Input-to-state stability for parameterized discrete-time
time-varying nonlinear systems with applications

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Abstract
Input-to-state stability (ISS) of a parameterized family of
discrete-time time-varying nonlinear systems is
investigated. A converse Lyapunov theorem for such
systems is developed. We consider parameterized
families of discrete-time systems and concentrate on
a semiglobal practical property that naturally arises
when an approximate discrete-time model is used to
design a controller for a sampled-data system. Ap-
lication of our main result to time-varying periodic
systems is presented. This is then used to design a
semiglobal practical ISS (SP-ISS) control law for the
model of a wheeled mobile robot.

Keywords: Converse Lyapunov theorem; Time-
varying; Discrete-time; Input-to-state stability; Non-
linear systems.

1 Introduction

The prevalence of computer controlled systems and the
fact that nonlinearities that arise naturally in most
plants dynamics often cannot be neglected in con-
troller design, have driven people to study and in-
vestigate nonlinear sampled-data control systems. A
framework for discrete-time controller design via ap-
proximate models of the plant has been proposed in
[16]. Within this framework, a parameterized family
of discrete-time models of the plant is used to perform
the controller design, aiming at stabilizing the origi-
nal continuous-time plant. As indicated in [16], time-
variant models that are usually used in design are
often not adequate in practice. There is a class of con-
trollable nonlinear systems that may not be stabiliz-
able using time-invariant control, but there exist time-
varying controls to stabilize such systems [2, 21]. Since
there are many systems in applications that belong to
this class, the stabilization problem using time-varying
control has become an important topic of study. In
[19], a systematic design of time-varying controllers for
a class of controllable systems without drift has been
proposed. Stabilization using sinusoids for nonholo-
nomic systems in power form was studied in [26]. A
number of more recent work were based on these early
results, e.g. [4] that studied exponential stabilization
using Lyapunov approach, [12] in which exponential
stabilization for homogeneous systems were thoroughly
investigated.

Among the results that are available in the literature,
there is hardly any that considered input-to-state sta-
bilization using time-varying control. Input-to-state
stability (ISS) is a type of robust stability for nonlin-
ear systems with inputs (see [20, 22]). Indeed, ISS
is very important, especially when dealing with sys-
tems in the presence of disturbances. The first papers
presenting Lyapunov characterization of ISS for time-
varying nonlinear systems are [3, 10]. The authors of
[15] have studied the problem in a different way, using
the averaging technique as the main tool. All the men-
tioned work considered continuous-time systems. To
the best of the authors knowledge, the only results on
discrete-time systems are given in [6, 14], where asymp-
totic stability for discrete-time time-varying systems is
studied. In [5], the same authors have used the results
of [6] to prove a converse Lyapunov theorem for ISS
for discrete-time time-invariant systems.

The importance of ISS and the scarcity of existing
results considering this property in the context of
discrete-time time-varying systems, have motivated
the authors to study the Lyapunov characterization of
ISS in a semiglobal practical sense, for discrete-time
time-varying systems. We consider a general parame-
terized family of discrete-time time-varying nonlinear
systems, which commonly appears when performing
sampled-data control design as discussed in [16]. Our
main result is a converse Lyapunov theorem that can
be seen as a discrete-time counterpart of the result of
[3], at the same time as a generalization of the results
of [5, 6]. We also present an application of our main
result to time-varying periodic systems and use this to
design a SP-ISS controller of a mobile robot [7, 19].

The paper is organized as follows. In Section 2, we
present preliminaries, to introduce notation and defi-
nitions. The main result is given in Section 3. Section
4 is dedicated to an application followed by example.
The paper is concluded with a summary in Section 5.
The sets of real and natural numbers (including 0) are denoted by \( \mathbb{R} \) and \( \mathbb{N} \), respectively. A function \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) is of class \( K \) if it is continuous, strictly increasing and zero at zero. It is of class \( K_\infty \) if it is of class \( K \) and unbounded. Functions of class \( K_\infty \) are invertible. A continuous function \( \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is of class \( K \) if \( \beta(\cdot, \tau) \) is of class \( K \) for each \( \tau \geq 0 \) and \( \beta(s, \cdot) \) is decreasing to zero for each \( s > 0 \). Given two functions \( \alpha(\cdot) \) and \( \gamma(\cdot) \), we denote their composition and multiplication by \( \alpha \circ \gamma(\cdot) \) and \( \alpha(\cdot) \times \gamma(\cdot) \), respectively.

In this paper, we consider a general parameterized family of discrete-time systems with input:

\[
x(k+1) = F_T(k, x(k), d(k))
\]

where \( x \in \mathbb{R}^n \), \( d \in \mathbb{R}^m \) are respectively the states and exogenous inputs to the system, and the parameter \( T > 0 \) is the sampling period. Systems having the form \( (1) \) commonly appear as a result of discretizing a nonlinear system

\[
\dot{x} = f(t, x(t), d(t))
\]

and letting the sampling period \( T \) as a free parameter to be chosen. Assume that \( f \) is locally Lipschitz and \( f(0,0,0) = 0 \). Without loss of generality, we may assume the same conditions for \( F_T \).

For any inputs \( d : \mathbb{N} \to \mathbb{R}^m \), we define \( \|d\|_\infty := \sup_{k \in \mathbb{N}} |d(k)| \). We use the notation \( U_B \), for the set of inputs \( d \) such that \( \|d\|_\infty \leq 1 \). We define \( x_0 := x(k_0) \), \( k_0 := k(0) \geq 0 \), and \( \text{Id} \) for the identity function, i.e., \( \text{Id}(s) = s \), and for any function or variable \( h \) we use the simplified notation \( h(k, \cdot) := h(kT, \cdot) \).

