Input-to-state stability for parameterized discrete-time time-varying nonlinear systems with applications

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Abstract

Input-to-state stability (ISS) of a parameterized family of discrete-time time-varying nonlinear systems is investigated. A converse Lyapunov theorem for such systems is developed. We consider parameterized families of discrete-time systems and concentrate on a semiglobal practical property that naturally arises when an approximate discrete-time model is used to design a controller for a sampled-data system. Application of our main result to time-varying periodic systems is presented. This is then used to design a semiglobal practical ISS (SP-ISS) control law for the model of a wheeled mobile robot.

Keywords: Converse Lyapunov theorem; Time-varying; Discrete-time; Input-to-state stability; Nonlinear systems.

1 Introduction

The prevalence of computer controlled systems and the fact that nonlinearities that arise naturally in most plants dynamics often cannot be neglected in controller design, have driven people to study and investigate nonlinear sampled-data control systems. A framework for discrete-time controller design via approximate models of the plant has been proposed in [16]. Within this framework, a parameterized family of discrete-time models of the plant is used to perform the controller design, aiming at stabilizing the original continuous-time plant. As indicated in [16], time-invariant models that are usually used in design are often not adequate in practice. There is a class of controllable nonlinear systems that may not be stabilizable using time-invariant control, but there exist time-varying controls to stabilize such systems [2, 21]. Since there are many systems in applications that belong to this class, the stabilization problem using time-varying control has become an important topic of study. In [19], a systematic design of time-varying controllers for a class of controllable systems without drift has been proposed. Stabilization using sinusoids for nonholonomic systems in power form was studied in [26]. A number of more recent work were based on these early results, e.g. [4] that studied exponential stabilization using Lyapunov approach, [12] in which exponential stabilization for homogeneous systems were thoroughly investigated.

Among the results that are available in the literature, there is hardly any that considered input-to-state stabilization using time-varying control. Input-to-state stability (ISS) is a type of robust stability for nonlinear systems with inputs (see [20, 22]). Indeed, ISS is very important, especially when dealing with systems in the presence of disturbances. The first papers presenting Lyapunov characterization of ISS for time-varying nonlinear systems are [3, 10]. The authors of [15] have studied the problem in a different way, using the averaging technique as the main tool. All the mentioned work considered continuous-time systems. To the best of the authors knowledge, the only results on discrete-time systems are given in [6, 14], where asymptotic stability for discrete-time time-varying systems is studied. In [5], the same authors have used the results of [6] to prove a converse Lyapunov theorem for ISS for discrete-time time-invariant systems.

The importance of ISS and the scarcity of existing results considering this property in the context of discrete-time time-varying systems, have motivated the authors to study the Lyapunov characterization of ISS in a semiglobal practical sense, for discrete-time time-varying systems. We consider a general parameterized family of discrete-time time-varying nonlinear systems, which commonly appears when performing sampled-data control design as discussed in [16]. Our main result is a converse Lyapunov theorem that can be seen as a discrete-time counterpart of the result of [3], at the same time as a generalization of the results of [5, 6]. We also present an application of our main result to time-varying periodic systems and use this to design a SP-ISS controller of a mobile robot [7, 19].

The paper is organized as follows. In Section 2, we present preliminaries, to introduce notation and definitions. The main result is given in Section 3. Section 4 is dedicated to an application followed by example. The paper is concluded with a summary in Section 5.

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2 Preliminaries

The sets of real and natural numbers (including 0) are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and zero at zero. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Functions of class $\mathcal{K}_\infty$ are invertible. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Given two functions $\alpha(\cdot)$ and $\gamma(\cdot)$, we denote their composition and multiplication by $\alpha \circ \gamma(\cdot)$ and $\alpha(\cdot) \times \gamma(\cdot)$, respectively.

In this paper, we consider a general parameterized family of discrete-time systems with input:

$$x(k+1) = F_T(k, x(k), d(k)) \ , \tag{1}$$

where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ are respectively the states and exogenous inputs to the system, and the parameter $T > 0$ is the sampling period. Systems having the form (1) commonly appear as a result of discretizing a nonlinear system

$$\dot{x} = f(t, x(t), d(t)) \ , \tag{2}$$

and letting the sampling period T as a free parameter to be chosen. Assume that f is locally Lipschitz and $f(0,0,0) = 0$. Without loss of generality, we may assume the same conditions for F_T.

For any inputs $d : \mathbb{N} \to \mathbb{R}^m$, we define $\|d\|_\infty := \sup_{k \in \mathbb{N}} |d(k)|$. We use the notation $\mathcal{U}_{\bar{B}}$, for the set of inputs d such that $\|d\|_\infty \leq 1$. We define $x_\circ := x(k_\circ)$, $k_\circ := k(0) \geq 0$, and Id for the identity function, i.e. $\mathrm{Id}(s) = s$, and for any function or variable h we use the simplified notation $h(k, \cdot) := h(kT, \cdot)$.

We emphasize that, for nonlinear systems, the exact discrete-time model $F_T^e(k, x(k), d(k))$ is usually not known, since it requires solving a nonlinear initial value problem which is almost impossible in general (see [13] for more details). Throughout the paper, we assume that (1) is obtained by approximating the exact discrete-time model of (2). As a result of the approximation, there is a mismatch between the exact and the approximate solutions of the system. To guarantee that (1) is a good discrete-time approximate model of (2), we assume that F_T satisfies the following consistency property that is used to limit the mismatch.

Definition 2.1 (One-step consistency) *[13] The family of approximate discrete-time models F_T is said to be one-step consistent with the exact discrete-time models F_T^e if given any strictly positive real numbers Δ_x, Δ_d, there exist a function $\varrho \in \mathcal{K}_\infty$ and $T^* > 0$ such that*

$$|F_T^e - F_T| \leq T\varrho(T) \tag{3}$$

holds for all $k \geq k_\circ$, $T \in (0, T^*)$, all $|x_\circ| \leq \Delta_x$, $\|d\|_\infty \leq \Delta_d$. ∎

The one-step consistency property is commonly used in numerical analysis literature (see for instance [9, 13, 17, 25]). We emphasize that although F_T^e is not known, the consistency property is checkable. Conditions that can be used to check this property for nonlinear time-invariant systems are presented in [13], which are extendable to use for nonlinear time-varying systems. Moreover, since we consider a semiglobal property, we assume that F_T^e and F_T are globally defined for small T.

