Integrated design of discrete-time controller for an active suspension system

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***Abstract*— A novel approach to solve a stabilization problem of an active suspension system using a quarter car model is presented. We apply a combination of our results for the framework of the approximate based direct discrete-time design and the Euler based discrete-time backstepping technique. This stabilization problem is very interesting since utilizing a simple quadratic Lyapunov function brings the system into a LaSalle type stability, which makes the design more complicated. To handle this problem, we use our result on changing supply rates lemma for LaSalle type stability condition, to construct a composite Lyapunov function that can be used for the design within our framework.**

I. INTRODUCTION

Most control systems nowadays are sampled-data in nature. A controller is usually implemented digitally and it is inter-connected with a continuous-time plant via ADC and DAC. In this paper, we study the problem of stabilizing an active suspension system, which is used to enable a car to run smoothly on a rough road for comfortable driving. Presently, active suspension systems are controlled using a hydraulic controller. In view of space limitation in a vehicle, it is most appropriate to use digital device to control the active suspension system, as it requires much less space. Since the active suspension module itself is a mechanical - therefore analog - plant, designing a digital controller for this system is a sampled-data system design.

Recently, a general uni£ed framework for controller design based on approximate discrete-time models was presented in [10] and further generalized in [7] for the input to state stabilization problem. In particular, the results provide suf£cient conditions for the continuous-time plant model, the controller and the approximate discrete-time model, to guarantee that the controller input-to-state stabilizes the exact discrete-time plant model, provided it stabilizes the approximate discrete-time plant model.

We design a discrete-time controller to asymptotically stabilize the active suspension system, using the Euler based backstepping technique [9]. Backstepping is a popular technique in nonlinear control design (see [4]). It is then shown that the Euler based discrete-time controller outperforms the emulation controller. This active suspension design problem is very interesting and motivating since the system enjoys a LaSalle type stability when using a simple quadratic Lyapunov function. In [3], where continuous-time stabilization for the same system was considered, stability analysis was done using LaSalle's invariance principle. Unfortunately, LaSalle's invariance principle is in most cases not applicable when approximate based discrete-time design is used, since semiglobal type of stability is usually achieved. In this situation, we apply our result from [6] to construct a composite Lyapunov function that can be used to characterize stability property of the system.

II. PRELIMINARIES

The set of real and natural numbers (including 0) are denoted respectively by $\mathbb{R}$ and $\mathbb{N}$. $\mathcal{SN}$ denotes the class of smooth nondecreasing functions $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which satisfy $q(t) > 0$ for all $t > 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{G}$ if it is continuous, nondecreasing and zero at zero. It is of class $\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing; and it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Functions of class $\mathcal{K}_\infty$ are invertible. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Given two functions $\alpha(\cdot)$ and $\gamma(\cdot)$, we denote their composition and multiplication respectively as $\alpha \circ \gamma(\cdot)$ and $\alpha(\cdot) \cdot \gamma(\cdot)$.

We consider a parameterized family of discrete-time nonlinear systems of the following form:

$$x(k+1) = F_T(x(k), u(k)), \qquad y(k) = h(x(k)) \ , \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ are respectively the state, input and output of the system. Note that the input u can be a control signal or an exogenous disturbance. It is assumed that F_T is well de£ned for all x, u and suf£ciently small T, $F_T(0, 0) = 0$ for all T for which F_T is de£ned, $h(0) = 0$ and F_T and h are continuous. $T > 0$ is the sampling period, which parameterizes the system and can be arbitrarily assigned. The following de£nitions are used to state results presented later in this section.

De£nition 2.1: The system (1) is semiglobally practically input-output to state stable (SP-IOSS), if there exist functions $\underline{\alpha}$, $\overline{\alpha}$, $\alpha \in \mathcal{K}_\infty$, and λ, $\sigma \in \mathcal{G}$, and for any triple of strictly positive real numbers $(\Delta_x, \Delta_u, \nu)$, there exists $T^* > 0$ and for all $T \in (0, T^*)$ there exists a smooth function $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ the following holds:

$$\underline{\alpha}(|x|) \leq V_T(x) \leq \overline{\alpha}(|x|) \tag{2}$$

$$\begin{aligned} V_T(F_T(x, u)) - V_T(x) \leq & -T\alpha(|x|) + T\lambda(|y|) \\ & + T\sigma(|u|) + T\nu \ . \end{aligned} \tag{3}$$

