of B which correspond to the requisite behavior (3.4) with $Q = N$ in the far field.

Alternatively, consider the shooting problem from the far field. Fixing the value of $Q = N$ (by virtue of the condition of continuity across $t = 0$) and picking a particular value of Q_2 one might expect that for some small but nonzero H,

$$\begin{aligned} H \sim & \left(\frac{m+q}{2q^2}B\right)^{-1/q} (\phi - B)^{1/q} \\ & + P_2 \exp\left(\left(\frac{B}{2}\right)\frac{m+q}{m-q}\left(\frac{m+q}{2q^2}B\right)^{m/q}(\phi - B)^{-(m-q)/q}\right). \end{aligned} \tag{3.10}$$

Hence the solution deviates from (3.3) unless $P_2 = 0$. Therefore it is expected that there are at most countably many values of Q_2 that correspond to $P_2 = 0$.

4. A numerical shooting scheme. In this section a numerical algorithm is proposed to construct solutions to (1.1) local to the interface and local to the reversing time. The approach is to use the asymptotic behaviors (2.4) and (2.5) to formulate an initial value problem for the ODE (2.2). A similar method is then used for the asymptotic behaviors (3.3) and (3.9) and the ODE (3.2). These can then be integrated numerically using the ODE45 package in MATLAB [22]. The details of the method are set out below.

- The solution for $t < 0$ can be determined using two different shooting schemes: (i) using the form shown in (2.4) to construct initial values for H and H' so that (2.2) can be integrated from $\phi = A$ toward the far field or (ii) using the form shown in (2.5) to form initial values for H and H' so that (2.2) can be integrated from the far field toward $\phi = A$. However, shooting in either of the aforementioned directions is inherently unstable. In case (i) this is due to the exponentially large term in (2.11) and in case (ii) is due to the constant M_2 in (2.9). Hence, in practice it is necessary to shoot both from $\phi = A$ adjusting A as a shooting parameter and from the far field adjusting N as a shooting parameter. By adjusting A and N correctly two solution curves (one using shooting method (i) and the other using method (ii)) can be obtained that agree (to within some small error tolerance) on a significant range of ϕ between A and the far field; see, for example, Figure 4.1. It is then reasonable to assume the required solution is very well approximated by solution (i) for ϕ near A and by (ii) in the far field.
- The solution for $t < 0$ is then fully determined. It is now necessary that the behavior of the solution as $\phi \to +\infty$ does not change as t passes through zero (and hence the concentration, h, is continuous) so a solution for $t > 0$ must be found with $Q = N$.
- The solution for $t > 0$ can be determined using two different shooting schemes: (i) using (3.3) to construct initial values for H and H' so that (3.2) can be integrated from $\phi = B$ toward the far field or (ii) using (3.9) to construct initial values for H and H' so that (3.2) can be integrated from the far field toward $\phi = B$. Due to the presence of the exponentially large term in (3.6), shooting in direction (ii) is inherently unstable. Therefore, in practice it may be necessary to shoot both from $\phi = B$ adjusting B as a shooting parameter

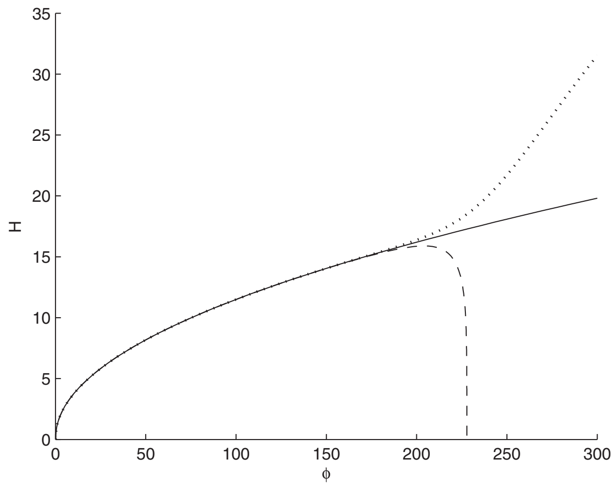

FIG. 4.1. *Plot of H vs ϕ for $t < 0$ for (1.1) with $m = 3$ and $q = 1$. The dashed and dotted curves show the solution computed by integrating (2.2) from $\phi = A$ with $A = 0.14397765 \pm 5 \times 10^{-8}$ respectively. It is noted that the dashed curve shows H tending to infinity in the negative direction whilst the dotted curve shows H tending to infinity in the positive direction corresponding to corresponding to negative and positive values of N_2. Therefore it is anticipated, although not proven, that the exact value of A which corresponds to $N_2 = 0$ lies in the range $0.14397765 \pm 5 \times 10^{-8}$. The solid line shows the solution computed from integrating from $\phi = 300$ with a value of $N = 1.1435$.*