We will use the following definitions and technicalities to construct and prove our main results. Note that these definitions are modifications of those given in [5, 6].

Definition 2.2 (Semiglobal practical ISS) *The family of systems (1) is semiglobally practically input-to-state stable (SP-ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that for any strictly positive real numbers $\Delta_x, \Delta_d, \delta$, there exists $T^* > 0$ such that the solutions of the system satisfy*

$$|x(k, k_\circ, x_\circ, d)| \leq \beta(|x_\circ|, (k - k_\circ)T) + \gamma(\|d\|_\infty) + \delta \ , \tag{4}$$

for all $k \geq k_\circ$, $T \in (0, T^*)$, all $|x_\circ| \leq \Delta_x$ and $\|d\|_\infty \leq \Delta_d$. Moreover, if the input $d = 0$, the system is semiglobally practically asymptotically stable (SP-AS). ∎

Definition 2.3 (SP-ISS Lyapunov function) *A family of continuous functions $V_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a family of SP-ISS Lyapunov functions for the family of systems (1) if there exist functions $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and for any strictly positive real numbers $\Delta_x, \Delta_d, \nu_1, \nu_2$, there exists $T^* > 0$, such that the following inequalities*

$$\underline{\alpha}(|x|) \leq V_T(k, x) \leq \overline{\alpha}(|x|) \ , \tag{5}$$

$$|x| \geq \chi(|d|) + \nu_1 \quad \Rightarrow$$

$$V_T(k+1, F_T) - V_T(k, x) \leq -T\alpha(|x|) \ , \tag{6}$$

$$V_T(k+1, F_T) \leq V_T(k, x) + \nu_2 \ , \tag{7}$$

hold for all $k \geq k_\circ$, $T \in (0, T^*)$, all $|x| \leq \Delta_x$ and $|d| \leq \Delta_d$. Moreover, if $d = 0$, the function V_T is called a SP-AS Lyapunov function. V_T is called a smooth Lyapunov function if it is smooth in $x \in \mathbb{R}^n$. ∎

Remark 2.1 By continuity of solutions, condition (7) is not needed in the continuous-time context, whereas it is required in the SP-ISS Lyapunov characterization to guarantee boundedness of trajectories, particularly for the case when $|x| < \chi(|d|) + \nu_1$ (see [16] for more details). ∎

Definition 2.4 (Δ-UBIBS) *The family of systems (1) is Δ- uniformly bounded input bounded state (Δ-UBIBS) if there exist functions $\sigma_1, \sigma_2 \in \mathcal{K}$, and for* *any strictly positive real numbers $\Delta_x, \Delta_d, \nu_b, \delta_b$, there exists $T^* > 0$ such that the following inequality:*

$$\sup_{k \geq k_\circ} |x(k, k_\circ, x_\circ, d)| \leq \max\{\sigma_1(|x|) + \nu_b, \sigma_2(\|d\|_\infty)\} + \delta_b \ , \quad (8)$$

holds for all $k \geq k_\circ$, $T \in (0, T^*)$, all $|x_\circ| \leq \Delta_x$ and $\|d\|_\infty \leq \Delta_d$. By causality, (8) is equivalent to $\sigma_1(s) \geq s$ and

$$|x(k, k_\circ, x_\circ, d)| \leq \max_{0 \leq j \leq k-1} \{\sigma_1(|x_\circ|) + \nu_b, \sigma_2(|d(j)|)\} + \delta_b \ . \quad (9)$$

■

Remark 2.2 Instead of (8), we could write

$$\sup_{k \geq k_\circ} |x(k, k_\circ, x_\circ, d)| \leq \max\{\sigma_1(|x_\circ|), \sigma_2(\|d\|_\infty)\} + \delta \ , \quad (10)$$

where $\delta := \nu_b + \delta_b$ (similarly for (9)). However, we have chosen to use (8) (respectively (9)) for convenience in proving our main result. ■

Definition 2.5 ($\mathcal{K}$-asymptotic gain) *The family of systems (1) has a $\mathcal{K}$-asymptotic gain if there exists a function $\gamma_a \in \mathcal{K}$ and for any strictly positive real numbers Δ_x, Δ_d, π, there exists $T^* > 0$, such that*

$$\overline{\lim_{k \to \infty}} |x(k, k_\circ, x_\circ, d)| \leq \gamma_a \left(\overline{\lim_{k \to \infty}} |d(k)| \right) + \pi \ , \quad (11)$$

for all $k \geq k_\circ$, $T \in (0, T^*)$, all $|x_\circ| \leq \Delta_x$, and $|d| \leq \Delta_d$. ■

Definition 2.6 (SP Robust stability) *The system (1) is semiglobally practically robustly stable (SPRS), if there exists a function $\rho \in \mathcal{K}_\infty$ and for any strictly positive real numbers $\Delta_x, \Delta_d, \delta$, there exists $T^* > 0$, such that for all $k \geq k_\circ$, $T \in (0, T^*)$, all $|x_\circ| \leq \Delta_x$, and $d \in \mathcal{U}_{\bar{B}}$ such that $\|d\rho(|x|)\|_\infty \leq \Delta_d$, the function*

$$x(k+1) = F_T(k, x, d\rho(|x|)) =: G_T(k, x, d) \quad (12)$$

is SP-AS. ■

Lemma 2.1 *[6] For any $\mathcal{KL}$ function β, there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that*

$$\beta(s, r) \leq \rho_1(\rho_2(s)e^{-r}) \ , \quad \forall s \geq 0 \ \ \forall r \geq 0 \ . \quad (13)$$