We emphasize that, for nonlinear systems, the exact discrete-time model \( F_T^e(k, x(k), d(k)) \) is usually not known, since it requires solving a nonlinear initial value problem which is almost impossible in general (see [13] for more details). Throughout the paper, we assume that \( (1) \) is obtained by approximating the exact discrete-time model of \( (2) \). As a result of the approximation, there is a mismatch between the exact and the approximate solutions of the system. To guarantee that \( (1) \) is a good discrete-time approximate model of \( (2) \), we assume that \( F_T \) satisfies the following consistency property that is used to limit the mismatch.

**Definition 2.1 (One-step consistency)** [13] The family of approximate discrete-time models \( F_T^e \) is said to be one-step consistent with the exact discrete-time models \( F_T^e \) if given any strictly positive real numbers \( \Delta_x, \Delta_d \), there exist a function \( \varrho \in K_\infty \) and \( T^* > 0 \) such that

\[
|F_T^e - F_T| \leq T \varrho(T) \tag{3}
\]

holds for all \( k \geq k_0, T \in (0, T^*) \), all \( |x_0| \leq \Delta_x \), \( \|d\|_\infty \leq \Delta_d \). The one-step consistency property is commonly used in numerical analysis literature (see for instance [9, 13, 17, 25]). We emphasize that although \( F_T^e \) is not known, the consistency property is checkable. Conditions that can be used to check this property for nonlinear time-invariant systems are presented in [13], which are extendable to use for nonlinear time-varying systems. Therefore, we consider a semiglobal property, we assume that \( F_T^e \) and \( F_T \) are globally defined for small \( T \).

We will use the following definitions and technicalities to construct and prove our main results. Note that these definitions are modifications of those given in [5, 6].

**Definition 2.2 (Semiglobal practical ISS)** The family of systems \( (1) \) is semiglobally practically input-to-state stable (SP-ISS) if there exist \( \beta \in K_L \) and \( \gamma \in K, \) such that for any strictly positive real numbers \( \Delta_x, \Delta_d, \delta \), there exists \( T^* > 0 \) such that the solutions of the system satisfy

\[
|x(k, k_0, x_0, d)| \leq \beta(|x_0|, (k - k_0)T) + \gamma(||d||_\infty) + \delta
\]

(4)

for all \( k \geq k_0, T \in (0, T^*) \), all \( |x_0| \leq \Delta_x \) and \( ||d||_\infty \leq \Delta_d \). Moreover, if the input \( d = 0 \), the system is semiglobally practically asymptotically stable (SP-AS).

**Definition 2.3 (SP-ISS Lyapunov function)** A family of continuous functions \( V_T : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+ \geq 0 \) is a family of SP-ISS Lyapunov functions for the family of systems \( (1) \) if there exist functions \( \underline{\alpha}, \overline{\alpha}, \alpha \in K_\infty \), \( \chi \in K \) and for any strictly positive real numbers \( \Delta_x, \Delta_d, \nu_1, \nu_2 \), there exists \( T^* > 0 \), such that the following inequalities

\[
\begin{align*}
\underline{\alpha}(|x|) & \leq V_T(k, x) \leq \overline{\alpha}(|x|) \	ag{5} \\
|x| & \geq \chi(||d||) + \nu_1 \Rightarrow \\
V_T(k + 1, F_T) - V_T(k, x) & \leq -T\alpha(|x|) \	ag{6} \\
V_T(k + 1, F_T) & \leq V_T(k, x) + \nu_2, \quad \tag{7}
\end{align*}
\]

hold for all \( k \geq k_0, T \in (0, T^*) \), all \( |x| \leq \Delta_x \) and \( ||d|| \leq \Delta_d \). Moreover, if \( d = 0 \), the function \( V_T \) is called a SP-AS Lyapunov function. \( V_T \) is called a smooth Lyapunov function if it is smooth in \( x \in \mathbb{R}^n \).

**Remark 2.1** By continuity of solutions, condition (7) is not needed in the continuous-time context, whereas it is required in the SP-ISS Lyapunov characterization to guarantee boundedness of trajectories, particularly for the case when \( |x| < \chi(||d||) + \nu_1 \) (see [16] for more details).

**Definition 2.4 (\( \Delta \)-UBIBS)** The family of systems \( (1) \) is \( \Delta \)-uniformly bounded input bounded state (\( \Delta \)-UBIBS) if there exist functions \( \sigma_1, \sigma_2 \in K \), and for
any strictly positive real numbers $\Delta_x, \Delta_d, \nu_b, \delta_b$, there exists $T^* > 0$ such that the following inequality:
\[
\sup_{k \geq k_0} |x(k, k_0, x_0, d)| 
\leq \max\{\sigma_1(|x|) + \nu_b, \sigma_2(|d|)\} + \delta_b, \tag{8}
\]
holds for all $k \geq k_0$, $T \in (0, T^*)$, all $|x_0| \leq \Delta_x$ and $|d| \leq \Delta_d$. By causality, (8) is equivalent to $\sigma_1(s) \geq s$ and
\[
|x(k, k_0, x_0, d)| 
\leq \max_{0 \leq j \leq k-1} \{\sigma_1(|x_0|) + \nu_b, \sigma_2(|d(j)|)\} + \delta_b. \tag{9}
\]

Remark 2.2 Instead of (8), we could write
\[
\sup_{k \geq k_0} |x(k, k_0, x_0, d)| 
\leq \max\{\sigma_1(|x_0|), \sigma_2(|d|)\} + \delta, \tag{10}
\]
where $\delta := \nu_b + \delta_b$ (similarly for (9)). However, we have chosen to use (8) (respectively (9)) for convenience in proving our main result.