and from the far field adjusting Q_2. By adjusting B and Q_2 correctly, two solution curves (one using shooting method (i) and the other using method (ii)) can be obtained that agree on a significant range of ϕ between B and the far field. As before, it is then reasonable to assume that the required solution is very well approximated by the solution (i) near $\phi = A$ and the solution (ii) in the far field.

4.1. The solution for $m = 3$ and $q = 1$: A spreading viscous film with evaporation. To demonstrate the method proposed in section 4, the example that originally motivated this study is used, that is, (1.1) with $m = 3$ and $q = 1$. In this case the forms (2.4), (2.5), (3.3), and (3.4) reduce to

$$H \sim (6A)^{1/3}(\phi - A)^{1/3} \quad \text{as} \quad \phi \to A^+ \quad \text{for} \quad t < 0, \tag{4.1}$$

$$H \sim N\phi^{1/2} \quad \text{as} \quad \phi \to +\infty \quad \text{for} \quad t < 0, \tag{4.2}$$

$$H \sim \frac{1}{2B}(\phi - B) \quad \text{as} \quad \phi \to B^+ \quad \text{for} \quad t > 0, \tag{4.3}$$

$$H \sim Q\phi^{1/2} \quad \text{and} \quad \text{as} \quad \phi \to +\infty \quad \text{for} \quad t > 0. \tag{4.4}$$

Using the numerical algorithm outlined above with the default settings in the ODE45 suite in MATLAB the following values are determined:

$$A \approx 0.1440, \ \ B \approx 0.0958, \ \ \text{and} \ \ N = Q \approx 1.1435. \tag{4.5}$$

As the absolute error tolerances were reduced below the ODE45 default value of 10^{-6} there were no appreciable changes in the computed solutions. The results of the numerical computations are shown in Figures 4.1 and 4.2. Figure 4.3 has been included to show that the condition of continuity across $t = 0$ has been satisfied. Other numerical computations were carried out for a large range of values of the

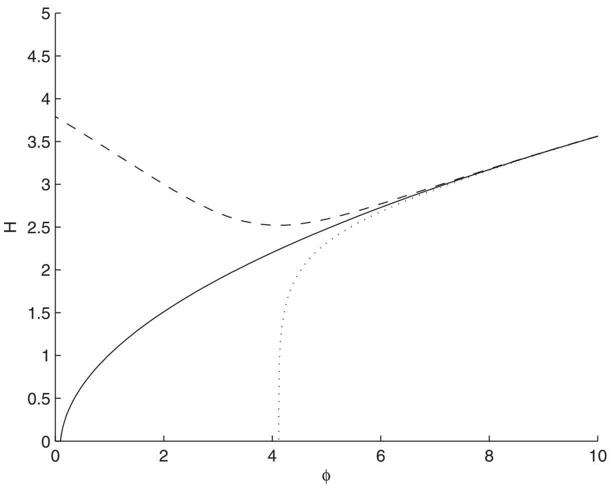

FIG. 4.2. *Plot of H vs. ϕ for $t > 0$ for (1.1) with $m = 3$ and $q = 1$. The dashed curve shows the solution computed by integrating (2.2) from $\phi = 30$ with $Q = 1.1435$ and $Q_2 = -16507$. The dotted curve shows the solution computed by integrating (2.2) from $\phi = 30$ with $Q = 1.1435$ and $Q_2 = -16508$. Although the dotted curve may appear to have the requisite behavior as H becomes small, the solution computed by integrating (2.2) using the corresponding value of A diverges from the required behavior in the far field. The solid line shows the solution computed from integrating from $\phi = B$ with a value of $B = 0.0958$.*