■

Lemma 2.2 (Comparison Principle) *[6] For any $\mathcal{K}$-function α, there exists a $\mathcal{KL}$-function $\beta_\alpha(s,t)$ with the following property: if $y : \mathbb{N} \to [0, \infty)$ is a function satisfying*

$$y(k+1) - y(k) \leq -\alpha(y(k)) \quad (14)$$

for all $0 \leq k < k_1$ for some $k_1 \leq \infty$, then

$$y(k) \leq \beta_\alpha(y(0), k), \quad \forall k < k_1 \ . \quad (15)$$

■

3 Main Result

In this section, we state and prove our main result, namely a converse Lyapunov theorem for SP-ISS for parameterized family of discrete-time time-varying nonlinear systems. We provide a necessary and sufficiency conditions for which a parameterized family of discrete-time time-varying nonlinear systems is input-to-state stable in a semiglobal practical sense. This result is a discrete-time counterpart of [3], and it generalizes the main results of [5, 6].

The technique used in proving our results is similar to the technique that has been used in [6]. However, there are more technicalities needed to treat the semiglobal practical property we consider. This is also the first proof of a converse Lyapunov theorem for stability property in a semiglobal practical sense. In the next section, we present an engineering example, which shows the usefulness of our results from a practical point of view, since we very often have to deal with semiglobal practical property when designing a discrete-time controller for a continuous-time plant.

We are now ready to state our main result.

Theorem 3.1 *The parameterized family of discrete-time time-varying systems (1) is SP-ISS if and only if it admits a smooth SP-ISS Lyapunov function V_T.* ■

Before we proceed with proving Theorem 3.1, we first state and prove the following lemmas, which are instrumental in constructing the proof of the theorem. The proofs of Lemmas 3.2 and 3.3 are given in [6].

Lemma 3.1 *If the family of systems (1) is SP-ISS, then it is Δ-UBIBS and it admits a $\mathcal{K}$-asymptotic gain. Moreover, the system is SPRS, and hence SP-AS.* ■

Lemma 3.2 *[6, Lemma 2.7] If there exists a continuous SP-ISS Lyapunov function V_T with respect to a compact set $\mathbb{X}$, then there exists also a smooth one, W_T, with respect to the same set. Moreover, if V_T is periodic with period $\lambda > 0$, then W_T can be chosen to be periodic with the same period.* ■

Lemma 3.3 *[6, Lemma 2.8] Assume that system (1) admits a SP-ISS Lyapunov function V_T. Then there exists a smooth function $\rho \in \mathcal{K}_\infty$ such that $W_T = \rho \circ V_T$ is also a SP-ISS Lyapunov function of (1), and (6) holds for some $\alpha \in \mathcal{K}_\infty$.* ■

Proof of Lemma 3.1:

SP-ISS ⇒ Δ-UBIBS + $\mathcal{K}$-asymptotic gain. Suppose that the system (1) is SP-ISS. Let $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ be as in Definition 2.2. By the property of $\mathcal{KL}$ functions, if we fix the second argument, then β is a $\mathcal{K}$ function in its first argument. Hence, the Δ-UBIBS property is directly implied. Also, by definition, the function γ is the $\mathcal{K}$-asymptotic gain of system (1).

Δ-UBIBS + $\mathcal{K}$-asymptotic gain $\Rightarrow$ SPRS $\Rightarrow$ SP-AS. Suppose that the system (1) is Δ-UBIBS and it admits a $\mathcal{K}$-asymptotic gain. Let $\sigma_1, \sigma_2 \in \mathcal{K}$ be as in Definition 2.4. Given any strictly positive numbers $\Delta_x, \Delta_d, \nu_b, \delta_b$, there exist $T^* > 0$, such that (8) holds for all $k \geq k_\circ$, all $T \in (0, T^*)$, $|x| \leq \Delta_x$, $\|d\|_\infty \leq \Delta_d$. Without loss of generality, let the $\mathcal{K}$-asymptotic gain

$$\gamma = \sigma_2 \ , \tag{16}$$

and $\pi = \delta_b$. Let the positive numbers ν_c and ν_d be such that

$$\nu_c \geq \nu_b + \delta_b \ , \tag{17}$$

$$\nu_d \leq \min_{s \in [0,1)} (\sigma_2(|\rho(|x_\rho(k,k_\circ)|)|) - \sigma_2(|s\rho(|x_\rho(k,k_\circ)|)|)) \ , \tag{18}$$

and

$$\nu_c - \nu_d < \nu_b \ . \tag{19}$$

We have, from Definition 2.4, that $\sigma_1 \geq \mathrm{Id}$ for all $s \geq 0$. Pick any function $\rho \in \mathcal{K}_\infty$ such that

$$\gamma \circ \rho(s) \leq s/2 \ , \quad \forall s \geq 0 \ . \tag{20}$$

We will show that with the correct choice of ρ, the system (12) is SP-AS.

Pick any initial condition such that $|x_\circ| \leq \Delta_x$. Let $x_\rho(k)$ denote the corresponding trajectory of system (12). We use the following claim:

Claim. $\sigma_2 \circ \rho(|x_\rho(k,k_\circ)|) \leq \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c$, *for all* $k \geq 0$. ■

Proof of Claim. Trivially the claim is true for $x_\circ = 0$. Suppose now we have nonzero initial states, $x_\circ \neq 0$. It is then obvious that the claim is true for $k = 0$, since

$$\begin{aligned}\sigma_2(\rho(|x_\rho(0)|)) = \gamma(\rho(|x_\rho(0)|)) &\leq \frac{1}{2}|x_\circ| \\ &\leq \frac{1}{2}\sigma_1(|x_\circ|) \leq \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c \ .\end{aligned} \tag{21}$$