The function V_T is called a SP-IOSS Lyapunov function. If the system is SP-IOSS with $\lambda = 0$, we say that the system is semiglobally practically input to state stable (SP-ISS) and V_T is called a SP-ISS Lyapunov function. If $\lambda = 0$ and the system (1) is an input-free system ($\sigma = 0$), the system is semiglobally practically asymptotically stable (SP-AS) and V_T is called a SP-AS Lyapunov function. Moreover, for SP-ISS, if the argument of $\alpha(\cdot)$ is the norm of the output y, which consists of only partial states, we have semiglobal practical quasi ISS (SP-qISS). ■

De£nition 2.2: [9] Let $\hat{T} > 0$ be given and for each $T \in (0, \hat{T})$ let the functions $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $u_T : \mathbb{R}^n \to \mathbb{R}$ be de£ned. We say that the pair (u_T, V_T) is a semiglobally practically asymptotically (SPA) stabilizing pair for F_T if there exist $\underline{\alpha}$, $\overline{\alpha}$, $\alpha \in \mathcal{K}_\infty$, such that for any pair of strictly positive real numbers (Δ, ν) there exists a triple of strictly positive real numbers (T^*, L, M), with $T^* \leq \hat{T}$, such that for all $x, z \in \mathbb{R}^n$ with $\max\{|x|, |z|\} \leq \Delta$, and $T \in (0, T^*)$ we have:

$$\underline{\alpha}(|x|) \leq V_T(x) \leq \overline{\alpha}(|x|) \tag{4}$$

$$V_T(F_T(x, u)) - V_T(x) \leq -T\alpha(|x|) + T\nu. \tag{5}$$

$$|V_T(x) - V_T(z)| \leq L\,|x - z| \tag{6}$$

$$|u_T(x)| \leq M\ . \tag{7}$$

■

III. Design tools

A. Framework for approximate based direct discrete-time design

In this subsection we present a result from [7] on input to state stabilization via approximate discrete-time models. Consider a continuous-time nonlinear plant

$$\dot{x}(t) = f(x(t), u(t), w(t)), \qquad y(t) = h(x(t))\ , \tag{8}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^p$ and $y \in \mathbb{R}^l$ are respectively the state, control input, disturbance and output.

We assume that for any given x_0, $u(\cdot)$ and $w(\cdot)$ the differential equation in (8) has a unique solution de£ned on its maximal interval of existence $[0, t_{\max})$. This may be guaranteed, for instance, by requiring f in (8) to be locally Lipschitz. The control is taken to be a piecewise constant signal $u(t) = u(kT) =: u(k),\ \forall t \in [kT, (k+1)T),\ k \in \mathbb{N}$, where $T > 0$ is the sampling period, and we suppose that the disturbance $w(\cdot)$ is constant during sampling intervals, that is $w(t) = w(k), \forall t \in [kT, (k+1)T)$. We assume that some combination (output) or all of the states ($x(k) := x(kT)$) are available at sampling instant $kT, k \in \mathbb{N}$. The exact discrete-time model for the plant (8), which describes the plant behavior at sampling instants kT, is obtained by integrating the initial value problem

$$\dot{x}(t) = f(x(t), u(k), w(t))\ , \tag{9}$$

with given $w(k)$, $u(k)$ and $x_0 = x(k)$, over the sampling interval $[kT, (k+1)T]$. If we denote by $x(t)$ the solution of the initial value problem (9) at time t with given $x_0 = x(k)$, $u(k)$ and $w(k)$, then the exact discrete-time model of (8) can be written as:

$$\begin{aligned} x(k+1) &= x(k) + \int_{kT}^{(k+1)T} f(x(\tau), u(k), w(k))d\tau \\ &=: F_T^e(x(k), u(k), w(k))\ . \end{aligned} \tag{10}$$

Since F_T^e is not known in most cases (see [7]), we use an approximate discrete-time model of the plant

$$x(k+1) = F_T^a(x(k), u(k), w(k))\ . \tag{11}$$

to design a discrete-time controller for the original plant (8). For instance, the Euler approximate model is $x(k+1) = x(k) + Tf(x(k), u(k), w(k))$.