Definition 2.5 ($\mathcal{K}$-asymptotic gain) The family of systems (1) has a $\mathcal{K}$-asymptotic gain if there exists a function $\gamma_a \in \mathcal{K}$ and for any strictly positive real numbers $\Delta_x, \Delta_d, \pi$, there exists $T^* > 0$, such that
\[
\lim_{k \to \infty} |x(k, k_0, x_0, d)| \leq \gamma_a \left(\lim_{k \to \infty} |d(k)|\right) + \pi, \tag{11}
\]
for all $k \geq k_0$, $T \in (0, T^*)$, all $|x_0| \leq \Delta_x$, and $|d| \leq \Delta_d$.

Definition 2.6 (SP Robust stability) The system (1) is semiglobally practically robustly stable (SPRS), if there exists a function $\gamma_a \in \mathcal{K}$ and for any strictly positive real numbers $\Delta_x, \Delta_d, \delta$, there exists $T^* > 0$, such that for all $k \geq k_0$, $T \in (0, T^*)$, all $|x_0| \leq \Delta_x$, and $d \in \mathcal{U}_g$ such that $\|d(k)\|_{\infty} \leq \Delta_d$, the function
\[
x(k+1) = F_T(k, x, d\{x\}) := G_T(k, x, d) \tag{12}
\]
is SP-AS.

Lemma 2.1 [6] For any $\mathcal{KL}$ function $\beta$, there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that
\[
\beta(s, r) \leq \rho_1(\rho_2(s)e^{-r}), \quad \forall s \geq 0, \forall r \geq 0. \tag{13}
\]

Lemma 2.2 (Comparison Principle) [6] For any $\mathcal{K}$-function $\alpha$, there exists a $\mathcal{KL}$-function $\beta_\alpha(s, t)$ with the following property: if $y : \mathbb{N} \to [0, \infty)$ is a function satisfying
\[
y(k+1) - y(k) \leq -\alpha(y(k)) \tag{14}
\]
for all $0 \leq k < k_1$ for some $k_1 \leq \infty$, then
\[
y(k) \leq \beta_\alpha(y(0), k), \quad \forall k < k_1. \tag{15}
\]

3 Main Result

In this section, we state and prove our main result, namely a converse Lyapunov theorem for SP-ISS for parameterized family of discrete-time time-varying nonlinear systems. We provide a necessary and sufficiency conditions for which a parameterized family of discrete-time time-varying nonlinear systems is input-to-state stable in a semiglobal practical sense. This result is a discrete-time counterpart of [3], and it generalizes the main results of [5, 6].

The technique used in proving our results is similar to the technique that has been used in [6]. However, there are more technicalities needed to treat the semiglobal practical property we consider. This is also the first proof of a converse Lyapunov theorem for stability property in a semiglobal practical sense. In the next section, we present an engineering example, which shows the usefulness of our results from a practical point of view, since we very often have to deal with semiglobal practical property when designing a discrete-time controller for a continuous-time plant.

We are now ready to state our main result.

Theorem 3.1 The parameterized family of discrete-time time-varying systems (1) is SP-ISS if and only if it admits a smooth SP-ISS Lyapunov function $V_T$. ■

Before we proceed with proving Theorem 3.1, we first state and prove the following lemmas, which are instrumental in constructing the proof of the theorem. The proofs of Lemmas 3.2 and 3.3 are given in [6].

Lemma 3.1 If the family of systems (1) is SP-ISS, then it is $\Delta$-UBIBS and it admits a $\mathcal{K}$-asymptotic gain. Moreover, the system is SPRS, and hence SP-AS. ■

Lemma 3.2 [6, Lemma 2.7] If there exists a continuous SP-ISS Lyapunov function $V_T$ with respect to a compact set $\mathcal{X}$, then there exists also a smooth one, $W_T$, with respect to the same set. Moreover, if $V_T$ is periodic with period $\lambda > 0$, then $W_T$ can be chosen to be periodic with the same period. ■

Lemma 3.3 [6, Lemma 2.8] Assume that system (1) admits a SP-ISS Lyapunov function $V_T$. Then there exists a smooth function $\rho \in \mathcal{K}_\infty$ such that $W_T = \rho V_T$ is also a SP-ISS Lyapunov function of (1), and (6) holds for some $\alpha \in \mathcal{K}_\infty$. ■

Proof of Lemma 3.1:

SP-ISS $\Rightarrow \Delta$-UBIBS + $\mathcal{K}$-asymptotic gain. Suppose that the system (1) is SP-ISS. Let $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ be as in Definition 2.2. By the property of $\mathcal{KL}$ functions, if we fix the second argument, then $\beta$ is a $\mathcal{K}$ function in its first argument. Hence, the $\Delta$-UBIBS property is directly implied. Also, by definition, the function $\gamma$ is the $\mathcal{K}$-asymptotic gain of system (1).
\(\Delta\)-UBIBS + \(K\)-asymptotic gain \(\Rightarrow\) SPRS \(\Rightarrow\) SP-AS. Suppose that the system (1) is \(\Delta\)-UBIBS and it admits a \(K\)-asymptotic gain. Let \(\sigma_1, \sigma_2 \in \mathcal{K}\) be as in Definition 2.4. Given any strictly positive numbers \(\Delta_x, \Delta_d, \nu_b, \delta_b\), there exist \(T^* > 0\), such that (8) holds for all \(k \geq k_o\), all \(T \in (0, T^*)\), \(|x| \leq \Delta_x, \|d\|_\infty \leq \Delta_d\). Without loss of generality, let the \(K\)-asymptotic gain

\[
\gamma = \sigma_2 ,
\]

and \(\pi = \delta_b\). Let the positive numbers \(\nu_c\) and \(\nu_d\) be such that

\[
\nu_c \geq \nu_b + \delta_b ,
\]

\[
\nu_d \leq \min_{s \in (0,1)} \left(\sigma_2(|x_c(k, k_c)|) - \sigma_2(|s| x_c(k, k_c)|)\right) ,
\]

(17)

and

\[
\nu_c - \nu_d < \nu_b .
\]

We have, from Definition 2.4, that \(\sigma_1 \geq \text{Id}\) for all \(s \geq 0\). Pick any function \(\rho \in \mathcal{K}_\infty\) such that

\[
\gamma \circ \rho(s) = s/2 , \quad \forall s \geq 0 .
\]

We will show that with the correct choice of \(\rho\), the system (12) is SP-AS.