The last part to prove is for $k > 0$. Let

$$k_1 = \min\left\{k \in \mathbb{N} | \sigma_2 \circ \rho(|x_\rho(k,k_\circ)|) \geq \frac{\sigma_1(|x_\circ|)}{2} + \nu_c\right\},$$

and note that $k_1 > 0$. Suppose that the claim is false and hence $k_1 < \infty$. For $0 \leq k \leq k_1 - 1$, it holds that $\sigma_2 \circ \rho(|x_\rho(k,k_\circ)|) \leq \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c$. From (18) and (19), we have that

$$\begin{aligned}\sigma_2(|d(k)\rho(|x_\rho(k,k_\circ)|)|) &\leq \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c - \nu_d \\ &\leq \frac{1}{2}\sigma_1(|x_\circ|) + \nu_b\end{aligned} \tag{22}$$

for $0 \leq k \leq k_1 - 1$. Consequently, it follows from the Δ-UBIBS property of the system, in particular from (8), that

$$\begin{aligned}|x_\rho(k_1)| \leq \max_{0 \leq j \leq k_1 - 1} &\{\sigma_1(|x_\circ|) + \nu_b, \\ &\sigma_2(|d(j)\rho(|x_\rho(j)|)|)\} + \delta_b \\ \leq\ &\sigma_1(|x_\circ|) + \nu_b + \delta_b \\ \leq\ &\sigma_1(|x_\circ|) + \nu_c \ ,\end{aligned} \tag{23}$$

which, by (16) and (20), implies that

$$\begin{aligned}\sigma_2(\rho(|x_\rho(k_1)|)) \leq \frac{1}{2}|x_\rho(k_1)| &\leq \frac{1}{2}\sigma_1(|x_\circ|) + \frac{\nu_c}{2} \\ &< \frac{1}{2}\sigma_1(|x_\circ|) + \nu_c \ ,\end{aligned} \tag{24}$$

which contradicts the definition of k_1. Hence, the claim is true.

An immediate consequence of the claim is that (23) holds for all $k \in \mathbb{N}$ and that $\overline{\lim}_{k\to\infty} |x_\rho(k)|$ is finite. Using (16), and taking the limits on both sides of (11), we have

$$\begin{aligned}\overline{\lim_{k\to\infty}} |x_\rho(k)| &\leq \overline{\lim_{k\to\infty}} \gamma(|d(k)\rho(|x_\rho(k)|)|) + \pi \\ &\leq \overline{\lim_{k\to\infty}} |x_\rho(k)|/2 + \nu_c + \pi \ ,\end{aligned} \tag{25}$$

which shows that $\overline{\lim}_{k\to\infty} |x_\rho(k)| \leq 2(\nu_c + \pi)$, which is bounded for each trajectory, for all $k \geq k_\circ$. This shows that (12) is SP-AS. Hence, this completes the proof of Lemma 3.1. ■

Proof of Theorem 3.1 The proof follows closely the steps used in proving the converse Lyapunov theorem in [11], combined with the proof of Theorem 1 of [5] (see also [23]).

Proof of sufficiency. From the statement of the theorem, suppose that for any strictly positive real numbers $\Delta_x, \Delta_d, \nu_1, \nu_2$, there exists $T^* > 0$ such that for all $T \in (0, T^*)$, $|x| \leq \Delta_x$, $\|d\|_\infty \leq \Delta_d$, a smooth radially unbounded continuous function $V_T(k,x)$ is a SP-ISS Lyapunov function for the family of systems (1). Let the functions $\underline{\alpha}, \overline{\alpha}, \alpha$ and χ be as in Definition 2.3 of SP-ISS Lyapunov function. Let $\delta > 0$ be such that

$$\max_{s \in (0,\Delta_d)} \{\underline{\alpha}^{-1}(\overline{\alpha}(\chi(s)+\nu_1)) - \underline{\alpha}^{-1}(\overline{\alpha}(\chi(s)))\} \leq \delta \ . \tag{26}$$

We consider two cases:

Case 1: $|x| \geq \chi(|d|) + \nu_1$

Using (5) and (6), it is obvious that we can write

$$\begin{aligned}&V_T(k,x) \geq \tilde{\chi}(|d|) + \tilde{\nu}_1 \quad \Rightarrow \\ &\quad V_T(k+1, F_T) - V_T(k,x) \leq -T\tilde{\alpha}(V_T(k,x)) \ ,\end{aligned} \tag{27}$$

by choosing $\tilde{\chi} = \underline{\alpha} \circ \chi$ and $\tilde{\alpha} = \alpha \circ \overline{\alpha}^{-1}$. Note that from Lemma 3.3, since V_T is a smooth Lyapunov function, we can have $\alpha \in \mathcal{K}_\infty$. Applying the comparison principle of Lemma 2.2, there exists a $\mathcal{KL}$-function β_α, such that

$$\begin{aligned}&V_T(k,x) \geq \tilde{\chi}(|d|) + \tilde{\nu}_1 \quad \Rightarrow \\ &\qquad V_T(k,x) \leq \beta_{\tilde{\alpha}}(V_T(k_\circ, x_\circ), k) \ .\end{aligned} \tag{28}$$

Therefore, for all $k \geq k_\circ$, we can write

$$V_T(k, x(k+k_\circ, k_\circ, x_\circ, d)) \leq \beta_{\tilde{\alpha}}(V_T(k_\circ, x_\circ), k) \ . \quad (29)$$

Further, using (5) we obtain

$$\begin{aligned} |x(k+k_\circ, k_\circ, x_\circ, d)| &\leq \underline{\alpha}^{-1} \circ \beta_\alpha(V_T(k_\circ, x_\circ), k) \\ &\leq \underline{\alpha}^{-1} \circ \beta_\alpha(\overline{\alpha}(|x_\circ|), k) \quad (30) \\ &=: \beta(|x_\circ|, k) \ . \end{aligned}$$

Hence, we can write

$$|x(k, k_\circ, x_\circ, d)| \leq \beta(|x_\circ|, (k-k_\circ)T) \ . \quad (31)$$

Case 2: $|x| < \chi(|d|) + \nu_1$
From (5), we have that

$$\underline{\alpha}(|x|) \leq V_T(k,x) \leq \overline{\alpha}(|x|) \leq \overline{\alpha}(\chi(|d|) + \nu_1) \ , \quad (32)$$

which implies that

$$\begin{aligned} |x(k, k_\circ, x_\circ, d)| &\leq \underline{\alpha}^{-1}(\overline{\alpha}(\chi(|d|) + \nu_1)) \\ &\leq \gamma(|d|) + \delta \quad (33) \\ &\leq \gamma(\|d\|_\infty) + \delta \ , \end{aligned}$$

where $\gamma := \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi$.
Combining (31) and (33), we have that for any $|x| \leq \Delta_x$, $\|d\|_\infty \leq \Delta_d$ the following holds:

$$|x(k, k_\circ, x_\circ, d)| \leq \beta(|x_\circ|, (k-k_\circ)T) + \gamma(\|d\|_\infty) + \delta \ , \quad (34)$$

and this completes the proof of sufficiency.