We consider a family of dynamic feedback controllers

$$\begin{aligned} z(k+1) &= G_T(x(k), z(k)) \\ u(k) &= u_T(x(k), z(k))\ , \end{aligned} \tag{12}$$

where $z \in \mathbb{R}^{n_z}$. We emphasize that if the controller (12) input to state stabilizes the approximate model (11) for all small T, this does not guarantee that the same controller would input to state stabilize the exact model (10) for all small T (see [1], [2], [10]). The following result provides a framework for controller design via approximate discrete-time models.

Theorem 3.1: [7] Suppose that there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$, and for any strictly positive real numbers $(\Delta_1, \Delta_2, \Delta_3, \nu)$ there exist $\varrho \in \mathcal{K}_\infty$, strictly positive real numbers T^*, L, M such that for all $T \in (0, T^*)$ there exists a function $V_T : \mathbb{R}^{n_x+n_z} \to \mathbb{R}_{\geq 0}$ such that for all $|(x, z)| \leq \Delta_1$, $|u| \leq \Delta_2$, $|w| \leq \Delta_3$, $T \in (0, T^*)$ we have: 1. SP-ISS Lyapunov conditions for closed-loop approximate; 2. consistency between F_T^a and F_T^e; 3. uniform local boundedness of u_T (see [7] for detail de£nitions). Then, there exists $\beta \in \mathcal{KL}, \gamma \in \mathcal{G}$ such that for any strictly positive real numbers $(\widetilde{\Delta}_1, \widetilde{\Delta}_2, \widetilde{\nu})$ there exists $\widetilde{T} > 0$ such that for all $|(x(0), z(0))| \leq \widetilde{\Delta}_1$, $\|w\|_\infty \leq \widetilde{\Delta}_2$ and $T \in (0, \widetilde{T})$ the solutions of (10), (12) satisfy SP-ISS of closed-loop exact. ■

We emphasize that the consistency condition in Theorem 3.1 is checkable although F_T^e is not known in general. De£nitions and lemmas that give suf£cient conditions for consistency condition are stated in [7].

B. Euler based discrete-time backstepping design

In this subsection, a result from [9] is cited. The Euler model is used, since it preserves the strict feedback structure of the plant that is needed for a backstepping design and it satis£es the consistency property required by Theorem 3.1.

Consider a continuous-time plant of the strict feedback form:

$$\dot{x} = f(x) + g(x)\xi \tag{13}$$

$$\dot{\xi} = u\ . \tag{14}$$

The Euler approximate model of (13),(14) is:

$$x(k+1) = x(k) + T(f(x(k)) + g(x(k))\xi(k)) \qquad (15)$$

$$\xi(k+1) = \xi(k) + Tu(k)\ . \qquad (16)$$

Under certain properties and conditions (see [9]), there exists a SPA stabilizing pair (u_T, V_T) for the Euler model (15),(16). In particular, we can take:

$$u_T = -c(\xi - \alpha_T(x)) - \frac{\widetilde{\Delta W}_T}{T} + \frac{\Delta\alpha_T}{T}\ , \qquad (17)$$

where $c > 0$ is arbitrary, $\xi = \alpha_T(x)$ asymptotically stabilizes (13) and

$$\Delta\alpha_T = \alpha_T(x + T(f + g\xi)) - \alpha_T(x) \qquad (18)$$

$$\widetilde{\Delta W}_T = \begin{cases} \frac{\overline{\Delta W}_T}{(\xi - \alpha_T(x))}, & \xi \neq \alpha_T(x) \\ T\frac{\partial W_T}{\partial x}(x + T(f + g\xi))g, & \xi = \alpha_T(x) \end{cases} \qquad (19)$$

$$\overline{\Delta W}_T = W_T(x(k+1)) - W_T(x + T(f + g\alpha_T)) \qquad (20)$$

and the Lyapunov function $V_T = W_T + \frac{1}{2}(\xi - \alpha_T(x))^2$.