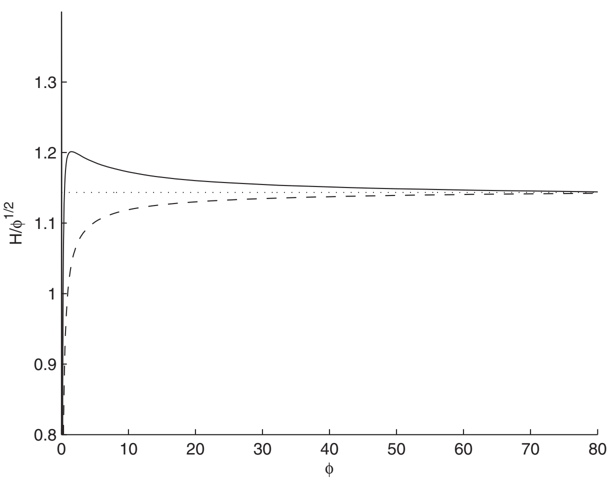

FIG. 4.3. *Plot of $H/\phi^{1/2}$ vs. ϕ for (1.1) with $m = 3$ and $q = 1$. The solid curve shows the solution for $t > 0$, the dashed curve shows the solution for $t < 0$, and the dotted curve shows $N\phi^{1/2}$ vs. $\phi^{1/2}$.*

shooting parameters A and N. It was found in all cases that the deviation from the required behavior was monotone in the shooting parameter. Although this is not a formal proof, the numerical evidence strongly suggests that the above values of A, B, N, and Q are unique. Figures 4.4 and 4.5 have been included to show the evolution of the solution $h(x, t)$ and the position of the interface during this evolution.

4.2. The solution for $m = 2$ and $q = 1$: A population with constant death rate. For a second demonstration of the numerical scheme proposed in section 4 a population model is considered. Many population studies use (1.1) with

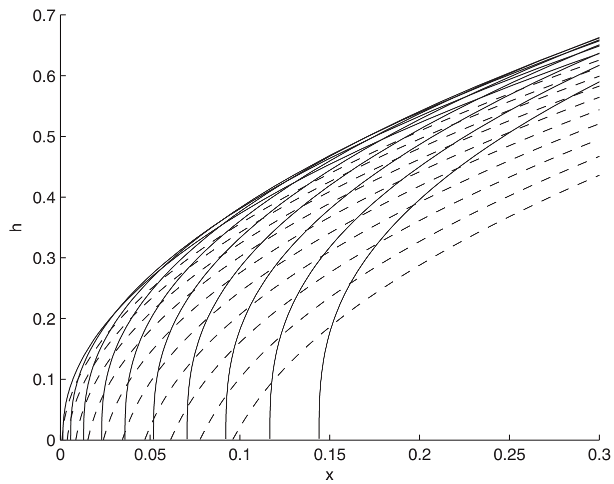

FIG. 4.4. *Plot of h vs. x for* (1.1) *with $m = 3$ and $q = 1$. The solution has been plotted at* 20 *equally spaced times between $t = -1$ and $t = 1$. The solid curves show the solution for $t < 0$ and the dashed curves for $t > 0$.*

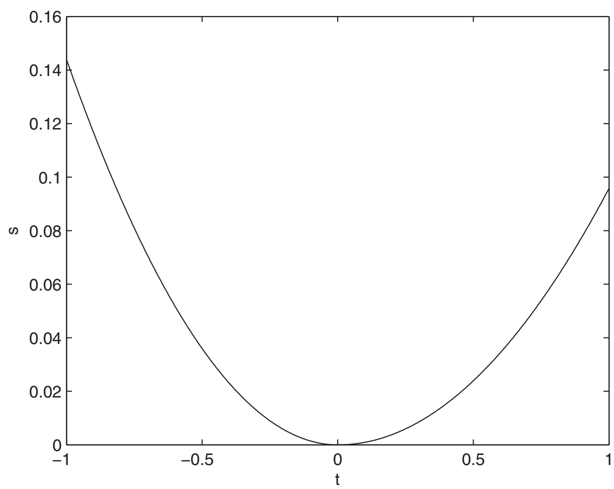

FIG. 4.5. *Plot of s vs. t for* (1.1) *with $m = 3$ and $q = 1$. The position of the interface as a function of time.*