Proof of necessity. Suppose that the system (1) is SP-ISS. Given any arbitrary strictly positive numbers $\Delta_x, \Delta_d, \tilde{\delta}$, let the numbers generate $T_1^* > 0$ and let $T^* := \min(1, T_1^*)$, such that (4) holds for all $k \geq k_\circ$, $T \in (0, T^*)$, $|x| \leq \Delta_x$, $\|d\|_\infty \leq \Delta_d$. We have shown in Lemma 3.1 that SP-ISS implies SPRS with input $d\rho(|x|)$, where $d \in \mathcal{U}_{\bar{B}}$ and $\rho \in \mathcal{K}_\infty$. This further implies that the system is SP-AS. By Lemma 3.1, let the numbers $\Delta_x, \Delta_d, \tilde{\delta}$ generate $\delta > 0$, such that for all $|x| \leq \Delta_x$, $d \in \mathcal{U}_{\bar{B}}$, $k \geq k_\circ$ and all $T \in (0, T^*)$ the following holds:

$$|x(k+k_\circ, k_\circ, x_\circ, d\rho(|x|))| \leq \beta(|x_\circ|, k) + \delta \ . \quad (35)$$

By Lemma 2.1, there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that

$$|x(k+k_\circ, k_\circ, x_\circ, d\rho(|x|))| \leq \rho_1(\rho_2(|x_\circ|)e^{-k}) + \delta \ . \quad (36)$$

Define $\omega := \rho_1^{-1}$, and let $\delta_\rho > 0$ be such that

$$\max_{s \in [0, \Delta_x]} \left(\omega(\rho_1(\rho_2(s)e^{-k}) + \delta) - \rho_2(s)e^{-k} \right) \leq \delta_\rho . \quad (37)$$

From (36) and (37), we obtain

$$\omega(|x(k+k_\circ, k_\circ, x_\circ, d\rho(|x|))|) \leq \rho_2(|x_\circ|)e^{-k} + \delta_\rho \ . \quad (38)$$

Since ω and ρ_2 are $\mathcal{K}_\infty$ functions, we can always find $\tilde{\rho}_2 \in \mathcal{K}_\infty$ such that

$$\begin{aligned} &\omega(\,|x(k+k_\circ, k_\circ, x_\circ, d\rho(|x|))|) \\ &\qquad \leq \tilde{\rho}_2(|x_\circ|)e^{-k} \leq \rho_2(|x_\circ|)e^{-k} + \delta_\rho \ . \end{aligned} \quad (39)$$

Define

$$\begin{aligned} &V_{0T}(k_\circ, x_\circ, d\rho(|x|)) \\ &\qquad = \sum_{k=0}^{\infty} \omega(|x(k+k_\circ, k_\circ, x_\circ, d\rho(|x|))|) \ . \end{aligned} \quad (40)$$

It then follows from (39) that

$$\begin{aligned} \omega(|x_\circ|) \leq V_{0T}(k_\circ, x_\circ, d\rho(|x|)) &\leq \sum_{k=0}^{\infty} \tilde{\rho}_2(|x_\circ|)e^{-k} \\ &\leq \frac{e}{e-1}\tilde{\rho}_2(|x_\circ|) \ . \end{aligned} \quad (41)$$

This shows that the series in (41) is convergent, uniformly on $x_\circ$ with $|x_\circ| \leq \Delta_x$ and on $d \in \mathcal{U}_{\bar{B}}$. Since for each $k_\circ \in \mathbb{N}$, ω is continuous uniformly on $d \in \mathcal{U}_{\bar{B}}$, then so is V_{0T}. Define V_T by

$$V_T(k_\circ, x_\circ) = \sup_{d \in \mathcal{U}_{\bar{B}}} V_{0T}(k_\circ, x_\circ, d\rho(|x|)) \ . \quad (42)$$

It then follows immediately from (41) that

$$\omega(|x_\circ|) \leq V_T(k_\circ, x_\circ) \leq \frac{e}{e-1}\tilde{\rho}_2(|x_\circ|) \ . \quad (43)$$

Hence, by taking $\underline{\alpha}(s) := \omega(s)$ and $\overline{\alpha}(s) := \frac{e}{e-1}\tilde{\rho}_2(s)$ we show that (5) holds.

To prove the continuity of the Lyapunov function $V_T(k,x)$, we use Lemma 4.4 of [6] that is directly valid for our case.

In the following, we show that V_T admits a desired decay estimate as in (6).

