Addendum to quasi-ideal transversals of abundant semigroups and spined products

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It has been brought to our attention that a very similar result to that of [2, Theorem 2.8] has recently been published in [5] by Xiangjun Kong.

Theorem 1 ([5, Theorem 3.1]) *Let L and R be abundant semigroups with a common adequate transversal S^0. Suppose that S^0 is a right ideal of L and a left ideal of R. Let $R \times L \to S^0$ described by $(a, x) \mapsto a * x$ be a mapping such that for any $x, y \in L$ and any $a, b \in R$*

1. $(a * x)y = a * xy$ *and* $b(a * x) = ba * x$;
2. *if* $x \in S^0$ *or* $a \in S^0$, *then* $a * x = ax$;
3. *For any* $b_1, b_2 \in R^1, y_1, y_2 \in L^1$ *if* $x_1 \mathcal{R}^* x_2$ *in* L, *then* $y_1(b_1 * x_1) = y_2(b_2 * x_1)$ *if and only if* $y_1(b_1 * x_2) = y_2(b_2 * x_2)$; *if* $a_1 \mathcal{L}^* a_2$ *in* R, *then* $(a_1 * y_1)b_1 = (a_1 * y_2)b_2$ *if and only if* $(a_2 * y_1)b_1 = (a_2 * y_2)b_2$.

Note that we have changed the notation slightly as Kong's use of R (resp. L) coincides with our use of L (resp. R). Although the results appear at first glance to be slightly different, it is possible to recover Kong's result from ours and vice-versa. We briefly demonstrate how to do this below. As mentioned in [2] we have based our approach on a similar result for inverse transversals given by Saito and described in [6], together with the spined product outline by Blyth in [4, Page 37] and the basic properties of R, L and quasi-ideals found in [1]. Kong's approach is similar to the split band approach of Blyth and McFadden in [3], which in turn is quoted by Saito in [6] and may be the inspiration for the previously mentioned outline in [4].

Lemma 2

1. *Let S be an abundant semigroup with an adequate transversal S^0. Then S^0 is a right ideal of S if and only if S^0 is a quasi-ideal of S and S is left adequate.*
2. *Let S^0, R and L be as in the statement of Theorem 1. Then for all $y, z \in L, a, b \in R$ with $\overline{y} = \overline{b}$, $(a * y)f_b * z = a * e_y(b * z)$.*
3. *Let S^0, R and L be as in the statement of [2, Theorem 2.8]. Then*
   (a) *for any $x, y \in L$ and any $a, b \in R$ $(a * x)y = a * xy$ and $b(a * x) = ba * x$;*
   (b) *for any $b_1, b_2 \in R^1, y_1, y_2 \in L^1$ if $x_1 \mathcal{R}^* x_2$ in L, then $y_1(b_1 * x_1) = y_2(b_2 * x_1)$ if and only if $y_1(b_1 * x_2) = y_2(b_2 * x_2)$; if $a_1 \mathcal{R}^* a_2$ in R, then $(a_1 * y_1)b_1 = (a_1 * y_2)b_2$ if and only if $(a_2 * y_1)b_1 = (a_2 * y_2)b_2$.*

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Proof.

1. Suppose that S^0 is a right ideal of S. Then $S^0SS^0 \subseteq S^0S^0 \subseteq S^0$ and so S^0 is a quasi-ideal of S. Now if $x \in S$ then $x = e_x\overline{x}f_x \in LS^0S \subseteq LS^0 \subseteq L$ and so from [1, Theorem 3.14] it follows that S is left adequate. Conversely, if S is left adequate and S^0 is a quasi-ideal of S then from [1, Corollary 3.6 and Theorem 3.14] we have $S^0S = S^0LR = S^0SS^0 \subseteq S^0$ as required.

2. Let $y, z \in L, a, b \in R$ with $\overline{y} = \overline{b}$. Then $(a*y)f_b*z = (a*y)(f_b*z) = (a*e_y\overline{y})(f_b*z) = (a*e_y)\overline{y}(f_b*z) = (a*e_y)(\overline{y}f_b*z) = (a*e_y)(\overline{b}f_b*z) = (a*e_y)(b*z) = a*e_y(b*z)$.

3. (a) From [2, Lemma 2.10] we see that $a*x = (a*x)f_x = (a*x)f_{\overline{x}}$ and so $(a*x)y = (a*x)*y = (a*x)f_{\overline{x}}*y = a*e_x(\overline{x}*y) = a*(e_x\overline{x}y) = a*xy$.

   (b) The proof follows a similar argument to that of [2, Lemma 2.9], making use of [1, Lemma 1.6] and is left as an exercise.

■

Consequently we can see that the conditions in [2, Theorem 2.8] imply those given by Kong in Theorem 1 and vice-versa. Moreover we see that property (3) of Theorem 1 is actually unnecessary and, from the proof of (2) above, that property (1) of Theorem 1 can be replaced by

$(1')$ for any $a \in R, x \in L, b, y \in S^0, (a*x)y = a*xy$ and $b(a*x) = ba*x$.

In addition we can use the lemma above to construct variants of the results in [2, Sections 3 & 4] by replacing property (1) with the corresponding property (1) of Theorem 1 (or $(1')$ if preferred), or by applying Lemma 2(1) above.

References

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