$m = 2$ to model the movement of a species; see, for example, [4] and the bibliography therein. The derivation of such models is based on the assumption that the dispersion of organisms in a species is prevalent in regions that are densely populated. For simplicity the case of a population subject to a constant death rate is considered. This gives rise to (1.1) with $q = 1$. In this case the forms (2.4), (2.5), (3.3), and (3.4) reduce to

$$H \sim (3A)^{1/2}(\phi - A)^{1/2} \quad \text{as} \quad \phi \to A^+ \quad \text{for} \quad t < 0, \tag{4.6}$$

$$H \sim N\phi^{2/3} \quad \text{as} \quad \phi \to +\infty \quad \text{for} \quad t < 0, \tag{4.7}$$

$$H \sim \frac{2}{3B}(\phi - B) \quad \text{as} \quad \phi \to B^+ \quad \text{for} \quad t > 0, \tag{4.8}$$

$$H \sim Q\phi^{2/3} \quad \text{and} \quad \text{as} \quad \phi \to +\infty \quad \text{for} \quad t > 0. \tag{4.9}$$

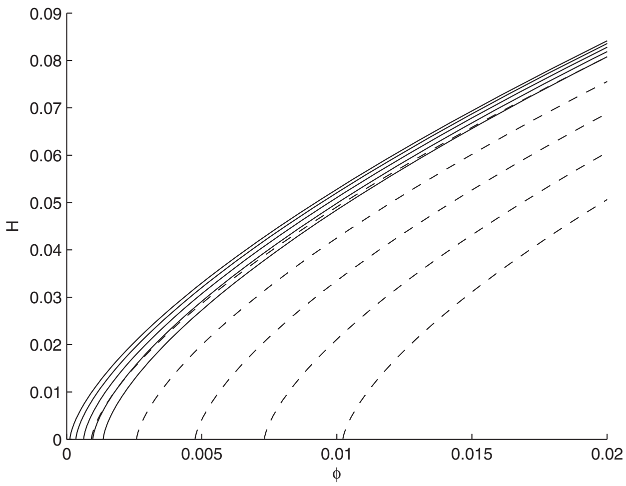

FIG. 4.6. *Plot of h vs. x for* (1.1) *with $m = 2$ and $q = 1$. The solution has been plotted at* 10 *equally spaced times between $t = -1$ and $t = 1$. The solid curves show the solution for $t < 0$ and the dashed curves for $t > 0$.*

Using the numerical algorithm outlined in section 4 and the default settings in ODE45, the following values are determined:

$$A \approx 0.001354, \quad B \approx 0.01022, \quad \text{and} \quad N = Q \approx 1.1445. \tag{4.10}$$

The convergence and uniqueness checks in section 4.1 were carried out. A plot of the solution is shown in Figure 4.6.

4.3. The solution for $m = 4$ and $q = 1$: Nonlinear heat conduction with absorption. For a further demonstration of the numerical scheme set up in section 4 a nonlinear heat conduction problem is considered. Previous authors have modeled the dissipation of heat in media where heat flow is due to radiation and the material is optically thick. For example, Zel'dovich and Raizer studied a problem in which the relevant model was (1.1) with $m = 4$ and $q = 1$ [23]. In this case the forms (2.4), (2.5), (3.3), and (3.4) reduce to

$$H \sim (10A)^{1/4}(\phi - A)^{1/4} \quad \text{as} \quad \phi \to A^+ \quad \text{for} \quad t < 0, \tag{4.11}$$

$$H \sim N\phi^{2/5} \quad \text{as} \quad \phi \to +\infty \quad \text{for} \quad t < 0, \tag{4.12}$$

$$H \sim \frac{2}{5B}(\phi - B) \quad \text{as} \quad \phi \to B^+ \quad \text{for} \quad t > 0, \tag{4.13}$$

$$H \sim Q\phi^{2/5} \quad \text{and} \quad \text{as} \quad \phi \to +\infty \quad \text{for} \quad t > 0. \tag{4.14}$$

Using the numerical algorithm outlined in section 4 and the default settings in ODE45, the following values are determined:

$$A \approx 0.3859, \quad B \approx 0.3405, \quad \text{and} \quad N = Q \approx 1.0101. \tag{4.15}$$

The convergence and uniqueness checks in section 4.1 were carried out. A plot of the solution is shown in Figure 4.7.