Pick any $k_\circ$, $x_\circ$ such that $|x_\circ| \leq \Delta_x$, and any $\mu \in \mathcal{U}_{\bar{B}}$. Let the exact solution $x_f := F_T^e(k_\circ, x_\circ, \mu\rho(|x|))$ and the approximate solution $x_F := F_T(k_\circ, x_\circ, \mu\rho(|x|))$, with $\mu := d(k_\circ)$. Since F_T is one-step consistent with F_T^e, we have that

$$|x_f - x_F| \leq T\varrho(T) \ , \quad \varrho \in \mathcal{K}_\infty \ . \quad (44)$$

Let $T^* \leq 1$ be sufficiently small such that by the continuity of V_T and the one-step consistency property of F_T, we may assume the existence of $\tilde{\varrho} \in \mathcal{K}_\infty$ such that the following holds for all $T \in (0, T^*)$

$$|V_T(k_\circ+1, x_F) - V_T(k_\circ+1, x_f)| \leq T\tilde{\varrho}(T) \ . \quad (45)$$

Let $\nu > 0$ be such that

$$\tilde{\varrho}(T^*) \leq \nu \ . \quad (46)$$

By uniqueness of exact solutions, we can see that for any $d \in \mathcal{U}_{\bar{B}}$ such that $d(k_\circ) = \mu$, it holds that

$$\begin{aligned} &x(k+k_\circ+1, k_\circ+1, x_f, d\rho(|x|)) \\ &\qquad = x(k+k_\circ+1, k_\circ, x_\circ, d\rho(|x|)) \ , \end{aligned} \quad (47)$$

for all $k \geq 0$. Hence, using (46) and $T^* \leq 1$, we have

$$\begin{aligned}
&V_T(k_\circ+1, x_F) \\
&= V_T(k_\circ+1, x_f) + V_T(k_\circ+1, x_F) - V_T(k_\circ+1, x_f) \\
&\leq V_T(k_\circ+1, x_f) + T\tilde{\varrho}(T) \\
&\leq \sum_{k=0}^{\infty} \omega(|x(k+k_\circ+1, k_\circ+1, x_f, d\rho(|x|))|) + T\nu \\
&\leq \sum_{k=0}^{\infty} \omega(|x(k+k_\circ+1, k_\circ, x_\circ, d\rho(|x|))|) + T\nu \qquad (48) \\
&\leq \sum_{k=1}^{\infty} \omega(|x(k+k_\circ, k_\circ, x_\circ, d\rho(|x|))|) + T\nu \\
&\leq \sum_{k=0}^{\infty} \omega(|x(k+k_\circ, k_\circ, x_\circ, d\rho(|x|))|) \\
&\quad - \omega(|x(k_\circ, k_\circ, x_\circ, d\rho(|x|))|) + T\nu \\
&\leq V_T(k_\circ, x_\circ) - \omega(|x_\circ|) + T\nu \\
&\leq V_T(k_\circ, x_\circ) - T\omega(|x_\circ|) + T\nu \ .
\end{aligned}$$

This shows that

$$\begin{aligned}
V_T(k_\circ+1, F_T(k_\circ, x_\circ, d\rho(|x|)) - V_T(k_\circ, x_\circ)& \\
\leq -T\omega(|x_\circ|) + T\nu \ , &\qquad (49)
\end{aligned}$$

for all $|x| \leq \Delta_x$ and all $d \in \mathcal{U}_{\bar{B}}$. Observe that this is equivalent to

$$\begin{aligned}
|u| \leq& \rho(|x|) \quad \Rightarrow \\
&V_T(k_\circ+1, F_T(k_\circ, x_\circ, u)) - V_T(k_\circ, x_\circ) \qquad (50) \\
&\qquad \leq -T\omega(|x_\circ|) + T\nu \ ,
\end{aligned}$$

and further it is obvious that it is also equivalent to

$$\begin{aligned}
|x| \geq& \chi(|u|) + \nu_1 \quad \Rightarrow \\
&V_T(k_\circ+1, F_T(k_\circ, x_\circ, u)) - V_T(k_\circ, x_\circ) \qquad (51) \\
&\qquad \leq -T\alpha(|x_\circ|) \ ,
\end{aligned}$$

by defining $\chi := \rho^{-1}$ and $\alpha := \frac{3}{4}\omega$ and $\nu_1 \leq \omega^{-1}(4\nu)$. Hence, (6) is satisfied.

Note however that the continuous Lyapunov function obtained in the proof is not necessarily smooth. To show the existence of a smooth Lyapunov function for (1) and to show that $\alpha \in \mathcal{K}_\infty$, we use Lemmas 3.2 and 3.3. Using Lemma 3.2, we can show the existence of a smooth Lyapunov function W_T as a continuous Lyapunov function V_T exists and using Lemma 3.3 it can be shown that if the Lyapunov function is smooth, there exists $\alpha \in \mathcal{K}_\infty$ such that (6) holds.

The last thing to show is that (7) holds. We have assumed that F_T is globally defined for small T, so that F_T is finite for all $k \geq k_\circ$, all $|x_\circ| \leq \Delta_x$ and $|d| \leq \Delta_d$. Then there exists $c > 0$ such that

$$|F_T - x_\circ| \leq c \ , \quad \forall k \geq k_\circ \ . \qquad (52)$$

Moreover, by Lemma 3.2 we may assume that V_T is smooth. Then using (52) and the smoothness of V_T, we obtain

$$\begin{aligned}
V_T(k_\circ+1, F_T(k_\circ, x_\circ, u)) - V_T(k_\circ, x_\circ)& \\
\leq L\,|F_T - x_\circ|& \qquad (53) \\
\leq Lc := \nu_2 \ ,&
\end{aligned}$$

with L is the Lipschitz constant of V_T. Hence (7) holds, and this completes the proof of necessary. Therefore, the proof of Theorem 3.1 is complete. ■

4 Application and example on periodic systems

4.1 Application to periodic systems

In this section we focus on a particular class of time-varying nonlinear systems, namely time-varying nonlinear periodic systems, which include a large class of systems. This class of systems is very important in various applications, particularly in tracking control problems (see for instance [12, 19, 24, 26]).

