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On Hadrons and Inflatons:
The Holography of
Strongly Coupled Processes

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ON HADRONS AND INFLATONS: THE HOLOGRAPHY OF STRONGLY COUPLED PROCESSES

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Strong coupling presents problems for traditional quantum field theory as it escapes the purview of perturbation theory. The last decade has seen the emergence of a new tool to tackle such problems, arising from a new string duality that was discovered in the last years of the twentieth century. Gauge/gravity duality (or the AdS/CFT Correspondence) translates weakly coupled supergravity to strongly coupled gauge theory. The duality is holographic in that the field theory can be considered to reside on the boundary of the bulk space that hosts the gravitational theory. In this thesis, I will provide background to this theory, reviewing the theory behind QCD, string theory and the AdS/CFT Correspondence. I will then present two new applications of the Correspondence.

Firstly, I will present a model of hadronization (a problem of strongly coupled QCD), in which separating quarks are envisioned holographically as the endpoints of a string which lie on a D7-brane – the brane is embedded in a dilaton-flow geometry that exhibits such QCD-like properties as confinement. Upon breaking the string at its midpoint and reattaching it to the brane, a kink is imposed, which propagates to the end of the string and quickly jerks the endpoint. This in turn leads to the radiation of a worldvolume gauge field, which is dual to rho mesons in the field theory.

In the second work, I will apply the Correspondence to the problem of cosmological inflation. In particular, I will consider the case of an inflaton formed as a composite scalar in a strongly coupled gauge theory. Holography then allows us to study the time dependence of the development of this condensate and to see which aspects of running coupling are conducive to slow-roll inflation.
To my family, in memory of my grandfather; he would have loved to have seen this.
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Author’s Declaration

The original work presented in this thesis was carried out at the University of Southampton with my supervisor Prof. Nick Evans and other collaborators. Some of the material presented has previously been published as follows:


Section 5.5 is original to this thesis.

Chapters 1, 2 and 3 review the history and theoretical foundations of the key concepts considered in this thesis. A wide selection of texts [3][4][5][6][7][8][9][10][11] were instrumental in the production of these chapters; all background material is properly cited and no claims of originality are made with regards to their content.
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Finally, and most importantly, I must thank my parents for their unwavering love and support since year dot. To them, to my brothers and to my friends at home (who have never really understood what I do), I can only apologise for not including in the following thesis the blueprints for a lightsabre or a time machine.
Chapter 1

Introduction

The twentieth century saw prodigious advances in our understanding of the natural world, with the development of relativity on the one hand, and quantum theory on the other. Casting aside the most basic assumptions of Newtonian physics, these theories revolutionized our perception of the nature of space and time, and of forces and matter. Rapid progress on the experimental side, with particle accelerator energy increasing by an order of magnitude every couple of decades, combined with a powerful new theoretical armoury to allow insights into scales that were previously unimaginable. The theoretical tools that set the framework for this new understanding came from the synthesis of special relativity and quantum mechanics – quantum field theory. By the end of the 1970s, the field content and interactions of sub-atomic physics had become sufficiently well-established to be known as the “Standard Model” of particle physics and it is a paradigm that remains broadly intact to this day.

The biggest challenge for the Standard Model to accommodate has been the incorporation of gravity – to form a quantum theory consistent with not just special relativity, but also general relativity. Naive attempts to formulate a quantum theory of gravity founder on the rocks of non-renormalizability. To this end, various new approaches have been attempted, by far the most popular of which has been string theory, which started life modelling hadronic physics, but was later reborn as a theory of everything. String theory replaces point particles with extended

1
one-dimensional string-like objects, whose excitations provide a spectrum of particles. Since the infinities of renormalization stem from integrating over all scales, string theory evades this problem by providing a natural cut-off (i.e. the string length).

In 1997, string theory found a strange new application with Juan Maldacena’s conjecture of a correspondence between strings in a particular background and a gravity-less quantum field theory in a lower dimensional space. The origins of such a correspondence go back to Gerard ’t Hooft, who had shown that the large $N$ limit of an $SU(N)$ gauge theory should be describable by some form of string theory. Later, he and Leonard Susskind would propose a “Holographic Principle”\cite{12,13} – based on the observation that black hole entropy scales with the surface area of the event horizon\cite{14}, it was proposed that a quantum theory of gravity should be holographic, in that it should have a description in terms of a field theory defined on the boundary of its space. Such ideas were brought to fruition in the gauge/gravity duality of the AdS/CFT Correspondence and theories using the Correspondence are thus called “holographic”.

This particular duality brought string theory back to its origins in hadronic physics, as the limit in which the duality arises is the strong coupling limit of the field theory. The popularity of this area stems from its ability to tackle this other thorny area of quantum field theory – for in the strong coupling regime, the standard tools of perturbation theory are rendered useless since an expansion in the coupling coefficient no longer stable. In this thesis, I use the AdS/CFT Correspondence to tackle topics that are problematic in field theory since they involve strong coupling. Hadronization, the process by which colourless hadrons are formed in particle colliders, necessarily involves the low-$Q^2$ regime of QCD, where it is strongly coupled. In Chapter 4 I will find that a holographic string picture of hadronization suggests a large role to be played by an accelerated string endpoint, which radiates a worldvolume gauge field, dual to rho mesons in the field theory.

In the other main work, presented in Chapter 5, I look at the dynamics of a strongly coupled condensate that breaks a $U(1)$ symmetry in the vacuum, with a
particular interest in application to the topic of cosmological inflation. I will try to draw lessons from holography about how the running of the gauge coupling influences the time it takes for this transition to take place.

I hope that this work will provide useful additions to the literature.

1.1 The Standard Model

The Standard Model of particle physics is constructed as a gauge theory based on three groups:

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$ (1.1)

The electroweak sector corresponds to the $SU(2) \times U(1)$ part, which is broken by the formation of a Higgs vev to the electromagnetic $U(1)$, while the $W^\pm$ and $Z$ bosons of the weak interaction become massive. Quantum Chromodynamics (QCD) is the remaining $SU(3)$ sector.

The Lagrangian of a generic $SU(N_c)$ gauge theory takes the form

$$-\frac{1}{4} G^{\alpha}_{\mu \nu} G^{\alpha \mu \nu} + \bar{\Psi}_i \gamma^{\mu} (i \delta_{ij} \partial_\mu + g A^a_\mu T^a_{ij}) \Psi_j + m \bar{\Psi}_i \Psi_i$$ (1.2)

where $i, j$ are labels called “colour” and take the values $1, 2, \ldots, N_c$ (for simplicity, we have restricted the Lagrangian to one flavour). The matrices $T^a$ are the $N_c^2 - 1$ generators of $SU(N_c)$, normalized so that $\text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab}$. The group’s Lie algebra defines its structure constants, $f^{abc}$:

$$[T^a, T^b] = i f^{abc} T^c$$ (1.3)

While the quark fields $\Psi_i$ transform in the fundamental group representation, the gauge fields $A^a_\mu$ take indices, $a = 1, \ldots, N_c^2 - 1$, and so transform in the adjoint representation of $SU(N_c)$, where the generators are the structure functions, $(T^a)_{bc} = i f^{abc}$. The gauge field tensor $G^{\alpha}_{\mu \nu}$ is defined by

$$G^{\alpha}_{\mu \nu} = \frac{i}{g} [D_\mu, D_\nu]^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$ (1.4)
where $\mathbf{D}_\mu$ is the covariant derivative $(\partial_\mu - ig\mathbf{A})$. The corresponding term in the Lagrangian is easily seen to be gauge invariant.

The Lagrangian (1.2) possesses a global $U(1)$ symmetry $\Psi \to e^{i\alpha} \Psi$ and, in the absence of the mass term, an axial $U(1)_A$, $\Psi \to e^{i\alpha \gamma^5} \Psi$, which acts differently on left- and right-handed fields. The symmetry may be enlarged to $U(N_f)_L \times U(N_f)_R$ if there are $N_f$ flavours of quark. The axial $U(1)_A$ symmetry and the chiral $SU(N_f)$ parts of this group are explicitly broken by the appearance of a mass term, but even in the massless limit, the gauge dynamics may conspire to spontaneously break the symmetry. This is the case in QCD, and it gives rise to an isotriplet of light mesons ($\pi^0, \pi^\pm$), corresponding to the generators of the broken $SU(2)$ symmetry between up and down quarks. The QCD vacuum breaks chiral symmetry by generating a vev for the quark bilinear $\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}_L^\dagger \psi_R + \bar{\psi}_R^\dagger \psi_L \rangle$, which is clearly not invariant under chiral transformations (since left- and right-handed transformations cannot be independent). This chiral condensate sets a natural scale for the theory (around 300 MeV). Because the up, down and strange quark masses are significantly below this, they may be treated as approximately massless, giving rise to the light octet of pseudo-Goldstone bosons (pseudo-Goldstone, because the quarks are not strictly massless, and the symmetry is not exact). It should be a nonet, since the broken $U(1)_A$ should also contribute another light state, but this symmetry is in fact anomalous in the full quantum theory (it is restored in the ’t Hooft limit: constant $g^2 N_c, N_c \to \infty$).

1.1.1 Asymptotic Freedom and Confinement

Classical conformal invariance is broken in the full quantum theory by loop corrections, which lead to the renormalization of various quantities within the theory. The exact dependence of the gauge coupling on energy can vary significantly between different theories and its running is determined by the beta function:

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} \quad (1.5)$$
where $\mu$ is the energy scale of interest. In the case of the Abelian $U(1)_{QED}$ with one fermion and coupling $e$, $\beta(e) = \frac{e^3}{12\pi^2}$, and in the large $\mu$ limit, the dependence of $\alpha (= e^2/4\pi)$ is

$$\alpha(\mu) = \frac{\alpha_0}{1 - (\alpha_0/3\pi) \ln \mu^2/\mu_0^2}$$  \hspace{1cm} (1.6)$$

where $\alpha_0 = \alpha(\mu_0)$. Clearly, when $\mu^2 \sim \mu_0^2 \exp (3\pi/\alpha_0)$, the coupling becomes divergent and we can no longer trust the perturbation theory used to derive the result. Known as a Landau Pole, this is so far beyond even the Planck scale, where gravitational physics becomes relevant, that it is not of concern for typical applications.

The story for non-Abelian gauge theories is somewhat different. The gauge boson self-interactions contribute extra terms to loop calculations, allowing the beta function to be negative. For a theory with $N_c$ colours and $N_f$ flavours in a representation $r$ of the gauge group, the one-loop beta function is

$$\beta(g) = -g^3 (4\pi)^2 \beta_0,$$

$$\beta_0 = \frac{1}{3} (11N_c - 4T(r)N_f),$$  \hspace{1cm} (1.7)

$$\beta_0 = \frac{1}{3} (11N_c - 4T(r)N_f),$$  \hspace{1cm} (1.8)

where $T(r)$ is a constant defined by $\text{Tr}(t^a_r t^b_r) = T(r)\delta^{ab}$ ($t^a_r$ are the group generators). For flavours in the fundamental representation, $T(r) = \frac{1}{2}$. So, for $N_f < \frac{11}{2}N_c$, the beta function is negative, and the coupling becomes weaker at higher energies. Equation (1.8) has the solution

$$\alpha(\mu^2) = \frac{\alpha_0}{1 + (\frac{\alpha_0}{12\pi})(11N_c - 2N_f) \ln \left(\frac{\mu^2}{\mu_0^2}\right)}$$  \hspace{1cm} (1.9)$$

and for $N_f < \frac{11}{2}N_c$, there is no longer a Landau pole at high energies; rather, the coupling decreases as $(\ln \mu^2)^{-1}$. This is the phenomenon of asymptotic freedom. Although there may appear to be two parameters here, $\mu_0^2$ and $\alpha(\mu_0^2)$, there is actually only one, as the equation can be reformulated as

$$\alpha(\mu^2) = \frac{12\pi}{(11N_c - 2N_f) \ln \left(\frac{\mu^2}{\Lambda^2}\right)}$$  \hspace{1cm} (1.10)$$
where $\Lambda$ is the characteristic scale of the theory, around which the gauge becomes strong (although by that point, we can no longer trust the derivation). For QCD, $N_c = 3$ and $N_f = 6$ (as of the time of writing), so it does feature asymptotic freedom. This property was first established by Politzer $[15]$ and Gross and Wilczek $[16]$, and successfully explained the apparently free nature of partons observed in high-energy collisions.

The two-loop beta function has an extra contribution $[17]$:

$$
\beta(g) = -\frac{g^3}{(4\pi)^2} - \frac{g^5}{(4\pi)^4} \beta_1,
$$

$$
\beta_1 = \frac{1}{3} \left( 34N_c^2 - 20T(r)N_cN_f - 12T(r)C_2(r)N_f \right),
$$

where $C_2(r)$ is the quadratic Casimir of the representation, such that $\sum_a t^a t^a = C_2(r)1_r$. We can therefore see that there is a range of theories with $\beta_1$ negative, which may develop an infra-red fixed point.

In QCD, the value of $\alpha_s$ rises from 0.1184 at the scale of the Z mass to around 0.5 at 1 GeV $[18]$ giving a value for $\Lambda$ of about 200 MeV, which corresponds to the confining scale in strong processes ($\sim 1$ fm). This is the scale at which QCD becomes non-perturbative and what happens to the coupling below that scale is a matter for debate. It is in this low-energy regime where we see another important aspect of QCD – confinement. After Gell-Mann and Zweig initially proposed the existence of quarks $[19, 20, 21]$, it was a while before the physics community began to see them as real physical objects existing within hadrons, instead of just helpful mathematical abstractions. The reason for this was that despite extensive searches, no isolated quarks had ever been observed. The evidence for their existence accumulated, until they were fully accepted into the conventional picture of particle physics, but a free quark was never seen and hasn’t been to this day – no matter how hard a hadron is hit, it merely fragments into other hadrons, keeping its fractionally-charged constituents confined within. This property is not fully understood within field theory, but the fact that $g_s$ becomes strong at low energies (or equivalently, large distances) means that we can qualitatively build a picture of a force that gets stronger, the further apart a pair of quarks is pulled. Indeed, we
can picture the QCD field between the quarks as forming a flux tube like a string, with constant energy per unit length. As the quarks are separated, the energy in the flux tube builds up until it becomes more energetically favourable to pair-create a quark and anti-quark to break the string, than to let it continue growing.

1.1.2 Hadronization

This process of string-breaking and the formation of new hadrons is known as hadronization (or fragmentation) and, in the factorization approach, can be separated from the perturbative process of pair production and gluon (or quark/antiquark) showering when considering the physics of jet formation in particle colliders. Clearly, the hadronization stage is one that lies outside the remit of perturbative QCD, since it is inherently a problem of strong coupling. Attempts to describe jets largely consist in producing phenomenological models that assign lists of four-vectors to sublists (i.e. jets) using model-dependent algorithms \[22\]. Models for hadronization include the cluster model \[23,24,25\] (wherein gluons split into colour singlet “clusters”, which decay isotropically) and the thermal model \[26\], in which hadrons are produced in the multiplicities that follow from a thermal gas of hadrons. This reflects the observation that hadron mass is the dominant factor, rather than quantum number, in the suppression of heavy particles \[22\]. Indeed, production yields are well fitted by a $e^{-M/T}$ Boltzmannian relation. This is also reflected in the UCLA and Lund string models, wherein a matrix element that goes as $e^{-A}$ determines hadron yields, where $A$, the spacetime area swept out by a string connecting the quarks, is proportional to the hadron mass-squared. This picture of a pair of separating quarks as being connected by a string which breaks, splitting off segments, has resonance with the heuristic model that will be presented in this thesis. Recently, a holographic approach to calculating hadron yields, by calculating the overlap of a Gaussian distribution with the holographic basis states corresponding to different mesons, achieved reasonable success with relatively few free parameters \[27\].
1.2 Supersymmetry

An extension of the Standard Model that plays an important role in string theory is supersymmetry. It arose out of the work of Ramond, Neveu and Schwarz [28][29][30], who tried to incorporate fermionic degrees of freedom into dual resonance (string) models of the early seventies. In 1971, these degrees of freedom were interpreted by Gervais and Sakita [31] (and others [32][33]) as coming from the addition of fermionic operators on the string worldsheet. In the following few years, this worldsheet supersymmetry would be brought to the world of four-dimensional field theory by (independently) Gol’fand and Likhtman [34], Volkov and Akulov [35], and Wess and Zumino [36], who also detailed the supersymmetry algebra. The first encouraging phenomenological application came in 1981 with the Minimal Supersymmetric Standard Model of Georgi and Dimopoulos [37]. Much work has been done on the subject in the three decades since and there are hopes that some sort of evidence may be found at the Large Hadron Collider (though, at the time of writing, the LHC has not been particularly obliging).

Supersymmetry is a symmetry of a quantum field theory that exchanges bosons and fermions with the use of a fermionic symmetry generator, $Q_\alpha$ (a two-component Weyl spinor). The super-Poincaré algebra has the following form:

\begin{align}
[P^\mu, Q_\alpha] &= [P^\mu, \bar{Q}^{\dot{\alpha}}] = 0 \\
[M^{\mu\nu}, Q_\alpha] &= -i(\sigma^{\mu\nu})_{\alpha}{}^{\beta} Q_\beta \\
[M^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] &= -i(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \bar{Q}^{\dot{\beta}} \\
\{Q_\alpha, Q_\beta\} &= \{Q_\alpha, \bar{Q}^{\dot{\beta}}\} = 0 \\
\{Q_\alpha, \bar{Q}^{\dot{\beta}}\} &= 2(\sigma^{\mu})_{\alpha}{}^{\beta} P_\mu
\end{align}

where $P^\mu$ generates translations, $M^{\mu\nu}$ is the generator of Lorentz transformations ($\sigma^{\mu\nu}$ in the Weyl spinor representation) and $\bar{Q}_\alpha = (Q_\alpha)^*$. A representation of the Poincaré algebra is labelled by its eigenvalues under the Casimir operators $-P^2$ and $W^2$ (where the Pauli-Lubanski four-vector $W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma$), which are $m^2$ (the mass-squared) and $m^2 j(j+1)$, where $j$ is the angular momentum eigenvalue.
For massless representations, $W^2 = P^2 = 0$, $W^\mu = \lambda P^\mu$ and states are labelled by the helicity, $\lambda$. It is possible to show that representations of the super-Poincaré algebra form a multiplet that consists of states of helicity $\lambda$ and $\lambda - 1/2$. Thus a “chiral” supermultiplet consists of a spin-1/2 quark (or lepton) and a spin-0 “squark” (or “slepton”) and a “vector” supermultiplet consists of a spin-1 gauge field and a spin-1/2 “gaugino”. The two states are obtained from one another by the action of the $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$ operators, so these operators transform fermions into bosons and vice versa. Since, from (1.13), $P^2$ commutes with $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$, we see that members of the same supermultiplet must have the same mass.

The supersymmetry may be enlarged by the addition of extra supercharges (extended supersymmetry). In the absence of a central charge $Z^{AB}$, where

$$\{Q^A, Q^B\} = \epsilon_{\alpha\dot{\beta}}Z^{AB},$$

there exists an internal (R-symmetry) group $U(N)$, where $N$ is the number of supercharges. These extra supercharges extend the supermultiplets to include states of helicity $\lambda, \lambda - 1/2, \ldots, \lambda - N/2$. The requirement that there be no states of helicity $|\lambda| > 1$ limits the number of supercharges to $N \leq 4$. However, for $N \geq 2$, fermions of opposite chirality are included in the same multiplet. Since the electroweak sector of the Standard Model couples differently to left- and right-handed fields (and fields in the same supermultiplet must transform in the same representation of a gauge group), extended supersymmetry cannot be a viable theory of nature, though it is nevertheless useful to explore.

The symmetry thus far described is a global one but when supersymmetry is rendered local, it becomes supergravity and the graviton acquires a superpartner – a gravitino. In 1981, Green and Schwarz showed that superstring theory had spacetime supersymmetry and in the low energy approximation could be described by ten-dimensional supergravity [38,39]. This will be detailed in the next chapter.

### 1.3 Inflation

The second topic of interest will be cosmological inflation. Here, I present some of the basics, with some more depth explored in Chapter 5.
Big Bang cosmology has met with considerable success over the past century. The prediction of cosmic background radiation (CMB) with a black-body thermal spectrum was confirmed spectacularly by COBE in 1990 [40]. The cosmological estimates for the age of the universe fit consistently with the limits inferred from clusters and the predictions of nucleosynthesis match well the observed abundances of the light elements. Despite these clear successes, in the years after the term “Big Bang” was coined, a few problems with the model started to surface. To understand these, it is useful to go over the basics of the relevant cosmology.

The universe is observed to be homogeneous and isotropic on the largest scales (indeed, observed isotropy and the Copernican principle lead one to assume homogeneity). The large scale structure of space-time is thus described by the *Friedmann-Lemaître-Robertson-Walker metric*:

\[
ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)\]

(1.18)

where \( k \) is the curvature of the three-dimensional manifold. If \( k = 0 \), the space is flat and Euclidean; if \( k > 0 \), the space is positively curved (“closed”) and globally a 3-sphere, while if \( k < 0 \) the space is negatively curved and “open”.

A solution to the Einstein equations is then the *Friedmann equation*:

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}
\]

(1.19)

where \( H \) is the *Hubble parameter* and \( \rho \) is the energy density. In the case of zero curvature \((k = 0)\), the density is equal to the *critical density*, defined by

\[
\rho_c = \frac{3H^2}{8\pi G}.
\]

(1.20)

Defining new density parameters, \( \Omega = \rho/\rho_c \) and \( \Omega_k = -k/(a^2H^2) \), the Friedmann equation can be rewritten in the form

\[
\Omega + \Omega_k = 1
\]

(1.21)
The most up-to-date measurements of the cosmic energy budget show that $\Omega$ is very close to one ($\Omega_k \ll 1$) and consistent with exact flatness. The problem raised with this observation is that such a situation is highly unstable, as is clear from examining the time dependence of the curvature density parameter:

$$\Omega_k = \frac{-k}{a^2 H^2} = \frac{-\dot{a}}{a^2}$$

(1.22)

In the standard cosmology $\ddot{a}$ is negative (since expansion is resisted by gravitational attraction), so $\Omega_k$ monotonically increases and $\Omega_k \ll 1$ is an unstable state (in a matter-dominated universe, it grows as $t^{2/3}$). If the curvature is close to zero now, it must have been much closer to zero further back in the past ($\sim 10^{-16}$ around $t \sim 1$ sec). This is not inconsistent with Big Bang cosmology, but it does pose uncomfortable questions about fine-tuning in initial conditions (it is not entirely clear, however, whether this “Flatness Problem” really is fine-tuning, as some authors have argued [41]).

Another problem arises from the finite speed of light. Given a fully described cosmological evolution, one can define a *cosmological horizon*, the furthest distance light can have travelled since $t = 0$:

$$d_H(t) = \int_0^t \frac{dt'}{a(t')}$$

(1.23)

where $d_H$ is measured in comoving coordinates. Since this is the distance a beam of light can travel in time $t$, it provides a limit on the causal separation of different regions of the universe. When we calculate the horizon distance (assuming the standard cosmology) at the time of decoupling (when the Cosmic Microwave Background was produced), we find that it subtends little more than one degree on the night sky – regions separated by more than that could not have been in causal contact when the CMB was produced, so why is it that we observe near-perfectly isotropic thermal equilibrium?

These were questions Alan Guth found himself attempting to answer when, in 1980, he proposed a new model of early cosmology he called *inflation* [42][11]. Guth
had, at the time, been motivated by another problem: how to reconcile the observed absence of magnetic monopoles with the high abundance of monopoles predicted by Grand Unified Theories. Guth realised that a phase transition in the early universe could lead to a brief period of de Sitter expansion, which would dilute the monopole density to vanishing degrees. A de Sitter universe is one with a positive cosmological constant, which enters the Einstein equation as

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \]  

(1.24)

Clearly, it acts as an effective positive vacuum energy (with pressure, \( p = -\rho = -\frac{\Lambda}{\kappa} \)). The solution to (1.19) with constant \( \rho \) is \( a(t) \sim \exp Ht \). Such a period of evolution will see accelerated exponential expansion, which rapidly dilutes away unwanted relics. Guth soon realised it solved the horizon problem as well, since its effect was to take regions of the universe in causal contact and stretch their separation by enormous factors – in other words, shrinking the horizon:

\[ dH = \int_0^t \frac{dt'}{a(t')} = \int \frac{d\ln a}{\ddot{a}} \]  

(1.25)

With suitable choices of parameters, a situation of positive \( \ddot{a} \) can force \( dH \) to be arbitrarily small. As noted above, accelerated expansion pushes \( \Omega_k \rightarrow 0 \), solving the flatness problem.

Clearly, the defining condition of inflation is \( \ddot{a} > 0 \). From the second Friedmann equation,

\[ \frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3} (3p + \rho), \]  

(1.26)

our condition for an inflationary period must be \( 3p < -\rho \) (we assume a positive energy density). What must be the physical content of a theory with such conditions? Ordinary matter and radiation do not fit the bill. The simplest candidate is a scalar field, \( \phi \), which has energy density and pressure given by

\[ \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi) \]  

(1.27)

where we have assumed isotropy (\( \partial_x \phi(t, x) = 0 \)). Spatial gradient terms are
exponentially suppressed by the growing scale factor and would destroy inflation if they were dominant anyway, since they would lead to $p = -\rho/3$ and no accelerated expansion. Guth’s original idea was to have $\phi$ (called an *inflaton*) left in a metastable vacuum by a supercooled phase transition, leading to perfect de Sitter conditions. It was soon found that this model’s predictions could not be married to observation, so *new inflation* was born, which simply had a scalar field whose kinetic energy was dominated by its potential $\left[43, 44\right]$. With the relations (1.27), the condition $3p < -\rho$ becomes $\dot{\phi}^2 < V(\phi)$. A commonly used assumption is the *slow-roll approximation*, $\dot{\phi}^2 \ll V(\phi)$. In addition to slow-rolling, inflation must last long enough to accomplish enough flattening and dilution to satisfy its motivating requirements. These two conditions are equivalent to the smallness of the two slow-roll parameters, 

$$
\epsilon = \frac{m_p^2}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2, \quad \eta = -m_p^2 \left( \frac{V''(\phi)}{V(\phi)} \right)
$$

(1.28)

In Chapter 5 I will explore the possibility of constructing the inflaton as a strongly-coupled composite mesonic operator that develops from a symmetric state into one that breaks a $U(1)$ chiral symmetry. Using this setup, I shall investigate what aspects of the gauge coupling running provide for a long roll time.
Chapter 2

A Brief Introduction to Strings and Branes

2.1 Historical Overview

String theory arose in the late sixties when much theoretical work was focussed on the phenomenological S-matrix approach to hadronic physics. In 1969, Gabriele Veneziano [45] turned his attention to the problem of finding a phenomenological description of hadron scattering, which was symmetric in the $s$- and $t$-channel and reflected the poles along the Regge trajectories (families of resonances with the relation $J = \alpha' M^2 + \alpha_0$, where $J$ is spin, $M$ is the mass of the resonance and $\alpha'$ is called the Regge slope parameter). The Veneziano amplitude was an Euler beta function with arguments $s$ and $t$:

$$B(s, t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)} \quad (2.1)$$

This was soon interpreted by Nambu, Susskind and Fairlie and Nielsen [46,47,48] as the amplitude for the scattering of two open strings. The theory now developed as a model of hadrons in which the recently proposed quarks were found in mesons, attached to the ends of a small section of string (of length $\sim 1 \text{ GeV}^{-1}$) such that their energy scale was that of strong interactions. A great deal of activity would
follow in the five years following Veneziano’s work, but despite many new insights and the development of new additions to field theory, by the mid seventies the theory was struggling. While the theory had the promise to fulfill its purpose, it was unclear how to deal with certain aspects of the theory – what to do, for example, with the massless spin-2 particle that was a clear prediction of the closed string? Moreover, Lovelace had shown that unitarity was preserved only if spacetime had 26 dimensions to it [49]. By introducing extra degrees of freedom, it was possible to lower this number, but there was no clear consistent 3+1-dimensional theory to work with. The deep inelastic scattering experiments at SLAC had revealed a hadron structure with hard, point-like partonic constituents, which was difficult to square with the string picture. Most damagingly, by the middle of the seventies, the Standard Model of non-Abelian gauge theories had emerged as renormalizable and perturbative in the high-energy limit and successful in its union of the electromagnetic and weak interactions – the status of QCD as the best description of strong interactions became unassailable.

String theory appeared dead by this point, but it was to have another life. Just as QCD soared to prominence, strings came to be seen in a new light. Scherk and Schwarz [50,51], and independently Yoneya [52], observed that the theory had ambitions beyond hadrons alone. The spin-2 particle had the necessary properties to be a graviton (indeed, a massless spin-2 field could only be a graviton) and the vector and spinor fields could be identified as fundamental gauge bosons and “leptons” – string theory was, in other words, a candidate theory of everything, including quantum gravity. Now, the string scale was pushed back, potentially all the way to the Planck scale. The extra dimensions still had to be taken care of, but compactification became considered more seriously [53]. The theory was reborn, but it would be ten years before it properly matured, when Michael Green and John Schwarz showed that ten-dimensional superstring theory was free of anomalies [54], sparking the “First Superstring Revolution”. From this point on, much work would be pursued on the subject of strings as fundamental objects in a candidate theory of everything.