Pick any initial condition such that \(|x_0| \leq \Delta_x\). Let \(x_\rho(k)\) denote the corresponding trajectory of system (12). We use the following claim:

**Claim.** \(\sigma_2 \circ \rho(x_\rho(k, k_c)) \leq \frac{1}{2}\sigma_1(|x_0|) + \nu_c , \quad \text{for all} \quad k \geq 0\).

**Proof of Claim.** Trivially the claim is true for \(x_0 = 0\). Suppose now we have nonzero initial states, \(x_0 \neq 0\). It is then obvious that the claim is true for \(k = 0\), since

\[
\sigma_2(\rho(|x_\rho(0)|)) = \gamma(\rho(|x_\rho(0)|)) \leq \frac{1}{2} |x_0|
\]

\[
\leq \frac{1}{2} \sigma_1(|x_0|) \leq \frac{1}{2} \sigma_1(|x_0|) + \nu_c .
\]

(21)

The last part to prove is for \(k > 0\). Let

\[
k_1 = \min \left\{ k \in \mathbb{N} | \sigma_2 \circ \rho(x_\rho(k, k_c)) \geq \frac{1}{2} \sigma_1(|x_0|) + \nu_c \right\} ,
\]

and note that \(k_1 > 0\). Suppose that the claim is false and hence \(k_1 < \infty\). For \(0 \leq k \leq k_1 - 1\), it holds that \(\sigma_2 \circ \rho(x_\rho(k, k_c)) \leq \frac{1}{2} \sigma_1(|x_0|) + \nu_c\). From (18) and (19), we have that

\[
\sigma_2(|d(k)| \rho(|x_\rho(k, k_c)|)) \leq \frac{1}{2} \sigma_1(|x_0|) + \nu_c - \nu_d
\]

\[
\leq \frac{1}{2} \sigma_1(|x_0|) + \nu_b
\]

for \(0 \leq k \leq k_1 - 1\). Consequently, it follows from the \(\Delta\)-UBIBS property of the system, in particular from

(8), that

\[
|x_\rho(k_1)| \leq \max_{0 \leq j < k_1 - 1} \{ \sigma_1(|x_0|) + \nu_b, \sigma_2(|d(j)| \rho(|x_\rho(j)|))\} + \delta_b
\]

\[
\leq \sigma_1(|x_0|) + \nu_b + \delta_b
\]

\[
\leq \sigma_1(|x_0|) + \nu_c ,
\]

(23)

which, by (16) and (20), implies that

\[
\sigma_2(\rho(|x_\rho(k_1)|)) \leq \frac{1}{2} |x_\rho(k_1)| \leq \frac{1}{2} \sigma_1(|x_0|) + \frac{\nu_c}{2}
\]

\[
< \frac{1}{2} \sigma_1(|x_0|) + \nu_c ,
\]

(24)

which contradicts the definition of \(k_1\). Hence, the claim is true.

An immediate consequence of the claim is that (23) holds for all \(k \in \mathbb{N}\) and that \(\lim_{k \to \infty} |x_\rho(k)|\) is finite. Using (16), and taking the limits on both sides of (11), we have

\[
\lim_{k \to \infty} |x_\rho(k)| \leq \lim_{k \to \infty} \gamma(|d(k)| \rho(|x_\rho(k)|)) + \pi
\]

\[
\leq \lim_{k \to \infty} |x_\rho(k)|/2 + \nu_c + \pi ,
\]

(25)

which shows that \(\lim_{k \to \infty} |x_\rho(k)| \leq 2(\nu_c + \pi)\), which is bounded for each trajectory, for all \(k \geq k_o\). This shows that (12) is SP-AS. Hence, this completes the proof of Lemma 3.1.

**Proof of Theorem 3.1** The proof follows closely the steps used in proving the converse Lyapunov theorem in [11], combined with the proof of Theorem 1 of [5] (see also [23]).

**Proof of sufficiency.** From the statement of the theorem, suppose that for any strictly positive real numbers \(\Delta_x, \Delta_d, \nu_1, \nu_2\), there exists \(T^* > 0\) such that for all \(T \in (0, T^*)\), \(|x| \leq \Delta_x, \|d\|_\infty \leq \Delta_d\), a smoothly unbounded continuous function \(V_T(k, x)\) is a SP-ISS Lyapunov function for the family of systems (1). Let the functions \(\alpha, \pi, \alpha, \chi\) be as in Definition 2.3 of SP-ISS Lyapunov function. Let \(\beta > 0\) be such that

\[
\max_{s \in (0, \Delta_\Delta)} \{ \alpha^{-1}(\pi(\chi(s) + \nu_1)) - \alpha^{-1}(\pi(\chi(s)))\} \leq \delta .
\]

(26)

We consider two cases:

**Case 1:** \(|x| \geq \chi(|d|) + \nu_1\)

Using (5) and (6), it is obvious that we can write

\[
V_T(k, x) \geq \tilde{\chi}(|d|) + \tilde{\nu}_1 \quad \Rightarrow \quad V_T(k + 1, F_T) - V_T(k, x) \leq -T\tilde{\alpha}(V_T(k, x)) ,
\]