We consider a family of parameterized periodic discrete-time time-varying systems. The system (1) is called a periodic system if F_T is periodic in k with period $\lambda > 0$, and hence we have the following

$$F_T(k+m\lambda, x, d) = F_T(k, x, d) \ , \quad m \in \mathbb{N} \ . \qquad (54)$$

By Theorem 3.1 we conclude that if the system is SP-ISS then it is Δ-UBIBS and it admits a $\mathcal{K}$-asymptotic gain. This further implies that for some function $\rho \in \mathcal{K}_\infty$ the corresponding system is SPRS and hence SP-AS. For a periodic system such that (54), we can show that the map

$$G_T(k, x, d) := F_T(k, x, d\rho(|x|))$$

is also periodic in k with the same period as F_T. Moreover, we can show that there exists a SP-ISS Lyapunov function V_T that is periodic with period λ, that satisfies

$$V_T(k_\circ + m\lambda, x) = V_T(k_\circ, x) \ , \qquad (55)$$

as has been proved in [6]. Hence, the following corollary follows directly from Theorem 3.1.

Corollary 4.1 *The parameterized family of time-varying periodic system (1) with period λ is SP-ISS if and only if it admits a smooth SP-ISS periodic Lyapunov function with the same period λ.* ■

The proof of Corollary 4.1 follows the same steps as the proof of Theorem 3.1 (see also [6]), hence it is not presented in the paper.

4.2 Example

Consider the model of a simple mobile robot moving on a plane, with two independent rear motorized wheels as illustrated in Figure 1 [7, 19]:

$$\begin{aligned}\dot{x} &= v\cos\theta + d\sin\theta \\ \dot{y} &= v\sin\theta - d\cos\theta \\ \dot{\theta} &= \omega\ ,\end{aligned} \qquad (56)$$

with v the forward velocity, ω the steering velocity, (x, y) the Cartesian position of the center of mass of the robot, θ the heading angle from the horizontal axis, and d a disturbance force perpendicular to the forward direction. The system (56) is a benchmark example of systems which are not stabilizable using continuous feedback [1].

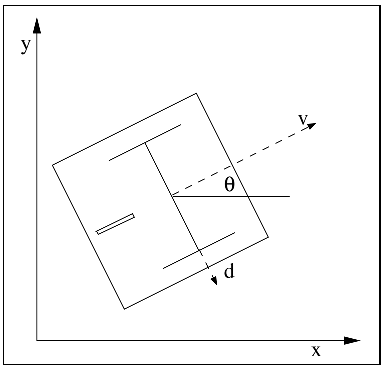

Figure 1: A two-wheeled drive mobile robot

Using the coordinates transformation

$$\begin{aligned}x_1 &= x\cos\theta + y\sin\theta \\ x_2 &= x\sin\theta - y\cos\theta \\ x_3 &= \theta\ ,\end{aligned} \qquad (57)$$

we obtain the dynamic model of system (56) in power form:

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_1u_2 + d \\ \dot{x}_3 &= u_2\ ,\end{aligned} \qquad (58)$$

where $u_1 := v - \omega x_2$, and $u_2 := \omega$.

The stabilization problem for system (58) in the absence of disturbances has been studied in [19]. Using the Lyapunov function

$$V(t, x) = \frac{1}{2}(x_1 + (x_2^2 + x_3^2)\cos t)^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2\ , \quad (59)$$

which is a time-varying periodic function, the controller

$$u_1 = (x_2^2 + x_3^2)\sin t - (x_1 + (x_2^2 + x_3^2)\cos t) \qquad (60)$$

$$\begin{aligned}u_2 = &-2(x_1 + (x_2^2 + x_3^2)\cos t)(x_1x_2 + x_3)\cos t \\ &- (x_1x_2 + x_3)\end{aligned} \qquad (61)$$

has been designed. From the time derivative of the Lyapunov function

$$\begin{aligned}\dot{V}(t,x) = -\Big(&2(x_1 + (x_2^2 + x_3^2)\cos t)(x_1x_2 + x_3)\cos t \\ &+ (x_1x_2 + x_3)\Big)^2 - (x_1 + (x_2^2 + x_3^2)\cos t)^2,\end{aligned} \quad (62)$$

and using La Salle Invariance Principle, it follows that in the case $d = 0$, the closed-loop system (58), (60), (61) is uniformly globally asymptotically stable.

We consider now the case when we have a nonzero additive disturbance entering the second equation. We are interested in a particular step of the stabilization of system (58), using a discrete-time time-varying periodic controller that is designed based on the approximate model of the system. In particular, we use the Euler model of the system (58), namely

$$\begin{aligned}x_1(k+1) &= x_1(k) + Tu_1(k) \\ x_2(k+1) &= x_2(k) + T(x_1(k)u_2(k) + d(k)) \\ x_3(k+1) &= x_3(k) + Tu_2(k)\ .\end{aligned} \qquad (63)$$

We emphasize that the Euler approximate model satisfies the one-step consistency we assume in constructing the results in this paper. We also need to point out that in this example we are not aiming to achieve SP-ISS for the system (58), but for the approximate model (63). However, it can be shown, following directly as what have been proved in [13, 18] for the time-invariant case, that under certain conditions the stability of the controlled exact discrete-time model is implied from the stability of the controlled approximate model, and the stability of the sampled-data system follows from the stability of the exact discrete-time models and boundedness of solutions.

We then apply our result, particularly Corollary 4.1, to check the SP-ISS property of the system (63) with a controller that is designed using the idea from [19]. Notice that for the rest of the paper, we drop the discrete-time argument k for simplicity.

It was shown by (62) that the derivative of the Lyapunov function (59) is negative semidefinite. Unfortunately, while we can apply La Salle Invariance Principle for systems without disturbance, we do not have such kind of tool for systems with inputs. Hence, (59) cannot be used to show input-to-state stability of the closed-loop system.

Using a similar idea as in [8], we construct another Lyapunov function that can be used to show ISS. We use the Lyapunov function

$$V_T = \varrho_1(V_{1T}) + \varrho_2(V_{2T})\ , \quad \varrho_1, \varrho_2 \in \mathcal{K}_\infty\ , \qquad (64)$$

where $V_{1T} = V$ and

$$V_{2T} = V_{1T} - \epsilon x_1(x_2^2 + x_3^2)\sin t\ , \qquad (65)$$

with $\epsilon > 0$ sufficiently small to guarantee that $V_T \geq 0$. We have chosen the $\mathcal{K}_\infty$ functions $\varrho_1 = \varrho_2 = \mathrm{Id}$. From

(63) and (64) it is easy to show that conditions (5) and (7) hold.