However, the link between string theory and strong interactions was down but
not out. In the 1970s, Gerard ’t Hooft had examined the large $N_c$ limit of a gauge theory at fixed ’t Hooft coupling $\lambda (= g^2 N_c$, where $g$ is the gauge coupling) and he had found that the theory could be described in terms of a string theory [55]. Many years later, in 1997, Juan Maldacena proposed his famous “AdS/CFT Conjecture”, linking particular formulations of string theory with strongly-coupled gauge theory [56]. The relevance of strings to hadronic physics, which could previously be viewed as the result of stringy flux tubes, could now be seen literally in terms of superstrings, just as in the theory of the early seventies. The particular duality introduced involved $\mathcal{N} = 4$ super-Yang-Mills, a conformal theory far removed from QCD. Much work has nevertheless proceeded on the basis that by introducing deformations that bring the theory closer to QCD, lessons about hadronic physics can be learned by studying the string theory.

A strength of the AdS/CFT Correspondence is also its weakness – the gravitational side is tractable when the curvature of space is small, which corresponds on the field theory side to the ’t Hooft coupling, $\lambda$, being large (i.e. strong coupling). This makes it difficult to test across the range of interaction strengths, but enough consistency checks have been performed to give the physics community confidence in its effectiveness, and it is now among the most active of research areas in high energy theory.

2.2 The Theoretical Foundations

As in any modern dynamical theory, the starting point for string theory is to construct an appropriate action. It is instructive to consider first the case of a 0+1-dimensional point particle, from which we may extrapolate to the 1+1-dimensional string.

The relativistic action for a point particle is given by the invariant length of its worldline:

$$S_{pp} = -m \int ds = -m \int \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau$$

(2.2)

where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ and $\tau$ is some monotonic function of $s$, which parameterizes the...
worldline. Note this preserves the full Poincaré invariance of the bulk space in which the particle moves. In addition, it has invariance under diffeomorphisms of the worldsheet coordinate, \( \tau \rightarrow \tau'(\tau) \). This form of the action suffers from two drawbacks: the square root in the integrand renders it difficult to quantize and it is clearly incapable of describing massless particles. The solution to this problem is to substitute an alternative action which is equivalent at the classical level. This can be achieved with the introduction of an auxiliary field (or \textit{einbein}) \( e(\tau) \):

\[
S'_{pp} = \frac{1}{2} \int d\tau (e^{-1} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - m^2 e) \quad (2.3)
\]

It is simple to derive the equation of motion for \( e \), substitute the solution back into \( S'_{pp} \) and see that it reduces to \( S_{pp} \).

The natural extension of the point-particle worldline action to a string should be the invariant area of the two-dimensional “worldsheet” swept out by the string (called the \textit{Nambu-Goto action} [57]):

\[
S_{NG} = -T \int d\sigma d\tau \sqrt{-\det \gamma_{ab}} \quad (2.4)
\]

Here, \( \gamma_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \), where \( X^\mu \) is the embedding of the string in the target space (\( \mu = 0, 1, \ldots, D \)) and \( \xi^a \) (\( a = 1, 2 \)) is a worldsheet coordinate, \( \sigma \) or \( \tau \). In the point particle case, the parameter \( m \) was the particle mass. Similarly, here, \( T \) is the energy per unit length of the string. It may be related to the Regge slope parameter by \( T = 1/(2\pi\alpha') \).

The Nambu-Goto action may have the advantage of naturalness, but it suffers the same drawbacks as does \( S_{pp} \). One may overcome the difficulty of the square root by trying a different action, akin to the second we tried in the case of the point particle. Instead of a single function \( e(\tau) \), we introduce a worldsheet tensor field to act as an auxiliary worldsheet metric, \( h_{ab} \).

\[
S_P = -\frac{T}{2} \int d^2 \sigma \sqrt{-h_{ab}} \gamma_{ab} \quad (2.5)
\]
where $h = \det h_{ab}$. We can derive the equations of motion for $h_{ab}$ by its variation:

$$
0 = \frac{\delta S}{\delta h_{ab}}
$$

(2.6)

$$
= -\frac{T}{2} \sqrt{-h} (\gamma_{ab} - \frac{1}{2} h_{ab} h^{cd} \gamma_{cd})
$$

(2.7)

$$
\Rightarrow \gamma_{ab} = \frac{1}{2} h_{ab} h^{cd} \gamma_{cd}
$$

(2.8)

using $\delta h = -h h_{ab} \delta h^{ab}$. From this, we derive $\frac{1}{4} \det \gamma_{ab} = \frac{1}{4} h (h^{cd} \gamma_{cd})^2$ and, plugging that back into (2.5),

$$
S_P = -\frac{T}{2} \int d^2 \sigma \sqrt{-h} h^{cd} \gamma_{cd}
$$

(2.9)

$$
= -\frac{T}{2} \int d^2 \sigma 2 \sqrt{-\det \gamma_{ab}} h^{cd} \gamma_{cd}
$$

(2.10)

$$
= -T \int d^2 \sigma \sqrt{-\det \gamma_{ab}}
$$

(2.11)

$$
= S_{NG}
$$

(2.12)

Thus, this action (usually called the Polyakov action) is classically equivalent to the Nambu-Goto action and we may use it for further purposes.

There are three important symmetries, two local and one global, under which the Polyakov action is invariant:

- **Diffeomorphism symmetry:**

  $$
  \sigma^a \rightarrow \sigma'^a, \quad h_{ab} \rightarrow \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} h_{cd}
  $$

- **Weyl symmetry:**

  $$
  \sigma^a \rightarrow \sigma^a, \quad h_{ab} \rightarrow e^\phi h_{ab}
  $$

- **Poincaré symmetry:**

  $$
  X^\mu \rightarrow X^\mu + a^\mu + \omega^\mu_\nu X^\nu
  $$

The last symmetry, with constant $a^\mu$ and $\omega^\mu_\nu$ ($\omega^\mu_\nu = -\omega^\nu_\mu$) is the same as in the Nambu-Goto case, and while $S_{NG}$ also had the diffeomorphism (or
reparameterization) symmetry, it is enlarged by the addition of the auxiliary metric. We can make use of these symmetries to eliminate spurious degrees of freedom in the metric and simplify the mathematical problem. Since the metric is a real symmetric two-dimensional tensor, it has only three degrees of freedom. The two diffeomorphisms of the worldsheet coordinates, \( \tau \) and \( \sigma \), remove two of those degrees and we can use the Weyl invariance to remove the third and gauge fix the metric to be flat, Minkowskian (up to global parameters):

\[
h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\] (2.13)

As a result, the action becomes

\[
S = -\frac{T}{2} \int d^2 \sigma \eta^{ab} \partial_a X \cdot \partial_b X
\] (2.14)

and the equation of motion for \( X^\mu \) is simply the wave equation:

\[
\Box X^\mu = 0
\] (2.15)

However, unlike in normal variational problems, where we may reasonably assume the fields go to zero at the boundary of our integration region, in the string we must be more careful and properly consider the boundary conditions. In addition to the wave equation, the variational problem also requires the vanishing of the boundary term

\[
n^a \delta X^\mu \partial_a X^\mu |_{\partial \omega}
\] (2.16)

where \( \partial \omega \) is the boundary of our integration region and \( n^a \) is a worldsheet vector perpendicular to that boundary. By convention, we simply consider the boundary of the space-like coordinate, \( \sigma \), and set it to run from 0 to \( \pi \):

\[
[\delta X^\mu X^\mu ]_{\sigma=\pi} - [\delta X^\mu X^\mu ]_{\sigma=0} = 0
\] (2.17)

This may be achieved by imposing periodic boundary conditions:

\( X^\mu (\sigma = \pi) = X^\mu (\sigma = 0) \). In this case the string is closed and the worldsheet has
the topology of a cylinder. Alternatively, we may have open strings which have Neumann boundary conditions: \(X'^\mu(\sigma = \pi) = X'^\mu(\sigma = 0) = 0\). Lastly, we may impose Dirichlet boundary conditions: \(\delta X^\mu = 0\), so that the endpoints are fixed:

\[
X^\mu(\tau, 0) = X_0^\mu \quad X^\mu(\tau, \pi) = X_\pi^\mu
\] (2.18)

This last option manifestly breaks Poincaré invariance, so it was ignored for many years. In 1989, Dai, Leigh and Polchinski (and independently, Hořava) discovered that such endpoint conditions necessarily arose in the context of T-duality, implying the existence of another class of objects in string theory: D-branes \([58,59]\). These will be important in the work that follows.

To derive equations of motion, we must include equations of motion for the worldsheet metric (which imply the vanishing of the energy-momentum tensor). Since we have gauge-fixed the metric, these equations form constraints on the form of \(X^\mu\):

\[
T_{ab} = \frac{2}{T} \frac{1}{\sqrt{-h}} \delta S \delta h_{ab}
\]

\[
= \partial_a X \cdot \partial_b X - \frac{1}{2} h_{ab} h^{cd} \partial_c X \cdot \partial_d X = 0
\] (2.19)

\[
\Rightarrow T_{00} = T_{11} = \frac{1}{2} (\dot{X}^2 + X'^2) = 0
\] (2.20)

\[
T_{01} = T_{10} = \dot{X} \cdot X' = 0
\] (2.21)

From (2.20) and the Neumann boundary conditions \((X' = 0)\), we see that open string endpoints must move at the speed of light.

### 2.2.1 String modes

The string equation of motion is the simple one-dimensional wave equation and may be generically solved by an expansion of Fourier modes, plus terms linear in \(\tau\) and \(\sigma\). The closed string boundary conditions of periodicity in \(\sigma\) remove the term linear in \(\sigma\) and discretize the Fourier modes. The solution may be divided into
left-moving and right-moving modes, which look like:

\[ X^\mu_L(\tau, \sigma) = \frac{1}{2} x^\mu + \frac{1}{2} l_s^2 p^\mu (\tau + \sigma) + \frac{i}{2} l_s \sum_{n \in \mathbb{Z} - \{0\}} \frac{\alpha_n^\mu}{n} e^{-in(\tau + \sigma)} \]  
(2.22)

\[ X^\mu_R(\tau, \sigma) = \frac{1}{2} x^\mu + \frac{1}{2} l_s^2 \tilde{p}^\mu (\tau - \sigma) + \frac{i}{2} l_s \sum_{n \in \mathbb{Z} - \{0\}} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\tau - \sigma)} \]  
(2.23)

where \( l_s = \sqrt{2\alpha'} \) is called the string length. Periodicity in \( \sigma \) ensures that \( p^\mu = \tilde{p}^\mu \) and that \( n \) be an integer, giving the full solution

\[ X^\mu(\tau, \sigma) = X^\mu_L(\tau, \sigma) + X^\mu_R(\tau, \sigma) \]  
(2.24)

\[ = x^\mu + \frac{l_s^2}{2} \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{\alpha_n^\mu e^{-in(\tau + \sigma)}}{n} + \tilde{\alpha}_n^\mu e^{-in(\tau - \sigma)} \right) \]  
(2.25)

In these expressions, \( x^\mu \) represents the initial (\( \tau = 0 \)) centre of mass coordinate of the string, as may be verified from

\[ X^\mu_{CM} = \frac{1}{\pi} \int_0^\pi X^\mu d\sigma = x^\mu + \frac{l_s^2}{2} p^\mu \tau \]  
(2.26)

The conjugate momentum to \( X^\mu \) is given by \( P^\mu(\tau, \sigma) = \frac{\delta S}{\delta \dot{X}^\mu} = T \dot{X}^\mu \). The parameter \( p^\mu \) represents the total centre-of-mass momentum:

\[ p^\mu_{CM} = \int_0^\pi P^\mu d\sigma = T \int_0^\pi \dot{X}^\mu d\sigma = \frac{1}{\pi l_s^2} \int_0^\pi l_s^2 \dot{p}^\mu d\sigma = p^\mu \]  
(2.27)

(Note: \( p^\mu \) is a conserved charge for both open and closed strings, since
\[ \frac{dp^\mu}{d\tau} = T \int \dot{X}^\mu d\sigma = T \int \partial^2_{\sigma^2} X^\mu d\sigma = T(X'(\sigma = \pi) - X'(\sigma = 0)) = 0. \] Here, we’ve used the equations of motion and the boundary conditions for closed strings and open Neumann strings. Clearly, for Dirichlet boundary conditions, there is a flow of momentum off the ends of the string.)

Imposing a reality condition on \( X_L \) and \( X_R \) means that \( x^\mu \) and \( p^\mu \) must be
real and that positive and negative modes are conjugate to one another:

\[
\alpha_{-n} = (\alpha_{n})^* \quad \tilde{\alpha}_{-n} = (\tilde{\alpha}_{n})^* \tag{2.28}
\]

For open strings, we have a similar decomposition into left- and right-moving modes, but the boundary condition \(X^\mu(\sigma = 0) = 0\) provides constraints on \(\alpha_{n}^\mu\) and \(\tilde{\alpha}_{n}^\mu\):

\[
X^\mu(0) = X_L^\mu(0) + X_R^\mu(0) = \frac{l_s^2}{2}(p^\mu - \tilde{p}^\mu) + \frac{l_s}{2} \sum_{n \neq 0} (\alpha_{n}^\mu - \tilde{\alpha}_{n}^\mu)e^{-in\tau} \tag{2.29}
\]

which demands \(p^\mu = \tilde{p}^\mu\) and \(\alpha_{n}^\mu = \tilde{\alpha}_{n}^\mu\), as one would expect for standing waves on a string. Similar boundary conditions at the other end (\(\sigma = \pi\)) again restrict \(n\) to be an integer. The full solution is thus

\[
X^\mu(\tau, \sigma) = x^\mu + \frac{l_s^2}{2}p^\mu \tau + il_s \sum_{n \in \mathbb{Z} - \{0\}} \frac{\alpha_{n}^\mu}{n} e^{-in\tau} \cos n\sigma \tag{2.30}
\]

where \(x^\mu\) and \(p^\mu\) have the same interpretations as before.

Concentrating on closed strings, the classical Poisson brackets are

\[
\{X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')\}_{P.B.} = \eta^{\mu\nu}\delta(\sigma - \sigma') \tag{2.31}
\]

\[
\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\}_{P.B.} = \{P^\mu(\tau, \sigma), P^\nu(\tau, \sigma')\}_{P.B.} = 0 \tag{2.32}
\]

We can invert the \(X^\mu\) expansion to get expressions for \(\alpha_{n}^\mu\) and \(\tilde{\alpha}_{n}^\mu\),

\[
\alpha_{n}^\mu = \frac{1}{2\pi l_s} \int_0^\pi e^{2i(\tau + \sigma)}(\dot{X}^\mu - i2nX^\mu) d\sigma \tag{2.33}
\]

\[
\tilde{\alpha}_{n}^\mu = \frac{1}{2\pi l_s} \int_0^\pi e^{2i(\tau - \sigma)}(\dot{X}^\mu - i2nX^\mu) d\sigma, \tag{2.34}
\]

with similar expressions for the open string. We can thus derive Poisson brackets for the position, momentum and the oscillators:

\[
\{x^\mu, p^\nu\} = \eta^{\mu\nu}, \quad \{\alpha_{n}^\mu, \tilde{\alpha}_{n}^\nu\} = 0 \tag{2.35}
\]

\[
\{\alpha_{m}^\mu, \alpha_{n}^\nu\} = \{\tilde{\alpha}_{m}^\mu, \tilde{\alpha}_{n}^\nu\} = -im\delta_{m+n}\eta^{\mu\nu} \tag{2.36}
\]
We proceed to quantize by promoting $X^L$ and $X^R$ to operators and the Poisson brackets to commutators. The operators $x^\mu$ and $p^\mu$ return the position and momentum, while Fourier coefficients $\alpha^\mu_{-n}$ and $\alpha^\mu_n$ become creation and annihilation operators for their respective oscillatory modes. The Poisson bracket to commutator prescription is $\{\ldots\}_P.B. \to -i[\ldots]$, so the operators acquire the commutation relations

\[
x^\mu, p^\nu = i\eta^{\mu\nu}, \quad [\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0
\]  
(2.37)

\[
[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}
\]  
(2.38)

By introducing new operators, $a^\mu_n$ and $\tilde{a}^\mu_n$, defined by

\[
a^\mu_n = \frac{\alpha^\mu_m}{\sqrt{n}}, \quad \tilde{a}^\mu_n = \frac{\tilde{\alpha}^\mu_n}{\sqrt{n}},
\]  
(2.39)

we have the commutation relations for the raising and lowering operators of an infinite number of simple harmonic oscillators. The single-particle Hilbert space for the string is then the Fock space constructed by acting with creation operators ($a^\mu_{-n}, \tilde{a}^\mu_{-n}$, where $n > 0$) on the vacuum state $|0\rangle$, defined by $a^\mu_{-n}|0\rangle = \tilde{a}^\mu_{-n}|0\rangle = 0$, $\forall n > 0$ (more generally, the oscillator vacuum state is a momentum eigenstate $|0, p^\mu\rangle$). Hence a state may be represented by

\[
|\phi\rangle = a^\mu_{-n}a^\nu_{-m}\ldots a^\rho_{-k}|0\rangle
\]  
(2.40)

This presents an immediate problem familiar from the quantization of gauge theories in ordinary quantum field theory - the commutation relations (2.38) mean there is a state with negative norm:

\[
|a^0_{-1}|0\rangle|^2 = \langle 0|a^0_{-1}a^0_{-1}|0\rangle = -1
\]  
(2.41)

This problem may be addressed by returning to the classical constraints (2.20) and imposing them as quantum conditions. In terms of left- and right-moving modes,
the energy-momentum tensor constraints are:

\[
T_{++} = \partial_+ X \cdot \partial_+ X = \frac{1}{4}(\dot{X}^\mu + X'^\mu)^2 = 0 \quad (2.42)
\]

\[
T_{--} = \partial_- X \cdot \partial_- X = \frac{1}{4}(\dot{X}^\mu - X'^\mu)^2 = 0 \quad (2.43)
\]

If we decompose this into a sum of contributions from each mode, we identify \(L_m\) and \(\tilde{L}_m\), a series of Fourier coefficients (called Virasoro generators), which must each vanish separately:

\[
L_m = \frac{1}{4\pi\alpha'} \int_0^\pi e^{i2m(\tau+\sigma)}T_{++}d\sigma = \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n = 0 \quad (2.44)
\]

\[
\tilde{L}_m = \frac{1}{4\pi\alpha'} \int_0^\pi e^{i2m(\tau-\sigma)}T_{--}d\sigma = \frac{1}{2} \sum_n \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n = 0 \quad (2.45)
\]

The Hamiltonian, \(H\), has the following expansion:

\[
H = \int (\dot{X} \cdot P - \mathcal{L})d\sigma \quad (2.46)
\]

\[
= \frac{1}{4\pi\alpha'} \int \dot{X}^2 + X'^2 d\sigma \quad (2.47)
\]

\[
= \frac{1}{2\pi\alpha'} \int T_{++} + T_{--} d\sigma \quad (2.48)
\]

\[
= \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \quad (2.49)
\]

Evidently, for closed strings, \(H = 2(L_0 + \tilde{L}_0)\) and for open strings, \(H = L_0\).

When we quantize the theory, the Hamiltonian and Virasoro generators become operators and we are obliged to normal order their expressions. For most Virasoro operators, this is not a problem, since \(\alpha_{m-n}\) and \(\alpha_n\) commute. However, for \(L_0\), we must take account of the commutation relation between \(\alpha_n\) and \(\alpha_{-n}\) and replace \(L_0\) with \(L_0 - a\), where \(a\) is a constant. It is also possible to show that the classical constraints \(L_m = 0\) cannot be imposed as operator conditions, \(L_m|\phi\rangle = 0\), but just as weak operator constraints - their expectation value on physical states vanishes. Detailed analysis reveals that ghost states (those with negative norm) disappear with the choice \(a = 1\) and the dimensionality of the target space \(D = 26\) (one of various methods that may be used to determine such choices).
The expressions for \( L_0 \) and \( \tilde{L}_0 \) are now

\[
L_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_n = \sum_{n > 0} \alpha_{-n} \cdot \alpha_n + \frac{1}{2} \alpha_0^2 + a \tag{2.50}
\]

\[
\tilde{L}_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = \sum_{n > 0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \frac{1}{2} \tilde{\alpha}_0^2 + a \tag{2.51}
\]

and, defining level operators,

\[
N = \sum_{n > 0} \alpha_{-n} \cdot \alpha_n, \quad \tilde{N} = \sum_{n > 0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \tag{2.52}
\]

we have

\[
0 = L_0 + \tilde{L}_0 - 2a = N + \tilde{N} + \frac{1}{4} l_s p^\mu p_\mu - 2a \tag{2.53}
\]

where we have used \( \alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2} l_s p^\mu \). We then obtain an formula for the mass of the string:

\[
M^2 = -p^\mu p_\mu = \frac{2}{\alpha'} (N + \tilde{N} - 2a), \tag{2.54}
\]

and for the open string, the analogous formula is

\[
M^2 = -p^\mu p_\mu = \frac{1}{\alpha'} (N - a). \tag{2.55}
\]

Though not developed in this introduction to bosonic string theory, the preceding work may be formulated in the light cone gauge, which removes two spurious degrees of freedom associated with a residual gauge freedom in the action. When doing so, the physical operators become \( \alpha_{i}^i \), where \( i = 1, \ldots, D - 2 \), and the first excited state is then \( \alpha_{i-1}^{i} | p^\mu \rangle \), a spacetime vector with \( D - 2 \) degrees of freedom, suggesting that it must be massless and that \( a = 1 \), confirming the previous assertion. Furthermore, the first excited level of the closed string is \( \alpha_{i-1}^{i} \tilde{\alpha}_{j-1}^{j} | p^\mu \rangle \).

This may be decomposed into representations of the transverse rotation group \( SO(D - 2) \), the traceless symmetric \( \left( \alpha_{i-1}^{i} \alpha_{j-1}^{j} - \frac{1}{D-2} \delta^{ij} \alpha_{-1}^{k} \alpha_{-1}^{k} \right) | p^\mu \rangle \) (identified with a graviton, \( h_{ij} \)), the antisymmetric \( \alpha_{i-1}^{i} \alpha_{j-1}^{j} | p^\mu \rangle \) (the Kalb-Ramond field, \( B_{ij} \)) and the trace \( \frac{1}{D-2} \delta^{ij} \alpha_{-1}^{k} \alpha_{-1}^{k} | p^\mu \rangle \) (identified as a scalar dilaton, \( \Phi \)).

The dilaton is notable, since it is incorporated into the action as

---

26
\[ \frac{1}{4\pi} \int d^2 \sigma \sqrt{h} \Phi R, \] where \( R \) is the worldsheet Ricci scalar. In two dimensions, the Ricci scalar is a total derivative and its integral on a closed orientable surface is a topological invariant: \[ \frac{1}{4\pi} \int d^2 \sigma \sqrt{h} R = 2 - 2g, \] where \( g \), the genus, counts the number of handles of the surface, corresponding to the number of loops on string worldsheet. Thus, a worldsheet with \( n \) loops will enter into the path integral as \( e^{-(2-2n)\Phi_0} \), where \( \Phi_0 \) is the vacuum expectation value for the dilaton. This leads us to identify \( e^{\Phi_0} \) as the closed string coupling constant, \( g_s \).

When the theory is developed in the light cone gauge, manifest covariance is lost. By carefully checking that the Lorentz algebra does indeed still close, it may be shown, again, that \( D \) must be 26. The non-appearance of most of those predicted 26 dimensions may be explained by compactification at a scale above that which experiments have yet probed. Two glaring issues remain outstanding though – the lack of spacetime fermions and the appearance of a tachyonic mode at the zeroth excited level of both open and closed strings. As it happens, both problems may be handled by introducing fermionic degrees of freedom to the worldsheet.

2.2.2 Fermions in String Theory

The theory thus far described suffers from at least major drawback – it only contains bosonic excitations of the string and so cannot account for the existence of quarks and leptons. To remedy this situation, we could enlarge the target space to \( D \)-dimensional superspace, the coordinates of which become worldsheet fermions. This approach, developed by Green and Schwarz, has the advantages of manifest spacetime supersymmetry as well as automatically removing the tachyon from the string spectrum. We will, however, focus on the first formalism developed (by Ramond, Neveu and Schwarz) to deal with the problem – introduce supersymmetry on the worldsheet (promoting it to two-dimensional superspace) with fermionic degrees of freedom, \( \psi^a(\sigma^\mu) \).

Our superstring action now contains extra terms and appears as (in
superconformal gauge and with \( l_s^2 = 1 \):

\[
S = -\frac{1}{2\pi} \int d^2 \sigma (\partial_a X^\mu \partial^a X_\mu - i \bar{\psi}^\mu \rho^a \partial_a \psi_\mu)
\]  

(2.56)

We have introduced worldsheet Dirac matrices, \( \rho^a \),

\[
\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\]  

(2.57)

satisfying the Clifford algebra:

\[
\{\rho^a, \rho^b\} = -2\eta^{ab}
\]  

(2.58)

The spinors have target space indices, so they transform as a target space vector, but from the perspective of the worldsheet, they are spinors, with an internal quantum number, \( \mu \).

If we write the worldsheet spinors as

\[
\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}
\]  

(2.59)

the fermionic part of the action takes the form

\[
S_f = \frac{i}{\pi} \int d^2 \sigma (\psi_- \cdot \partial_+ \psi_- + \psi_+ \cdot \partial_- \psi_+) \]  

(2.60)

leading to equations of motion \( \partial_- \psi_+ = \partial_+ \psi_- = 0 \) (we have left the spacetime indices implicit) and the mode expansions

\[
\psi_- = \sum b_n e^{-in\sigma_-} \]  

(2.61)

\[
\psi_+ = \sum b_n e^{-in\sigma_+} \]  

(2.62)

(where \( \sigma_+ = \tau + \sigma \) and \( \sigma_- = \tau - \sigma \)). We must again consider the boundary conditions.
as we did for the bosonic string. These are

\[ [\psi_\delta \delta \psi_+ - \psi_\delta \delta \psi_+] = 0 \]  \hspace{1cm} (2.63)

For the open string, these conditions may be satisfied by requiring \( \psi_+ = \pm \psi_- \) at both ends. Since the overall normalization is a matter of convention, we can define \( \psi_+(\sigma = 0) = \psi_-(\sigma = 0) \) and then take one of two choices:

\[
\begin{align*}
\psi_+(\pi) &= \psi_-(\pi) \quad \text{Ramond} \\
\psi_+(\pi) &= -\psi_-(\pi) \quad \text{Neveu-Schwarz}
\end{align*}
\]  \hspace{1cm} (2.64)

These conditions translate into conditions on the exponents in the mode expansions (2.61) and (2.62), so that in the Ramond sector

\[
\begin{align*}
\psi_- &= \sum_{n \in \mathbb{Z}} d_n e^{-i n \sigma_-} \\
\psi_+ &= \sum_{n \in \mathbb{Z}} d_n e^{-i n \sigma_+}
\end{align*}
\]  \hspace{1cm} (2.66)

and in the Neveu-Schwarz sector,

\[
\begin{align*}
\psi_- &= \sum_{r \in \mathbb{Z} + 1/2} b_r e^{-i n \sigma_-} \\
\psi_+ &= \sum_{r \in \mathbb{Z} + 1/2} b_r e^{-i n \sigma_+}
\end{align*}
\]  \hspace{1cm} (2.67)

where again \( d_n^\mu = (d_n^\mu)^* \) and \( b_r^\mu = (b_r^\mu)^* \). From now on, we will assume \( n \in \mathbb{Z} \) and \( r \in \mathbb{Z} + 1/2 \). Imposing the canonical anti-commutation rules (in the same way as was done for the bosonic case), we obtain the standard fermionic anti-commutation laws:

\[
\begin{align*}
\{ \psi_+^\mu (\tau, \sigma), \psi_-^\nu (\tau, \sigma') \} &= \pi \eta^{\mu\nu} \delta(\sigma - \sigma') \\
\{ \psi_-^\mu (\tau, \sigma), \psi_-^\nu (\tau, \sigma') \} &= \pi \eta^{\mu\nu} \delta(\sigma - \sigma') \\
\{ \psi_-^\mu (\tau, \sigma), \psi_+^\nu (\tau, \sigma') \} &= 0
\end{align*}
\]  \hspace{1cm} (2.70)
Upon quantization, these translate into anti-commutation relations

\[ \{d_\mu^m, d_\nu^n\} = \delta_{m+n} \eta^{\mu\nu} \]  

(2.73)

in the Ramond sector, and

\[ \{b_\mu^r, b_\nu^s\} = \delta_{r+s} \eta^{\mu\nu} \]  

(2.74)

in Neveu-Schwarz.