(27)

by choosing \(\tilde{\chi} = \alpha \circ \chi\) and \(\tilde{\alpha} = \alpha \circ \pi^{-1}\). Note that from Lemma 3.3, since \(V_T\) is a smooth Lyapunov function, we can have \(\alpha \in \mathcal{K}_\infty\). Applying the comparison principle of Lemma 2.2, there exists a \(\mathcal{K}\)-function \(\beta_\alpha\), such that

\[
V_T(k, x) \geq \tilde{\chi}(|d|) + \tilde{\nu}_1 \quad \Rightarrow \quad V_T(k, x) \leq \beta_\alpha(V_T(k_o, x_0, k) , k) .
\]

(28)
Therefore, for all \( k \geq k_0 \), we can write
\[
V_T(k, x(k + k_0, k_0, x_0, d)) \leq \beta_0(V_T(k_0, x_0), k) .
\] (29)
Further, using (5) we obtain
\[
|x(k + k_0, k_0, x_0, d)| \leq \alpha^{-1} \circ \beta_0(V_T(k_0, x_0), k) \\
\leq \alpha^{-1} \circ \beta_0(\pi(x_0), k) \\
=: \beta(|x_0|, k) .
\] (30)
Hence, we can write
\[
|x(k, k_0, x_0, d)| \leq \beta(|x_0|, (k - k_0)T) .
\] (31)

Case 2: \(|x| < \chi(|d|) + \nu_1\)
From (5), we have that
\[
\alpha(|x|) \leq V_T(k, x) \leq \pi(|x|) \leq \pi(\chi(|d|) + \nu_1) ,
\] (32)
which implies that
\[
|x(k, k_0, x_0, d)| \leq \alpha^{-1}(\pi(\chi(|d|) + \nu_1)) \\
\leq \gamma(|d|) + \delta \\
\leq \gamma(||d||_{\infty}) + \delta ,
\] (33)
where \( \gamma := \alpha^{-1} \circ \pi \circ \chi. \)
Combining (31) and (33), we have that for any \(|x| \leq \Delta_x, \ ||d||_{\infty} \leq \Delta_d \) the following holds:
\[
|x(k, k_0, x_0, d)| \leq \beta(|x_0|, (k - k_0)T) + \gamma(||d||_{\infty}) + \delta ,
\] (34)
and this completes the proof of sufficiency.

**Proof of necessity.** Suppose that the system (1) is SP-ISS. Given any arbitrary strictly positive numbers \( \Delta_x, \Delta_d, \delta \), let the numbers generate \( T^*_1 > 0 \) and let \( T^* := \min(1, T^*_1) \), such that (4) holds for all \( k \geq k_0, T \in (0, T^*), \ |x| \leq \Delta_x, \ ||d||_{\infty} \leq \Delta_d \). We have shown in Lemma 3.1 that SP-ISS implies SPRS with input \( dp(x_0) \), where \( d \in U_B \) and \( \rho \in \mathcal{K}_{\infty} \). This further implies that the system is SP-AS. By Lemma 3.1, let the numbers \( \Delta_x, \Delta_d, \delta \) generate \( \delta > 0 \), such that for all \(|x| \leq \Delta_x, \ d \in U_B, \ k \geq k_0 \) and all \( T \in (0, T^*) \) the following holds:
\[
|x(k + k_0, k_0, x_0, dp(|x|)| \leq \beta(|x_0|, k + \delta) .
\] (35)
By Lemma 2.1, there exist \( \rho_1, \rho_2 \in \mathcal{K}_{\infty} \) such that
\[
|x(k + k_0, k_0, x_0, dp(|x|)| \leq \rho_1(\rho_2(|x|))e^{-k} + \delta.
\] (36)
Define \( \omega := \rho_1^{-1}, \) and let \( \delta > 0 \) be such that
\[
\max_{s \in [0, \Delta_x]} \left( \omega(\rho_1(\rho_2(s))e^{-k}) + \delta - \rho_2(s)e^{-k} \right) \leq \delta. \] (37)
From (36) and (37), we obtain
\[
\omega(|x(k + k_0, k_0, x_0, dp(|x|)|) \leq \rho_2(|x_0|)e^{-k} + \delta.
\] (38)
Since \( \omega \) and \( \rho_2 \) are \( \mathcal{K}_{\infty} \) functions, we can always find \( \hat{\rho}_2 \in \mathcal{K}_{\infty} \) such that
\[
\omega(|x(k + k_0, k_0, x_0, dp(|x|)|) \leq \hat{\rho}_2(|x_0|)e^{-k} \leq \rho_2(|x_0|)e^{-k} + \delta.
\] (39)
Define
\[
V_{\hat{\omega}}(k, x_0, dp(|x|)) = \sum_{k=0}^{\infty} \omega(|x(k + k_0, k_0, x_0, dp(|x|)|). \] (40)
It then follows from (39) that
\[
\omega(|x_0|) \leq V_{\hat{\omega}}(k_0, x_0, dp(|x|)) \leq \sum_{k=0}^{\infty} \hat{\rho}_2(|x_0|)e^{-k} \\
\leq \frac{e}{e - 1} \hat{\rho}_2(|x_0|).
\] (41)
This shows that the series in (41) is convergent, uniformly on \( x_0 \) with \( |x_0| \leq \Delta_x \) and on \( d \in U_B \). Since for each \( \hat{\omega} \in \mathcal{K}_{\infty} \), \( \omega \) is continuous uniformly on \( d \in U_B \), then so is \( V_{\hat{\omega}} \). Define \( V_T \) by
\[
V_T(k, x_0) = \sup_{d \in U_B} V_{\hat{\omega}}(k_0, x_0, dp(|x|)). \] (42)
It then follows immediately from (41) that
\[
\omega(|x_0|) \leq V_T(k, x_0, dp(|x|)) \leq \frac{e}{e - 1} \hat{\rho}_2(|x_0|).
\] (43)
Hence, by taking \( \omega(s) := \omega(s) \) and \( \bar{\omega}(s) := \frac{e}{e - 1} \hat{\rho}_2(s) \) we show that (5) holds.