4.4. Others values of q. It has been shown that there is a similarity reduction to (1.1) capable of describing reversing behavior for parameters in the range $m > 0$,

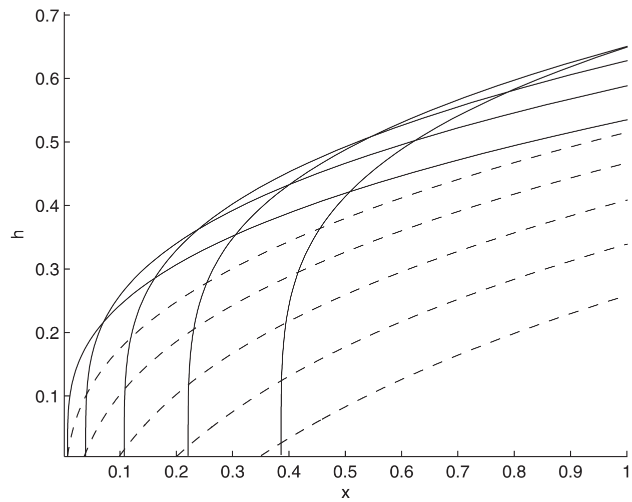

FIG. 4.7. *Plot of h vs. x for* (1.1) *with $m = 4$ and $q = 1$. The solution has been plotted at* 10 *equally spaced times between $t = -1$ and $t = 1$. The solid curves show the solution for $t < 0$ and the dashed curves for $t > 0$.*

$q > 0$, and $m-q > 0$. The work in sections 4.1, 4.2, and 4.3 has demonstrated that this reduction leads to meaningful reversing similarity solutions in the cases when $m = 3$ and $q = 1$, when $m = 2$ and $q = 1$, and when $m = 4$ and $q = 1$. Further numerical experimentation has indicated that it is also possible to find reversing solutions for all values of $m > 1$ and $q = 1$.

However, whether this is the case for all pairs of values m and q in the range $m > 0$, $q > 0$, and $m - q > 0$ remains an open question. Hence, this section discusses the practicalities of finding solutions to the ODEs (2.2) and (3.2) that satisfy the condition of continuity across $t = 0$ when $q \neq 1$.

To explore this question we first set $m = 3$ and allow q to increase above 1. Carrying out the numerical scheme outlined in section 4 has revealed that when $1 < q < q_3$ (where $q_3 \approx 1.1$) it is possible to find solutions to (2.2) and (3.2) with nonzero values of the parameters A and B and with $Q = N$ in the far field. Hence, this leads to solutions that describe reversing behavior of solutions to (1.1); see Figure 4.8. However, when $q > q_3$ it appears that the value of $B = 0$ and hence the only solution to the problem for $t > 0$ is the exact solution (2.3); see Figure 4.9. Furthermore, it is not possible to find a solution to (2.2) that satisfies the condition of continuity across $t = 0$ (i.e., it is not possible to find solutions with $Q = N$). Therefore, it is conjectured, but not proved, that the self-similar solution ceases to exist for $q > q_3$.

Other interesting behavior has been observed by holding $m = 3$ and decreasing q below 1. In this case it appears that the only solution to (2.2) is the exact solution (2.3). Furthermore, when carrying out the numerical scheme outlined in section 4 it has not been possible to find a solution to (3.2) with $Q = N$ and a nonzero value of B. Therefore, it is conjectured, but not proved, that the self-similar solution ceases to exist for $q < 1$.