For closed strings, we can impose Ramond and Neveu-Schwarz conditions as periodicity or anti-periodicity on left- and right-moving modes:

\[ \psi_\pm^\mu(\tau, \sigma) = \pm \psi_\pm^\mu(\tau, \sigma + \pi) \]  

(2.75)

and we gain new right-moving modes, \( \tilde{d}_\mu^m \) and \( \tilde{b}_r^\mu \), with anticommutation relations

\[ \{\tilde{d}_\mu^m, \tilde{d}_\nu^n\} = \delta_{m+n} \eta^{\mu\nu}, \quad \{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \delta_{r+s} \eta^{\mu\nu} \]  

(2.76)

\[ \{d_\mu^m, \tilde{d}_\nu^n\} = \{b_\mu^r, \tilde{b}_s^\nu\} = 0 \]  

(2.77)

The fermionic part of the Hamiltonian becomes

\[ H = \frac{i}{2\pi} \int (\psi_+ \psi'_+ - \psi_- \psi'_-) d\sigma = \sum_n n(d_{-n} \cdot d_n + \tilde{d}_{-n} \cdot \tilde{d}_n) \]  

(2.78)

\[ = 2(L_0^d + \tilde{L}_0^d) \]  

(2.79)

in the Ramond sector, and

\[ H = \sum_r r(b_{-r} \cdot b_r + \tilde{b}_{-r} \cdot \tilde{b}_r) = 2(L_0^b + \tilde{L}_0^b) \]  

(2.80)
in the Neveu-Schwarz sector. The fermionic Virasoro operators are

\[ L_m = \frac{1}{2\pi} \int \! d\sigma e^{i2m\sigma} T_{++} \]  
\[ = \frac{i}{4\pi} \int \! d\sigma e^{i2m\sigma} \psi_+ \cdot \partial_+ \psi_+ \]  
\[ = \frac{1}{2} \sum_n \left( n - \frac{m}{2} \right) d_{m-n} \cdot d_n \]  
\[ \text{or} \]  
\[ = \frac{1}{2} \sum_r \left( r - \frac{m}{2} \right) b_{m-r} \cdot d_r \]

The open string on-shell mass condition is \( L_0 |\phi\rangle = a |\phi\rangle \) and it now becomes

\[ \alpha' p^2 + N^\alpha + N^d - a_R = 0 \]  
\[ \alpha' p^2 + N^\alpha + N^b - a_{NS} = 0 \]

where we have introduced fermionic level operators analogous to the bosonic ones.

By methods similar to those used in the bosonic string, we find that \( a_R = 0 \) and \( a_{NS} = 1/2 \). The Lorentz algebra is closed if the number of spacetime dimensions is ten.

We note that in the Neveu-Schwarz case, a unique string vacuum can be consistently defined by \( \alpha_\mu^0 |0\rangle = b_\mu^0 |0\rangle = 0 \) \( \forall n, r > 0 \). The first excited state, \( b_{-1/2}^\mu |0\rangle \), is massless and transforms as a spacetime vector. We construct additional states by acting with operators with spacetime indices, so all NS states are bosonic.

We define the Ramond ground state in a similar way \( (d_\mu^n |0\rangle = 0 \) \( \forall n > 0 \)). However, all operators anti-commute with \( d_0^\mu \), so we may act on the ground state with \( d_0^\mu \) without changing its mass. From the anti-commutation relation

\[ \{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}, \]  
we see that the operators form a representation of the Clifford algebra under the identification \( \Gamma^\mu = i\sqrt{2}d_0^\mu \):

\[ \{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu} \]

and so the degenerate ground state is a spinor representation of \( SO(8) \). Additional
states are created by acting on this fermionic ground state with vector operators, so R states are spacetime fermions.

We are again faced with the problem of a tachyonic ground state in the Neveu-Schwarz sector. This can be eliminated by truncating the spectrum using the GSO projection \[ GSO \]

\[
P = \frac{1}{2} \left( 1 - (-1)^F \right) \quad \text{NS} \tag{2.88}
\]

\[
P = \frac{1}{2} \left( 1 - \eta(-1)^F \right) \quad \text{R} \tag{2.89}
\]

where the fermion parity operator \((-1)^F = \exp(i\pi \sum b_i r_i)\) and \(\eta = \pm 1\) is the chirality of the Ramond ground state spinor, which can be chosen independently for left and right movers. Hence, only states with odd numbers of fermionic operators contribute to the Neveu-Schwarz spectrum, eliminating the tachyonic mode. Moreover, this projection leaves equivalent numbers of spacetime fermionic degrees of freedom at each level as there are bosonic degrees of freedom, meaning that the theory has spacetime supersymmetry as well as worldsheet supersymmetry.

When we consider closed strings, we may consider the left-moving and right-moving sectors separately and in the same way we have looked at the open string spectrum. The massless states are

\[
|0\rangle_R \otimes b_{-1/2}^i |0\rangle_L \quad \text{R-NS} \tag{2.90}
\]

\[
b_{-1/2}^i |0\rangle_R \otimes |0\rangle_L \quad \text{NS-R} \tag{2.91}
\]

\[
|0\rangle_R \otimes |0\rangle_L \quad \text{R-R} \tag{2.92}
\]

\[
b_{-1/2}^i |0\rangle_R \otimes b_{-1/2}^j |0\rangle_L \quad \text{NS-NS} \tag{2.93}
\]

The first two states are spacetime fermions, while the second two are spacetime bosons. The NS-NS state divides up as before into the irreducible representations of \(SO(8)\) - the symmetric \(35 (G_{\mu\nu})\), the antisymmetric \(28 (B_{\mu\nu})\) and the trace \(1 (\Phi)\).

The R-R states depend on the version of superstring theory we employ. In Type IIA (which only contains closed strings), we choose the chirality of the left
movers to be different to that of the right movers, whereas in Type IIB, the chirality is the same. The Type IIA fields consist of a 1-form $C_i$ and a 3-form $C_{ijk}$. The Type IIB fields are the scalar $C$, 2-form $C_{ij}$ and the self-dual 4-form $C_{ijkl}$.

The R-NS and NS-R fields are spinors and both sectors consist of an 8 (dilatino) and a 56 (gravitino), the chiralities of the two sectors being opposite or equal depending on whether it is Type II A or B being considered. The fact that left-moving fermion modes are distinct from right-movers means that there are two gravitinos and the theory thus has $N = 2$ spacetime supersymmetry (hence the appellation “Type II”).

The status of R-R fields as antisymmetric $n$-forms means they should be considered as rank $n$ gauge fields (or potentials). Such gauge fields must be sourced by some objects within the theory, but not perturbative states, since their vertex operators are the field strengths, not the gauge fields themselves – the non-perturbative objects which source the RR fields will appear as the D-branes discussed in the next section.

### 2.2.3 T-Duality and D-Branes

The necessity of having a spacetime with ten or twenty-six dimensions clearly poses a phenomenological problem for string theory, since the world around certainly appears to have just three spatial and one time dimension. A solution to this problem could be to insist that the visible world is constructed from degrees of freedom that are confined to a 3+1-dimensional subspace of the full 9+1-dimensional bulk space. A more popular solution is the one first proposed by Oskar Klein in 1926, that the extra dimensions are topologically compact and that the radius of compactification lies well below the scales yet probed in the highest energy experiments. With one extra dimension, the simplest scenario would be to take a five-dimensional Minkowski space and periodically identify the fifth coordinate:

$$x^5 \sim x^5 + 2\pi R$$  \hspace{1cm} (2.94)
The 4+1d manifold therefore has the topology $\mathbb{R}^4 \times S^1$. With the full ten
dimensions of string theory, the target space should have the topology of $\mathbb{R}^4 \times M^6$,
where $M^6$ is some six-dimensional manifold with the appropriate geometric and
topological properties. If part of this manifold consists of an $n$-torus (the product of
$n$ $S^1$s), there appears extra interesting structure in the theory that is missing in
non-compact space. This is the phenomenon of $T$-duality.

Consider bosonic string theory and just identify the twenty-fifth dimension,
$x^{25} \sim x^{25} + 2\pi R$. We then have a possibility that we lacked before – closed strings
can wrap around this dimension. A string wrapped around the $S^1$ $n$ times is
characterised by its winding number, $n$, which cannot change without interactions
with another string. Our previous boundary conditions, $X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma)$
must be altered to $X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + n2\pi R$. Thus, we must now relate
the left- and right-moving momentum parameters as $p^{25} - \tilde{p}^{25} = \frac{2nR}{\alpha'}$. Moreover,
the string wave function will include the factor $e^{ip \cdot x}$ and in order for the
wavefunction to be single-valued, the string momentum must be quantized:
\[ \frac{1}{2}(p^{25} + \tilde{p}^{25}) = \frac{m}{R}, \]
where $m$ is an integer.

Therefore, the closed string left- and right-moving mode expansions now take
the form
\[ X^{25}_L(\tau, \sigma) = \frac{1}{2}x^{25} + \left(\frac{ma'}{R} + nR\right)(\tau + \sigma) + \text{oscillators} \quad (2.95) \]
\[ X^{25}_R(\tau, \sigma) = \frac{1}{2}x^{25} + \left(\frac{ma'}{R} - nR\right)(\tau - \sigma) + \text{oscillators} \quad (2.96) \]
and we can identify
\[ p^{25} = \frac{m}{R} + \frac{nR}{\alpha'} \quad (2.97) \]
\[ \tilde{p}^{25} = \frac{m}{R} - \frac{nR}{\alpha'}. \quad (2.98) \]

What effect does this have on the mass spectrum of the theory? Returning to the
equation (2.53)
\[ 0 = L_0 + \tilde{L}_0 - 2 = N + \tilde{N} + \frac{\alpha'}{4}(2p^\mu p_\mu + (p^{25})^2 + (\tilde{p}^{25})^2) - 2 \quad (2.99) \]
where $\mu = 0, 1, \ldots, 24$, so

$$M^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(N + \tilde{N} - 2) + \frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2} \quad (2.100)$$

Each term has a clear physical interpretation – contributions to the mass come from the excitations of the string, as before, but also from the momentum in the compact dimension (as in normal field theory). The last term counts the energy stored in the string as it is stretched $m$ times around the circle.

There is an evident symmetry here, where interchanging $m$ and $n$ and changing $R \to \tilde{R} = \frac{\alpha'}{R}$ leaves the mass spectrum invariant. This is a symmetry known as $T$-duality. Physically, it means that compactification on a circle of radius $R$ is indistinguishable from compactification on a circle of radius $\tilde{R}$. The limit of large $R$ sees winding modes grow heavy and decouple, while the momentum spectrum becomes a continuum; the opposite limit $R \to 0$ sees momentum modes decouple and winding modes approach a continuum, as if a new dimension (of length $\tilde{R}$) is opening up.

The action of this symmetry is to send $p \to p$ and $\tilde{p} \to -\tilde{p}$. This leads us to formulate the string theory in terms of a new field,

$$\tilde{X}(\tau, \sigma) = X_L - X_R, \quad (2.101)$$

which will help us to consider T-duality in the context of open strings.

With the structure so far developed, there is no way for open strings to transform in the same way as closed strings, since we cannot assign a conserved winding number to freely-moving open strings – any open string wrapped around a dimension can contract to a point. It seems that T-duality does not apply to open strings. However, any theory of interacting open strings must necessarily also contain closed strings (since the endpoints must be able to join up). The only difference between the two species of string lies in the open string endpoints and it is here that the different behaviour must appear. If we are to impose winding number on the string, we must do it by keeping the endpoints fixed. This is indeed
what happens when T-duality is applied to the open string expansion:

\[ \tilde{X} = \tilde{X}_L + \tilde{X}_R = X_L - X_R = x + pl_s^2 \sigma + il_s \sum_{n} \alpha_n e^{-in\sigma} \sin n\sigma \quad (2.102) \]

Evidently, Neumann boundary conditions have been swapped for Dirichlet boundary conditions. When we first discussed the boundary conditions for the open string, we mentioned Dirichlet conditions, but dismissed them for manifestly violating Poincaré invariance. Here, we find them forced upon us, by consistency of the theory. We have spoken of compactification and T-duality in one dimension, but in principal we could apply this to any number of dimensions, and have a mix of Dirichlet and Neumann conditions on the string boundaries. Having Dirichlet condition on \( n \) dimensions means fixing the string endpoints to a \((10 - n)\)-dimensional submanifold of the full 10D spacetime. If this were an arbitrary division of the spacetime, it would indeed violate Poincaré invariance, but not if the submanifold itself is realised as a dynamical and physical object in its own right. This is the \( D\)-brane – an object of \( p \) spatial dimensions (specifically referred to as a \( Dp\)-brane) to which open string endpoints are anchored (the \( D \) stands for \( \text{Dirichlet} \)). Since we can impose Dirichlet boundary conditions on any number of dimensions, we can have anything from a \( D(-1)\)-brane (an instanton) to a space-filling D9-brane. With this last possibility, we may think of all open strings (even those with all ten Neumann endpoint conditions) as attached to D-branes, if we consider the bulk space as a D9.

With both ends attached to a Dp-brane, a string is confined to move in the brane’s \( p + 1 \) spacetime dimensions, so we may consider its physics to be that of fields existing in the worldvolume of the brane. A Dp-brane breaks the full \( SO(1,9) \) Lorentz symmetry of the bulk to the subgroups \( SO(1,p) \times SO(9 - p) \) and the fields \( X^\mu \) must break up into representations under the respective groups: the parallel oscillations \( X^\alpha \) \( (\alpha = 0, 1, \ldots, p) \) and the transverse oscillations \( X^I \) \( (I = p + 1, \ldots, 9) \). The former must transform as the components of an \( SO(1,p) \) vector in the brane worldvolume, and represent a gauge field \( A^\alpha \), while the latter are scalars, \( \Phi^I \) (they form vectors under \( SO(9 - p) \), but this is now an internal
symmetry group). These scalar fields have a natural interpretation as the transverse displacements of the brane.

Although we have established the existence of D-branes as objects to fix the endpoints of strings, they are dynamical objects (as they must be in general relativity) and we must find an action to describe these dynamics. Taking a similar approach to the one we adopted to find the Nambu-Goto action, we assume an action that is the invariant worldvolume of the brane:

\[
S_{D\text{-brane}} = -T_p \int d^{p+1}\xi \sqrt{-\det \gamma} \quad (2.103)
\]

where \(T_p\) is the Dp-brane tension, \(\xi^\alpha\) are the \(p+1\) coordinates parameterising the brane volume and \(\gamma\) is the metric pullback to the brane:

\[
\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} \eta_{\mu\nu} \quad (2.104)
\]

This action was first written down by Dirac for a spherical model of the electron \[61\]. We need to expand this action to include the effects of the gauge field \(A^\alpha\), for which we turn to an action due to Born and Infeld \[62\], which accounts for higher order terms in a non-linear gauge theory:

\[
S_{BI} \sim \int d\xi^{p+1} \sqrt{-\det(\eta_{\mu\nu} + 2\pi \alpha' F_{\mu\nu})} \quad (2.105)
\]

When we include the effects of the NS-NS background fields, \(G_{\mu\nu}\), \(B_{\mu\nu}\) and \(\Phi\), we get the more general Dirac-Born-Infeld action,

\[
S_{DBI} = -T_p \int d\xi^{p+1} e^{-\Phi} \sqrt{-\det(\gamma_{\alpha\beta} + B_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta})} \quad (2.106)
\]

where

\[
\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} G_{\mu\nu} \quad B_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} B_{\mu\nu} \quad (2.107)
\]

The \(B\)-field is the two-form Kalb-Ramond gauge field mentioned earlier. When we consider the effects on the string of both this field and the brane gauge field \(A_\alpha\), we see that a gauge transformation \(B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu\) can be compensated by
the transformation, \( A_\alpha \rightarrow A_\alpha - \frac{1}{2\pi\alpha'} C_\alpha \). Thus, the combination \( B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta} \) is gauge-invariant.

The brane tension has the formula

\[
T_p = \frac{1}{(2\pi)^p \alpha'^{p+1} g_s}
\]

which contains a \( g_s^{-1} \), from the dilaton contribution to the action (2.106).

When we consider a combination of \( N \) D-branes, we can have strings with their endpoints attached to two different branes. If all the branes are coincident, the \( U(1) \) gauge symmetry (represented in \( F_{\alpha\beta} \)) gets promoted to a \( U(N) \) symmetry, the group indices \( a \) and \( b \) on \( A_\mu \overset{a b}{\rightarrow} \) labelling the endpoints of strings by their holding brane. If we separate off \( n \) of the branes, the gauge group is broken to \( U(n) \times U(N-n) \) and the off-diagonal W-bosons gain a mass \( \sim (x_1^\mu - x_2^\mu)^2 \), where \( x_1^\mu \) and \( x_2^\mu \) are the coordinates of the two brane stacks (there is a clear generalisation to the case of separating various numbers of branes).

We noted in the previous section that there exist \( n \)-form gauge fields which must have sources. These sources are indeed the D-branes discovered in this section. Consider first an ordinary point particle coupled to an electromagnetic gauge field. This gives rise to a term in the action of the form

\[
S_A = e \int A = e \int A_\mu dx^\mu = e \int A_\mu \frac{dx^\mu}{d\tau} d\tau
\]

where \( \tau \) parameterizes the worldline. The 0-dimensional point particle couples to a 1-form gauge field. If we generalise this to an \( (p+1) \)-form gauge field, the action must take the form of a coupling to a \( p \)-dimensional object:

\[
S_{A_{p+1}} = \mu_p \int A_{p+1} = \mu_p \frac{1}{(p+1)!} \int A_{\mu_1...\mu_{p+1}} \frac{\partial x^{\mu_1}}{\partial \xi^{\alpha_1}} \ldots \frac{\partial x^{\mu_{p+1}}}{\partial \xi^{\alpha_{p+1}}} d\xi^{\alpha_1} \wedge \ldots \wedge d\xi^{\alpha_{p+1}}
\]

As established by Polchinski \[63\], this object is simply a Dp-brane, with charge \( \mu_p \).
As with electromagnetism, an $n$-form field gives rise to an $(n + 1)$-form field strength, $F_{n+1} = \text{d}A_n$. The charge on a D$p$-brane can be calculated by integrating $\int \star F$ over a $(D - p - 2)$-sphere surrounding the brane, where $\star F$ is the Hodge (or magnetic) dual of $F$. The magnetic dual of a $p$-brane is a $(D - p - 4)$-brane, which electrically sources the magnetic field due to the $p$-brane. In ten dimensions, therefore, a 3-brane is self-dual.

Since a $p$-brane carries a conserved charge, $\mu_p$, it is stable. The existence of a 1-form and 3-form in Type IIA superstring theory means that 0-branes and 2-branes exist as stable objects, which couple electrically to the fields, and 4-branes and 6-branes, which couple magnetically. In a similar way, Type IIB theory has stable $-1$, 1-, 3-, 5- and 7-branes. Although not coupling to any dynamical gauge fields, there also exist 8- and 9-branes in Type IIA and IIB respectively.

The $p$-brane couples directly to a background $C_{p+1}$, but in the presence of background $B$-fields or worldvolume gauge fields, it can also couple to $n$-forms of lower rank. The total contribution to the action is given by

$$S_C = \mu_p \sum_n \int_{p+1} C_n e^{B + 2\pi \alpha' F}$$

where the integral is taken over the $p + 1$ dimensions of the brane world-volume.
Chapter 3

The AdS/CFT Correspondence

The fields of string theory can also be derived from a low-energy effective action approximation, wherein they appear as fields in 10D supergravity. Type IIA and Type IIB supergravity comprise the fields present in the corresponding superstring theory. Just as we find black holes as isotropic solutions to ordinary general relativity, we may identify certain solitonic solutions to the supergravity field equations as $p$-branes, sources for both gravitational NS-NS fields and the R-R fields appropriate to each theory. In general, the metric sourced by a stack of $N$ coincident $p$-branes in 9+1 dimensions is:

$$
\text{d}s^2 = H_p^{-1/2} \text{d}x_\mu \text{d}x^\mu + H_p^{1/2} \text{d}y_i \text{d}y^i
$$

where $x^\mu$ are coordinates parallel to the brane and $y^i$ are those transverse to the brane. The harmonic function $H_p$ is given by

$$
H_p = 1 + \left( \frac{R}{r} \right)^{7-p}
$$

where $r$ is the radial coordinate of the transverse space and

$$
\left( \frac{R}{2\pi l_s} \right)^{7-p} = g_s N \frac{\Gamma(\frac{9-p}{2})}{7 - p \frac{\Gamma(9-p)^2}{2}}
$$

where $R$ may be considered the “radius” of the space (we now use $l_s = \sqrt{\alpha'}$).
The $p$-brane sources a $(p+1)$-form gauge field, $C_{p+1}$,

$$C_{p+1} = C_{0..p}dx^0 \wedge \ldots \wedge dx^p \quad (3.4)$$

with $C_{0..p} = 1 - H_p^{-1}$. The dilaton is

$$e^{2\phi} = g_s^2 H_p^{(3-p)/2} \quad (3.5)$$

In 1995, Polchinski discovered that D$p$-branes were precisely the $p$-branes that appeared in these supergravity solutions [63]. A consequence of this is that we can look at the physics of such branes from two points of view. One is in terms of their stringy physics. With a D$p$-brane background, the physics consists of both open and closed string excitations. The open strings have their endpoints attached to the brane and they describe the excitations of the brane; the closed strings describe excitations of the bulk.

We take the special case of D3-branes. Here, the dilaton is a constant. As noted earlier, if we have $N$ coincident D-branes, the low-energy excitations are described by a $U(N)$ gauge theory. This can be seen from the DBI action:

$$S = -T_p \int d\xi^{p+1} \sqrt{-\det(P[G_{\alpha\beta}] + 2\pi \alpha' F_{\alpha\beta})} \quad (3.6)$$

$$\supset T_p \int d\xi^{p+1} \sqrt{-g} \left( \frac{(2\pi \alpha')^2}{4} \frac{1}{\text{Tr}}(F^a_{\alpha\beta}F^a_{\beta\alpha}) + \ldots \right) \quad (3.7)$$

where $g$ is the determinant of the induced metric $g_{ab}$. The trace is taken over the $U(N)$ group indices, conventionally normalised to $\text{Tr}(F^a F^b) = \frac{1}{2} \delta^{ab}$. For the particular case of $p = 3$, the brane tension $T_3 = \left( (2\pi \alpha')^3 \alpha'^2 g_s \right)^{-1}$. If we are to identify this with a conventional $U(N)$ Yang-Mills gauge theory with kinetic term

$$S = - \int d^4x \frac{1}{4g_{YM}^2} F^a_{\alpha\beta}F^{a\alpha\beta} \quad (3.8)$$

we can relate the string coupling to the gauge coupling as $g_s^2 = g_Y^2$. Thus, the low energy dynamics of $N$ coincident D3-branes are those of $SU(N)$ super-Yang-Mills gauge theory (the $U(1)$ component of $U(N)$ corresponds to the
centre of mass of the brane stack and it decouples). Since the brane dynamics derive from Type IIB superstring theory, the full DBI action should be supersymmetric, with spinorial degrees of freedom. In addition to the worldvolume gauge field, there are six scalars $\Phi^I$, which correspond to the transverse oscillations of the brane, and there are four Weyl fermions, $\lambda^a$. The brane breaks the $SO(1,9)$ isometry of the bulk to $SO(1,3) \times SO(6)$, and the $SO(6)$ turns up in the field theory as an $R$-symmetry, rotating the scalars amongst themselves (and the fermions, which transform in the $SU(4)$ representation). All fields transform in the adjoint representation of the gauge group. Since there are four gauginos, there are four supercharges and the field content is that of $\mathcal{N} = 4$ super-Yang-Mills theory. This particular theory is unusual in that has vanishing beta function and as such is a conformal field theory.

The closed string sector consists of gravitational and Ramond-Ramond fields. If we expand the gravitational field as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $\kappa$ is the 10D gravitational constant ($\sim l_s^2$), the expansion of the gravitational action goes as

$$S = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} R$$

(3.9)

$$\sim \int d^{10}x (\partial h)^2 + \kappa h (\partial h)^2 + \ldots$$

(3.10)

All higher order terms come with positive powers of $\kappa$, so in the low energy limit (equivalently, $l_s \to 0$), these terms decouple and we are left with free Type IIB supergravity propagating on flat space. Interaction terms also come with factors of $\kappa$ and so they too decouple from the spectrum in the low-energy limit. We are left with the physics of four-dimensional $SU(N)$ super Yang-Mills theory and free Type IIB supergravity and no interactions between the two.

We can also consider the physics of D3-branes from the point of view of an observer at infinity. The branes act as sources for bulk fields as noted above. The metric created by the branes is analogous to the Schwarzschild metric in standard 4D general relativity and as in that case, near-horizon fields have their energy
red-shifted when measured at infinity:

$$E_\infty = H(r)^{-1/4} E_r$$  \hfill (3.11)$$

Any near-horizon field will appear to have low energy to an observer at infinity and these fields face a potential barrier which prevents their escaping to the boundary. In addition, the observer will see other low energy fields – the long-wavelength mass bulk fields, propagating in the flat space far from the centre of the geometry and non-interacting with the near-horizon region. In the near-horizon ($r \rightarrow 0$) limit, the metric becomes

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2$$  \hfill (3.12)$$

$$= \frac{R^2}{z^2} (dx_\mu dx^\mu + dz^2) + R^2 d\Omega_5^2$$  \hfill (3.13)$$

where the latter form has been rendered by the change of variables $z = \frac{R^2}{r}$, which makes it clear that the first part of the geometry is that of five-dimensional anti-de Sitter space while the second is a five-sphere. This is the metric probed by the near-horizon fields.

Thus, we have two different perspectives on the same physics; both consist of two decoupled sectors, one of which is free Type IIB supergravity on a flat background. It is natural to wonder if we may identify the other sectors. This is what led Juan Maldacena in 1997 to propose the $\textbf{AdS/CFT}$ conjecture $[56]$: $4\text{D } \mathcal{N} = 4 \text{ SU}(N)$ super Yang-Mills theory is equivalent to Type IIB supergravity on an $\text{AdS}_5 \times \text{S}_5$ background.

From (3.3), we can derive the relation

$$\frac{R^4}{l_s^4} = 4\pi g_s N = g_{YM}^2 N = \lambda.$$  \hfill (3.14)$$

The supergravity approximation is appropriate when the bulk space is weakly-curved, i.e. $R$ is large. From (3.14), we see that this requires the ’t Hooft coupling $\lambda$ also to be large – so the field theory is strongly coupled. It is in this that
the power of the Correspondence lies, since it translates intractable problems into tractable ones. Of course, it also means that it is difficult to easily verify with perturbative calculations on either side, so other checks must be performed. One is to see that both theories contain the same symmetries – we find that that is indeed the case. The isometry group of $AdS_{d+1}$ space is $SO(2,d-1)$ and that of an $n$-sphere is $SO(n+1)$. The $N=4$ super Yang-Mills theory is a conformal theory, invariant under the conformal group $SO(2,4)$ and contains the $R$-symmetry group $SO(6)$, which map to the isometry groups of $AdS_5$ and $S^5$ respectively. Since the field theory is conformally invariant, we should see a similar invariance in the bulk theory under a rescaling of Minkowski coordinates. Clearly, the metric (3.7) is invariant under rescalings $x \rightarrow ax$ if $r$ is inversely scaled: $r \rightarrow r/a$. Hence, $r$ scales as energy, and bulk physics at different radii correspond to different energy scales in the field theory. We note that for $p = 3$, (3.5) tells us that the dilaton is constant at all $r$, confirming that the coupling does not vary at different energy scales.

If we are to connect these two different physical theories, we need a dictionary to translate objects in one theory to objects in the other. The prescription we follow is that of Witten [64] and Gubser, Klebanov and Polyakov [65]:

$$W_J[\phi_0, \mathcal{O}]_{CFT} = S[\phi_0 = \lim_{\epsilon \rightarrow 0} \phi(x, z = \epsilon)]_{bulk}$$

where $W_J$ is the field theory connected generating functional and $S_{bulk}$ is the bulk gravitational action; $\phi_0$ corresponds to the boundary limit of the bulk field $\phi$.

Let us take a simple example, that of a free scalar propagating in the bulk:

$$S = -\frac{1}{2} \int d^4x \sqrt{-g}(g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2)$$

with equations of motion $(\nabla_M \nabla^M - m^2)\phi = 0$. In $AdS_{d+1}$, this is

$$z^2 \phi'' + (1 - d)z \phi' - (z^2 k^2 + m^2 R^2)\phi = 0$$
where $\phi' = \partial_z \phi$ and $\partial_\mu \partial^\mu \phi = -k^2 \phi$. In the small $z$ limit, this has the solution

$$
\phi(r) = A z^{\Delta_-} + B z^{\Delta_+} \tag{3.18}
$$

where $\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2}$ and $A$ and $B$ are constants to be explained. Stable fields in $AdS_{d+1}$ may have negative $m^2$ as long they satisfy the Breitenlohner-Freedman bound, $m^2 R^2 > -\frac{d^2}{4}$ \cite{66}. If the field saturates this bound, $\Delta_+ = \Delta_-$. For all values of $m^2$, $\Delta_+ > 0$, so that part of the solution goes to zero at the boundary. We adjust the boundary condition to extract the finite part of the solution: $\phi_0 = \lim_{\epsilon \to 0} \epsilon^{\Delta_-} \phi(\epsilon)$, and we identify $A$ as the source term $\phi_0$. If we consider the field theory action as the bulk action near the boundary, it will contain terms quadratic in $\phi$:

$$
S \sim \int_{\epsilon \to 0} d^d x \sqrt{\gamma} A B e^{(\Delta_+ + \Delta_-)} \tag{3.19}
$$

where $\gamma$ is the determinant of the induced metric on the boundary, $\sqrt{\gamma} = \left(\frac{R}{\epsilon}\right)^d$. This is finite as $\epsilon \to 0$, since $\Delta_+ + \Delta_- = d$, and we are left with an action of the form $\int d^d x A(x) B(x)$. Since we have identified $A$ as a source term for an operator, it natural to identify $B$ as the expectation value for the operator $B(x) = O(x)$, with scaling dimension $\Delta_+$.