C. A LaSalle criterion for SP-ISS

The result from [6] on changing supply rates for SP-ISS discrete-time systems, provides a recipe for constructing a composite Lyapunov function to solve LaSalle type stability problem in sampled-data system. Consider the system (1). Using Corollary 5.1 of [6], we show that if the functions $V_{1T} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $V_{2T} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are respectively a SP-qISS Lyapunov function and a SP-IOSS Lyapunov function of the system (1), and

$$\limsup_{s\to+\infty} \frac{\lambda_2(s)}{\alpha_1(s)} < +\infty, \qquad (21)$$

Then, the function $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that satis£es

$$V_T = V_{1T} + \rho(V_{2T})\ . \qquad (22)$$

where $\rho(s) := \int_0^s q(\tau)d\tau$, with $q \in \mathcal{SN}$ and $\rho \in \mathcal{K}_\infty$, is a SP-ISS Lyapunov function of the system (1).

IV. Control of an active suspension system

A. Car suspension system modeling

We use the quarter car model as the mathematical description of the suspension system, following the model used in [3]. The schematic diagram of the model is shown in Figure 1. In this model, the suspension actuator is taken to be a force actuator acting between the car body (the sprung mass) and the axle of the car. The tire is an ideal, undamped spring between the axle and the ground. Finally, the axle and wheel assemblies are represented as a mass (the unsprung mass) connected to the ground via the tire spring. As shown in Figure 1, the suspension force also reacts against the unsprung mass.

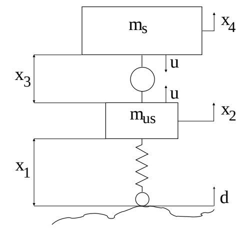

Fig. 1. The quarter car suspension model

A linear time invariant dynamic model of the system is represented as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 - d, & \dot{x}_3 &= -x_2 + x_4 \\ \dot{x}_2 &= -\omega^2 x_1 + \rho u, & \dot{x}_4 &= -u \end{aligned} \qquad (23)$$

where x_1 - tire de¤ection (m), x_2 - unsprung mass velocity (m/sec), x_3 - suspension de¤ection (m) and x_4 - sprung mass velocity (m/sec). The parameter ω is the unsprung mass natural frequency, ρ is the sprung to unsprung mass ratio and assume that the travel limit of the suspension is $\pm D$. In other words, as long as the suspension de¤ection x_3 satis£es $-D < x_3 < D$, the suspension will not bottom out. Following [3], we use the parameters $\omega = 2\pi \cdot 10$ rad/sec, $\rho := m_s/m_{us}$ =10, D =0.1 m.

B. Discrete-time backstepping controller design

To obtain a strict feedback form, the state equations are reordered using the following diffeomorphism:

$$\begin{aligned} z_1 &= x_1 + \tfrac{\rho}{\rho+1}x_3, & z_3 &= x_3 \\ z_2 &= \tfrac{1}{\rho+1}x_2 + \tfrac{\rho}{\rho+1}x_4, & z_4 &= -x_2 + x_4 \end{aligned}$$

The model is then rewritten in the following form

$$\dot{z}_1 = z_2 - d \qquad (24)$$

$$\dot{z}_2 = -\frac{\omega^2}{\rho+1}z_1 + \frac{\rho\omega^2}{(\rho+1)^2}z_3 \qquad (25)$$

$$\dot{z}_3 = z_4 \qquad (26)$$

$$\dot{z}_4 = \omega^2 z_1 - \frac{\rho\omega^2}{\rho+1}z_3 - (1+\rho)u = \tilde{u} \qquad (27)$$

The Euler model of the system in a strict feedback form is written as follow:

$$z_1(k+1) = z_1(k) + T(z_2(k) - d) \qquad (28)$$

$$z_2(k+1) = z_2(k) + T(\frac{-\omega^2 z_1(k)}{\rho+1} + \frac{\rho\omega^2 z_3(k)}{(\rho+1)^2}) \qquad (29)$$

$$z_3(k+1) = z_3(k) + Tz_4(k) \qquad (30)$$

$$\begin{aligned} z_4(k+1) = z_4(k) + T(\omega^2 z_1(k) - \frac{\rho\omega^2 z_3(k)}{\rho+1} \\ - (1+\rho)u(k)) = z_4(k) + T\tilde{u}(k) \end{aligned} \qquad (31)$$

In the design, the disturbance d is taken to be zero, which is a reasonable approach since the disturbances affecting the system are nearly impulsive and thus correlate to nonzero initial conditions. Therefore, the problem is simpli£ed to an asymptotic stabilization problem. We follow similar design steps to those done in [3], applying the Euler based backstepping design [9] as cited in Subsection 3.2. Due to space limitation, some trivial steps are omitted.