To prove the continuity of the Lyapunov function \( V_T(k, x) \), we use Lemma 4.4 of [6] that is directly valid for our case.

In the following, we show that \( V_T \) admits a desired decay estimate as in (6).
Pick any \( k_0, x_0 \) such that \( |x_0| \leq \Delta_x \), and any \( \mu \in U_B \). Let the exact solution \( x_f := F_T^*(k_0, x_0, \mu dp(|x|)) \) and the approximate solution \( x := F_T(k_0, x_0, \mu dp(|x|)) \), with \( \mu := d(k_0) \). Since \( F_T \) is one-step consistent with \( F_T^* \), we have that
\[
|x_f - x| \leq T\eta(T), \ \nu \in \mathcal{K}_{\infty}.
\] (44)
Let \( T^* \leq 1 \) be sufficiently small such that by the continuity of \( V_T \) and the one-step consistency property of \( F_T \), we may assume the existence of \( \bar{\eta} \in \mathcal{K}_{\infty} \) such that the following holds for all \( T \in (0, T^*) \)
\[
|V_T(k_0 + 1, x_f) - V_T(k_0 + 1, x_f)| \leq T\bar{\eta}(T).
\] (45)
Let \( \nu > 0 \) be such that
\[
\bar{\eta}(T^*) \leq \nu .
\] (46)
By uniqueness of exact solutions, we can see that for any \( d \in U_B \) such that \( d(k_0) = \mu \), it holds that
\[
x(k + k_0 + 1, k_0 + 1, x_f, dp(|x|)) = x(k + k_0 + 1, k_0, x_0, dp(|x|)).
\] (47)
for all \( k \geq 0 \). Hence, using (46) and \( T^* \leq 1 \), we have
\[
V_T(k_0 + 1, x_F) = V_T(k_0 + 1, x_F) + V_T(k_0 + 1, x_F) - V_T(k_0 + 1, x_F) \\
\leq V_T(k_0 + 1, x_F) + T\beta(T) \\
\leq \sum_{k=0}^{\infty} \omega(|x(k + k_0 + 1, k_0 + 1, k_0 + 1, x_F, dp(|x|)|) + T\nu \\
\leq \sum_{k=0}^{\infty} \omega(|x(k + k_0 + 1, k_0, x_o, dp(|x|)|) + T\nu \\
\leq \sum_{k=0}^{\infty} \omega(|x(k + k_0, k_0, x_o, dp(|x|)|) + T\nu \\
\leq \sum_{k=0}^{\infty} \omega(|x(k + k_0, k_0, x_o, dp(|x|)|) \\
\quad \quad \quad - \omega(|x(k_0, k_0, x_o, dp(|x|)|) + T\nu \\
\leq V_T(k_0, x_0) - \omega(|x_0|) + T\nu \\
\leq V_T(k_0, x_0) - T\omega(|x_0|) + T\nu .
\]

This shows that
\[
V_T(k_0 + 1, F_T(k_0, x_0, dp(|x|))) - V_T(k_0, x_0) \\
\leq -T\omega(|x_0|) + T\nu ,
\]
for all \( |x| \leq \Delta_x \) and all \( d \in \mathcal{U}_B \). Observe that this is equivalent to
\[
|u| \leq \rho(|x|) \Rightarrow V_T(k_0 + 1, F_T(k_0, x_0, u)) - V_T(k_0, x_0) \leq -T\omega(|x_0|) + T\nu ,
\]
and further it is obvious that it is also equivalent to
\[
|x| \geq \chi(|u|) + \nu_1 \Rightarrow V_T(k_0 + 1, F_T(k_0, x_0, u)) - V_T(k_0, x_0) \leq -T\alpha(|x_0|) ,
\]
by defining \( \chi := \rho^{-1} \) and \( \alpha := \frac{4}{3} \omega \) and \( \nu_1 \leq \omega^{-1}(4\nu) \). Hence, (6) is satisfied.

Note however that the continuous Lyapunov function obtained in the proof is not necessarily smooth. To show the existence of a smooth Lyapunov function for (1) and to show that \( \alpha \in \mathcal{K}_{\infty} \), we use Lemmas 3.2 and 3.3. Using Lemma 3.2, we can show the existence of a smooth Lyapunov function \( W_T \) as a continuous Lyapunov function \( V_T \) exists and using Lemma 3.3 it can be shown that if the Lyapunov function is smooth, there exists \( \alpha \in \mathcal{K}_{\infty} \) such that (6) holds.

The last thing to show is that (7) holds. We have assumed that \( F_T \) is globally defined for small \( T \), so that \( F_T \) is finite for all \( k \geq k_0 \), all \( |x_0| \leq \Delta_x \) and \( |d| \leq \Delta_d \). Then there exists \( c > 0 \) such that
\[
|F_T - x_0| \leq c , \quad \forall k \geq k_0 .
\]

Moreover, by Lemma 3.2 we may assume that \( V_T \) is smooth. Then using (52) and the smoothness of \( V_T \), we obtain
\[
V_T(k_0 + 1, F_T(k_0, x_0, u)) - V_T(k_0, x_0) \leq L |F_T - x_0| \leq Lc := \nu_2 ,
\]
with \( L \) is the Lipschitz constant of \( V_T \). Hence (7) holds, and this completes the proof of necessary. Therefore, the proof of Theorem 3.1 is complete.

\section{4 Application and example on periodic systems}

\subsection*{4.1 Application to periodic systems}

In this section we focus on a particular class of time-varying nonlinear systems, namely time-varying nonlinear periodic systems, which include a large class of systems. This class of systems is very important in various applications, particularly in tracking control problems (see for instance [12, 19, 24, 26]).