### 3.1 Adding Flavour

The theory as it stands consists solely of glue (to borrow terminology from QCD) and matter in the adjoint representation of the gauge group. We would like to expand the scope and bring it closer to QCD-like physics by introducing matter in the fundamental representation. All fields so far considered are adjoint, because they are open strings with both ends attached to the D3 stack; one end is charged in the $N$ representation of $SU(N)$, while the other is in the $\tilde{N}$. In order to introduce fundamental degrees of freedom, we need to provide another brane for the strings to end on, so that the string as a whole is solely in the $N$ or $\tilde{N}$ if only one end is on the D3 stack. This can be accomplished with the introduction of $N_f$
D7-branes \textsuperscript{[67]}, which provide for $\mathcal{N} = 2$ hypermultiplets in the field theory\textsuperscript{[1]}. Of course, introducing another set of branes will alter the geometry of the bulk space, but if $N_f \ll N_c$, we are introducing flavour branes in the probe limit and their backreaction can be ignored (this corresponds to the quenched approximation in lattice physics – ignoring quark loops). The embedding of a D7-brane is determined by the DBI action:

$$S_{DBI} = -T_7 \int d\xi^8 \sqrt{-\det(G_{\alpha\beta})} \quad (3.20)$$

It will help to rewrite the second half of the metric \textsuperscript{(3.7)} as

$$\frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 = \frac{R^2}{r^2} (dr^2 + r^2 d\Omega_5^2) \quad (3.21)$$

$$= \frac{R^2}{r^2} \sum_{i=1}^{6} dw_i^2 \quad (3.22)$$

since the part in brackets is $\mathbb{R}^6$. It is useful to divide this space further:

$$\sum_{i=1}^{6} dw_i^2 = d\rho^2 + \rho^2 d\Omega_3^2 + dw_5^2 + dw_6^2 \quad (3.23)$$

where $\rho^2 = w_1^2 + w_2^2 + w_3^2 + w_4^2$. The action is then greatly simplified if we choose the static gauge, identifying the first eight brane coordinates with the first eight spacetime coordinates, while the remaining spacetime coordinates become functions of the brane worldvolume coordinates. This breaks the $SO(6)$ symmetry of the transverse space to $SO(4) \times SO(2) \simeq SU(2)_L \times SU(2)_R \times U(1)_R$ in the gauge theory. We have two functions to vary over the volume of the brane, $w_5(\xi^\alpha)$ and $w_6(\xi^\alpha)$, but we will use the $SO(2)$ symmetry between $w_5$ and $w_6$ to work with one function, $L = \sqrt{w_5^2 + w_6^2}$. Thus, the action is

$$S_{DBI} = -T_7 \int d\xi^8 \sqrt{\rho^4 (1 + L'(\rho)^2)} \sim \int d\rho \rho^3 \sqrt{1 + L'(\rho)^2} \quad (3.24)$$

\textsuperscript{1}If we had chosen to use a supersymmetric embedding of D5s, the strings stretched between the branes would have been restricted to two of the three spatial dimensions on the D3s. A D9 would not allow for separation between the branes, which (as we will see) is necessary for the fundamental matter to be massive.
From this action, the embedding equation is
\[ \partial_\rho \left( \frac{\rho^3 L'(\rho)}{\sqrt{1 + L'(\rho)^2}} \right) = 0 \] (3.25)

Clearly, this admits the solution \( L(\rho) = m \), where \( m \) is a constant. In such an embedding, a string stretched between the branes will have a minimum energy, \( \frac{m}{2\pi \alpha} \), and this energy appears as the mass of the fundamental supermultiplet. A non-zero \( m \) explicitly breaks the \( SO(2) \) symmetry of the transverse space. This corresponds to a mass term explicitly breaking the chiral \( U(1)_R \) symmetry in the field theory.

In the large \( \rho \) limit, (5.18) has the solution \( L \sim m + c/\rho^2 \). The parameter \( c \) must be zero absenting any chiral symmetry breaking terms in the action. Indeed, from (5.18), we have that
\[ \frac{\rho^3 L'(\rho)}{\sqrt{1 + L'(\rho)^2}} = -2c. \]
Regularity of the solution ensures that \( L'(\rho = 0) = 0 \), meaning that in pure AdS, the parameter vanishes. In fact, from the field-operator dictionary noted at the end of the last section, we see that \( c \) is proportional to the vev for a chiral quark condensate, \( \langle \bar{q}q \rangle \). Its presence indicates the spontaneous breaking of the chiral \( U(1)_R \) symmetry, since in the massless case, \( m = 0 \), a non-zero \( c \) again breaks the \( SO(2) \) spacetime symmetry.

Since chiral symmetry breaking implies supersymmetry breaking, this requires the addition of some fields which break supersymmetry (in both the bulk and the field theory). This can be achieved with the introduction of a non-trivial dilaton that back-reacts against the geometry – examples include the Constable-Myers [68] geometry and the dilaton-flow geometry of Kehagias and Sfetsos [69] and Gubser [70]. The latter will be employed in Chapter 4 of this thesis. In string frame, the metric is
\[ ds^2 = e^{\phi/2} \left( \frac{r^2}{R^2} A^2(r) dx_{3+1}^2 + \frac{R^2}{r^2} dr_6^2 \right), \] (3.26)
with
\[ A(r) = \left( 1 - \left( \frac{r_w}{r} \right)^8 \right)^{1/4}, \quad e^{\phi} = \left( \frac{1 + (r_w/r)^4}{1 - (r_w/r)^4} \right)^{\sqrt{3}/2}. \] (3.27)

Note that \( e^{\phi} \) goes asymptotically as \( 1 + \sqrt{6} r_w^4 / r^4 \), so it corresponds to the
appearance of a vev for a dimension 4 operator – that is, $\text{Tr} F^2$ (in \[70\] and \[71\] this was argued to be an induced effect produced by giving the scalars an $SO(6)$-invariant mass).

The singularity that occurs at $r = r_w$ acts as a repulsion around the centre of the geometry and forces any brane embedding to warp around the centre to avoid the geometry. This is shown in Fig. (3.2), where the embedding $L(\rho)$ for a D7 brane is shown ($L(\rho \to \infty) \to 0$, so the corresponding quarks are massless). This warping spontaneously breaks the $SO(2)$ symmetry in the bulk and the embedding develops a non-zero $c$ – the quark condensate develops a vev and chiral symmetry is broken.

The repulsive nature of this geometry can be observed in the fact that the singularity in the dilaton causes any string frame action to blow up:

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma e^{\phi/2} \sqrt{-\eta^{ab} \partial_a X^M \partial_b X^N G_{MN}}. \quad (3.28)$$

The equations of motion for a string or brane are analogous to the case of a point particle in general relativity:

$$\partial_a \partial^a X^\mu + \Gamma^\mu_{\kappa\nu} \partial_\kappa X^\nu \partial^a X^\nu = 0 \quad (3.29)$$

where $\Gamma^\mu_{\kappa\nu} = \frac{1}{2} G^\mu_{\rho\nu} (\partial_\kappa G_{\rho\nu} + \partial_\nu G_{\rho\kappa} - \partial_\rho G_{\kappa\nu})$ is the ten-dimensional Christoffel symbol. The small velocity limit of this equation for a point particle is

$$\ddot{x} = \frac{1}{2} G^{xx} \partial_x G_{tt} = -\partial_x \Phi(x) \quad (3.30)$$

where we identify a Newtonian “gravitational potential”, $\Phi(x)$. The radial dependence of this potential is therefore

$$\Phi(r) = -\frac{1}{2} \int_{r_0}^r G^{tt} \partial_r G_{tt} dr' = -\frac{1}{2} \int_{r_0}^r -2 \frac{r'^3}{L^4} dr' \sim r^4 \quad (3.31)$$

for ordinary AdS, with a suitable choice of $r_0$. Thus, AdS is always attractive toward $r = 0$ (and, for example, a string attached to a brane will hang down toward
the centre of the geometry). With the metric (3.26), the potential is

\[
\Phi(r) = \frac{1}{2} \int_{r_0}^{r} e^{-\phi/2} r'^2 \partial_{r'} \left( e^{\phi/2} A(r')^2 r'^2 \right) dr'
\]  

(3.32)

This potential is plotted in Fig. 3.1, where we can see the minimum at \( r = r_0 = 1.18 \). When we look at the embedding of a brane with zero-mass quarks (Fig. 3.2), we find that the point of closest approach is \( r = L(\rho = 0) = 1.23 \), close to the “potential” minimum \( r_0 \).

The naked singularity in this set up makes it somewhat difficult to work with, particularly if, as in Chapter 5, we wish to start with a flat, symmetric embedding that goes through the origin. A softer divergence is required and for that purpose we can introduce a background magnetic field or a hand-chosen dilaton flow that is not back-reacted to the geometry and is chosen to mimic an interesting coupling.
running. The magnetic field corresponds to imposing on the D7 worldvolume a constant non-zero $F_{xy}$. From (2.106), this gives an action

$$S_B \sim \int d\rho^3 \sqrt{1 + \frac{B^2}{\rho^2 + L(\rho)^2}} \sqrt{1 + L'(\rho)^2}$$

(3.33)

We may continue the brane embedding all the way to $r (= \sqrt{\rho^2 + L(\rho)^2}) = 0$, since that is the location of the divergence.

Another QCD-like feature that the dilaton-flow geometries possess is confinement. A quark/antiquark pair may be pictured as a string that hangs down from a brane. In pure AdS, the energy of such a configuration may be calculated, and it goes as $1/d$, where $d$ is the separation of the endpoints (quarks). This is to be expected, since in the conformal case, $d$ provides the only dimensionful parameter and scale. Within the dilaton-flow geometry, the endpoints can only separate so far before the middle of the string will encounter the wall and be forced to lie flat along it. The energy now increases linearly with the separation and confinement is induced, as the quarks are not free to separate to infinity. More properly, the Wilson loop (as prescribed in [72,73,74]) exhibits area law scaling. The confining property will be seen clearly in the time dependent simulation presented in Chapter 4.

### 3.2 Meson Fluctuations

If fundamental matter ("quarks") are represented as strings stretched between the D7s and D3s, with one colour and one flavour index, then mesons (colourless and in the adjoint of the flavour group) must be represented by strings with both ends on a D7 (we consider one flavour for convenience) [75]. The dynamics of these mesonic strings are controlled by the DBI action of the brane, since they are represented by fluctuations of the coordinates on the brane (worldvolume scalar and vector fields).
Expanding (3.20) in small fluctuations about \( L = m \),

\[
\mathcal{L} \simeq -T_7 \sqrt{\det(g_{ab})} \left( 1 + \frac{1}{2} (2\pi\alpha')^2 \left( \frac{g^{ab} R^2}{\ell^2} \partial_a \phi \partial_b \phi + \frac{1}{2} F^{ab} F_{ab} \right) \right)
\]

(3.34)

where \( g_{ab} \) is the induced metric for the flat embedding and \( \phi \) is the scalar fluctuation \( (L = m + 2\pi\alpha' \phi) \). The equation of motion for the scalars is

\[
\frac{R^4}{(\rho^2 + m^2)^2} \partial^\mu \partial_\mu \phi + \frac{1}{\rho^2} \partial_\rho (\rho^2 \partial_\rho \phi) + \frac{1}{\rho^2} \nabla_i \nabla_i \phi = 0,
\]

(3.35)

where \( \nabla_i \) is the covariant derivative on the internal 3-sphere. This has solutions of the form

\[
\phi = R(\rho) e^{ik \cdot x} Y^l(S^3),
\]

(3.36)

where \( Y^l(S^3) \) are the spherical harmonics on the the \( S^3 \), solutions to

\[
\nabla_i \nabla_i Y^l = -l(l + 2) Y^l,
\]

(3.37)

and \( R(\rho) \) (not to be confused with the AdS radius, \( R \)) has the form

\[
R(\rho) = \frac{\rho^2}{(\rho^2 + m^2)^{n+l+1}} F \left( -(n + l + 1), -n; l + 2; -\rho^2/m^2 \right),
\]

\[
R(\rho) = \frac{4\pi m_q}{\sqrt{\lambda}} \sqrt{(n + l + 1)(n + l + 2)},
\]

(3.38)

where \( m_q \) is the mass of the fundamental hypermultiplet.

This solution is set by the requirements of reality, regularity and normalizability, which also ensure a discrete mass spectrum,

\[
M(n, l) = \frac{2m}{R^2} \sqrt{(n + l + 1)(n + l + 2)} \quad (3.39)
\]

\[
= \frac{4\pi m_q}{\sqrt{\lambda}} \sqrt{(n + l + 1)(n + l + 2)} \quad (3.40)
\]

where \( m_q \) is the mass of the fundamental hypermultiplet.

There are also vector fluctuations that come from the brane Maxwell field, with equation of motion

\[
\partial_a (\sqrt{-g} F^{ab}) = 0.
\]

(3.41)

The solutions come in various types, but for the ones that will interest us later (i.e.
those with non-zero $A_\mu$) this reduces to the same equation of motion as for the scalar excitations and they have the same solutions.

Note that there is a mass gap in the spectrum and the masses take their scale from the brane separation, $m$. For $m \to 0$, there are no solutions, reflecting the unbroken conformal symmetry which prevents the formation of bound states. Also note that unlike in QCD, for the large $\lambda$ regime, the mesons are much lighter than the bare quarks, by a factor $\sim 1/\sqrt{\lambda}$. Suitable deformations to the AdS geometry that induce chiral symmetry breaking achieve results that are rather closer to those observed in QCD (e.g. [76]).
Chapter 4

Hadronization at the AdS Wall

4.1 Introduction

Hadronization, the conversion of quark pair production into jets of hadrons is traditionally a thorny problem in QCD. With the AdS/CFT Correspondence, we have a new tool to explore strongly-coupled gauge theories – it is therefore interesting to model this particular process with the Correspondence.

We should stress before we begin that theories that lie close to the $\mathcal{N} = 4$ gauge theory will necessarily include physics that is unlike that in QCD since the glue dynamics is strongly coupled into the far UV. It is not our intention therefore to provide an exhaustive analysis of hadronization in such theories, although we will develop computations that would contribute to such an endeavour. Instead we wish to explore a few characteristic hadronization events, seeking clues to new processes that might be of relevance to QCD. Our main conclusion is that hadronization events very probably contain the fast acceleration of slow moving or static quarks and that, in the AdS/CFT description, that acceleration induces radiation of hadronic modes. We will concentrate on $\rho$ meson production, the simplest example in AdS duals, where that radiation is described by an electromagnetic problem. The radiation is generated by the acceleration of the flavour charge. This dependence on the quark’s acceleration profile and the importance in the computation of the hadronic wave function both suggest how one might incorporate
new physics into QCD beyond, for example, the Lund string model \cite{77}. Whether such insights will survive as one interpolates between theories close to $\mathcal{N} = 4$ Yang Mills and QCD is, of course, unclear but we simply seek to raise new ideas here. Of course, irrespective of its relevance to QCD, it is interesting that there appear to be hadronization mechanisms beyond the Lund model in these models with AdS duals.

The combination of asymptotic freedom and confinement in QCD ensures that the road from quark production to jets, as in a high-energy accelerator such as LEP, is broken up into three overlapping stages. This is the statement of factorization. For a recent review, see \cite{78}. The first, pair production, typically occurs at high energies and so the relevant degrees of freedom are a bare quark/anti-quark pair. As they fly apart, these quarks begin to radiate soft gluons and quark/anti-quark pairs - the process known as showering - so that the relevant degrees of freedom are dressed quarks. Showering can be computed at high energies in QCD with effective theories including soft collinear effective theory. After the dressed quark/anti-quark pair have propagated a distance of order $1/\Lambda_{\text{QCD}}$ apart, the strong coupling and confining dynamics of QCD become relevant: the quark pair forms a highly excited colourless hadron that can and does fragment. This last conversion is known as hadronization.

The middle process of showering has been studied in different contexts via the Correspondence. A number of authors \cite{79,80} have computed the evolution after the injection of $R$ charge into the $\mathcal{N} = 4$ Super Yang-Mills theory. The $\mathcal{N} = 4$ theory is strongly coupled at all scales and so there is no suppression of the emission of large transverse momentum - rather than seeing jets, the events fill the entire space surrounding the initial pair creation. This is the first example of how theories close to $\mathcal{N} = 4$ Yang Mills differ from QCD. Our philosophy will simply be to take no lessons away from AdS duals in such circumstances and instead seek processes that might plausibly survive the transition to QCD.

In the AdS/CFT Correspondence, strings with their ends tied to a D7 brane represent a quark/anti-quark pair in the dual field theory. The dual quarks have a constituent mass that goes to zero in the limit that the D7 brane fills the entire
AdS space. The evolution of such pairs in the massless quark limit with separating end points was first studied in \[81\]. They worked in the quenched $\mathcal{N} = 2$ gauge theory obtained by placing probe D7 branes in $\text{AdS}_5 \times S^5$. Both the endpoints and the string between them fall indefinitely into the radial direction of the AdS space representing the spreading of baryon charge and energy respectively in the dual field theory as the state evolves. These states are anisotropic and their evolution naturally describes the showering of flavour. Hadron (rho meson) production can only occur on the D7 world-volume through the motion of the endpoints of the string. The production for these states was computed \[81\] and jet-like structures were seen.

Much attention has focused on heavy quark propagation at finite temperature \[82,83\]. The $\mathcal{N} = 4$ theory is perhaps more like QCD in its non-conformal high temperature phase which is described by an AdS-Schwarzschild geometry \[72\]. These computations have also been inspired by attempts to reproduce jet quenching observed in heavy ion collision data at the Relativistic Heavy Ion Collider.

In this chapter we wish to return to study the second and third phases of hadronization described above at zero temperature. To find a dual gravity description that behaves like QCD one must seek a non-conformal version of the AdS/CFT Correspondence that at least incorporates confinement. There has been a considerable body of work in this direction. We will work in the context of deformed AdS geometries \[84,85,86,87,88,89,69,70,68\] which have the benefit of being 3+1 dimensional at all energy scales and having naive perturbative dimensions for a class of operators in the UV (of course the UV is not perturbative but conformal, strongly coupled, and highly supersymmetric). Simple deformations such as the introduction of masses for the $\mathcal{N} = 4$ matter content that break the supersymmetry to $\mathcal{N} = 2$ \[85\], $\mathcal{N} = 1$, \[86,87,88\] or less \[89,69,70,68\] have been studied. These dualities are not as well understood as the original Correspondence but a number of features seem generic. In particular, the geometries dual to confining theories develop a wall structure in the interior that stops fields from penetrating to energy scales (radial distances in AdS) below the mass gap of the
theory. Such a repulsive wall leads to linear potentials between quarks since they are described by the ends of a fundamental string in the geometry - typically the centre of these strings fall on to and lie close to the repulsive wall with their energy then just scaling with their length.

In fact, the imposition of a simple hard wall cut off on AdS is now frequently used to model a mass gap \([90, 91, 92]\). That will not be sufficient for us since we wish to evolve the strings onto the wall and therefore need the repulsion to be described by the geometry. Given that these dualities are not fully understood we will take the simplest example of a deformation with such a hard wall. So called “dilaton flow” geometries \([69, 70, 68]\) have a non-trivial dilaton profile in a deformed AdS space. The dilaton, or on the field theory side the gauge coupling, blows up at some infra-red scale generating the wall (effectively the scale \(\Lambda_{\text{QCD}}\)). The wall is repulsive to strings and also to the D7 branes they are attached to, which dynamically induces both confinement \([93, 70]\) and chiral symmetry breaking \([94, 95]\). Using D7 branes to introduce quarks has many advantages - the set up is simple, it is easy to introduce an explicit quark mass and follow its renormalization in the presence of chiral symmetry breaking that induces a constituent mass, and the Gell-Mann Oakes Renner relation is simply reproduced \([94, 95]\). Such a construction is a decent first stab at describing a QCD-like gauge theory.

Equally we must stress that there is clearly an important and outstanding problem in the field of understanding the structure of hard walls in such theories but we don’t seek to address that here - our philosophy mirrors that of AdS/QCD \([90, 91, 92]\) and the great deal of subsequent work, in simply imposing a repulsive wall and seeing the consequences for phenomenology. The reader, of course, should not trust any quantitative statements around the wall (actually AdS/QCD is remarkably quantitatively successful although there can be no systematic estimates of errors) but we nevertheless hope to extract correct qualitative behaviour.

Having found a model for such a theory, we follow the programme of \([81]\) and evolve strings in this geometry. In particular, we consider numerical solutions of the
Polyakov string action for strings with ends tied to a D7 brane and point-like initial conditions to represent the pair creation of a quark/anti-quark pair. We allow the end points to separate quickly leaving a string falling into the geometry between. It is fairly easy (though computer-time consuming) to follow this evolution with pre-built numerical integrator functions such as \texttt{NDSolve} in Mathematica. As one might expect, in the hard wall setup the string falls onto and bounces off the wall in the radial direction - the centre of the string then oscillates, falling on to the wall repeatedly. Whilst this happens, the endpoints move apart and a longer string grows, taking up the kinetic energy near the end points. Formally, we study this process at infinite $N$ and so there is no string breaking; instead, the string continues to evolve. Since the string can only extend so far, the endpoints eventually stop and their motion reverses. The string bounces in the three dimensional space. The model therefore correctly reproduces expectations from field theory in the large $N$ limit.

For real QCD, string breaking is presumably greatly favoured, occurring soon after there is of order $\Lambda_{\text{QCD}}$ worth of potential energy in the string. In our gravity description, string breaking is captured by stringy $1/N$ effects: there will be some transition amplitude between our initial string and a final state and the actual evolution will involve a sum over all possible intermediate states, including those with two string segments. Attempts have been made to compute this breaking probability in \cite{96,97,98,99} which we do not add to here. We choose instead to study the evolution after string breaking since we will find additional radiation resulting from the evolution after that breaking. We simply break our strings by hand and consider particular states with two string segments. In particular, we locally insert a static quark/anti-quark pair (presumably the most likely vacuum state) and then evolve these new particular strings. We simulate the inserted pair with a static string that falls straight down into the AdS space from the D7 brane. At a time of our choosing (usually once the initial string grows to a size of order $1/\Lambda_{\text{QCD}}$), we break the initial string in the middle by hand and join each half to a static segment. We then follow this broken string as it evolves.

This configuration again evolves as one might expect - the static endpoint
(anti-quark) is dragged by the fast moving endpoint (quark) that it has been attached to and the endpoints (quark/anti-quark pair) of the original string are free to separate to infinity. The broken strings have a kink at the point where the static string was attached which evolves to the endpoints. The kink then causes a rapid jerk in the motion of the inserted static endpoint (anti-quark). The important conclusion we wish to take away from this computation is that hadronization events are liable to contain rapid acceleration of slow moving quarks.

In the AdS/CFT Correspondence, mesons made of the quarks are associated with open strings tied to the D7 brane [75] - in fact, in the large $N$ and strong coupling limit, rho mesons are the dominant such mode, described by a dual gauge field on the D7 world-volume. The string endpoint is a source for that gauge field. The rapid jerk of an endpoint provides a mechanism for copious production of the rho mesons, in the same way that an accelerating charge radiates in classical electromagnetism. We show how to compute this production.

The rho meson and each of its radially excited states is associated with a normalizable mode in the radial direction on the D7 brane. The full production of the gauge field on the D7 world-volume can therefore be computed mode by mode. In fact the radial dynamics simply encodes the mass of the physical state and the strength with which a particular source couples to that mode. For a single mode, the calculation is just of the usual form for a moving source coupled to a massive vector field in 4D and is straightforwardly computed. A static quark (string endpoint) has a cloud of mesons around it with their density decaying with distance as an exponential of the mass. When an endpoint accelerates, it also radiates waves which correspond to the production of rho mesons in the hadronization event. We explore this production for the rho meson and its excited states.

The solutions we present for moving string solutions provide a qualitative understanding of how hadronization may occur in QCD or related theories. In particular the new physics we highlight is the rho meson radiation generated by the acceleration of quarks, which depends on the quarks’ acceleration profile and the hadronic wave functions. We do not wish to make any claims that these
computations are numerically accurate for QCD, but we hope that the mechanisms revealed will provide thought for the future modelling of hadronization in QCD. We discuss the prospects in the final section.

4.2 Geometries and Quarks

The vacuum structure of $\mathcal{N} = 4$ SU($N$) super Yang-Mills gauge theory is encoded in the AdS$_5 \times S^5$ geometry

$$\text{d}s^2 = \frac{r^2}{R^2} \text{d}x_{3+1}^2 + \frac{R^2}{r^2} \text{d}r_6^2$$

where $R^4 = 4\pi g_s N \alpha'^2$, $g_s$ is the string coupling, and $\alpha'$ sets the string length. The dilaton is constant and there is a four form $C_{(4)} = r^4 / R^4 \text{d}x^0 \wedge \ldots \text{d}x^3$. Note that, in these coordinates, $r$ (the radial direction in the six dimensional $r_6$ space) corresponds to the gauge theory energy scale ($r = 0$ is the infra-red). The AdS space has a boundary at $r = \infty$ and the dual gauge theory “lives” on that boundary with a metric conformal to the Minkowski space metric $\text{d}x_{3+1}^2$.

To introduce a single quark flavour (really a single $\mathcal{N} = 2$ hypermultiplet), we place a D7 brane lying in the $x^0 - x^3$ and $r^1 - r^4$ directions (see Figure 4.1). In pure AdS, such a brane lies flat with $r_3^2 + r_4^2 = r_0^2$. The separation $r_0$ sets the scale.

**Figure 4.1:** The basic D3-D7 system showing the coordinate labelling we use - note the D3 lies at $r = 0$. We also show two possible moving string configurations - one lies off the plane of closest approach between the D3 and the D7 and lies in a 3D space, the string with end points separating along that plane lies in just 2D. We study moving strings of the latter type.
of the constituent quark mass \(m_q = \frac{r_0}{2\pi\alpha'}\), related to the energy of an open string with one end at the bottom of the D7 brane and the other at the AdS horizon \(r = 0\). In contrast, a fundamental string with each end on the D7 brane represents a quark/anti-quark pair and the interaction energy between them. As shown in [74], for very heavy static quarks (i.e. with their ends tied to a D7 brane at \(r = \infty\)), the dual string hangs further and further into the bulk as the quarks are separated. The total energy of the state scales as one over the quark separation.

To provide a gravity description of a confining gauge theory, we will consider a geometry that is asymptotically AdS (the UV contains the same degrees of freedom as the \(\mathcal{N} = 4\) theory) but which has a back-reacted hard wall in the IR. The simplest such example we know is a dilaton-controlled deformation [69,70]. The metric in Einstein frame for this deformed geometry is

\[
ds^2 = \frac{r^2}{R^2} A^2(r) \, dx_{3+1}^2 + \frac{R^2}{r^2} \, dr^2,
\]

(4.2)

with

\[
A(r) = \left(1 - \left(\frac{r_w}{r}\right)^8\right)^{1/4}, \quad e^\phi = \left(\frac{1 + (r_w/r)^4}{1 - (r_w/r)^4}\right)^{\sqrt{3}/2},
\]

(4.3)

and the four form is as in the pure AdS solution.

Note that the dilaton (dual to the gauge coupling) and metric have a singularity at \(r = r_w\) which can be loosely interpreted as the scale at which the gauge coupling diverges, \(\Lambda_{\text{QCD}}\). In [71] it was argued that the best interpretation of this geometry might be as dual to the low energy theory below some UV cut off where some highly irrelevant supersymmetry-breaking but \(SO(6)\)-preserving coupling becomes important in the \(\mathcal{N} = 4\) gauge theory (a scalar mass term is an example of such an operator).

For our purposes in this paper though, the geometry is simply playing the role of a back-reacted hard wall to include confinement and the precise physics is unimportant. As a result, we cannot hope to make quantitative predictions near the wall in our analysis but we hope to extract important qualitative results.

Quarks can generically be added to the theory by including a D7 brane as in
the AdS case. The hard wall is repulsive to the D7 brane and so induces chiral symmetry breaking through the formation of a quark condensate - the details are discussed in \[94, 95\]. In Figure 4.2 we sketch the form of the D7 embedding in the geometry for the case of zero bare quark mass - the D7 warps around the origin so that there is never a zero length string between the D3 and D7: the quarks have acquired a constituent mass.