Step 1: From the continuous-time model, it can be seen that if $z_3 \equiv 0$, then subsystem (28), (29) is marginally stable. We design a virtual feedback control law $z_{3d}(z_1, z_2)$ which is bounded between $-D$ and D and renders the origin of the closed-loop (z_1, z_2) subsystem SP-AS. A control that satis£es this is

$$z_{3d} = -D \tanh(\frac{k_1 z_2}{D}), \quad k_1 > 0 \ . \tag{32}$$

Unfortunately, the candidate Lyapunov function

$$V_{0T_1}(z_1, z_2) = \frac{1}{2}\frac{\omega^2}{\rho+1} z_1^2 + \frac{1}{2} z_2^2 \ , \tag{33}$$

which was used in the continuous-time design [3], gives

$$\Delta V_{0T_1} \leq -TMz_2 \tanh(z_2) + T\nu_{01} \tag{34}$$

with $z_3 = z_{3d}$, which is negative semide£nite with small offset $\nu_{01} > 0$.

While we can apply LaSalle Invariance Principle for the continuous-time case, we cannot do the same for the sampled-data design when semiglobal stability condition occurs. The Euler based backstepping [9] we use does not facilitate this condition, and the candidate Lyapunov function V_{0T_1} does not satisfy the £rst condition of Theorem 3.1. To solve this problem, we apply Corollary 5.1 of [6] to construct a SP-AS Lyapunov function for subsystem (28), (29). It has been shown earlier that $V_{0T_1}(z_1, z_2)$ is in fact a SP-qISS Lyapunov function for the subsystem. The surface plots of V_{0T_1} and ΔV_{0T_1} are shown in Figure 2.

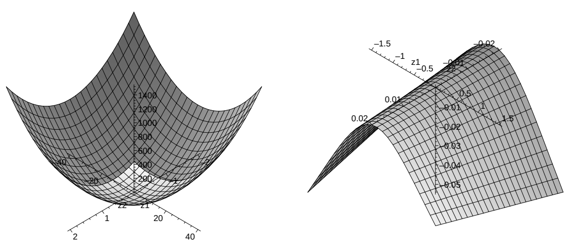

Fig. 2. Surface plots for V_{0T_1} (l) and ΔV_{0T_1} (r), with $T = 0.001$ sec.

To show that the subsystem is SP-AS, we introduce another function

$$V_{0T_2}(z_1, z_2) = \frac{1}{2}\frac{\omega^2}{\rho+1} z_1^2 + \frac{1}{2} z_2^2 + \varepsilon \frac{z_1 z_2}{(1+z_1^2)^{3/4}} \ . \tag{35}$$

For small sampling period $T > 0$ and small $\varepsilon > 0$, the Lyapunov difference of ΔV_{0T_2} satis£es

$$\begin{aligned} \Delta V_{0T_2} \leq & -TMz_2 \tanh(z_2) - TM_1 z_1^2 \\ & + TM_3(z_2^2 + \tanh^2(z_2)) + T\nu_{02} \end{aligned} \tag{36}$$

with some $M, M_1, M_3 > 0$. Hence, V_{0T_2} is a SP-IOSS Lyapunov function for the £rst two subsystems.

From (34) and (36), it is obvious that the condition (21) is satis£ed, and hence all conditions of Corollary 5.1 of [6] holds. Hence, we can conclude that for some $\rho \in \mathcal{K}_\infty$, the function V_{0T_a} that satis£es

$$V_{0T_a} = V_{0T_1} + \rho(V_{0T_2}) \tag{37}$$

is a SP-AS Lyapunov function for the £rst two subsystems. The surface plots of V_{0T_2} and ΔV_{0T_2} are shown in Figure 3. Suppose we are given a set of initial conditions, such that the SP-AS property of the subsystem (28), (29) is guaranteed with $T = 0.001$ sec. For a £x $\varepsilon = 0.1$, choosing an appropriate $\rho \in \mathcal{K}_\infty$, then we can use formula (37) to combine Figure 2 and Figure 3 after scaling V_{0T_2} with the function ρ, to show the SP-AS Lyapunov surface V_{0T_a} and the SP-AS difference ΔV_{0T_a} of the subsystem (28), (29), for the given set of initial conditions. Choosing $\rho(\cdot) = \mathrm{Id}(\cdot)$ results in a Lyapunov function