Further numerical experimentation has been carried out for values of $m \neq 3$. It appears that the lack of solutions for $q < 1$ is generic for all values of m. It also appears that for any given value of m there is a critical value of q, q_m, say, such that

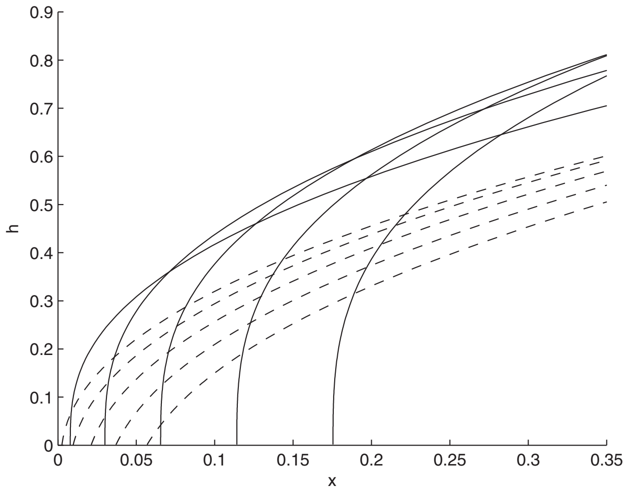

FIG. 4.8. *Plot of h vs. x for* (1.1) *with $m = 3$ and $q = 1.05$. The solution has been plotted at* 10 *equally spaced times between $t = -1$ and $t = 1$. The solid curves show the solution for $t < 0$ and the dashed curves for $t > 0$.*

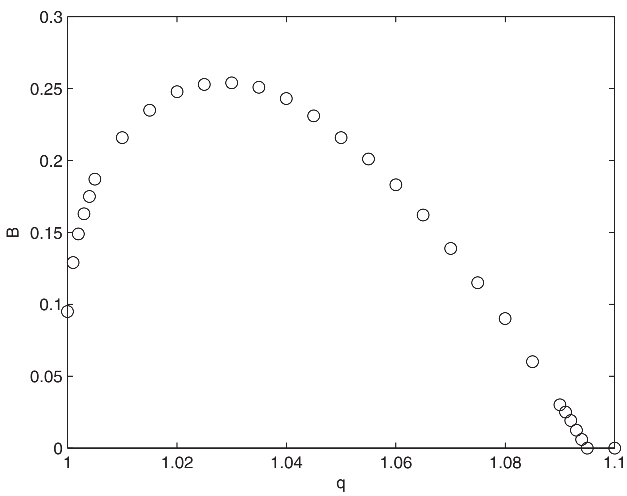

FIG. 4.9. *Plot of B vs. q for $m = 3$.*

if $q > q_m$, then the self-similar solution does not exist. These conjectures are based on numerical evidence and as such do not constitute rigorous proofs.

5. Discussion and conclusions. This study has been concerned with solutions to the family of equations (1.1) with $m > 0$, $q > 0$, and $m - q > 0$. These restrictions were placed on the exponents m and q to ensure that the interfaces of the solution travel with a finite velocity, that receding interfaces can exist, that an advancing interface moves due to diffusion, and that a receding interface moves due to absorption. In particular, solutions to these equations local to the interface and local to the reversing time which give the generic behavior of a reversing interface have been examined. By doing this, an analytical explanation of how an interface reversal occurs has been given. Self-similar reductions were made to the governing equation (1.1) to derive ODEs for the dependent self-similar variable H as a function of ϕ. The analysis was split into two parts, one for the solution prior to the reversing time and the other for the solution after the reversing time. In each case the asymptotic behavior of solutions as $H \to 0^+$ and as $\phi \to +\infty$ were studied. A numerical algorithm which made use of the determined asymptotic behaviors was then put forward as a method for furnishing solutions to (1.1) local to $x = 0$ and $t = 0$. This algorithm was then demonstrated using the examples of $m = 3$ and $q = 1$, $m = 2$ and $q = 1$, and $m = 4$ and $q = 1$. Finally, some remarks have been made on the existence and practicalities of finding solutions for any pair of values of m and q in the range under consideration.

In section 2 the solution prior to the reversing time was studied. By making a self-similar reduction to the governing equation a second order nonlinear ODE was derived for the dependent self-similar variable H. By studying the asymptotic behaviors of this ODE as $\phi \to A^+$ and as $\phi \to +\infty$ it was demonstrated that close to the interface the solution takes the form

$$h(x,t) \sim \left(\frac{m+q}{2q}Am\right)^{1/m} (-t)^{(m-q)/2mq} \left(x - A(-t)^{(m+q)/2q}\right)^{1/m}$$
$$\text{as} \quad x \to A(-t)^{(m+q)/2q}. \tag{5.1}$$

For large negative time this has the form of a traveling wave with velocity proportional to $t^{(m-q)/2q}$. From (5.1) it can also be seen that h increases proportional to $x^{1/m}$ close to this interface. It was also shown that the far field behavior of the solution is

$$h(x,t) \sim Nx^{2/(m+q)} \quad \text{as} \quad x(\pm t)^{-(m+q)/2q} \to +\infty. \tag{5.2}$$

Hence, local to the reversing time and close to, but away from, the interface, the solution is stationary with behavior proportional to $x^{2/(m+q)}$.