As discussed in \[93, 70\] a string with ends tied to a D7 brane, corresponding to a quark pair, is repelled by the hard wall at \( r = r_w \) and the string lies close to that wall - there is a linear potential between the quarks.

**Conventions**

Throughout this work, we consider fields living on four different spaces: the full 10D geometry, the worldvolume of a D7 brane, the worldsheet of a fundamental string, and on the Minkowski spacetime of the dual field theory. To simplify some of this chaos, we choose some conventions for how we label indices and these fields. First, we will use the upper-case Latin letters \((M, N,..)\) to refer to 10D spacetime indices (e.g. \( G_{MN} dx^M dx^N \) is the metric on the 10D space), while lower-case Latin letters \((a, b,..)\) refer to worldsheet and worldvolume indices. Finally, we use lower-case Greek letters \((\mu, \nu,..)\) to index the Minkowski directions in both the 10D geometry and D7 worldvolume (i.e. \( x^\mu, \mu = 0, 1, 2, 3 \)).
4.3 Strings in Pure AdS

A study of string evolution in pure AdS was presented in [81]. Here we repeat that framework, which we will use below, and summarize the form of the results.

We work with the Polyakov action in Einstein frame

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma e^{\phi/2} \sqrt{-h} h^{ab} \partial_a X^M \partial_b X^N G_{MN},$$

where $h_{ab}$ is the world sheet metric, $\sigma^a$ labels the worldsheet coordinates, $X^M$ are the embedding functions, and $G_{MN}$ is the background metric. One is free to use the symmetries of the theory to fix the world sheet metric and we will pick the form

$$h_{ab} = \begin{pmatrix} -\Sigma(x,r) & 0 \\ 0 & \frac{1}{\Sigma(x,r)} \end{pmatrix}. \quad (4.5)$$

We pick and choose the form of the stretching function $\Sigma$ problem by problem in order to keep our numerics stable. Also, we label the timelike worldsheet coordinate as $\tau$ and the spacelike as $\sigma$ (i.e. $d^2\sigma = d\tau d\sigma$).

The equations of motion are

$$\partial_a [e^{\phi/2} \sqrt{-h} h^{ab} G_{MN} \partial_b X^N]$$

$$- \frac{1}{2} \partial_a X^P \partial_b X^N \frac{\partial}{\partial X^P} (e^{\phi/2} \sqrt{-h} h^{ab} G_{PN}) = 0. \quad (4.6)$$

Note that the derivative in the final term acts on $h^{ab}$ depending on the choice of $\Sigma$.

The open string boundary condition applied at the string end points $\sigma = \sigma_*$ is

$$e^{\phi/2} \sqrt{-h} \sigma b G_{MN} \partial_b X^N = 0. \quad (4.7)$$

However, if the string is attached to a D-brane localized at $x^M = c^M$, then this condition is replaced in the directions transverse to the brane by the Dirichlet condition $X^M = c^M$. 
There are also world sheet constraints from variation with respect to $h_{ab}$:

$$
\partial_a X^M \partial_b X^N G_{MN} = \frac{1}{2} h_{ab} h^{cd} G_{MN} \partial_a X^M \partial_d X^N,
$$

which, with the form of $\eta_{ab}$ above become

$$
\dot{X} \cdot X' = 0, \quad (4.9)
$$

$$
\dot{X}^2 + \Sigma^2 X'^2 = 0, \quad (4.10)
$$

where $\dot{X}$ indicates $\partial_\tau X$ and $X'$ denotes $\partial_\sigma X$.

We will also need to compute the energy of the strings that we generate. That energy can be determined from the 10D space-time stress energy tensor

$$
T^{MN} = \frac{2}{\sqrt{-G}} \frac{\delta S_p}{\delta G_{MN}(x)} = -\frac{1}{2\pi \alpha'} \int d^2 \sigma e^{\phi/2} \sqrt{-h} h^{ab} \partial_a X^M \partial_b X^N \frac{\delta^{10}(x - X)}{\sqrt{-G}}. \quad (4.11)
$$

The conserved energy is then given by the integral over the 9D space of $-T^0_0$:

$$
E = \frac{1}{2\pi \alpha'} \int d^2 \sigma e^{\phi/2} \sqrt{-h} h^{ab} \partial_a X^0 \partial_b X_0 \delta(t - X^0). \quad (4.12)
$$

If we rewrite the delta function as

$$
\delta(t - X^0) = \frac{\delta(\tau - \tau_s(\sigma))}{|\partial_\tau X^0(\tau_s(\sigma), \sigma)|}, \quad (4.13)
$$

where $\tau_s(\sigma)$ is the solution to $X^0(\tau_s(\sigma), \sigma) = t$, we can then integrate over $\tau$ to find

$$
E = \frac{1}{2\pi \alpha'} \int d\sigma e^{\phi/2} \sqrt{-h} \frac{h^{ab} \partial_a X^0 \partial_b X_0}{|\partial_\tau X^0|} |_{\tau = \tau_s(\sigma)}. \quad (4.14)
$$
4.3.1 Separating Quark Solutions

In [81] falling strings in AdS with endpoints separating in the Minkowski directions $x^\mu$ were studied - those solutions describe states with massless quarks in the basic $\mathcal{N} = 2$ theory described by the D3-D7 system. For massless quarks, the D3 lies within the D7 and the strings with ends on the D7 are free to come arbitrarily close to the D3s at $r = 0$. The solutions they found had the endpoints separating in $x$ and, at late times, falling in $r$ along geodesics. The centre of the string fell faster in $r$ leaving a curved string between the endpoints. They showed that kinks in the initial condition of the string were quickly smoothed out by the inflation-like growth of the string.

We now wish to begin to make contact to QCD where, although the current quark masses are small (for the light quarks), they nevertheless obtain a large constituent mass from the chiral symmetry breaking dynamics of the theory. For this reason, we will work with the D3-D7 system in a configuration where the D3 and D7 do not coincide (i.e. the dual quarks are massive). Later, we will consider the setup where we embed the D7 in the deformed AdS of Eq. (4.2).

As shown in Figure 4.1, the motion of a string ending on the D7 is generically a three dimensional problem. The string can move in the Minkowski directions $x^\mu$ and the $r^{1-4}$ directions on the D7 whilst the centre of the string may droop into the remaining two $r^{5-6}$ directions of the geometry. For simplicity, we will restrict the motion to two dimensions - that is, we will look at strings lying at the point of closest approach between the D3 and the D7, with ends separating in the $x^\mu$ directions. There is no hard computational block to performing the more generic computation, but we believe we can pick out the crucial physics within the simplified setup.

So that our analysis can be compared to that in [81], we will follow their example and define an inverse radial coordinate

$$z^2 = \frac{R^4}{r_5^2 + r_6^2}. \tag{4.15}$$
The D7 brane embeddings are more complicated in these coordinates, but we will only work with strings at the point of closest approach to the D3, $z_0$. The strings we study therefore lie at $r_1^2 + r_2^2 + r_3^2 + r_4^2 = 0$. The constituent mass of the quarks is then $m_q = \frac{R^2}{2\pi\alpha'z_0}$. The AdS boundary is now located at $z = 0$ and the IR (or the D3 branes) is at $z = \infty$. The string then falls in the 3d space parameterized by the Minkowski directions $x^0 \equiv t, x^1 \equiv x$ and radial coordinate $z$ with metric

$$ds^2 = \frac{R^2}{z^2} \left( - (dx^0)^2 + (dx^1)^2 + dz^2 \right). \quad (4.16)$$

The general initial conditions for such a string are parameterized by specifying the embedding functions $t, x, z$ and their derivatives at worldsheet time $\tau = 0$ as a function of $\sigma$. As an example, consider the initial condition

$$t(\sigma, 0) = 0, \quad x(\sigma, 0) = 0, \quad z(\sigma, 0) = z_0. \quad (4.17)$$

Note that this initial condition corresponds to a configuration where the string is initially contracted to a point. Since $X'(\sigma, 0) = 0$, the first world sheet constraint Eq. (4.9) is satisfied and the second Eq. (4.10) is simply $\dot{X}^2 = 0$.

Of course, the point-like initial conditions Eq. (4.17) are not sufficient to specify the subsequent string evolution. We must also specify the $\tau$-derivatives at $\tau = 0$ consistent with the constraint equations Eq. (4.8). Consistent point-like initial conditions are then specified by two free functions, $\dot{x}(\sigma, 0)$ and $\dot{z}(\sigma, 0)$. The last derivative, $\dot{t}(\sigma, 0)$, is fixed by the constraints to be

$$\dot{t}(\sigma, 0) = \sqrt{\dot{x}(\sigma, 0)^2 + \dot{z}(\sigma, 0)^2}. \quad (4.18)$$

As an example, we consider strings with initial derivatives

$$\dot{z}(\sigma, 0) = Cz_0 \sin \sigma, \quad \dot{x}(\sigma, 0) = Dz_0 \cos \sigma, \quad (4.19)$$

where the string endpoints are located at $\sigma_* = 0, \pi$. This set of initial conditions has two free parameters $C$ and $D$ that essentially set the string’s momentum in the
z and x directions respectively (the total string’s x momentum obviously vanishes -
D sets the size of the momentum in the x direction of either half of the string).
Strings generated from large D and small C will have end points that quickly
separate in the x direction, so that the dual field theory state contains a
back-to-back quark/anti-quark pair with large momenta.

The string generated from these initial conditions has some energy. Using Eq.
(4.14), and the initial value of \( \dot{i}(\sigma, 0) \),

\[
\dot{i}(\sigma, 0) = z_0 \sqrt{C^2 \sin^2 \sigma + D^2 \cos^2 \sigma},
\]

we find that the total energy of this string is

\[
E = \frac{1}{2\pi\alpha'} \int d\sigma g_{00} \frac{\partial_\tau X^0}{\Sigma} \bigg|_{\tau=0} = \frac{R^2}{2\pi\alpha'z_0} \int d\sigma \sqrt{C^2 \sin^2 \sigma + D^2 \cos^2 \sigma},
\]

where we have normalized the stretching function \( \Sigma \) by \( \Sigma(z_0) = 1 \). The constant C
lets us choose the initial speed of the centre of the string in the z direction while D
(in the large D/C limit) picks the total energy of the string.

With these initial conditions we can solve the three equations of motion (4.6)
provided we also impose the end-point boundary conditions

\[
\tau'(\sigma_*, \tau) = 0, \quad x'(\sigma_*, \tau) = 0, \quad z(\sigma_*, \tau) = z_0.
\]

Note that our choices of initial time derivatives above are also consistent with these
boundary conditions. Finally, since the initial conditions satisfy the constraint
equations Eq. (4.8), the evolved string satisfies them at all times.

A judicious choice of the stretching function allows NDSolve in Mathematica
to follow the evolution for a considerable time. Here we pick

\[
\Sigma = \frac{z_0^2}{z^2}
\]
To check the consistency of the solutions along the time evolution we monitor the total energy of the configuration using Eq. (4.14) - for all our solutions energy is conserved at, at least, better than the 1% level through the evolution. For the simple solutions in AdS the conservation is much better than 1%.

In Figure 4.3 we show the evolution of such a configuration with $C$ and $D = 1$ and $z_0 = 0.5R$. As the end-points separate, the string droops down into the AdS bulk. Increasing $C$ only serves to push the middle of the string down into AdS faster. Overall, this picture is consistent with the usual expectations from static strings in AdS and with the results described at zero mass in [81]. In particular, the fact that the string can penetrate indefinitely into the bulk of AdS reflects the conformal $1/r$ potential between two quarks. Of course, this behaviour is closer to the asymptotically free regime of QCD than the confining phase. We will shortly turn to modelling the latter.

It is first worth returning to consider the more complicated problem where the initial string configuration might be expected to be created on the D7 brane at large $r$. From the analysis of [81] and our computations above we can deduce the behaviour of the string. The end-points would separate in $x$ and fall in the $r$ direction. The string between those end-points would fall off the D7 brane towards $r = 0$ as shown in Figure 4.1. Eventually the end-points would fall down to the point of closest approach between the D3 and D7 branes and the evolution would then at long times merge onto the behaviour we have computed. We expect that for all of the behaviours we study henceforth the long term evolution would place the string at the closest approach point. We can therefore obtain the qualitative information we need from just considering the more limited but computationally tractable problem where the string ends are at $z_0$ for all time.

4.4 Strings with a hard wall

We will now study the motion of strings in the deformed-AdS geometry Eq. (4.2) to represent quark/anti-quark pair production in a confining gauge theory. As for the
Figure 4.3: The evolution of a string with end points on a D7 brane at \( z_0 = 0.5R \) in pure AdS with the initial conditions Eq. (4.17, 4.19) and parameters \( C = D = 1 \). The top plot shows the string worldsheet evolution and the bottom a series of constant time slice shots of the string’s motion at \( t = 0.4R, 0.8R, 1.2R, \) and \( 1.6R \).
strings we studied in the last section, we will again set the string in motion in the $x$ direction whilst localised in the $r^{1-4}$ directions at the point of closest approach of the D7 to the D3 brane - see Figure 4.2. The embedding function of the D7 brane will not therefore play any role in the computation other than determining that closest approach point, $z_0$. In practice, even $z_0$ would be a phenomenological parameter that would need to be fitted to the current quark mass - for the qualitative analysis here we set $z_w/z_0 = 2$ but the precise value is not important (if one made the ratio very large then the quarks would become heavy relative to the scale $\Lambda_{QCD} \sim 1/z_w$ representing a large bare and constituent quark mass).

We can now consider the evolution of strings in this geometry using initial conditions like those we used above in pure AdS - that is, we look at a point-like string with separating end points. For simplicity, we will take almost exactly the same initial conditions. We maintain the conditions in Eqs. (4.17), (4.19) but to satisfy the constraint Eq. (4.10) with the new metric, we need a new initial condition for $\dot{t}(\sigma, 0)$:

$$\dot{t}(\sigma, 0) = z_0 \sqrt{(C^2/A^2(z_0)) \sin^2 \sigma + D^2 \cos^2 \sigma}. \quad (4.24)$$

Since $A(z)$ rapidly becomes unity away from the wall $z = z_w$, these strings are generated from essentially the same initial conditions as in the pure AdS analysis. We use the same stretching function $\Sigma$ as in the AdS case here, Eq. (4.23). We plot the resulting string evolution in Figure 4.4.

The solution plotted in Figure 4.4 shows many of the properties one would expect. In particular, as the centre of the string approaches the hard wall, it is repelled and a straight section of string builds against the wall - this is the formation of the naive QCD string. The string action diverges at $z_w$ so the repulsion acts to stop the string ever actually reaching $z_w$. The precise maximum value of $z$ the string reaches depends on the kinetic energy in the $z$ direction of the string’s midpoint, controlled in our ansatz (4.19) above by the parameter $C$. We are working at infinite $N$ where string breaking is forbidden and so, rather than breaking, the string continues to evolve. The centre of the string begins to bounce
Figure 4.4: String evolution in a hardwall geometry. Here we set $C = 2.5, D = 0.5$ in the initial conditions Eqs. (4.17), (4.19), and choose $z_0 = 0.5 z_w$ to set the quark mass. The top plot shows the world sheet evolution. The bottom plot shows time slices through the world sheet at times $t = 0.2 z_w, 0.5 z_w, 0.8 z_w$, and $1.1 z_w$. 

Figure 4.5: Sketch of the insertion of a quark/anti-quark pair into the string evolution to represent string breaking.

off the hard wall and starts oscillating between the position of the D7 brane and the hard wall. Meanwhile, as the potential energy in the string grows and the quarks separate, the quarks’ kinetic energy begins to be sucked into the string - they slow. Eventually the quarks are brought to a halt. The energy in the string near the wall is linear $93,70$ so the maximum length the string attains is proportional to the kinetic energy in the $x$ direction, i.e. $D$ in the original ansatz (4.19) which enters the energy as given in (4.21). Next the string between the quarks begins to contract, reversing the quarks’ motion until they pass through each other. This oscillation will continue indefinitely in the absence of string breaking.

We conclude that the inclusion of a hard wall does indeed begin to move the description of quark pair production closer to the expected behaviour in QCD. In the next section we will discuss how to include string breaking and thereby allow two separated jets to emerge.

4.5 String Breaking

Our analysis of hadronization so far lacks one crucial ingredient with respect to QCD - there is no string breaking and therefore no hadronization! String breaking is an inherently $1/N$ effect and so absent from the AdS/CFT Correspondence in the tractable limit we study. Some progress has been made in studying this process though - if the string lies within the D7 worldvolume where it can break directly the rate is computable $96,97,98,99$ and is of order $1/N$. When the string lies away from the D7 there must be a quantum fluctuation that brings the string back to the D7 where it can break - estimates for this process can be found in $98,99$. Clearly
it is more likely that the string breaks near its end points than in the centre in Figure 4.4 because the string must fluctuate further back to the D7 in the centre (this is also the case when extra quark flavours with different masses are introduced, as string breaking occurs where the string intersects a brane, which is most likely near the endpoints). This is possibly why the AdS/CFT dual provides a separate description of small strings breaking from the end of the string as radiation on the D7 brane which we will compute in detail below. To compute the probability of a fluctuation in the central regions would require a full treatment of quantized strings in AdS. Here we do not wish to add to the computations of that process in [96,97,98,99] but will instead concentrate on the end-point radiation that we can compute. In QCD we know that with probability one the string will break at least once. We will assume one breaking and then evolve the string to understand the consequences for the radiation from the end points.

An unpaired quark corresponds to a string stretched from the D7 brane to the D3 branes. A static solution of this form with a new choice of stretching function ($\Sigma = A(z)/A(z_0)$) is

$$t(\tau, \sigma) = \frac{2(z_0 - z_s)}{\pi A(z_0)} \tau$$

$$z(\tau, \sigma) = \frac{2}{x} (z_0 - z_s) \sigma + 2z_s - z_0,$$

(4.25)

where $x$ is constant. This solution satisfies the equations of motion, the constraint equations and the boundary conditions for all $\tau$ and $\sigma$. It is a straight string stretching from $z_0$ (at $\sigma = \pi$) through the point $z_s$ (at $\sigma = \pi/2$).

We can therefore split our string in the centre and complete it to the D7 brane with a straight string segment as shown in Figure 4.5. We believe this is a sensible description of a typical hadronization event for the following reasons. The typical energy and momenta on the original string half segment will be large, say at LEP of order the Z mass. On the other hand the energy of a typical pair of quarks created in the vacuum will be only of order $\Lambda_{QCD}$ - relative to the rest of the jet they will be slow or as we model it essentially static. With only $\Lambda_{QCD}$ of energy
available it is hard to imagine the half string segment being greatly changed since
that would involve considerable re-arrangement of energy and momenta on it. In
addition to change the shape of the half string segment would be very non-local and
so further suppressed. The kink-like join between the two segments would
presumably be preferred as the jet energy went to infinity so that energies of order
$\Lambda_{QCD}$ would be a negligible change. In practice in any particular hadronization
event one would expect some smoothing of the junction. The kink is useful to us
though for two reasons. Firstly, it allows us to simply match string segments since
for our ansatz there is never energy or momentum flow across the central point and
hence we don’t introduce any jumps in these derivatives. Secondly though it makes
it easy to follow the content of the numerical outputs - the kink provides an easy
guide for the eye to the motion of the exchange of energy and momentum between
the two quarks along the string.

As an example we will split the string solution shown in Figure 4.4 at the
point where the centre of the string starts to bounce off the wall ($t = 1.13z_w$) - the
QCD string has roughly just formed at this point in the time evolution. At the
splitting time $t_s$ ($\tau_s$ in world sheet time) when $t(\tau_s, \pi/2) = t_s$ and $z(\tau_s, \pi/2) = z_s$ we
use the new initial conditions for the range $\pi/2 < \sigma \leq \pi$.

Strictly inserting the extra string length does not conserve energy for the half
string piece. In the infinite jet energy limit this error will become negligible though.
The solution we display below has sufficient initial energy that the straight string is
slightly less than a 10% correction to the total energy. We don’t expect the
non-conservation to have any great effect on the qualitative behaviour of the
solutions. Indeed it is possible to numerically follow more energetic strings with
much more initial momentum in the $x$ directions (ie larger $D$ in eqn. (4.17)) but the
resulting configurations are rather flat and extended in the $x$ direction making it
harder to extract physics visually from the plots - there is no qualitative change in
behaviour though. More energy can also be included in the initial half string
segment by increasing $C$, or the $z$ momentum - those cases we have found harder to
follow numerically because the string bounces in the $z$ direction. In fact since we
only study the evolution of one half of the initial string configuration one could

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imagine that some asymmetry in the distribution of energy between the two halves might be induced in the string breaking in any case. One could also imagine splicing out a straight string section in the middle of the initial string before inserting the vertical string pieces back to the D7 brane. To do this would require finding solutions for slow moving ends and then matching the position of the string and the energy and momentum derivatives at the junction. Given that following the evolution of these strings is a computer time intensive activity though this would be a very protracted procedure. Since we again would expect to see little qualitative change in the behaviour of the solutions we work with the simple central join described.

The time evolution of the half string segment can now be followed and we show such a numerical solution in Figure 4.6. The solution again behaves in accordance with naive expectations - the fast moving end point of the string continues to move, whilst two kinks in the string induced by the breaking propagate to the end points. When the kink arrives at the static end point, it is jerked (along with the entire string segment) in the direction of motion of the fast end point. Similar evolution of broken strings may be found in [100]. Following the evolution without further string breaking leads to oscillations of the end points in the segment’s centre of mass frame, which separates infinitely far away from the other broken string segment.

The crucial (and perhaps unsurprising) lesson we wish to take from this analysis is that the fast acceleration of quarks (string end points) is to be expected in a typical hadronic event. We plot in Figure 4.7 the motion of the two end-points of the string through the evolution above. The string end-points are special because they source a gauge field on the D7 brane world volume which corresponds to rho meson production - the strength of their acceleration determines how copious the emission rate is. We will compute this production in the next section.
Figure 4.6: We show a string worldsheet evolution after imposing string breaking. The initial condition involves half of the string from Figure 4.4 at time $t = 1.13z_w$ broken and extended by a straight string back to the D7 brane at $\sigma = \pi/2$ to $\sigma = \pi$. The top plot is again the world sheet evolution and the bottom plot slices taken at $t = 1.4z_w, 1.8z_w, 2.3z_w,$ and $3.2z_w$. 
4.6 Rho meson production

To understand how the string solutions above radiate energy into hadronic modes, one must study the theory on the surface of the D7 brane. We are interested in making contact to QCD so we will concentrate on the production of the rho meson states. These states are associated with an electromagnetic field on the D7 which couples to the flavour charges on the string end-points.

To introduce the formalism we will first work in the $\mathcal{N}=2$ gauge theory described by a D7 brane in pure AdS since in this case the meson solutions are known analytically. However, the formalism is much more general and we will see that the rho meson emission computation can be easily updated to include the full problem with a curved D7 in a more complex space.

In the $\mathcal{N}=2$ theory the D7 embedding is flat and extends down to $r = r_0$ with induced metric

$$ P[G] \equiv g = \frac{\rho^2 + r_0^2}{R^2} dx_4^2 + \frac{R^2}{\rho^2 + r_0^2} (d\rho^2 + \rho^2 d\Omega_3^2). \quad (4.26) $$

Here, $\rho$ is the radial direction on the worldvolume of the D7 so that

$$ \rho^2 = r_1^2 + r_2^2 + r_3^2 + r_4^2. $$

The end-points of the string act as electrically charged sources for the gauge field that lives on the worldvolume of the D7 brane. The equation of motion for that
gauge field follows from the variation of the electromagnetic action on the brane,
\[ S_{EM} = -\frac{1}{4} \int d^8 x \sqrt{-g} F_{ab} F^{ab} + \int d^8 x \sqrt{-g} j^a A_a. \] (4.27)

The variation of the gauge field gives both the equation of motion and the boundary action,
\[ \delta S_{EM} = \int d^8 x \sqrt{-g} \delta A_b \left[ \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} F^{ba}) + j^b \right] + \delta S_{bdy}, \] (4.28)

so that the equations of motion are just Maxwell’s equations,
\[ \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} F^{ba}) = j^b, \] (4.29)

and there is a boundary action (at \( \rho = 1/\epsilon \to \infty \))
\[ \delta S_{bdy} = - \int_{\rho = 1/\epsilon} d^4 x d\Omega_3 \sqrt{-g} \delta A_a F^{a\alpha}. \] (4.30)

Let us first understand the solutions in the absence of a source.

4.6.1 Rho mesons

The propagating modes of the gauge field arrange themselves into multiplets of the \( SO(4) \) isometry group of the \( S^3 \). The resulting Kaluza-Klein fields each map to different operators of the gauge theory; in particular, the singlet on the \( S^3 \) maps to a conserved baryon current [75]. To study this current, we therefore give the solutions of Eq. (4.29) without sources for the modes that only have non-zero \( A_\mu \) and are singlets on the \( S^3 \). We impose the gauge choice \( \nabla_\mu A^\mu = 0 \). The equation of motion in the absence of sources is then [75]
\[ D A_\mu = -\frac{1}{\rho^3} \partial_\rho (\rho^3 \partial_\rho A_\mu) - \frac{R^4}{(\rho^2 + r_0^2)^2} \nabla_{(4)}^2 A_\mu = 0, \] (4.31)

where \( \nabla_{(4)}^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu \) is the scalar Laplacian in Minkowski space and \( D \) is a second-order differential operator. Fourier transforming in the Minkowski
directions, we can write the equation above as an eigenvalue equation in the radial coordinate with eigenfunctions \( f_n(\rho) \) given by

\[
f_n(\rho) = I_n \frac{\,\,^2F_1(-n,-n-1,2; -\rho^2/r_0^2)}{(\rho^2 + r_0^2)^{n+1}},
\]

where the \( I_n \) are normalization constants, \( n = 0, 1, 2, \ldots \), and eigenvalues

\[
M_n^2 = 4(n+1)(n+2)\frac{r_0^2}{R^4}
\]

The solutions to Eq. (4.31) are therefore given by the modes \( A_{\mu,n} = \epsilon_\mu f_n(\rho)e^{ik_n \cdot x} \) with \( k_n^2 = -M_n^2 \).

These states have a discrete mass spectrum and are identified with the rho mesons of the dual gauge theory. The factor of \( r_0^2/R^4 \sim m_q^2/\lambda \) indicates that the meson masses are much smaller than the quark mass at large 't Hooft coupling.

Moreover, the \( f_n \) (with appropriate choices of the \( I_n \) normalizations) are orthonormal functions (subject to the weight factor \( w \))

\[
\int d\rho \, w \, f_n f_m = \delta_{mn}, \quad w = \frac{\rho^3 R^4}{(\rho^2 + r_0^2)^2}.
\]

The corresponding choice of normalization is given by

\[
I_n^2 = 2(n+1)(n+2)(3+2n).
\]

### 4.6.2 Green’s Functions

To observe the emission of rho mesons by the string end points we will solve (4.29) by means of a Green’s function for the field \( A_\mu \). In Lorenz gauge, the relevant Green’s function satisfies

\[
\mathcal{D}G^{\nu\mu}_\mu = \frac{1}{\rho^3} \delta^{\nu\mu} \delta(\rho - \rho') \delta(x^\nu - x'^\nu),
\]
where $D$ is the differential operator defined in Eq. (4.31). Since the equation of motion for the gauge field in the presence of our source is given by

$$D A_\mu = \eta_{\mu\nu} j^\nu,$$

(4.37)

the full solution for an arbitrary current distribution $j^\mu$ follows from the convolution integral

$$A_\mu(x) = \int d^8 x' \sqrt{-g} \ G_{\mu'}(x, x') \eta_{\mu'\nu'} j^{\nu'}(x').$$

(4.38)

The actual current distribution will be localized on the worldline of the string end-point and will take the form $j^\mu = q \dot{x}^\mu \delta^8(x)$ where the dot represents differentiation with respect to proper time.

In order to obtain the Green’s function, let us expand in the basis of eigenfunctions describing the rho mesons used in Eq. (4.32) so that

$$G_{\mu'}(\rho, x^\nu; \rho', x'^{\nu'}) = \sum_n f_n(\rho) f_n(\rho') \tilde{G}_{n,\mu'}(x^\nu, x'^{\nu'}).$$

Inserting this form into Eq. (4.36), multiplying by $\rho^3 f_m$ and integrating over all space we find that the four-dimensional functions $\tilde{G}_n$ are just the Green’s functions for massive vectors in Minkowski spacetime with masses corresponding to the rho meson masses.