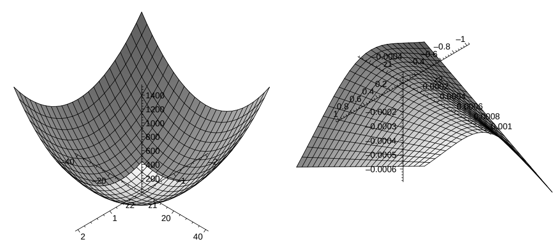

Fig. 3. Surface plots for V_{0T_2} (l) and ΔV_{0T_2} (r), with $T = 0.001$ sec and $\varepsilon = 0.1$.

$$V_{0T_a} = V_{0T_1} + V_{0T_2} \tag{38}$$

for the subsystem (28), (29), and we can immediately see that with $z_3 = z_{3d}$ the Lyapunov difference $\Delta V_{0T_a} = \Delta V_{0T_1} + \Delta V_{0T_2}$ is negative de£nite and hence the subsystem (28), (29) is SP-AS.

Remark 4.1: We emphasize that choosing $\rho = \mathrm{Id}$ in our case is possible since we are dealing with a semiglobal practical property. The V_{0T_a} obtained does not hold globally, because of the saturation coming from the tanh function in z_{3d}. We also have to choose a small ε to guarantee that ΔV_{0T_a} does not become positive for quite large z_2. ■

To continue the design procedure, for the purpose of simpler computation, we choose to use

$$V_{0T} = \frac{1}{2} V_{0T_a} = \frac{1}{2}(V_{0T_1} + V_{0T_2}) \ , \tag{39}$$

since it is allowable to scale a Lyapunov function with a constant. We also £x $\varepsilon = 0.1$ in this design. It is obvious that V_{0T} satis£es **the £rst condition** of Theorem 3.1.

Step 2: We de£ne $\xi_1 = z_3 - z_{3d}(z_2)$ and denote $\xi_2 := \dot{\xi}_1$ to obtain the following third order Euler model:

$$\begin{aligned} z_1(k+1) &= z_1 + Tz_2 \\ z_2(k+1) &= z_2 + T\left(-\frac{\omega^2}{\rho+1}z_1 + \frac{\rho\omega^2}{(\rho+1)^2}(z_{3d}+\xi_1)\right) \\ \xi_1(k+1) &= \xi_1 + T\xi_2 \ . \end{aligned} \quad (40)$$

Using a candidate Lyapunov function

$$V_{1T}(z_1,z_2,\xi_1) = V_{0T}(z_1,z_2) + k_2\frac{\xi_1^2}{2}, \quad k_2>0 \quad (41)$$

and choosing the stabilizing controller as

$$\xi_{2d} = -\frac{1}{k_2}\frac{\rho\omega^2}{(\rho+1)^2}(z_2 + \frac{\varepsilon z_1}{(1+(z_1+Tz_2)^2)^{3/4}}) - k_3\xi_1,$$

with $k_3 > 0$, we have

$$\Delta V_{1T} = \Delta V_{0T}|_{z_{3d}} - Tk_2k_3\xi_1^2 + T\nu_1 \ , \quad (42)$$

which is negative de£nite with a small offset $\nu_1 > 0$. Hence, the equilibrium $(z_1, z_2, \xi_1) = (0, 0, 0)$ is SP-AS. Since $z_{3d}(0) = 0$ we can conclude that the origin $(z_1, z_2, z_3) = (0, 0, 0)$ is also SP-AS.