We consider a family of parameterized periodic discrete-time time-varying systems. The system (1) is called a periodic system if \( F_T \) is periodic in \( k \) with period \( \lambda > 0 \), and hence we have the following
\[
F_T(k + m\lambda, x, d) = F_T(k, x, d) , \quad m \in \mathbb{N} .
\]

By Theorem 3.1 we conclude that if the system is SP-ISS then it is \( \Delta \)-UBIBS and it admits a \( \mathcal{K} \)-asymptotic gain. This further implies that for some function \( \rho \in \mathcal{K}_{\infty} \) the corresponding system is SPRS and hence SP-AS. For a periodic system such that (54), we can show that the map
\[
G_T(k, x, d) := F_T(k, x, dp(|x|))
\]
is also periodic in \( k \) with the same period as \( F_T \). Moreover, we can show that there exists a SP-ISS Lyapunov function \( V_T \) that is periodic with period \( \lambda \), that satisfies
\[
V_T(k_0 + m\lambda, x) = V_T(k_0, x) ,
\]
as has been proved in [6]. Hence, the following corollary follows directly from Theorem 3.1.

\begin{corollary}
The parameterized family of time-varying periodic system (1) with period \( \lambda \) is SP-ISS if and only if it admits a smooth SP-ISS periodic Lyapunov function with the same period \( \lambda \).
\end{corollary}

The proof of Corollary 4.1 follows the same steps as the proof of Theorem 3.1 (see also [6]), hence it is not presented in the paper.
4.2 Example
Consider the model of a simple mobile robot moving on a plane, with two independent rear motorized wheels as illustrated in Figure 1 [7, 19]:

\[
\begin{align*}
\dot{x} &= v \cos \theta + d \sin \theta \\
\dot{y} &= v \sin \theta - d \cos \theta \\
\dot{\theta} &= \omega,
\end{align*}
\]

with \(v\) the forward velocity, \(\omega\) the steering velocity, \((x, y)\) the Cartesian position of the center of mass of the robot, \(\theta\) the heading angle from the horizontal axis, and \(d\) a disturbance force perpendicular to the forward direction. The system (56) is a benchmark example of systems which are not stabilizable using continuous feedback [1].

Using the coordinates transformation

\[
\begin{align*}
x_1 &= x \cos \theta + y \sin \theta \\
x_2 &= x \sin \theta - y \cos \theta \\
x_3 &= \theta,
\end{align*}
\]

we obtain the dynamic model of system (56) in power form:

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= x_1 u_2 + d \\
\dot{x}_3 &= u_2,
\end{align*}
\]

where \(u_1 := v - \omega x_2\), and \(u_2 := \omega\).

The stabilization problem for system (58) in the absence of disturbances has been studied in [19]. Using the Lyapunov function

\[
V(t, x) = \frac{1}{2}(x_1 + (x_2^2 + x_3^2) \cos t)^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2,
\]

which is a time-varying periodic function, the controller

\[
\begin{align*}
u_1 &= (x_2^2 + x_3^2) \sin t - (x_1 + (x_2^2 + x_3^2) \cos t) \\
u_2 &= -2(x_1 + (x_2^2 + x_3^2) \cos t)(x_1 x_2 + x_3) \cos t \\
&\quad - (x_1 x_2 + x_3)
\end{align*}
\]

has been designed. From the time derivative of the Lyapunov function

\[
\dot{V}(t, x) = -\left(2(x_1 + (x_2^2 + x_3^2) \cos t)(x_1 x_2 + x_3) \cos t + (x_1 x_2 + x_3) \right)^2 - (x_1 + (x_2^2 + x_3^2) \cos t)^2,
\]

and using La Salle Invariance Principle, it follows that in the case \(d = 0\), the closed-loop system (58), (60), (61) is uniformly globally asymptotically stable.

We consider now the case when we have a nonzero additive disturbance entering the second equation. We are interested in a particular step of the stabilization of system (58), using a discrete-time time-varying periodic controller that is designed based on the approximate model of the system. In particular, we use the Euler model of the system (58), namely

\[
\begin{align*}
x_1(k + 1) &= x_1(k) + Tu_1(k) \\
x_2(k + 1) &= x_2(k) + T(x_1(k) u_2(k) + d(k)) \\
x_3(k + 1) &= x_3(k) + Tu_2(k)
\end{align*}
\]

We emphasize that the Euler approximate model satisfies the one-step consistency we assume in constructing the results in this paper. We also need to point out that in this example we are not aiming to achieve SP-ISS for the system (58), but for the approximate model (63). However, it can be shown, following directly as what have been proved in [13, 18] for the time-invariant case, that under certain conditions the stability of the controlled exact discrete-time model is implied from the stability of the controlled approximate model, and the stability of the sampled-data system follows from the stability of the exact discrete-time models and boundedness of solutions.

We then apply our result, particularly Corollary 4.1, to check the SP-ISS property of the system (63) with a controller that is designed using the idea from [19]. Notice that for the rest of the paper, we drop the discrete-time argument \(k\) for simplicity.

It was shown by (62) that the derivative of the Lyapunov function (59) is negative semidefinite. Unfortunately, while we can apply La Salle Invariance Principle for systems without disturbance, we do not have such kind of tool for systems with inputs. Hence, (59) cannot be used to show input-to-state stability of the closed-loop system.

Using a similar idea as in [8], we construct another Lyapunov function that can be used to show ISS. We use the Lyapunov function

\[
V_T = g_1(V_{1T}) + g_2(V_{2T}), \quad g_1, g_2 \in \mathcal{K}_\infty,
\]

where \(V_{1T} = V\) and

\[
V_{2T} = V_{1T} - \epsilon x_1(x_2^2 + x_3^2) \sin t,
\]

with \(\epsilon > 0\) sufficiently small to guarantee that \(V_T \geq 0\). We have chosen the \(\mathcal{K}_\infty\) functions \(g_1 = g_2 = \text{Id}\). From

\[Figure 1: A two-wheeled drive mobile robot\]
(63) and (64) it is easy to show that conditions (5) and (7) hold.