### 4.6.3 Boundary Data

The near-boundary behaviour of the gauge field is related to the one-point function of the dual conserved baryon current in the field theory. In particular, that one-point function is given as

$$\langle J^\mu(x^\nu) \rangle = \lim_{\epsilon \to 0} \frac{\delta S_{SUGRA}}{\delta A_\mu(x^\nu, 1/\epsilon)},$$

(4.39)

where $S_{SUGRA}$ is the on-shell bulk gravity action and the bulk gauge field $A_\mu$ is the singlet mode on the $S^3$. Using the variation of the bulk action in Eq. (4.28), the
boundary current is simply

\[
\langle J^\mu(x) \rangle = -\lim_{\epsilon \to 0} \rho^3 \eta^{\mu\nu} \partial_\rho A_\nu(x^\nu, \rho) |_{\rho = 1/\epsilon}.
\]  

(4.40)

We can therefore write a bulk-to-boundary Green’s function that relates the bulk source to the boundary current. In particular, we write

\[
\langle J^\mu(x^\nu) \rangle = \int d^8 x' \sqrt{-g} G_{\mu}^\mu(x^\nu; x'^\nu) f^\nu(x'),
\]

(4.41)

where we define the bulk-to-boundary Green’s function \( G \) as

\[
G_{\mu}^\mu(x^\nu; x'^\nu, \rho') \equiv \sum_n 2(-1)^n I_n f_n(\rho') \bar{G}_{n,\mu}^\mu(x^\nu, x'^\nu),
\]

(4.42)

where the \( f_n \) are the eigenfunctions in Eq. (4.32) and \( \bar{G}_n \) is the 4d Green’s function for a massive vector as before. The factor of \( 2(-1)^n I_n \) comes from the insertion of the near-boundary expansion of the \( f_n \)’s,

\[
f_n(\rho) = (-1)^{(n+1)} I_n \frac{1}{\rho^2} + O(\rho^{-3}),
\]

(4.43)

into the form of the boundary current in Eq. (4.40).

### 4.6.4 Retarded Potential

We have now reduced the problem to solving for each mode \( G_n \) the retarded potential for a massive field in flat space. The retarded potential takes the form [101]

\[
\bar{G}_{\mu}^\nu = \frac{1}{4\pi} \theta(t - t') (\delta(\sigma) + V(\sigma) \theta(-\sigma)) \delta_{\mu}^\nu.
\]

(4.44)

Here we use the Synge world-function \( \sigma = \frac{1}{2} \eta_{\mu\nu}(x - x')^\mu(x - x')^\nu \). The non-singular part of the solution is given by \( V(\sigma) = -\frac{M}{\sqrt{-2\sigma}} J_1(M_n \sqrt{-2\sigma}) \) where \( J_1 \) is the Bessel function of order 1.
4.6.5 Beyond the $\mathcal{N} = 2$ Theory

The above rho meson emission analysis was specific to the $\mathcal{N} = 2$ theory on the surface of a D7 brane in AdS. However, it is clear that it is easily updated to the spectrum on a generic D7 embedding in some deformed AdS space, such as the one we have been considering. Firstly the differential operator $\mathcal{D}$ in (4.37) would change. As a result the rho meson states would have different masses $M_n$ and there would be different hadronic wave functions $f_n(\rho)$. For examples of the numerical solutions in a dilaton flow geometry see [76]. In fact the hard wall is sufficiently localized that for D7 branes further out than about $2r_w$ the embeddings are flat [93] and well approximated by the $\mathcal{N} = 2$ theory’s results in any case. These are the only changes to the above analysis though - performing all of the steps above one again arrives at the need to solve the 4D retarded potential in (4.44) but with the new $M_n$ values. The wave functions $f_n(\rho)$ enter via their value at $\rho = 0$, $I_n$, through the source as we will see - again for a particular embedding the appropriate replacements can be made.

4.6.6 Static String End Point

As a first example of using this formalism we will compute the baryon density around a static quark. Consider such a charge at $x = 0$ and at $\rho = 0$ ($r = r_0$), the point of closest approach on the D7 brane. We will concentrate on the temporal component of the gauge field $A^0$ which is dual to the operator $\bar{\psi}\gamma^0\psi$, the quark density.

One seeks to evaluate the integral over the past trajectory of a point source moving with a constant speed in a ‘static gauge’ given by $x' = \beta t'$ - here $\beta$ is the speed of the end point in natural units. Doing the spatial integral using the fact that the source is located at a point in the space-like dimensions leaves one with the integral

$$\langle J^0(x_n') \rangle_n = \frac{2(-1)^n I_{n}^2 q}{4\pi} \int dt' \theta(t - t') (\delta(\sigma) + V(\sigma)\theta(-\sigma)) \frac{d\sigma}{dt'} \frac{dt'}{d\tau} \quad (4.45)$$
where we have used the fact that $f_n(\rho = 0) = I_n$.

The time component of the four-velocity is actually cancelled by a Lorentz factor coming from the splitting of Minkowski spacetime into space-like sections when integrating along the particle worldline. The two contributions to the integral are easily computed (for the non-singular piece due to the massive field using the integration variable $u \equiv M_n \sqrt{-2\sigma}$) giving the correctly Lorentz-covariant expression (the $\gamma$ is the usual boost factor)

$$
\langle J^0(x^\nu) \rangle_n = 2(-1)^n I_n^2 q \gamma e^{-M_n \left( \sqrt{\gamma^2(x-\beta t)^2 + y^2 + z^2} \right)} \frac{e^{-M_n \sqrt{\gamma^2(x-\beta t)^2 + y^2 + z^2}}}{\sqrt{\gamma^2(x-\beta t)^2 + y^2 + z^2}} (4.46)
$$

In the rest frame of the point source this reduces to the usual Yukawa form. The full solution is a sum over modes weighted by the $f_n$ normalizations $I_n^2$, see Eq. (4.35). There is a rapid rise in these normalizing factors with $n$ which is due to the end-point of the string being a delta function. Away from $N \to \infty$ one would expect the string to have some width and the expansion to truncate at some intermediate $n$. In any case this rise is not faster than the exponential fall off of the solutions so the physics away from the source is still dominated by the lightest modes. The Green’s function converges for all $|x| > 0$ due to the exponential factor in the Yukawa potential of each partial wave. The behaviour is dominated by the lighter modes at distances comparable to the Compton wavelength of the lightest mode. We interpret this Green’s function as the ‘dressing’ of an isolated quark by a cloud of mesons. Holography gives the relative amounts of each of the excited states in the cloud.

4.6.7 Radiation From String End Points

We now have a framework in which the emission of mesons can be modelled using the techniques of classical relativistic wave equations. Any acceleration of the string end point will result in some radiation but it will be greatest in the cases of hardest acceleration. In the large $N$ limit without string breaking the acceleration will be continuous as the string ends decelerate and then accelerate as the string bounces
as shown in Figure 4.4. Inputing a suitable acceleration profile below would determine this emission. We wish though to concentrate on the string breaking event in Figure 4.6 since this is most appropriate to QCD - here there seems to be generically a much more vigorous acceleration as the static string end is jerked into motion. We will compute for that case which would be expected to generate the most emission.

The retarded Green’s function is straightforwardly integrated over the past worldline of an accelerating endpoint to give, e.g., for a particle moving in the $x$-direction (the $u$ variable is as defined in the preceding section)

$$\langle J^0(x^\mu) \rangle_n = \frac{2(-1)^n J_n^2 q}{4\pi} \left[ \frac{1}{t - t'(\sigma = 0) - \frac{dx'}{dt'} (x - x'(\sigma = 0))} \right.$$  

$$+ \int_0^\infty du J_1(u) \frac{1}{t - t' - \frac{dx'}{dt'} (x - x')} \right]$$  

(4.47)

We will again plot the baryon number density which is holographically encoded by the sum over the $J_n^0$. It may be noted that the plots we obtain give the superposition of radiated baryon density and the static baryon density associated with the probe quark. An elementary prescription is available for computing the reaction force on the probe quark due to the radiation (by differencing the advanced and retarded potentials) but this is not what we are interested in here (it involves a negative counting of the non-causal advanced potential and so would not produce a plot resembling meson emission). In holographic scenarios (large $N$) the force exerted on the quark by the dynamics of the colour flux tube far exceeds the reaction force from meson emission anyway. In the case of an instantaneous acceleration it is possible to subtract the appropriate static and boosted solutions inside and outside of the particle’s light cone but we do not apply this here.

**Massless Meson Limit**

In the strict $\lambda \to \infty$ limit the meson masses are very small relative to the string mass (see (4.33)). At least for the lightest members of the tower, it is therefore
interesting to compute the radiation into a massless gauge field on the D7. For this case \( \sigma = \frac{1}{2} \left( -(t - t')^2 + (x - x'(t'))^2 \right) \). The first term in (4.44) then gives

\[
\langle J^0(x') \rangle_n = \frac{2(-1)^n l_n^2 q}{4\pi} \int_{-\infty}^{t} dt' \delta(\sigma)
\]

\[
= \frac{2(-1)^n l_n^2 q}{4\pi} \int d\sigma \frac{\delta(\sigma)}{(t - t') - (x - x') \frac{dx'}{dt'}}
\]

\[
= \frac{2(-1)^n l_n^2 q}{4\pi} \frac{1}{t - t'(\sigma = 0) - \dot{x}_0 (x - x'(\sigma = 0))}
\]

This is straightforward to evaluate for acceleration kicks such as those we found for the string end-points in Figure 4.7 above. For example the static end-point is accelerated quickly to a constant speed. Rather than use the particular output from earlier sections for this kick, that depends on the initial conditions we chose and the particulars of the hard wall, we will parameterize the acceleration more generally. The radiation computations we present here are much more general than the particular hard wall model presented above. As an example form for the function \( x'(t') \) that describes a stationary particle accelerating to a final speed \( a \) we take

\[
x'(t') = \frac{ab}{\pi} + at' \left( \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{t'}{b} \right) \right)
\]

where \( b \) controls the time interval over which the acceleration occurs - we plot some sample trajectories in Figure 4.8.

It is a simple matter to plot the resulting wave induced. Emission is typically a spherical shell radiating from the point of acceleration - there is an SO(2) symmetry in the \( y, z \) coordinates so we shall plot the intensity of the wave in the \( x, y \) plane at \( z = 0 \). Examples of the gauge field produced are shown in Figure 4.9. The radiative piece is visible along with the ‘hill’ of the boosted static potential. A clear, narrow emission wave is observable. For larger values of the final speed \( a \) the forward emission is typically enhanced relative to the backwards emission, and the overall emission is greater. For smaller acceleration times (smaller \( b \)) the wave front simply becomes narrower. For the accelerations of the string end-points in Figure
Plots of the function $x'(t')$ in (4.49) used to parameterize the motion of an accelerating point source. The parameter $a$ controls the final speed and is set to $a = 0.2$ here. $b$ controls the time scale of the acceleration and the plots show $b = 0.8$ (top), $b = 0.3$ (middle) and $b = 0.05$ (bottom).

In section 4.7 we expect precisely such emission of the lower mass members of the mesonic tower.

It is useful to also have analytic expressions for the angular distribution of the rho meson emission. In the massless limit we can just recall well known classical electromagnetic results [102]. For example, the angular dependence of the power spectrum for a charge linearly accelerating with acceleration $a$ is given by

$$\frac{dP}{d\Omega} = \frac{e^2 a^2}{4\pi c^3} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$  (4.50)

where $\theta$ is the angle between the normal to the observer from the charge and the direction of acceleration. As $\beta$ tends to one this emission is very forwards which seems to match the idea of a narrow jet formation.
Figure 4.9: The radiation of light mesons by a quark given an impulse in the positive $x$-direction, shown in the $z = 0$ plane. The plot shows the density of the emitted (radiative part of field) and bound (boosted static part) mesons. The top plot is for a terminal velocity of 0.2$c$ and below is 0.6$c$ (in both plots the parameter $b = 0.2$).

Massive Meson Limit

For members of the meson tower with masses close to the quark mass (very high $n$ at large 't Hooft coupling) we must compute the non-singular term in (4.47) which involves numerical integration of the Bessel function. In Figures 4.10 and 4.11 we show the effect of increased meson mass on the radiated mesons (the meson mass should be compared to the inverse time over which the string end point is accelerated). As the meson mass is increased we find emission of the more massive states are suppressed.

In the massive case the waves are dispersive and produce an interesting pattern which is not just a wave localized on the light-front. The ‘wavy’ emission of
Figure 4.10: The radiation of massive mesons by a quark given an impulse in the positive $x$-direction, plotted along the $x$-axis. The plot shows the density of the emitted mesons. The meson masses increase through the plots as $0, 10^{-2}, 10^{-1}, 1$. Note the background static field peak becomes narrower as the mass is increased.

Meson density can be considered to be arising from quantum-mechanical interference effects.

In principle we could sum over the emission of all of the meson states. At large 't Hooft coupling though there are many states lighter than the quark mass so the result would be unilluminating. The precise form of the meson masses and the coefficients $I_n^2$ are also model dependent. However, we believe that the computations we have made show how in principle the radiation could be computed and give a good understanding of the generic features of that meson radiation.
Figure 4.11: Emission of massive vector mesons \((m = \frac{1}{2})\). The parameters \(a = b = 0.2\).

4.7 Other Emission Processes

We have attempted to concentrate on the aspects of the deformed AdS dual that might have lessons for QCD. There are though many other emission processes in the full dilaton flow geometry dual. Firstly the strings can emit closed string states, or in the supergravity limit gravitational waves. A computation of this sort of emission at finite temperature can be found in [103]. Since the whole string can emit bulk modes one would expect this emission to be large in the full theory. In other words glueball emission is much stronger in theories close to \(\mathcal{N} = 4\) than in QCD since the glue is strongly coupled at all scales - this is another example of the non-QCD-like behaviour described in [79][80]. Our view is simply that one should not try to draw lessons for QCD for this aspect of emission since \(\mathcal{N} = 4\) is so far from QCD in this respect. For this reason we do not compute it here although it might be of more generic theoretical interest.

The gauge field on the D7 world volume also describes additional states [76]. There are physical states that are space-time scalars but have non-trivial wave functions on the \(S^3\) of the D7 world-volume. These scalars are R-charged and hence present in the \(\mathcal{N} = 4\) like theories but not QCD. Generically these too will be produced by the acceleration of string end points. Again we can hope to learn little for QCD, where these states do not exist, other than that end-point emission of all
available hadron states is expected. In fact, the wave functions \( f_n(\rho) \) for these states all vanish at \( \rho = 0 \) since the \( S^3 \) degenerates there. Thus for the particular string configurations we discussed above which lie only at \( \rho = 0 \) they can not be produced, more generally though their presence is expected.

4.8 Lessons for QCD

We have explored how a hadronization event happens in a gauge theory that has the degrees of freedom of \( \mathcal{N} = 4 \) super Yang-Mills in the UV but a deformation that leads to a back-reacted hard wall and confinement in the IR. This theory, of course, is not QCD but the generic picture that emerges may have some lessons for the construction of a phenomenological model of hadronization in QCD. We stress that we have tried to look at a typical event that behaves as in QCD and neglected emissions that are inappropriate to QCD.

In particular we have suggested a picture in which the initial quark anti-quark pair separate, growing a string between them that dips into a holographic radial direction. Initially the string’s energy represents a \( 1/r \) potential between the quarks reminiscent of the asymptotically free regime of QCD. The string then encounters a hard wall in the geometry and spreads out along it forming a traditional QCD-like string with energy growing with its length. We have proposed that this string will break quickly once there is sufficient energy in it to pair create a quark pair, breaking the string. The two sub-strings then separate. This process can be expected to generically involve the hard acceleration of quarks or string end-points and this is confirmed by our computation. The interesting ingredient that the AdS description provides to the emission process is that this acceleration is associated with rho meson production. That production can be cleanly computed as an electromagnetic problem. If this process survives to QCD, where there is no \( 1/N \) suppression, we would expect the theory to make good use of this emission channel. It is even possible that the majority of the hadronization particle production might come from this radiation so there would be no need for repeated string breakings. Of course, some mixture of the two may happen.
It is interesting to compare this picture to that of the Lund string model [77] which is one of the leading descriptions of hadronization used in accelerator Monte Carlos. The Lund model is based on the 1980s picture of the link between QCD and string theory. The strings between quarks live in the 3+1 dimensions of QCD. The model assumes that a very long string forms between the quark/anti-quark pair which then sequentially fragments with the fragments being assigned as various hadronic states. Attempts to compute this breaking probability in the AdS picture can be found in [96,97,98,99] - the centre of the string generically lies off the D7 brane suggesting a suppression in the probability of breaking which is presumably why the end point emission of open strings is described separately as the electromagnetic emission on the D7 world volume. It is possible to morph our picture into the Lund one by assuming that the D7 brane in Figure 4.2 lies very close to the hard wall so this suppression is removed - the fifth dimension then plays little role and the string almost lies in the same plane as the end points move. How long the string grows and how many times it then breaks are not things we have computed so we can not dispute the Lund model. The striking difference though between the models is that significant rho meson production occurs at the end points of the string moving in AdS, removing the need for repeated breakings of the initial string in the central region. Further though a clean description of this emission is provided based on an electromagnetic computation and crucially it depends on both the acceleration of the end points and the hadronic wave functions. Of course the differences in the AdS picture may be an artefact of a large $N$ expansion and not relevant to true QCD, but equally $N = 3$ is believed to not be so far from $N \rightarrow \infty$.

When comparing the evolution in our model, say in Figure 4.4 or 4.7, to QCD it is important to remember that these plots contain artefacts of the large $N$ expansion. In particular at infinite $N$ string breaking and end point radiation are suppressed. In Figure 4.7 we include one breaking but no more and no radiation. The long time evolution is therefore not appropriate to QCD and, for example, the lead quark is unlikely to be brought to a halt in the process. The first 2 time units or so of evolution would be appropriate to QCD - in that period the string breaks,
the static end accelerates hard and would radiate, generating the jet content. The remaining half string segment should then be considered as the lead particle in the jet (assuming it does not break again).

In theories close to the $\mathcal{N} = 4$ theory the rho mesons are special, in that they are associated with operators whose dimensions are protected from renormalization. This means they are present as supergravity modes in the DBI action of the D7 brane - other quark bound states would be represented by stringy states also tied to the D7 world volume. In QCD we would not expect a separation in character between these modes and all hadronic species should be produced at the end-point governed by the same end-point motion.

The idea of separating, radiating string fragments is reminiscent of another reasonably successful model of hadronization. Thermal models [26] have been proposed that treat the event as separating fireballs radiating hadrons in thermal equilibrium. Some discussion of how that thermal spectrum can emerge from a gauge theory event with many final states even at zero temperature can be found in [104,105] - it seems likely that the core idea is just that the energy of hadronization is freely available to all modes. The thermal model predicts that in the centre of mass frame of the jet emission is angularly uniform which is rather different from the very forward emission the electromagnetic computation predicts. It would be interesting to implement both our model and the thermal model in a full Monte Carlo where these different angular dependences could be tested.

Another major prediction of our framework is that the relative multiplicities of states will depend on the overlap integral of their holographic wave function and the string end-point source. At large $N$ that string end-point is a point charge but in QCD it would presumably broaden. A very toy model of holographic hadronization was presented, essentially based on these ideas, in [27] - there trial phenomenological wave functions for the different QCD states were overlapped with a Gaussian of width $\Lambda_{QCD}$. The model seemed to reproduce the jet species multiplicity data reasonably well too, suggesting that a more complete analysis
would not be immediately at odds with data.\footnote{Note to compare to data there is the need to impose an elaborate decay chain between the initially produced particles and the finally observed particles so the number of species for which there is data is much less than the total number of species in the initial yield, which may hide many evils.}

It is therefore interesting that our holographic model seems to include aspects of both the Lund string model and thermal models of hadronization. We hope that insights from AdS will lead to phenomenologically more successful models of hadronization in QCD in the future.
Chapter 5

Holography of a Composite Inflaton

5.1 Introduction

The AdS/CFT Correspondence has been extended in various different manners since the original duality was proposed, and tools have been introduced to accommodate fundamental fields in strongly coupled field theories which break chiral symmetries, as does QCD, by the formation of a quark condensate. In this chapter these tools will be used to make a first study of time dependence in such symmetry breaking theories. We expect strongly coupled gauge theories to undergo phase transitions between a high temperature chirally symmetric phase and a low temperature chiral symmetry breaking phase. Holography should for the first time allow us to analyse the evolution of such theories between these different phases. The simplest example of where such a transition might play an important role in the evolution of our Universe is in inflation and in this chapter we will concentrate on that possibility.

Inflation (see for example [106]) is now a key part of the standard cosmological model of the Universe, supported by many pieces of astrophysical data. The usual description involves one or more scalar fields slow-rolling from an unstable point in a potential to the true vacuum; however the origin for such a
scalar field remains unclear. Indeed, until evidence for supersymmetry is found in nature, fundamental scalar fields are formally unnatural in field theory because of the hierarchy problem (radiative corrections cause their mass to naturally grow to the Planck scale). This need be no obstacle to the paradigm though since many strongly coupled systems have scalar order parameters describing their dynamics that could play the same role. For example, in QCD the gauge interactions dynamically generate a quark bilinear condensate (⟨\bar{q}q⟩) that breaks chiral symmetries - one may think of that bilinear as the expectation value of a composite scalar field. The main reason to work with fundamental scalars in cosmology is simply that we have had no tools to study strong coupling problems such as the condensation in QCD. For the first time, however, the AdS/CFT Correspondence allows us to analyse just such a scenario.

We will concentrate on a particular duality which we believe to be the simplest example of holography with fundamental quark fields \[107, 108, 67, 75, 109\]. We wish to stress that we do not consider the specific degrees of freedom of the theory too crucial - it is some strongly coupled gauge theory that generates quark condensates. We hope, in the spirit of AdS/QCD models \[91, 92\], that it reflects broad aspects of many strongly coupled systems. The specific gauge theory is constructed from the D3/D7 system \[1\] in type IIB string theory which we will describe in detail below. The original theory is, as before, the large \(N_c\) \(\mathcal{N}=4\) \(U(N_c)\) gauge theory with a small number of quark hypermultiplets. There is a \(U(1)\) symmetry (a remnant of the \(SU(4)\) R-symmetry of the \(\mathcal{N}=4\) theory) which is broken when a quark condensate forms \[94\]. Several mechanisms for triggering this condensation have been explored. The cleanest is when a background magnetic field is introduced \[111, 112, 113, 114, 115, 116, 117, 118, 119\]. It was noted in the previous chapter that running of the coupling also causes quark condensation, as has been shown in back-reacted dilaton flow geometries \[95, 94\] and models with a phenomenologically imposed dilaton profile \[120\]. The quark condensate can be

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\(^1\)The D3/D7 system has been used to construct an inflation model in \[110\] and subsequent literature. The motivation in those models is rather different since they are not holographic descriptions of a strongly coupled gauge theory but instead assumed to describe weakly coupled fields with the extra dimensions compactified. The dilaton profiles and brane motions we will consider are very different, being inspired by the gauge duality and the lessons we will seek to extract are for strongly coupled gauge dynamics.
determined in these models and an effective IR quark mass is generated. The theories display a massless pion-like Goldstone field and a massive sigma field (since we are at large $N_c$ it is stable) that is the effective Higgs particle.

We will seek to learn about a strongly coupled counterpart to a “new inflation” or “small field inflation” model [44,43]. One assumes a high temperature phase where the effective Higgs scalar’s vev is zero and a low temperature phase where the vev is non-zero. A second order phase transition should connect these phases so that the vacuum is left at the unstable zero vev point of the low temperature theory and then rolls to the true vacuum. For inflation the potential needs to be very flat near the origin so the roll takes a long time. Of course, other possible inflationary scenarios exist with multiple scalar fields or where the scalar expectation value is initially large relative to the vacuum value and it would be interesting to investigate these ideas holographically too in the future.

In principle one could imagine taking a holographic model and tracking it through the finite temperature transition. On the gravity side temperature is introduced through a black hole in the AdS geometry. The D3/D7 model we investigate has been shown to display both first order (e.g. with a magnetic field [111] - [119]) and second order (e.g. with a magnetic field and chemical potential [116] - [119]) symmetry restoration transitions. Ideally one would work in a time dependent background describing a shrinking black hole. In fact such geometries are known [121] and a first study of D7 branes moving slowly near a potential minimum in those geometries can be found in [122]. Dealing with such branes when they touch the black hole is hard though - we hope to study this problem in a future publication.

The goal in this chapter is more limited though and similar to the usual simplest field theory analysis of inflation models. We will assume that below the phase transition the theory is well described by the $T = 0$ theory. We will invoke initial conditions that place the vacuum in the symmetric phase and then watch it roll to the true vacuum that breaks the symmetry. These initial conditions assume
the existence of a second order transition.\footnote{Also see \cite{123} for a time-dependent D5 brane embedding dynamics in the context of a quantum quench. Non-equilibrium dynamics has been studied also in holographic superconductor \cite{124}.} Our first results are numerical simulations of this roll in the theory where a magnetic field is inducing the symmetry breaking. As we said above this theory actually has a first order thermal transition rather than a second order one - the scenario serves to demonstrate the formalism though. The results would be relevant to the phase transition within a highly supercooled bubble. Here there is no fine tuning to make the potential particularly flat so the model would make a poor inflation model. It serves as a proof of the numerical techniques we use to study the problem and it is impressive that a relatively simple computation can track such a transition in a strongly coupled gauge theory.

One does not imagine that a magnetic field induced symmetry breaking is likely in a realistic model of inflation. We therefore turn to a more phenomenological model in which we embed the D7 brane in an AdS geometry but with a hand chosen and non-back-reacted dilaton profile. Whilst this is not a completely kosher string dual we hope that it allows us to understand broadly the dependence of a strongly coupled theory’s dependence on its running coupling. A similar approximation was used in \cite{120} to study walking gauge dynamics and found agreement between the gravity dual description and the usual expectations for the coupling dependence of the chiral symmetry breaking. The model is sufficiently simple that we can understand how the running of the gauge coupling affects the potential for the effective Higgs mode. We attempt to engineer models with a slow roll between the symmetric and broken phases. We also show that part of our success in slowing the roll time is due to additional dynamics in the holographic radial coordinate of the dual description. This dynamics reflects the strong coupling dynamics of the gauge theory which seems under some circumstances to favour inflation. From this analysis we conclude that gauge theories displaying a small increase in the gauge coupling, preferably over a wide energy regime, are liable to make good inflation models.

Finally we briefly speculate on how IR conformal fixed points may well be
common in strongly coupled gauge dynamics and suggest a wider set of theories that might display running of the coupling like that we find gives inflation. Obvious examples are walking gauge theories.

5.2 Inflation

Let us very briefly review the standard inflation model. Based on the cosmological principle we assume the Friedmann-Robertson-Walker metric of the form

\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right), \]  

(5.1)

and the energy-momentum tensor of a perfect fluid \( T^\mu_\nu = \text{diag}(-\varepsilon, p, p, p) \) (we use \( \varepsilon \) for the energy density to avoid confusion with the brane coordinate \( \rho \)). For simplicity we will work in the \( k = 0 \) case. Then the general Einstein equations describing the expansion of the Universe are reduced to two equations (in units such that \( 8\pi G_N = 1 \)):

\[ H^2 = \frac{1}{3} \varepsilon : \text{Friedmann equation}, \]  

(5.2)

\[ \dot{\varepsilon} + 3H(\varepsilon + p) = 0 : \text{Fluid equation}, \]  

(5.3)

where \( H(t) \equiv \dot{a}/a \) is the Hubble parameter and the scale factor \( a(t) \) is readily determined from \( H(t) \) as

\[ a(t) = a(0) \exp \left( \int_0^t H(t')dt' \right). \]  

(5.4)

Equations (5.2) and (5.3) can be combined to form a so called “acceleration equation”

\[ \frac{\dot{a}}{a} = -\frac{1}{6}(\varepsilon + 3p), \]  

(5.5)

from which the condition for inflation can be expressed in terms of the
energy-momentum tensor.

\[
\text{Inflation} \iff \ddot{a} > 0 \iff p < -\frac{\varepsilon}{3} .
\]

Thus a natural question for inflation is asking what kind of matter and dynamics can generate sufficient negative pressure (5.6).

For example let us consider a scalar field, with action

\[
S = \int d^4x \sqrt{-g} \frac{1}{2} R + S_M ,
\]

where \( R \) is the Ricci scalar and

\[
S_M = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^\mu\nu \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] .
\]

From a scalar matter action the energy-momentum tensor of a homogeneous field \( \phi(t) \) can be obtained as

\[
\varepsilon = -T^0_0 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = T^i_i = \frac{1}{2} \dot{\phi}^2 - V(\phi) .
\]

The inflation condition (5.6) is rephrased as

\[
p < -\frac{\varepsilon}{3} \iff \dot{\phi}^2 < V(\phi) .
\]

Whenever the kinetic term is small compared to the potential energy there will be inflation. Then by analogy to mechanics we may say \( \phi \) is rolling slowly.