Step 3: Backstepping ξ_2 through an integrator results in the dynamical system, whose Euler model can then be written as follow:

$$\begin{aligned} z_1(k+1) &= z_1 + Tz_2 \\ z_2(k+1) &= z_2 + T(-\frac{\omega^2}{\rho+1}z_1 + \frac{\rho\omega^2}{(\rho+1)^2}(z_{3d}+\xi_1)) \\ \xi_1(k+1) &= \xi_1 + T\xi_2 \\ \xi_2(k+1) &= \xi_2 + T\tilde{\tilde{u}} \ . \end{aligned} \quad (43)$$

At this step, we consider a candidate Lyapunov function

$$V_{2T}(z_1,z_2,\xi_1,\xi_2) = V_{1T}(z_1,z_2,\xi_1) + \frac{k_4}{2}(\xi_2-\xi_{2d})^2 \ ,$$

with $k_4 > 0$. We apply the formula (17) to obtain $\tilde{\tilde{u}}_T$ using the following terms:

$$\begin{aligned} \alpha_T &= \frac{-\rho\omega^2}{k_2(\rho+1)^2}(z_2 + \frac{\varepsilon z_1}{(1+(z_1+Tz_2)^2)^{3/4}}) - k_3\xi_1 \\ W_T &= \frac{k_2}{2k_4}\xi_1^2 \ , \end{aligned}$$

(it turns out that $\alpha_T := \xi_{2d}$) and get

$$\begin{aligned} \frac{\Delta\alpha_T}{T} &= \frac{\xi_{2d}(k+1)-\xi_{2d}(k)}{T} \\ &= -\frac{1}{k_2}\frac{\rho\omega^2}{(\rho+1)^2}\left(\zeta_2 + \frac{\varepsilon z_1}{(1+(z_1+2Tz_2+T^2\zeta_2)^2)^{3/4}}\right) \\ &\quad - \frac{1}{k_2T}\frac{\rho\omega^2}{(\rho+1)^2}\varepsilon z_1\left(\frac{1}{(1+(z_1+2Tz_2+T^2\zeta_2)^2)^{3/4}}\right. \\ &\quad \left. - \frac{1}{(1+(z_1+Tz_2)^2)^{3/4}}\right) - k_3z_4 \\ &\quad - \frac{1}{T}k_3D(\tanh(z_2+T\zeta_2) - \tanh(z_2)) \ , \end{aligned} \quad (44)$$

where $\zeta_2 := -\frac{\omega^2}{\rho+1}z_1 + \frac{\rho\omega^2}{(\rho+1)^2}z_3$, and

$$\begin{aligned} \overline{\Delta W}_T &= V_{1T}^+(\xi_1+T\xi_2) - V_{1T}^+(\xi_1+T\xi_{2d}) \\ &= \frac{k_2}{2k_4}(2T\xi_1\xi_2 - 2T\xi_1\xi_{2d} + T^2\xi_2^2 - T^2\xi_{2d}^2). \end{aligned}$$

Moreover, using (19) we have

$$\frac{\widetilde{\Delta W}_T}{T} = \frac{k_2}{k_4}\xi_1 + \frac{k_2T}{2k_4}(\xi_2+\xi_{2d}) \ . \quad (45)$$

Hence, we obtain $\tilde{\tilde{u}} = \tilde{\tilde{u}}_T$ by substituting (44),(45) to

$$\tilde{\tilde{u}}_T = -\frac{k_5}{k_4}(\xi_2-\xi_{2d}) - \frac{\widetilde{\Delta W}_T}{T} + \frac{\Delta\alpha_T}{T} \ , \quad k_5>0. \quad (46)$$

It can be shown that implementing $\tilde{\tilde{u}}_T$ to the system results in ΔV_{2T} negative de£nite with small offset $\nu_2 > 0$. This means that the equilibrium $(z_1, z_2, \xi_1, \xi_2) = (0, 0, 0, 0)$ is SP-AS. Since $z_{3d}(0) = 0$, then the origin $(z_1, z_2, z_3, z_4) = (0, 0, 0, 0)$ is also SP-AS.

We have seen earlier that V_{1T} satis£es the SP-ISS (in this case SP-AS) Lyapunov condition of Theorem 3.1. It is then obvious that with V_{2T} that **the £rst condition** of Theorem 3.1 still holds.

Step 4: Following exactly as in the continuous-time design, the resulting control law $u = u_T$ that SPA stabilizes the Euler model (28)-(31) has form

$$u_T = \frac{1}{\rho+1}(-\tilde{\tilde{u}}_T - \Delta\zeta_{3d} + \omega^2x_1) \quad (47)$$

with $\Delta\zeta_{3d} := \ddot{z}_{3d}$. Finally, by substituting the appropriate terms, we have u_T as a nonlinear control law parameterized by the sampling period T and £ve positive tuning parameters k_1, k_2, k_3, k_4 and k_5.