In section 3 the solution after the reversing time was studied. In a similar fashion to section 2 a self-similar reduction was made to the governing equation. By studying the asymptotic behavior of solutions as $\phi \to B^+$ and as $\phi \to +\infty$ it was shown that close to the interface

$$h(x,t) \sim \left(\frac{m+q}{2q^2}B\right)^{-1/q} t^{-(m-q)/2q^2} \left(x - Bt^{(m+q)/2q}\right)^{1/q}$$
$$\text{as} \quad x \to Bt^{(m+q)/2q}. \tag{5.3}$$

It can again be seen that for large positive time this takes the form of a traveling wave with velocity proportional to $t^{(m-q)/2q}$. Also, the concentration h close to the interface increases proportional to $x^{1/q}$. It was shown that the far field necessarily took the same form as the solution prior to the reversing time, that is, proportional to $x^{2/(m+q)}$. Furthermore, so that the concentration h was continuous as t passed through zero, the constant of proportionality was chosen so that the far field behavior was the same as that prior to the turning time.

In section 4 knowledge of the aforementioned asymptotic behaviors was used in order to formulate a numerical algorithm to furnish meaningful solutions to (1.1). This algorithm was demonstrated using the examples of (1.1) with $m = 3$ and $q = 1$, $m = 2$ and $q = 1$, and $m = 4$ and $q = 1$. In section 4.4 the practicalities of finding solutions for all pairs of values of m and q in the range of interest were considered. Based on numerical evidence it has been conjectured that the self-similar solution proposed in this study does not exist when $q < 1$. Furthermore, for each value of m there exists a q, q_m, say, such that for $q > q_m$ the self-solution does not exist.

Attention is also drawn to another notable result from the analysis that applies to any equation of the form (1.1) with $q = 1$. It appears that there is only one form of reversing behavior for each value of m. The authors note that the asymptotic forms (2.4), (2.5), (3.3), and (3.9) depend only on the values of m, A, B, N, Q, and Q_2. Numerical evidence suggests that all these values are uniquely determined for any given equation in the family (1.1) with $q = 1$. In other words, so long as the solution exhibits a reversing of an interface at some time in its evolution, the solution local to this interface and local to the reversing time is generic.

Finally, the authors discuss the implication of the results in the context of the physical problems: the spreading of a viscous film under gravity and subject to evaporation, the dispersion of a population with constant death rate, and nonlinear heat conduction with absorption. In each of these cases it has been shown that the behavior near the interface changes dramatically as t passes through zero. In particular, the slope at the interface is infinite before $t = 0$ and finite after $t = 0$. Furthermore, while the position of the interface is quadratic in time, the quadratic coefficient changes as t passes through zero. In the case of a spreading viscous film, the film thickness near an advancing interface is proportional to $(x - s(t))^{1/3}$ and the behavior of the solution is largely dominated by diffusive effects. However, near a receding interface the film thickness is proportional to $(x - s(t))$ and the behavior is largely dominated by evaporation. In the case of a dispersing population, the population density near an advancing interface is proportional to $(x-s(t))^{1/2}$ and the behavior is largely dictated by the effect of population pressure. However, near a receding interface the population density tends to zero linearly and the behavior is controlled by the constant death rate of the population. In the case of nonlinear heat conduction, the temperature near an advancing interface is proportional to $(x-s(t))^{1/4}$ and the behavior near the interface is dominated by diffusive effects. However, near a receding interface the temperature is proportional to $(x - s(t))$ and the behavior is largely dictated by the absorption of the heat into the medium. Any numerical scheme that attempts to solve (1.1) must capture these nontrivial changes accurately. Therefore, (5.1), (5.2), and (5.3) provide a way of validating such numerical scheme's accuracy near the interface.

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