The Lyapunov difference ΔV_T is obtained as follows:

$$\begin{aligned}
&\Delta V_T(k,x) = V_T(k+1,F_T) - V_T(k,x) \\
&= (x_1(k+1) + (x_2^2(k+1) + x_3^2(k+1))\cos((k+1)T))^2 \\
&\quad + x_2(k+1)^2 + x_3(k+1)^2 \\
&\quad - \epsilon x_1(k+1)(x_2^2(k+1) + x_3^2(k+1))\sin((k+1)T) \\
&\quad - (x_1(k) + (x_2^2(k) + x_3^2(k))\cos(kT))^2 - x_2(k)^2 \\
&\quad - x_3(k)^2 + \epsilon x_1(k)(x_2^2(k) + x_3^2(k))\sin(kT) \\
&= \Big(x_1 + Tu_1 + ((x_2 + T(x_1u_2 + d))^2 + (x_3 + Tu_2)^2) \\
&\qquad \times \cos((k+1)T)\Big)^2 - (x_1 + (x_2^2 + x_3^2)\cos(kT))^2 \\
&\quad + (x_2 + T(x_1u_2 + d))^2 - x_2^2 + (x_3 + Tu_2)^2 - x_3^2 \\
&\quad - \epsilon(x_1 + Tu_1)\Big((x_2 + T(x_1u_2 + d))^2 + (x_3 + Tu_2)^2\Big) \\
&\qquad \times \sin((k+1)T) + \epsilon x_1(x_2^2 + x_3^2)\sin(kT)\ .
\end{aligned}$$

Assuming that the sampling period T is sufficiently small ($0 < T < 1$), we use the following approximation

$$\cos((k+1)T) - \cos(kT) \approx T\sin(kT) \approx O(T^2)\ , \quad (66)$$

$$\sin((k+1)T) - \sin(kT) \approx T\cos(kT) \approx O(T)\ . \quad (67)$$

Assume also that ϵ is sufficiently small ($\epsilon = O(T)$). The Lyapunov difference can then be written as

$$\begin{aligned}
&\Delta V_T(k,x) \\
&\approx 2Tu_1\Big(x_1 + (x_2^2 + x_3^2)\cos((k+1)T) + 2T(x_1x_2 + x_3) \\
&\qquad \times u_2\cos((k+1)T) - \frac{\epsilon}{2}(x_2^2 + x_3^2)\sin((k+1)T)\Big) \\
&\quad + 2Tu_2(x_1x_2 + x_3)\Big(1 - \epsilon x_1\sin((k+1)T) \\
&\quad + 2(x_1 + (x_2^2 + x_3^2)\cos((k+1)T))\cos((k+1)T)\Big) \\
&\quad + 2Tdx_2\Big(1 + 2(x_1 + (x_2^2 + x_3^2)\cos((k+1)T)) \\
&\qquad \times \cos((k+1)T)\Big) + O(T^2)\ .
\end{aligned}$$

Applying a discrete-time controller

$$\begin{aligned}
u_{1T} = &-(x_1 + (x_2^2 + x_3^2)\cos((k+1)T)) \\
&- 2T(x_1x_2 + x_3)u_2\cos((k+1)T) \\
&+ \frac{\epsilon}{2}(x_2^2 + x_3^2)\sin((k+1)T)
\end{aligned} \quad (68)$$

$$\begin{aligned}
u_{2T} = &-(x_1x_2 + x_3)\Big(1 - \epsilon x_1\sin((k+1)T) \\
&+ 2(x_1 + (x_2^2 + x_3^2)\cos((k+1)T)) \\
&\times \cos((k+1)T)\Big)\ ,
\end{aligned} \quad (69)$$

that is very similar to (60), (61), we will show that the closed-loop system (63),(68),(69) is SP-ISS. Substituting (68), (69) into the Lyapunov difference, we obtain

$$\begin{aligned}
&\Delta V_T(k,x) \\
&\le -2T\epsilon^2((\sin((k+1)T))^2 + a)\Big[(x_1x_2 + x_3)^2x_1^2 \\
&\quad + \frac{(x_2^2 + x_3^2)^2}{4}\Big] - 2T(x_1 + (x_2^2 + x_3^2)\cos((k+1)T))^2 \\
&\quad - 2T(x_1x_2 + x_3)^2\Big(2(x_1 + (x_2^2 + x_3^2)\cos((k+1)T)) \\
&\qquad \times \cos((k+1)T) + 1\Big)^2 \\
&\quad + 2Tdx_2\Big(2(x_1 + (x_2^2 + x_3^2)\cos((k+1)T)) \\
&\qquad \times \cos((k+1)T) + 1\Big) + O(T^2)\ ,
\end{aligned}$$

after adding a small positive offset $a <<< T$ to avoid the first term on the right-hand side of the inequality to become zero at $(k+1)T = i\pi,\ i \in \mathbb{N}$. Finally, we use Young's inequality to arrive at

$$\begin{aligned}
\Delta V_T(k,x) \le &-T(M_1|x_1|^2 + M_2|x_2|^4 + M_3|x_3|^4) \\
&+ TM_4|d|^2 + O(T^2)\ ,
\end{aligned}$$

with $M_i > 0,\quad i \in \{1,\cdots,4\}$. Therefore, it is obvious that (6) holds and hence the closed-loop discrete-time model (63),(68),(69) is SP-ISS. Moreover, notice that the Lyapunov function V_T is a periodic function with period 2π, the same as the period of the closed-loop system (63),(68),(69).

5 Summary

We have presented a converse Lyapunov theorem for ISS for parameterized discrete-time time-varying systems. We have considered the ISS property of the systems in a semiglobal practical sense, which appears naturally in sampled-data design. We have also presented an application of our result to discrete-time time-varying periodic systems. Finally, by the provided example, we have illustrated the usefulness of our results from a practical point of view.

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