Expanding u_T in series representation, we can show that u_T satis£es **the third condition** of Theorem 3.1. Since all conditions of Theorem 3.1 hold, we can guarantee that u_T SPA stabilizes the closed-loop approximate model, and also stabilize the closed-loop exact model. We further use results from [8], to conclude the SP-AS for the sampled data system (23), (47).

C. Comparing the Euler-based controller with the Emulation controller

We have designed a discrete-time backstepping controller (47) that SPA stabilizes the active suspension system. Now, we implement the controller (47), and observe the performance of the closed-loop sampled-data system with the designed controller, and compare it with a controller that has form:

$$u = \frac{1}{\rho+1}(-\tilde{\tilde{u}} - \Delta\zeta_{3d} + \omega^2x_1) \quad (48)$$

where $\tilde{\tilde{u}} = \Delta\xi_{2d} - \frac{k_2\xi_1}{k_4} - \frac{k_5}{k_4}(\xi_2 - \xi_{2d})$, with $\Delta\xi_{2d} := \dot{\xi}_{2d}$. The controller (48) is obtained via emulation design, by holding the continuous control constant during every sampling period (using zero order hold). By applying Corollary 5.1 of [5], we can show that the discrete-time emulation controller (48) also SPA stabilizes the continuous-time plant (23).

We study the condition when there are small offsets to the initial states, in other words, when allowing nonzero initial states. We observe the responses of the system to bumps of different heights and compare the performance of each controller. The shape of the isolated bump [1] is chosen to be in the form that gives rise to the following velocity input:

$$d(t) = \begin{cases} 0, & t \leq 0 \\ 10\pi A \sin(20\pi t), & 0 < t \leq 0.1 \\ 0, & t > 0.1. \end{cases} \qquad (49)$$

We £rst run the Simulation *Road-1* (see Figure 4), when setting $T = 0.001$ sec, initial states $(0.01\ 0\ 0.01\ 0)^T$ and bump height $A = 0.01$ m.

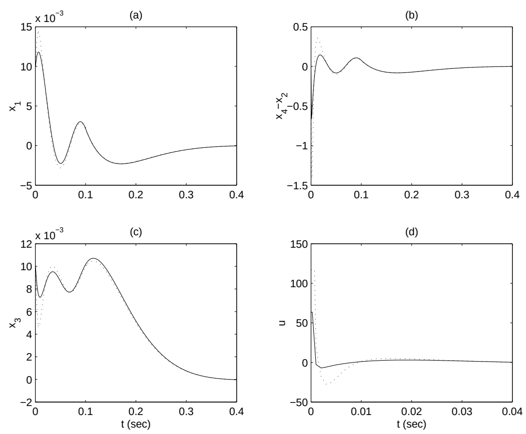

Fig. 4. Simulation *Road-1* for a low bump condition, $\cdots$ emulation and --- Euler.

In Simulation *Road-2* (see Figure 5), we set $A = 0.1$ meter, which is considered as a high bump. We set $T = 0.001$ sec and initial states $(0.01\ 0\ 0.01\ 0)^T$.

From the two sets of simulation, we see that the system can always achieve better performance with the Euler based controller than with the emulation controller.

[1] this kind of bump is a haversine of height A m and length $l = 2$ m, while assuming the vehicle is traversing the road at a speed of $v = 20$ m/s.

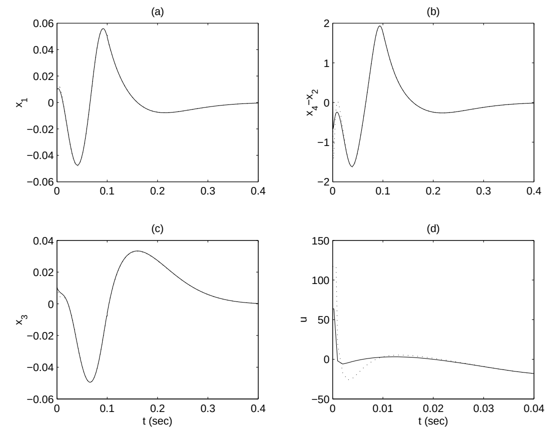

Fig. 5. Simulation *Road-2* for a high bump condition, $\cdots$ emulation and --- Euler.

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