Let us apply this slow roll condition to the equations of motion (5.2) and (5.3).

\[
H^2 = \frac{1}{3} \varepsilon = \frac{1}{3} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right) ,
\]

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0 ,
\]

where the fluid equation (5.12) is nothing but the equation of motion for \( \phi \), which can be obtained from (5.8).
If the scalar field stays in some part of the potential with $\dot{\varphi}^2 \ll V$ then
$H^2 \sim V(\varphi(t))/3$ from (5.11). Furthermore if $\ddot{\varphi}$ is sufficiently smaller than the other terms in (5.12) then $V(\varphi)$ can be considered as constant, say $V_0$, for a long time. Thus the scale factor (5.4) in the slow rolling range reads

$$a(t) \sim a(0)e^{Ht} = a(0)e^{\sqrt{8\pi V_0 / 3} H t / m_P t} ,$$

where we have reinserted the Planck mass. This is the de Sitter limit zeroth order approximation of an inflating Universe, which we will adopt in this paper. To have a graceful exit from inflation the Hubble parameter must be time dependent and fall to zero at late times. In an inflating regime the space is equivalent to de Sitter space with a cosmological constant $\Lambda = 8\pi m_P^{-2} V_0$, where $V_0$ is interpreted as the vacuum energy density. The amount of inflation is specified by the number of e-folds given by

$$N_e \equiv \log \frac{a(t_e)}{a(0)} = H t_e = \sqrt{8\pi V_0 / 3} t_e / m_P ,$$

where $t_e$ is the time when inflation ends (when slow roll conditions are violated) starting from $t = 0$. One needs $N_e$ to be more than 60 phenomenologically.

Note that a larger $H$ yields a more inflationary evolution. $H$ is nothing but a friction coefficient when we interpret (5.12) as an equation for the classical particle trajectory ($\varphi(t)$) under the potential $V$. Thus it is natural that a larger friction induces a slower rolling of the particle.

In this chapter we will replace the scalar field sector with a strongly coupled gauge theory. The gauge dynamics will generate a wine bottle shaped potential with a non-zero condensate of a quark bilinear at the minimum. We will phenomenologically adjust the running of the gauge theory’s coupling constant to control the shape of the potential. We hope to learn how a gauge theory’s coupling must run to generate inflation. Clearly our task is to compute the stress-energy tensor ($\varepsilon$ and $p$) of a strongly coupled gauge theory so we can substitute them into the Friedmann and fluid equations above. We must find the equivalent of the
Euler-Lagrange equation for the scalar field $\phi$ in the inflating background for the time evolution of the strongly coupled gauge dynamics. To do this we will turn to Gauge/Gravity duality.

### 5.3 Holographic Description of a Strongly Coupled Gauge Theory

#### 5.3.1 Brane Construction

First let us review the gravity dual description of the symmetry breaking behaviour of our strongly coupled gauge theory [95,94,120,125]. To begin with we will work in a time independent flat space (i.e. there is no inflation here and $a(t) = 1$).

The brane set-up is as before, such that the weak coupling picture for our D3/D7 set up is shown in Fig 5.1 - there are $N$ D3 branes and the lightest string states with both ends on that stack generate the adjoint representation fields of the $\mathcal{N} = 4$ gauge theory. Strings stretched between the D3 and the D7 are the quark fields lying in the fundamental representation of the $SU(N)$ group (they have just one end on the D3).

In the strong coupling limit the D3 branes in this picture are replaced by the geometry that they induce [56,64,72,65]. We will consider a gauge theory with a
holographic dual described by the Einstein frame geometry $AdS_5 \times S^5$

$$ds^2 = \frac{r^2}{R^2} dx_4^2 + \frac{R^2}{r^2} \left( d\rho^2 + \rho^2 d\Omega_3^2 + dw_5^2 + dw_6^2 \right), \quad (5.15)$$

where we have split the coordinates into the $x_4(x_{3+1})$ of the gauge theory, the $\rho$ and $\Omega_3$ which will be on the D7 brane world-volume and two directions transverse to the D7, $w_5, w_6$. As before, the radial coordinate, $r^2 = \rho^2 + w_5^2 + w_6^2$, corresponds to the energy scale of the gauge theory and the radius of curvature is given by $R^4 = 4\pi g_s N_c \alpha'^2$ with $N_c$ the number of colours. In addition we will allow an arbitrary dilaton as $r \to 0$ to represent the running of the gauge theory coupling

$$e^\Phi = g_s \beta(r^2) \sim g_{YM}^2(r^2), \quad (5.16)$$

where the function $\beta(r) \to 1$ as $r \to \infty$. The $r \to \infty$ limit of this theory is dual to the $\mathcal{N} = 4$ super Yang-Mills theory and $g_{UV}^2$ is the constant large $r$ asymptotic value of the gauge coupling.

We will introduce a single D7 brane probe into the geometry to include quarks - by treating the D7 as a probe we are working in a quenched approximation although we can reintroduce some aspects of quark loops through the running coupling’s form if we wish (or know how).

In the true vacuum at zero temperature ($T = 0$) the brane will be static. We must find the D7 embedding function e.g. $w_5(\rho) \equiv L(\rho)$ with $w_6 = 0$. The DBI action in Einstein frame is given by

$$S_{D7} = -T_7 \int d^8 \xi e^{\Phi} \sqrt{-\det P[G]_{ab}}$$

$$= -\tilde{T}_7 \int d^4 x d\rho \, \rho^3 \beta \sqrt{1 + L'^2}, \quad (5.17)$$

where $P[G]_{ab}$ is the pull back of the background metric onto the D7 and $L' \equiv \partial_\rho L(\rho)$. $T_7 = 1/(2\pi)^7 \alpha'^4$ and $\tilde{T}_7 = 2\pi^2 T_7 / g_s$ when we have integrated over the
3-sphere on the D7. The equation of motion for the embedding function is therefore

$$\rho \left[ \frac{\beta \rho^3 L'}{\sqrt{1 + L^2}} \right] - 2 L \rho^3 \sqrt{1 + L^2} \frac{\partial \beta}{\partial r^2} \frac{\partial (r^2)}{\partial r^2} = 0 \ . \quad (5.18)$$

The UV asymptotic of this equation, provided the dilaton returns to a constant so the UV dual is the $\mathcal{N} = 4$ super Yang-Mills theory, has solutions of the form

$$L = m + \frac{c}{\rho^2} + \ldots \ , \quad (5.19)$$

where we can interpret $m$ as the quark mass ($m_q = m/2\pi\alpha'$) and $c$ is proportional to the quark condensate.

The embedding equation (5.18) clearly has regular solutions $L = m$ when $g^2_{YM}$ is independent of $r$ - the flat embeddings of the $\mathcal{N} = 2$ theory [67,75]. Equally clearly, if $\partial \beta(r^2)/\partial r^2$ is non-trivial in $L$ then the second term in (5.18) will not vanish for a flat embedding.

There is always a solution $L = 0$ which corresponds to a massless quark with zero quark condensate ($c = 0$). In the high $T$ phase this is the true vacuum. In the symmetry breaking low $T$ geometry this configuration is a local maximum in the potential.

Note that in the particular case when

$$\beta = \sqrt{1 + \frac{B^2}{(\rho^2 + L^2)^2}} \ , \quad (5.20)$$

the DBI action for the D7 brane is that of the D3/D7 system with a background magnetic field $B$. This model has been extensively studied in [111] - [119]. The action manifestly grows as one approaches $L = \rho = 0$ so the D7 brane is repelled from that point. In Fig 5.2 we plot the D7 embedding in the magnetic field case.

An interesting phenomenological case is to consider a gauge coupling running with a step of the form

$$\beta = A + 1 - A \tanh [\Gamma(r - \lambda)] \ . \quad (5.21)$$
Of course in this case the geometry is not back-reacted to the dilaton and the model is a phenomenological one in the spirit of AdS/QCD. This form introduces conformal symmetry breaking at the scale $\Lambda = \lambda/2\pi\alpha'$ which triggers chiral symmetry breaking. The parameter $A$ determines the increase in the coupling across the step. If the coupling is larger near the origin then again the D7 brane will be repelled from the origin. The parameter $\Gamma$ spreads the increase in the coupling over a region in $r$ of order $\Gamma^{-1}$ in size - the effect of widening the step is to enhance the large $\rho$ tail of the D7 embedding.

We again display the embeddings for some particular cases in Fig. 5.3. Note that we have chosen parameters here that make the potential difference between the symmetric and symmetry broken phases the same in each case. This is crucial to ensure that we are comparing models that will generate the same cosmological constant and hence the same rate of inflation when the quark condensate is zero. The vacuum energy is given by the DBI action evaluated on the solution. In fact this energy is formally divergent corresponding to the usual cosmological constant problem in field theory. As usual we subtract the UV component of the energy - we do this by subtracting the energy of the symmetry breaking, lowest energy embedding in each case.

The symmetry breaking of these solutions is manifest. The U(1) symmetry corresponds to $SO(2)$ rotations of the solution in the $w_5$-$w_6$ plane. An embedding along the axis corresponds to a massless quark with the symmetry unbroken (this is the configuration that is preferred at high temperature and it has zero condensate...
The vacuum, curved configurations map onto that case at large $\rho$ (the UV of the theory) but bend off axis breaking the symmetry in the IR.

One can interpret the D7 embedding function as the dynamical self energy of the quark, similar to that emerging from a gap equation [126]. The separation of the D7 from the $\rho$ axis is the mass at some particular energy scale given by $\rho$ - in the $\mathcal{N} = 2$ theory where the embedding is flat the mass is not renormalized, whilst with the magnetic field or running coupling an IR mass forms.

### 5.3.2 Approximate Potentials

It is natural to want to plot the effective potential for the quark condensate using the holographic description. However, this is somewhat ambiguous. The embedding equation determines the D7 embeddings that correspond to the extrema of such a potential. In between these points we need to treat the condensate as a free parameter. The condensate is determined from the embedding by the asymptotic curvature parameter $c$. We must therefore allow the embedding to deviate from its vacuum form as we vary $c$.

In principle one would need to find the minimum action configuration that

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>$A$</th>
<th>$\lambda$</th>
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</thead>
<tbody>
<tr>
<td>Black</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Green</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Blue</td>
<td>1</td>
<td>1.8</td>
</tr>
<tr>
<td>Orange</td>
<td>1</td>
<td>1.5</td>
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<tr>
<td>Red</td>
<td>0.5</td>
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falloff in the UV as $c/\rho^2$ for arbitrary $c$ and has $L' = 0$ in the IR. This can’t be done by a simple numerical shooting because such configurations are not solutions of the Euler-Lagrange equations. A reasonable way to get qualitative results though is as follows - we simply take the vacuum embedding solution, say $L_0$, and consider the one parameter family of embeddings, $q \cdot L_0$, where $q$ is an arbitrary constant. This gives embeddings for all $L(0)$ and all values of $c$ i.e. $L(0)$ and $c$ are monotonic functions of $q$. For example the case $q = 1$ is the embedding corresponding to the global minimum of the wine bottle potential, and the case $q = 0$ is the flat embedding corresponding to the central maximum of the potential. We will assume that this form is appropriate for all values of $q$. This then lets us plot the potential, which is an on-shell action (5.17), as a function of $q$. Since $q$ has a one-to-one correspondence to $c$, we can plot the potential in terms of $c$ - the potential automatically takes the correct value at the maximum and minimum. See Fig 5.4.

It is worth noting from the dynamical analysis presented in Section 5.5 any embedding with the centre of the brane caught below the step will quickly adopt a canonical shape, whose subsequent evolution is $\rho$-independent, until it settles in the broken symmetry ground state. Thus, although this procedure for calculating the potential is somewhat ad hoc, it allows a first rough understanding of the potential shape since it is a reasonably faithful reflection of the actual behaviour of the brane. Indeed, from (5.9), one can calculate the potential as a function of the dynamical embedding and plot it against the condensate that corresponds to that embedding –
it provides very similar plots to those displayed in Fig. 5.4.

We can see that when we have symmetry breaking induced by a magnetic field, the potential is rather steep around the origin. The step function ansatz for $\beta$ is gentler there and by decreasing $A$ the curvature is further reduced. Note that as one decreases $A$ one must increase $\lambda$ in order to keep the value of the potential when $c = 0$ equal. In the extreme low $A$ limit the model is characterized by that potential value being much less than the characteristic scale of the running $\Lambda$. The quark condensate also grows in this limit. Comparing the blue curve and the red curve shows that varying $\Gamma$ also leads to a potential with a flatter and longer period near the origin. This is encouraging since for an inflation model we would want the potential to be as flat and as extended as possible around the origin.

The lowest $A$ curve (Orange) shows a new minimum at zero and an intermediate maximum in the potential but this is an artefact of the crude approximation being made here. Were the maximum to really form then there would be a new solution to the embedding Euler-Lagrange equation for the D7 brane - we have checked and no such solution exists. We are simply describing the non-vacuum configurations incorrectly. However, these parameters are close to ones which do indeed exhibit a new unstable symmetry-breaking configuration – in this case, the flat embedding is in a local minimum of the effective potential, and the symmetry is not broken spontaneously (see Section 5.5 for more). In any case, we stress that Fig 5.4 just to show qualitatively that a Higgs-like potential exists in the theory. Indeed we will see below that the time evolution of a rolling configuration depends crucially on the behaviour in the holographic directions of the description of the gauge theory, to which these plots are blind. To do a better job of understanding the dynamics of these models near the origin we turn to studying the full time dependent problem of a moving D7 brane.
5.4 Holography Of Inflation

5.4.1 Time Dependent Holography

We will now move on to look at time dependent evolution in the gauge theory described above. Strictly the AdS/CFT Correspondence describes our gauge theory in rigid flat space. To include inflation we will instead assume a background holographic geometry given by

$$ds^2 = \frac{r^2}{R^2} (g_{tt} dt^2 + g_{ij} dx^i dx^j) + \frac{R^2}{r^2} (d\rho^2 + \rho^2 d\Omega_3^2 + dw_5^2 + dw_6^2),$$  \hspace{1cm} (5.22)

where $g_{tt} = -1$ and $g_{ij} = a(t)^2 \delta_{ij}$. For this to be valid even as an approximation we must formally assume that $a(t)$ is growing much slower than the typical time for equilibration in the $\mathcal{N} = 4$ Yang-Mills plasma. How then can we hope to study inflation with exponential space-time growth? The formal answer is that we will study inflation driven by the flavour sector of our gauge theory. The contribution of the D7 branes to the full vacuum energy of the field theory is $1/N_c$ suppressed relative to the glue dynamics contributions. We will be able to study the theory where this perturbation to the vacuum energy drives inflation but formally in a limit where the inflation is still slow relative to the $\mathcal{N} = 4$ equilibration time. Our inflation is that relevant for the quark fields. This provides a formal justification for what follows.

In fact for a true model of inflation one would want exponential expansion relative to all time scales in the problem. The reader can therefore choose to view (5.22) as a phenomenological model. In practice for the results we wish to extract - what dilaton profile will generate slow rolling behaviour - one could just work in static $x_4$ flat space. However, it is useful to have the damping term in the scalar equation of motion in (5.12) to remove late time oscillations - this provides another justification for using (5.22).
The DBI action in such a geometry is given by

\[ S_M = S_{DBI} = \int d^4x \sqrt{-g} \mathcal{L} \]  

(5.23)

where

\[ \mathcal{L} = -\tilde{T}_7 R^4 \int d\rho \beta \rho^3 \sqrt{\frac{g^{00} \dot{L}^2}{(\rho^2 + L^2)^3} + 1 + L^2} \]  

(5.24)

Here \( g_{\mu\nu} \) are boundary field theory metric components and \( L = \omega_5(t, \rho) \) with \( \omega_5 = 0 \). \( L' = \delta_\rho L \) and \( \dot{L} = \delta_t L \). All variables \( (\rho, L, t, \Gamma, \lambda) \) in the integral (5.24) are rescaled by \( R \) and dimensionless. We will reinstate \( R \) when needed. Note that the 4D effective action derived from the DBI action can be written in a covariant form with the correct measure.

The integral form of the 4D Lagrangian (5.24) may be understood as the effective Lagrangian of some field theory quantity after integration over the extra direction \( \rho \). Since the bulk embedding dynamics is closely related to the spontaneous symmetry breaking in the boundary field theory, it is natural to associate an effective degree of freedom to an order parameter, the condensate \( c(t) \).

Thus we may consider (5.24) as an inflaton model of a composite scalar field \( c(t) \) with some potential \( V(c) \). The quark condensate \( c \) can be extracted numerically from the asymptotic large \( \rho \) form of the solution (5.19). Even though it’s not straightforward to read the form of \( V(c) \) from (5.24) it is not an obstacle to the study of inflation, since \( \varepsilon \) and \( p \) can be computed from (5.24) and we can apply the inflation condition (5.6). Furthermore our potential for the condensate has its origin in (5.24), so is determined in principle and not put in by hand.

We can obtain \( \varepsilon \) and \( p \) by computing the expectation value of the stress energy tensor of the 3+1D field theory from

\[ \langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}}. \]  

(5.25)

According to the AdS/CFT Correspondence the gravitational action is the Masterfield of the field theory [56,64,72,65] with the boundary values of
gravitational fields playing the roles of the sources. Here we have explicitly shown
the behaviour of the gravity action on the boundary metric so we may simply
compute. We find

$$\varepsilon(t) = \tilde{T}_7 R^4 \int d\rho \rho^3 \beta \frac{1 + L'^2}{\sqrt{1 + L'^2 - \frac{L^2}{(\rho^2 + L^2)^2}}} - \varepsilon_0$$

(5.26)

$$p(t) = -\tilde{T}_7 R^4 \int d\rho \rho^3 \beta \sqrt{1 + L'^2 - \frac{L^2}{(\rho^2 + L^2)^2}} - p_0$$

(5.27)

where we have renormalized by subtracting $\varepsilon_0, p_0$, which are the values of the
integrals with the asymptotic static symmetry breaking solution for $L$, say $L_s$.
Examples of $L_s$ are shown in Fig 5.3. There is no explicit $a(t)$ dependence in $\varepsilon$ and $p$, but its information is encoded in $L(t, \rho)$ since $L(t, \rho)$ is a solution of equation of motion of (5.23) which includes $a(t)$ in $\sqrt{-g}$.

Note that $p(0) = -\varepsilon(0)$ if $L = 0$. So if an initial condition has $L \sim 0$ then the
Universe will start inflating ($p < -\varepsilon/3$) regardless of an initial $L'$. However we
know that eventually the embedding will asymptote to $L_s$ because of the wine
bottle shaped potential, which implies $\varepsilon \to 0$ and $p \to 0$. Thus we expect $\varepsilon$ is
decreasing and $p$ is increasing with time. If $p$ increases faster than $\varepsilon$ then there will
be a time when $\varepsilon + 3p = 0$, ending the inflation. How long it takes depends on the
initial configuration $L(0, \rho)$ and the parameters of $\beta$, the dilaton profile. According
to small field inflation models it is natural to start with $L = 0$, which is a local
maximum configuration and the symmetric minimum at high temperature. Then
the inflation time or the number of e-folds ($N_e$) can be studied as a function of
coupling $\beta$. In our setup inflation always happens to some degree and the issue is
the amount of inflation.

Since we know $\varepsilon$ and $p$ we can proceed with the Friedmann (5.2) and fluid
(5.3) equations. Let’s start with the Friedman equation.

$$H^2 = \frac{1}{3\varepsilon}$$

(5.28)

where $\varepsilon$ is (5.26). This makes calculating the evoluton very complicated, since the
equation of motion for $L$ depends on $a(t)$ which is determined by $\varepsilon$, itself a time dependent function of $L$.

To make the computation tractable we start with the zeroth order slow rolling approximation by assuming $\varepsilon$ is almost constant for a long enough time, namely $\varepsilon = V_0$. It gives us a simple solution for $a(t)$

$$a(t) = e^{\sqrt{\frac{V_0}{2}} t},$$

(5.29)

where we set $a(0) = 1$. This solution must be consistent with the fluid equation, which is equivalent to the equation of motion for $L$. Thus we will solve the equation of motion resulting from (5.23) with a constant $V_0$ (5.29) and then plug the solution back into (5.26). If the calculated $\varepsilon$ changes slowly enough (slow rolling), then our solution is self-consistent. In our numerical computation $\varepsilon$ and $p$ are always rescaled as $\varepsilon \rightarrow \varepsilon T_7 R^4$ and $p \rightarrow p T_7 R^4$ so are dimensionless.

5.4.2 Rolling in a B field

As a first example of our formalism we will consider the case of symmetry breaking induced by a magnetic field. We do not expect such a model to be well suited to a realistic inflation model because the typical curvature of the potential (Fig 5.4) for the condensate $c$ is large. There is no particular fine tuning in the model. We will be able to track the time evolution of the brane configuration though from a symmetric to a symmetry breaking vacuum.

We will first study the time dependence of the model in a constantly inflating Universe i.e. with fixed cosmological constant $H$. In reality $H$ should be determined by the D7 energy density through the evolution ($H$ depends on the choices of $N_c$ and the ’t Hooft coupling - we will show some generic numerical results). In the early inflating stages of the evolution we are most interested in, assuming a constant $H$ is sound. For the evolution near the true vacuum $H$ should go to zero and the brane would be expected to oscillate about the true vacuum configuration.

Most of the solutions we will show will remain highly damped in the late time
regime which allows us to numerically test that the configuration indeed ends on
the static vacuum D7 embedding. We will though show some results for \( H = 0 \),
where those oscillations are observable, shortly. The high damping is due to our
unphysical constant \( H \) at all time. In a full computation it would be time dependent
and vanish at late time. If we could solve with a time dependent \( H \) we would find
the oscillation around the true vacuum at late time as hinted in the case \( H = 0 \).

In particular we will start with the initial conditions

\[
L(\rho, t = 0) = 0, \quad \dot{L}(\rho, t = 0) = ve^{-\rho^2},
\]

where \( \dot{L} \equiv \partial_t L \). The initial speed and \( \rho \) dependence of this ansatz is not picked for
any deep reason but is just illustrative of some initial condition that initiates the
roll down to the potential minimum. We have checked that none of our results are
qualitatively changed by varying for example the width of this initial condition. We
will typically pick \( v \) to be very small so that the roll time from the peak of the
potential is quite long. This also ensures that inflation happens at early times. The
early time inflationary period in the plots below are the result of this fine tuned
small initial condition and the large damping term and not a sign that the potential
is particularly flat near the origin.

A slow early roll will mean we can study the dynamics more easily
numerically and understand better how to lengthen that early roll period. Through
the roll we must ensure that \( L'(0, t) = 0 \) and that \( L(\infty, t) = 0 \). The evolution can in
fact be followed with these boundary conditions using the inbuilt numerical partial
differential equation solver in Mathematica.

In Fig 5.5 we show a sample plot of the numerical evolution. The figure is for
an inflating Universe with \( H = \sqrt{70/3}, B = 35.6 \). We show a three dimensional plot
for the time evolution of the D7 embedding - at early times the D7 is flat, \( L = 0 \),
but as soon as the kinetic energy of the brane begins to grow as it experiences the
curvature of the potential, it rapidly transitions to the vacuum embedding with
symmetry breaking.
Figure 5.5: We show the time evolution of the D7 brane with \( H = \sqrt{70}/3 \), \( B = 35.6 \) and initial velocity parameter \( v = 0.00001 \). The transition from the flat embedding (chirally symmetric phase) to the curved embedding (chirally broken phase) is apparent.

Figure 5.6: \( \varepsilon + 3p \) (dotted) and \( \varepsilon \) (solid) plotted against time for the time evolution of the configuration shown in Fig 4.
Fig 5.6 is a plot of \( \varepsilon + 3p \) and \( \varepsilon \) versus time. \( \varepsilon(t) \approx -p(t) \) at all time. \( \varepsilon \) shows a plateau until \( t \sim 80 \) and changes abruptly, while \( \varepsilon + 3p \) is negative. Thus our slow roll approximation is valid up to \( t \sim 80 \). Even though the solution after \( t \sim 80 \) is beyond our approximation we read off the inflation ending time as the time when \( \varepsilon + 3p = 0 \) as an estimate, which is \( t_e \sim 93 \). The end time will depend on \( B \) or \( H \), which are related.

We stress again that we have taken the Hubble parameter \( H \) constant through the brane motion in the results just presented. With this unphysical late time damping it can be seen that the solutions precisely match on to the time independent vacuum configuration at large time. To show the oscillatory behaviour one expects when the Hubble parameter is not present we can also solve for a similar configuration with \( H = 0 \). We show a plot in Fig 5.7 of the time dependence of the quark condensate \( c \) in such a scenario - \( c \) is extracted from the large \( \rho \) dependence of the solution through (5.19). With no damping at all the D7 moves to approximately the vacuum configuration, then overshoots, returns to the flat embedding before moving below the axis, etc. The oscillatory behaviour is clear and can be followed through many cycles. Note that there are “fine wrinkles” near \( c = 0 \) in this plot. These are due to a peculiarity of the magnetic field induced symmetry breaking - in particular there are an infinite set of meta-stable vacua near \( c = 0 \) in this theory as explored in [127]. We see their influence on this motion although they play no particular role in our analysis here. Our phenomenological dilaton profiles below do not generate such structure.

Let us briefly return to compare these simulations to what would be needed for inflation in our Universe. The energy density (5.23) is naively

\[
\varepsilon = \tilde{T}_7 R^4 E(B),
\]

(5.31)

where \( E(B) \) is a number obtained by the numerical integration in (5.24). Note the choice of the magnetic field here introduced the intrinsic scale of symmetry breaking.

\( \varepsilon \) is also a measure of the symmetry breaking scale of the theory and in flat

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space we could set it to one as the defining scale in the problem. When we include the inflationary dynamics through the damping term in (5.24) the Planck scale also enters. The ratio of the Planck scale to the fourth root of \( \varepsilon \) is a free parameter. We will associate the Planck mass with some distance in the holographic \( \rho \) direction, \( \gamma_p \). The physical Planck mass in the coordinates with which we compute (rescaled by \( R \) as discussed under (5.24)) is thus

\[
m_P = \frac{\gamma_p}{R}.
\]

(5.32)

It is then sensible to express the energy density in Planck units so

\[
\varepsilon = \varepsilon_0 m_P^4,
\]

(5.33)

where

\[
\varepsilon_0 \equiv \frac{1}{2(2\pi)^5 g_s \gamma_p^4 \alpha'^4} \frac{R^8}{\lambda_{tH} N} = \frac{\lambda_{tH} N}{(2\pi)^4 \gamma_p^4}, \quad \lambda_{tH} \equiv 4\pi g_s N.
\]

(5.34)

Note this is \( \alpha' \) independent as it should be since \( R^8/\alpha'^4 = \lambda_{tH}^2 \).

The roll times \( \tilde{t}_e \) we have computed are in dimensionless units and should be written in Planckian units too.

\[
t_e m_P = R \tilde{t}_e m_P = \gamma_p \tilde{t}_e.
\]

(5.35)
The scale factor in the metric is therefore given by

\[ a(t) = e^{H t} = e^{\sqrt{\frac{8\pi\varepsilon_0}{3}} \gamma_p \hat{t} e}. \]  

(5.36)

Our simulations have been with \( \sqrt{\frac{8\pi\varepsilon_0}{3}} \gamma_p = \sqrt{70}/3 \). Note one can realize this for any value of \( \varepsilon_0 \) - once \( \varepsilon_0 \) is fixed we can choose \( \gamma_p \) by setting the gauge theory parameters such as \( \lambda_{\text{eff}} \) and \( N_c \). The requirement that \( \varepsilon_0 \ll 1 \) (so that we are outside the realm of quantum gravity) means that \( \gamma_p \) should itself be large - in other words, the characteristic scale of our field theory is well below the Planck scale.

We conclude this section with the comment that it is quite remarkable that we can compute so straightforwardly the time evolution of a strongly coupled gauge theory!

### 5.4.3 Towards an Inflationary Dilaton Profile

Our next goal is to look to extend the period of slow roll in the holographic models by phenomenologically changing the dilaton profile or gauge coupling’s running. This is the equivalent of making the usual inflaton potential flatter around the origin.

To examine this analytically let us first consider the roll in static space \( H = 0 \). We linearise the equation of motion about the initial symmetry-preserving configuration \( L = 0 \)

\[ \ddot{L} = 3\rho^3 L' + \rho^4 L'' + \rho^4 \frac{\beta_0}{\beta_0} L' - \rho^3 \frac{\beta_0}{\beta_0} L, \]  

(5.37)

where \( \beta_0 \) is the coupling at \( L = 0 \) (for the step function form of the running we have \( \beta_0 = A + 1 - A \tanh[\Gamma(\rho - \lambda)] \) and \( \beta_0' = -A \Gamma \sech[\Gamma(\rho - \lambda)]^2 \)). If we consider a truly flat configuration so \( L' = L'' = 0 \) then only the last term on the right hand side contributes. Clearly the acceleration of the brane is localized around the point in \( \rho \) where there is a significant change in the coupling value. We show that this is indeed how the motion proceeds in Fig 5.8 where we show a plot of the early time evolution of the brane in the presence of a step in the value of the dilaton. A bump
Figure 5.8: The early time evolution of the D7 brane in the presence of a dilaton step with $A = 3$, $\lambda = 3.24$, $\Gamma = 1$ and for the initial condition $v = 0.00001$. $H = \sqrt{70/3}$. The acceleration occurs around the step.

grows around the point in $\rho$ where the dilaton changes. We also plot the full evolution with a cosmological damping parameter $H = \sqrt{70/3}$ showing the configuration move to the true vacuum in Fig 5.9.