The Lyapunov difference \( \Delta V_T \) is obtained as follows:

\[
\Delta V_T(k, x) = V_T(k+1, F_T) - V_T(k, x)
\]

\[
= (x_1(k+1) + (x_2(k+1) + x_3(k+1)) \cos((k+1)T))^2
\]

\[
+ x_2(k+1)^2 + x_3(k+1)^2
\]

\[
- \epsilon x_1(k)(x_2(k) + x_3(k)) \sin((k+1)T)
\]

\[
-(x_1(k) + (x_2(k) + x_3(k))) \cos((k+1)T)) - x_2(k)^2
\]

\[
- x_3(k)^2 + \epsilon x_1(k)(x_2(k) + x_3(k)) \sin(kT) \]

\[
= \left( x_1 + Tu_1 + ((x_2 + T(x_1u_2 + d))^2 + (x_3 + Tu_2)^2 )
\]

\[
\times \cos((k+1)T) \right)^2 - (x_1 + (x_2^2 + x_3^2)) \cos(kT))^2
\]

\[
+ (x_2 + T(x_1u_2 + d))^2 - x_2^2 + (x_3 + Tu_2)^2 - x_3^2
\]

\[
- \epsilon(x_1 + Tu_1)((x_2 + T(x_1u_2 + d))^2 + (x_3 + Tu_2)^2
\]

\[
\times \sin((k+1)T) + \epsilon x_1(x_2^2 + x_3^2) \sin(kT) .
\]

Assuming that the sampling period \( T \) is sufficiently small (0 < T < 1), we use the following approximation

\[
\cos((k+1)T) - \cos(kT) \approx T \sin(kT) \approx O(T^2), \quad (66)
\]

\[
\sin((k+1)T) - \sin(kT) \approx T \cos(kT) \approx O(T) . \quad (67)
\]

Assume also that \( \epsilon \) is sufficiently small (\( \epsilon = O(T) \)).

The Lyapunov difference can then be written as

\[
\Delta V_T(k, x)
\]

\[
\approx 2T u_1 \left[ x_1 + (x_2^2 + x_3^2) \cos((k+1)T) + 2T(x_1x_2 + x_3)
\]

\[
\times x_2 \cos((k+1)T) - \frac{\epsilon}{2} (x_2^2 + x_3^2) \sin((k+1)T) \right)
\]

\[
+ 2T u_2 (x_1 + x_3) \left( 1 - \epsilon x_1 \sin((k+1)T) \right)
\]

\[
+ 2(x_1 + (x_2^2 + x_3^2)) \cos((k+1)T)) \cos((k+1)T) \)
\]

\[
+ 2T u_2 (1 + 2(x_1 + (x_2^2 + x_3^2)) \cos((k+1)T))
\]

\[
\times \cos((k+1)T) + O(T^2) .
\]

Applying a discrete-time controller

\[
u_1 = - (x_1 + (x_2^2 + x_3^2) \cos((k+1)T))
\]

\[
- 2T(x_1x_2 + x_3)u_2 \cos((k+1)T)
\]

\[
\frac{\epsilon}{2} (x_2^2 + x_3^2) \sin((k+1)T) \quad (68)
\]

\[
u_2 = -(x_1x_2 + x_3) \left( 1 - \epsilon x_1 \sin((k+1)T) \right)
\]

\[
+ 2(x_1 + (x_2^2 + x_3^2)) \cos((k+1)T)) \)
\]

\[
\times \cos((k+1)T) \right) ,
\]

that is very similar to (60), (61), we will show that the closed-loop system (63),(68),(69) is SP-ISS. Substituting (68), (69) into the Lyapunov difference, we obtain

\[
\Delta V_T(k, x)
\]

\[
\leq -2T \epsilon^2 \left( \sin((k+1)T) \right)^2 + \frac{(x_1^2 + x_3^2)}{4} - 2T(x_1 + (x_2^2 + x_3^2) \cos((k+1)T))^2
\]

\[
- 2T(x_1x_2 + x_3)^2 \left( 2(x_1 + (x_2^2 + x_3^2) \cos((k+1)T))
\]

\[
\times \cos((k+1)T) + 1 \right)^2
\]

\[
+ 2T u_2 \left( 1 + 2(x_1 + (x_2^2 + x_3^2)) \cos((k+1)T))
\]

\[
\times \cos((k+1)T) + O(T^2) ,
\]

after adding a small positive offset \( a << T \) to avoid the first term on the right-hand side of the inequality to become zero at \( (k+1)T = i\pi, \ i \in \mathbb{N} \). Finally, we use Young’s inequality to arrive at

\[
\Delta V_T(k, x) \leq -T(M_1 |x_1|^2 + M_2 |x_2|^2 + M_3 |x_3|^4)
\]

\[
+ TM_4 |d|^2 + O(T^2) ,
\]

with \( M_i > 0, \ i \in \{1, \ldots, 4\} \). Therefore, it is obvious that (6) holds and hence the closed-loop discrete-time model (63),(68),(69) is SP-ISS. Moreover, notice that the Lyapunov function \( V_T \) is a periodic function with period 2\( \pi \), the same as the period of the closed-loop system (63),(68),(69).

5 Summary

We have presented a converse Lyapunov theorem for ISS for parameterized discrete-time time-varying systems. We have considered the ISS property of the systems in a semiglobal practical sense, which appears naturally in sampled-data design. We have also presented an application of our result to discrete-time time-varying periodic systems. Finally, by the provided example, we have illustrated the usefulness of our results from a practical point of view.

References