For the step configuration with large $\Gamma$, so we can consider it to be a sharp step, we have

$$\ddot{L}(\rho) \sim \lambda^3 \frac{A\Gamma}{A+1} L(\lambda) \delta(\rho - \lambda) ,$$

(5.38)

naively lowering $A$ or $\Gamma$ will reduce the acceleration. In fact though to keep the difference in the potential between the symmetric and symmetry breaking configurations equal as we reduce $A$ or $\Gamma$ we must increase $\lambda$. For the numerical values in Fig 5.4 increasing $\lambda$ wins so $\frac{A\Gamma}{1+\lambda^3}$ increases when $A$ decreases.

This analysis is overly naive though because as the D7 brane evolves, as shown in Fig 5.8, $L'$ and $L''$ will become important. Around the localized peak of the bump where $L' = 0$ $L''$ is negative and the equation of motion will be

$$\ddot{L}(\rho) \sim \left( \lambda^3 \frac{A\Gamma}{A+1} L(\lambda) + \lambda^4 L''(\lambda) \right) \delta(\rho - \lambda) .$$

(5.39)

Since $L''$ is negative the latter term will slow the acceleration in the $A \to 0, \lambda \to \infty$
Figure 5.9: The full time evolution of the D7 brane in the presence of a dilaton step with $A = 3$, $\lambda = 3.24$, $\Gamma = 1$ and for the initial condition $v = 0.00001$. $H = \sqrt{70}/3$.

limit. Thus we conclude that for models with fixed $H$ those with a larger value of $\lambda$ will provide a longer period of inflation. We want to stress that this conclusion is completely dependent on the holographic description through evolution associated with the $\rho$ direction - we are learning about the role of the strong interaction dynamics. One certainly can not, for example, deduce the motion from the simplistic approximate potentials we displayed in Fig 5.4. More generally we note that the holographic dependence on $\rho$ introduces considerably more complication to the evolution and explicit simulation is required.

It is clear that if we wish to prolong the early time roll period we need to reduce the rate of change of the dilaton and push $\lambda$ far above the vacuum energy $H$. There are two ways we can do that within our ansatz (5.21) - we can reduce $A$ for a fixed $\Gamma$ so that the step is smaller. We would expect this to increase the roll time. Also for fixed $A$ we can decrease $\Gamma$ so the change in $A$ occurs over a larger $\rho$ range. This reduces the dilaton derivative but spreads the region of $\rho$ over which the change is occurring so there could be no net change to the total rate of acceleration - we will need to test this case numerically.

In Fig 5.10 we plot $\varepsilon$ for the three different sets of parameters in $\beta$. Like Fig 5.6 $\varepsilon + 3p$ is always negative before it suddenly vanishes. Even though our computation is not valid when $\varepsilon$ starts changing fast we choose to take the time when $\varepsilon + 3p$ vanish as the end time of inflation, $t_e$. Since our purpose is to compare
We will now make a comparison of the roll time for a number of different dilaton step profiles. In each case the difference in vacuum energy between the symmetric and vacuum symmetry breaking configuration is the same. We also use the same initial velocity perturbation for the D7 brane (5.30). As an example of the differences we plot the energy density $\varepsilon$ against time in Fig 5.10 and the condensate $c$ against time for three configurations in Fig 5.11. First compare the curves for the dilaton parameters $[A = 3, \Gamma = 1, \lambda = 3.24]$ and $[A = 1.8, \Gamma = 1, \lambda = 4.325]$. Decreasing $A$ indeed increases the time the configuration takes to reach the tipping point to the true vacuum. Next we can change $\Gamma$ to try to further increase the roll
time - the final curve is for the configuration \([A = 1.8, \Gamma = 0.5, \lambda = 5.882]\) and indeed we find a further lengthening of the inflationary period.

Finally we show these trends in more detail in Fig 5.12 where we plot \(t_e\) against \(A\) for a sequence of values of \(\Gamma\) (1.5, 1, 0.5). There is a clear trend for decreasing both \(A\) and \(\Gamma\) increasing the roll time. As this roll time increases the parameters become more fine tuned reflecting the fine tuning we are making in the effective potential for \(c\), i.e. the usual fine tuning in inflation.

This analysis has been performed again in a heavily damped scenario which as discussed in (5.36) assumes a growing value of \(N\) in the gauge theory as the energy density is reduced relative to the Planck scale. We choose this regime primarily for computational convenience. Having high damping removes late time oscillations of the D7 motion. It also extends the roll time making changes in that roll time more easily apparent. To demonstrate that the effects we have observed are still present at lower values of the damping we finally display the behaviour of the condensate against time for the case \(H = \sqrt{1/3}\) and for steps with two different values of step height \(A\) in Fig 5.13. The late time oscillations are now apparent but the increase of the roll time with decreasing \(A\) is also maintained.
Figure 5.13: \( t_e \) against \( A \) for two cases sequence of values. Blue: \( [A = 27.3, \Gamma = 0.5, \lambda = 0.25] \), Red: \( [A = 2, \Gamma = 0.5, \lambda = 5.326] \). \( v = 0.0001 \). \( H = \sqrt{1/3} \).

5.5 Analysis of Linearized Regime

The previous section provided a heuristic and intuitive look at the brane dynamics. One may wonder how well the linear approximation fits the brane over the course of its evolution. The answer is, notably, almost all of it. Moreover, it is within the linear approximation that the field theory interpretation is clear and we can ask the question, is this single or multifield inflation? We return to the Lagrangian (5.17) and reintroduce spatial Minkowski derivatives:

At early times the brane lies close to \( L(\rho) = 0 \) and has small \( L'(\rho) \) so it is natural to consider a linearised regime. We take the full action,

\[
S = -T_7 \int d^4 x d\rho d\Omega a(t)^3 \rho^3 \beta \sqrt{1 + L'^2 - \frac{\dot{L}^2}{q^2 + L'^2}} \tag{5.40}
\]

and approximate it using a quadratic Lagrangian \( \mathcal{L}_{(0)} \):

\[
\mathcal{L}_{(0)} \sim \int d\rho \ a(t)^3 \left( \frac{\beta_0}{2\rho} \dot{X}^2 - \frac{\beta_0}{2} X'^2 - \frac{\beta'_0}{2} \rho^2 X^2 \right) \tag{5.41}
\]

where

\[
\beta_0 = A + 1 - A \tanh [\Gamma (\rho - \lambda)] \\
\beta'_0 = -A \Gamma \text{sech} [\Gamma (\rho - \lambda)]^2 \tag{5.42}
\]

Note we have written \( L = X \) here to denote the linearised solutions. This expansion
is not completely legitimate. The expansion may break down as $\rho \to 0$ in Eq. (5.41) since $X$ may not be smaller than $\rho$. Thus we are assuming either 1) $X(0) \to 0$ faster than $\rho \to 0$ at least for the slow rolling regime or 2) there is a IR cut-off $\rho_c$ so that $\rho_{IR} \gg X(\rho_c)$. We will adopt the second and introduce an IR cut-off, $\rho_c \equiv \epsilon$ - the justification for this will be retrospective based on its success as we discuss in detail below.

The linearised equation of motion is

$$ - \beta_0 \frac{\partial_t (a^3 \dot{X})}{a^3} = -\left(\beta_0 \rho^3 X'\right)' + \rho^2 \beta_0' X $$ \hspace{1cm} (5.43)

We can separate variables with the ansatz

$$ X(t, \rho) \equiv T(t)R(\rho) $$ \hspace{1cm} (5.44)

the equation of motion reads

$$ \ddot{T} + 3H \dot{T} + m^2 T = 0, $$ \hspace{1cm} (5.45)

$$ R'' + \left( \frac{3}{\rho} + \frac{\beta_0'}{\beta_0} \right) R' + \left( -\frac{\beta_0'}{\rho \beta_0} + \frac{m^2}{\rho^2} \right) R = 0. $$ \hspace{1cm} (5.46)

where $m^2$ is an arbitrary separation constant for now, but in principle will be determined by the embedding dynamics along the $\rho$-direction in the background of $\beta$. So $m^2$ is implicitly a function of $A, \Gamma, \lambda$ and can be written as

$$ m^2 = \frac{\rho}{\beta_0} \left( -\frac{\beta_0 \rho^3 R'}{R} + \rho^2 \beta_0' \right) $$ \hspace{1cm} (5.47)

from the separation of variables. Equation (5.46) is of course a Sturm-Liouville problem of the form

$$ \hat{O} R(\rho) = m^2 w(\rho) R(\rho). $$ \hspace{1cm} (5.48)
where
\[ \hat{O} R = \left( -\rho^3 \beta_0 + \rho^2 \beta_0' \right) R' + \rho^2 \beta_0' R \] (5.49)
\[ w(\rho) = \frac{\beta_0}{\rho} \] (5.50)

so \( \hat{O} \) is a Hermitian operator with infinite real eigenvalues \( m^2 \) and eigenfunctions

\[ \int d\rho \, w(\rho) R_a(\rho) R_b(\rho) = \delta_{ab}. \] (5.51)

Hence, (5.47) may also be interpreted as an eigenvalue equation

\[ m^2 = \langle R_a | \hat{O} | R_a \rangle = \int d\rho \left( \beta \rho^3 R_a^2 + \beta_0' \rho^2 R_a^2 \right). \] (5.52)

The natural way to proceed to the solutions is as follows: firstly one could solve (5.46) with \( R \to 0 \) at large \( \rho \) and subject to \( R'(0) = 0 \). One would expect a tower of solutions corresponding to discrete eigenvalues \( m^2 \). One could then solve for \( T(t) \) using (5.45). The solution of that equation is simply

\[ T(t) = T_0 e^{\alpha t}, \quad \alpha^2 + 3H\alpha + m^2 = 0 \] (5.53)

This only has solutions with \( \alpha \) positive if \( m^2 < 0 \). Thus after some evolution only the unstable mode (or potentially modes) would remain in the solution.

### 5.5.1 Linearized versus full solution

We can not complete the programme above in the linearised regime. The reason is that the linearised approximation is not valid at small \( \rho \) - we can not impose correctly \( R'(0) = 0 \). To circumvent this one has to return to the full solutions to show that the linearised approximation correctly captures the main physics of the D7’s roll and to compute \( m^2 \).

Let us show this procedure in a specific example. Figure 5.14 shows the full non-linear evolution of the embedding \( L(t, \rho) \) (for \( A = 3, \Gamma = 1 \) and \( \lambda = 3.24 \)) at
times $t = 1, 5, 18$ and 70. (The dotted curves are the solutions to (5.46) and will be explained shortly). From $t = 5$ onwards the $\rho$ profile of the solution stabilizes (initially at $\rho > 1$) which fits with a separation of variables in the solution.
Assuming that separation of variables, the function \( T(t) \) may be obtained from examining the time dependence of any one point on the embedding (in Fig. 5.15 we have chosen \( L(t, \lambda) \)). The solution to (5.45) is \( T(t) \sim \exp(\alpha t) \). If the linearised approximation is valid, the time-derivative of the logarithm of \( L(t, \lambda) \) should be a constant, \( \alpha \), and that is indeed what we find (Fig. 5.15).

![Figure 5.15: Exponential evolution of \( L(t, \lambda = 3.24) \).](image)

We can now determine \( m^2 \) for the most unstable mode by simply reading off the value of \( \alpha \) from (Fig. 5.15). In this specific case, \( \alpha = 0.198 \) and \( m^2 = -2.907 \). Note that there is only one mode contributing here.

We can now perform a further consistency check. We can take this value of \( m^2 \) and plug it into (5.46). We then solve (5.46) for \( R(\rho) \) numerically by shooting in from large \( \rho \). By hand we cut off the solution in the IR at the point where \( R(\rho) = 0 \) and assume that below this \( \rho_c \) the linearised solution is not valid but also that the physics below this small \( \rho \) value is not crucial to the dynamics. Note that \( \rho_c \) is not arbitrary and completely determined by solving (5.46) for a given parameters in \( \beta \) and \( m^2 \). In our specific example, \( \rho_c = 0.643 \).

For example in our specific case we find

\[
X = 4.47 \times 10^{-7} e^{0.198t} R(\rho),
\]

The pre-factor is fixed by matching to the full numerics at \( t = 18 \). We now plot these solutions (5.54) as the dotted curves in Figure 5.14. Clearly beyond \( t = 17 \) we
have captured the full evolution very well within the linearised approximation (and again it demonstrates that there is a single unstable mode dominating the evolution).

The embedding matches the linearised evolution until around $t = 80$ which is essentially the full roll period (see Fig 5.11). We also note that the peak in $R(\rho)$ is, as expected, centred around the point $\rho \approx \lambda = 3.24$ where the step change in the dilaton is found.

![Figure 5.16: $m^2 = -3$ numerical consistency check by plugging in a numerical solution $R(\rho)$ (5.47)](image)

### 5.5.2 The Role of Higher Eigenmodes

In principle, there are an infinite number of higher $m^2$ eigenvalue modes defined by the Sturm-Liouville operator (5.46), apart from this lowest $m^2$ we have found above (but none with a lower eigenvalue). If these all have positive $m^2$ (and hence negative $\alpha$) then their contribution to the evolution will die away exponentially fast with time (5.53). Could there be other sub-leading negative $m^2$ states, though? To attempt to answer this we numerically solve (5.46) starting with $R(\rho_c) = 0$ and seeking modes that fall to zero at large $\rho$. We show the results in Fig 5.17 and the corresponding eigenvalues. The unstable mode we have already discussed is the only negative mass squared state. Of course our IR boundary condition is somewhat ad hoc but the conclusion that there is just a single unstable mode seems strong.
5.5.3 A Single Effective Inflaton

Let us stress the conclusion of the analysis of the previous section. For the majority of the roll time in this strongly coupled gauge theory the dynamics is dominated by a single negative mass squared, linearised mode. Higher modes have positive mass and their contributions to the evolution are exponentially suppressed with time.

With a given solution $R(\rho)$ the Lagrangian (5.41) boils down to a standard scalar inflaton field:

$$\mathcal{L}_{(0)} = \sqrt{-g} \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 \right)$$

(5.55)

where $\sqrt{-g} = a(t)^3$ and

$$\phi(t) := T(t) \sqrt{\bar{T}_7} \int d\rho \frac{\beta_0}{\rho} R(\rho)^2 = \sqrt{\bar{T}_7} T(t).$$

(5.56)

Here we used (5.51) and (5.52).

This is nothing but a typical example of a single field inflaton model. We will now go on to study higher couplings of this effective instanton field by expanding the DBI action.
Let us assume
\[ L = X(t, \rho) + Y(t, x, y, z, \rho), \] (5.57)
where \( Y \) is a quantum fluctuation around the classical vacuum \( X(t, \rho) \).

We are interested in a slow roll time regime only, where \( L(\rho) \ll \rho \) for \( \rho \in [\rho_c, \infty) \) and \( \rho_c \) is a small IR cut-off. In this regime, \( \beta \), our dilaton, a function of \( r^2 = \rho^2 + L^2 \), may be expanded:
\[ \beta(\sqrt{\rho^2 + L^2}) \approx \beta(\rho) + \frac{L^2}{2\rho} \beta'(\rho) + \frac{L^4}{8\rho^3}(\rho\beta''(\rho) - \beta'(\rho)) + \ldots \] (5.58)
and similarly,
\[ \frac{1}{r^4} = \frac{1}{(\rho^2 + L^2)^2} \approx \frac{1}{\rho^4}(1 - 2\frac{L^2}{\rho^2} + \ldots) \] (5.59)
The integrand of (5.40), with spatial Minkowski derivatives reinstated,
\[ \rho^3\beta\sqrt{1 + L^2 + \frac{\partial_\mu L \partial^\mu L}{r^4}} \] (5.60)
is expanded up to fourth order in \( L \):
\[
\mathcal{L}_{DBI} = \rho^3\beta\sqrt{1 + L^2 + \frac{\partial_\mu L \partial^\mu L}{r^4}} \\
\approx \rho^3 \left( \beta(\rho) + \frac{L^2}{2\rho} \beta'(\rho) + \frac{L^4}{8\rho^3}(\rho\beta''(\rho) - \beta'(\rho)) \right) \sqrt{1 + L^2 + \frac{\partial_\mu L \partial^\mu L}{\rho^4}(1 - 2\frac{L^2}{\rho^2})} \\
\approx \rho^3 \left( \beta(\rho) + \frac{L^2}{2\rho} \beta'(\rho) + \frac{L^4}{8\rho^3}(\rho\beta''(\rho) - \beta'(\rho)) \right) \times \\
\left( 1 + \frac{1}{2} L^2 + \frac{\partial_\mu L \partial^\mu L}{2\rho^4} - \frac{1}{8} \left( \frac{L^2}{\rho^6}(\partial_\mu L)^2 + L^4 + 2\frac{L^2}{\rho^4}(\partial_\mu L)^2 + \frac{(\partial_\mu L)^4}{\rho^8} \right) \right) \\
\approx \rho^3 \beta \\
+ \frac{1}{2} \rho^3 \beta L^2 + \frac{\beta(\partial_\mu L)^2}{2\rho} + \rho^2 \frac{L^2}{2} \beta'(\rho) \\
+ \frac{-\rho^3}{8} \left( \frac{L^2}{\rho^6}(\partial_\mu L)^2 + L^4 + 2\frac{L^2}{\rho^4}(\partial_\mu L)^2 + \frac{(\partial_\mu L)^4}{\rho^8} \right) \\
+ \frac{\rho^3}{4} \left( L^2 L^2 + \frac{(\partial_\mu L)^2 L^2}{\rho^4} \right) + \frac{1}{8}(\rho\beta'' - \beta')L^4 \] (5.61)
We substitute $L \rightarrow X + Y$ and extract the second and third order $Y$ terms:

$$L^{(0)} \approx \frac{1}{2} \rho^3 \beta X'^2 + \frac{\beta (\partial_{\mu} X)^2}{2 \rho} + \rho^2 X^2 \beta' (\rho)$$

$$L^{(2)} \approx \frac{1}{2} \rho^3 \beta Y'^2 + \frac{\beta (\partial_{\mu} Y)^2}{2 \rho} + \rho^2 Y^2 \beta' (\rho)$$

$$L^{(3)} \approx -\frac{\rho^3 \beta}{8} \left(4 X' Y'^3 + \frac{4}{\rho^4} \left( X' Y' (\partial_{\mu} Y)^2 + Y'^2 (\partial_{\mu} X \partial_{\nu} Y) + \frac{4}{\rho^8} (\partial_{\mu} X \partial_{\nu} Y) (\partial_{\nu} Y)^2 \right) \right) + \left( \frac{\beta'}{2 \rho^2} - \frac{2 \beta}{\rho^3} \right) (XY (\partial_{\mu} Y)^2 + Y^2 (\partial_{\mu} X \partial_{\nu} Y)) + \frac{\rho^2 \beta'}{2} (XY Y'^2 + X' Y'^2) + \frac{1}{2} (\rho \beta'' - \beta') XY'^3$$

where we only kept the terms linear to $X$, since $X$ itself is very small. We did a double expansion: First for $X$ and then for $Y$ on top of $X$. $L^{(1)}$ vanishes by the equation of motion for $X$.

We assume the following: (1) The terms involving Minkowski derivatives $(\partial_{\mu} X)$ are non-renormalizable and presumably disregarded, (2) We assume $Y(x^\mu, \rho) = \varphi(x^\mu) R(\rho)$ where $X(t^\mu, \rho) = T(t) R(\rho)$. This latter form is an approximation that is correct in the large wavelength limit in the $x$-directions when $x$ derivatives are vanishingly small in the equation of motion - locally $Y$ is then just an added piece of $X$ and will have the same functional $\rho$ form.

Then the effective Lagrangian for $\varphi$ ($\phi$ denote inflaton and $\varphi$ denote a fluctuation around $\phi$) is

$$L_\varphi = \frac{1}{2} (\partial_{\mu} \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \Lambda \varphi^3$$

where $m^2$ is determined by (5.47) or (5.52) and $\varphi$ has been canonically normalized: $\tilde{T}_T \varphi \rightarrow \varphi$.

$$\Lambda = \frac{1}{\sqrt{T_T}} \int d\rho \frac{1}{2} \left\{ (\rho \beta'' - \beta') R(\rho)^4 + 2 \rho^2 \beta' R(\rho) R'(\rho)^2 - \rho^3 \beta R'(\rho)^4 \right\} T(t)$$

$$= \frac{1}{\sqrt{T_T}} \Lambda_0 T_0 e^{\alpha t}$$

(5.64)
Once $X(t, \rho)$ is determined by equation of motion, equivalently by $\mathcal{L}(1) = 0$, $\mathcal{L}(2)$ and $\mathcal{L}(3)$ will determine the dynamics of the fluctuation. It would be interesting to try to extract CMB features from these dynamics, such as the two-point function (power spectrum) from $\mathcal{L}(2)$ and the three-point function (bispectrum) from $\mathcal{L}(3)$.

In the Figures 5.18, 5.19 we show $m^2$ and $\Lambda$ against the parameter $A$ (at fixed $\Gamma = 1$). Note that $m^2$ goes to zero for small enough $A$. A positive $m^2$ would mean that the flat embedding is a local minimum and there is no spontaneous exponential (tachyonic) growth of the condensate. We might naturally have expected that large values of $A$ would render the symmetric configuration unstable, triggering symmetry breaking under any small perturbation. What we learn from this analysis is that there is a critical value of the coupling step below which the symmetric embedding is metastable and symmetry breaking requires either large disturbance or tunnelling. We know this to be unsuitable for any inflationary model, so $A$ is required to be small, but not too small (similar principles are noted in [128], where symmetry breaking is completely disfavoured if the energy cost of warping the brane is greater than that saved by avoiding the dilaton step – in this
5.6 Discussion

Inspired by small field inflation models, we have used holography to study out-of-equilibrium dynamics in a strongly coupled gauge theory with chiral symmetry breaking. We have shown that we can extract an approximate effective potential for the quark condensate. We have then been able to explicitly follow the roll of a configuration from the symmetric potential maximum to the symmetry-breaking potential minimum. We first performed this computation in a model with chiral symmetry breaking induced by a magnetic field. While the configuration lies near the potential maximum it generates inflation. Formally we worked in a probe limit that requires that the rate of inflation be small relative to the typical relaxation time of the background $\mathcal{N} = 4$ gauge dynamics.

We then studied a holographic model of a large $N_c$ gauge theory with an arbitrary running of the gauge coupling. It is a model in the spirit of AdS/QCD since we do not back-react the geometry to the dilaton profile. The dilaton profile we chose to study means the gauge theory has a strongly coupled conformal UV and deforms to an IR conformal theory with a higher value of the gauge coupling. That change in the coupling triggers chiral symmetry breaking for the quarks. We have found that when the change in the coupling is fine tuned to be small or slow-running the theory can give rise to inflationary dynamics. An important contribution to this slowing of the roll was made by the D7 dynamics in the radial direction of the geometry through (5.39) - this suggests that the strong dynamics of these gauge theories encourages inflation. Are there broad lessons to be learnt here about strongly coupled gauge theories or is this an oddity of a particular AdS setup?

There is a growing belief that many asymptotically free gauge theories indeed give rise to strongly coupled conformal regimes. Seiberg’s dualities for $\mathcal{N} = 1$ supersymmetric QCD [129] were the first hint - they show that SQCD flows to a non-trivial IR conformal theory in the range $N_c + 1 < N_f < 3N_c$. For much of that
range the UV degrees of freedom are strongly coupled. Near \( N_f = 3N_c \) and at large \( N_c \) these phases match onto the perturbative Banks Zaks fixed points \(^{130}\). Banks Zaks fixed points also exist in non-supersymmetric theories and it is reasonable to expect that IR conformal fixed points also exist for a considerable range of \( N_f \) in those theories (see \(^{131}\)\(^{132}\)\(^{133}\)\(^{134}\) for some speculation about these theories).

Theories with higher dimension representation matter fields would also be expected to generate strongly coupled IR fixed points. Recently there has begun to be lattice simulations of QCD with varying unquenched quark flavours \(^{135}\)\(^{136}\) and higher dimensional representation matter present \(^{137}\)\(^{138}\)\(^{139}\) - there is certainly encouragement in these results for the view that IR conformal theories exist in some of these cases.

If we believe that such fixed points are fairly common then we can imagine several ways to construct a theory with the profile for the running coupling we have studied in this paper. For example, one could begin with an SU(\(N_c\)) theory with sufficient non-fundamental matter to place it at a strongly coupled fixed point. If \( N_c \) is appropriately large then we can also add fundamental quark multiplets as a perturbation - their contributions to the beta function coefficients will be \( N_f/N_c \) suppressed and hence they will most likely generate just a small change in the fixed point’s coupling value. The fundamental quarks though can be used to dial a profile for the running coupling if they can be sequentially decoupled. For example, if they are vector-like one could just put in masses to adjust the running at the order \( N_f/N_c \) shifting the IR theory from one conformal point to another. This realizes the running we described above. It is not obvious how chiral symmetry breaking might happen in this scenario though. Usually one imagines an NJL-model-like critical coupling for chiral symmetry breaking \(^{126}\). One would need to tune the shift in the coupling to cross that critical value for the fundamental quarks. Naively higher dimension representation quarks might also be expected to condense at the same point or at a lower value of the coupling though.

Another possibility, which is sometimes discussed \(^{130}\), is that pure glue gauge theories in fact have an IR fixed point for the coupling. One could imagine perturbing that fixed point with some fundamental matter and again, by
appropriate decoupling, change the running to match that which we seek. The pure glue fixed point would again need to lie just above the critical coupling for chiral symmetry breaking.

Finally the other possibility we are led to is walking dynamics $^{141,142}$. A theory with fundamental quarks that is approaching an IR fixed point where the coupling is tuned just above the critical coupling might spend many RG decades at strong coupling but without triggering chiral symmetry breaking before finally reaching the critical value at a low scale. Equally the coupling might cross the critical value at a high scale but be so close to the critical value from above that the quarks’ dynamical mass is at a much lower scale so they don’t decouple and the fixed point is maintained. Such a theory would naturally realize both the small $A$ and small $\Gamma$ limit of our coupling ansatz, both of which led towards inflation. Walking is of course proposed as a solution of the mini-hierarchy problem in technicolour theories of electroweak symmetry breaking $^{143,144}$. It is certainly intriguing if the fine tuning already used to solve that problem also generates inflationary dynamics.

In conclusion it certainly seems possible that a range of asymptotically free gauge theories might realize the behaviour we have seen and generate effective dynamics that encourages inflation. The need for intrinsic fine tuning, as usual in small field inflation, seems unavoidable though.
Chapter 6

Conclusions and Outlook

The AdS/CFT Correspondence has provided fertile ground for theoretical work for over a decade now. Much work has gone into understanding the formal aspects of the Correspondence as well as applying it to concrete problems. Attempts to extract QCD physics include top-down approaches (such as those noted in this thesis), wherein the bulk theory is augmented with fields that correspond to relevant operators in the field theory, which break supersymmetry and conformality \[145,146,75,76\]. Bottom-up approaches (“AdS/QCD”) approaches, on the other hand, start from QCD and attempt to construct the holographic dual, without starting from a proper string set up \[91,92,147\]. In the last few years, there has been a great deal of interest in the application to condensed matter physics \[148\], with work being done on holographic superconductivity, superfluidity, Fermi and non-Fermi liquids and the quantum Hall effect \[149,150,151,152,153\]. Perhaps most notable has been the work on heavy ion physics, which has been of particular interest to those interpreting the results emanating out of RHIC \[154,83,155\]. We note that there have even been applications to the field of strongly coupled gravity, such as may be relevant for study of inflation \[156\] (note that this considers the Correspondence from the opposite direction to the one presented in this thesis).

In this thesis, we have presented two uses of the Correspondence related to these last two areas of research. The first was to look at the process of
hadronization from the viewpoint of a string expanding in the bulk, developing work started in [81]. The significant role played by radiative emission off the endpoints suggests that it would be interesting to see how that could be incorporated into current phenomenological hadronization schemes. Future work could involve inserting such a process into a scheme such as SHERPA [157] or HERWIG [25] (such a task is certainly not trivial, but could be fruitful). In addition, scalar modes could potentially be excited on the brane by modelling the string as a time dependent brane spike [158].

In the other work presented, we have looked at the time dependent evolution of a quark/antiquark condensate which occurs in bulk theories with a spontaneously broken $SO(2)$ symmetry. While the main geometric set up used was somewhat ad hoc, we believe it provided a useful phenomenological toy model for a wide class of theories with a UV and IR fixed point and we saw clearly that a small running of the gauge coupling was important for the slow development of the condensate from the symmetric initial conditions. Other work that considers a strongly coupled composite inflaton can be found in [159], which looked at walking technicolour with non-minimal gravitational coupling and found that the symmetry-breaking scale was necessarily that of Grand Unified Theories ($\sim 10^{16}$ GeV). While there is formally no coupling to gravity in our version of the Correspondence, it might nevertheless be interesting to consider such quantities as the bispectrum and see what connections could be made with [160,161], which discussed whether measurable CMB features could be signatures of a composite nature for the inflaton.
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