UNIVERSITY OF SOUTHAMPTON

On approximation properties of group C^*- algebras

by

Kannan Kankeyanathan

Thesis for the degree of Doctor of Philosophy

in the

School of Mathematics

July 2011

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

SCHOOL OF MATHEMATICS

Doctor of Philosophy

ON APPROXIMATION PROPERTIES OF GROUP C^*-ALGEBRAS

by

Kannan Kankeyanathan

In this thesis we study analytic techniques from operator theory that encapsulate geometric properties of a group. Rapid Decay Property (Property RD) provides estimates for the operator norm of elements of the group ring (in the left-regular representation) in terms of the Sobolev norm. Roughly, property RD is the noncommutative analogue of the fact that smooth functions are continuous. Our work then concentrates on a particular form of an approximation property for the reduced C^*- algebra of a group: the invariant approximation property. This statement captures a particular relationship between three important operator algebras associated with a group: the reduced C^*- algebra, the von Neumann algebra, and the uniform Roe algebra. The main result is the proof of the invariant approximation property for groups equipped with a conditionally negative length function. We prove also that the invariant approximation property passes to subgroups and then discuss the behaviour of the invariant approximation property with the respect to certain classes of extensions. We show that the invariant approximation property passes to direct products with finite group. We show that the invariant approximation property passes to extensions of the following form. If G is a discrete group and H is a finite index normal subgroup of G with IAP,

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

then G has IAP.

Author's declaration

I, Kannan Kankeyanathan, declare that the thesis entitled *On approximation properties of group* C^*- *algebras* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;

Signed: ______________________________

Date: ______________________________

Acknowledgment

I would like to express my gratitude to my supervisor Prof. Jacek Brodzki, University of Southampton, I am particularly, thankful to my supervisor for his valuable suggestions that have improved and encouraged my progress. I wish to express my warm and sincere thanks to Prof. Graham Niblo and Dr. Nick Wright.

I am deeply grateful to Prof. V. Tharmaratnam, Late Mr. P. Mahinan, Late Dr. M. Skandarajah and Dr. S. Srisatkunarajah, University of Jaffna, Sri Lanka for their guidance throughout my academic career. My special gratitude is due to Dr. David Schley, former visiting Lecturer, University of Jaffna, Sri Lanka for his loving support. I wish to thanks to Prof. S. Parvathi, Prof. M. Loganathan, and Prof. S. Sahadevan, Ramanujan institute of advanced study in Mathematics (RIASM), University of Madras, India.

I am very grateful to the University of Jaffna, Sri Lanka for financially supporting me for the first two years through the "Improving Relevance Quality of Undergraduate Education"(IRQUE) project funded by the World Bank. I would also like to thank Prof. K. Kandasamy, Prof. R. Kumaravadivel, and Prof. R. Vigneswaran, University of Jaffna, Sri Lanka for all their administrative support. I am very grateful to the University of Southampton for financially supporting the final year of my Ph.D programme.

Far too many people to mention individually have assisted in so many ways during my work at Southampton. In particular, I would like to thank my friends Ana Khukhro, Alex Bailey, Glenn Ubly, Giovanni Gandini, Ramanan and Martin Fluch at University of Southampton.

Finally, I owe special gratitude to my family for continuous and unconditional support: To my parents for the interest they showed in my studies and the motivation they gave me during those hard times in my Ph.D programme. I owe my loving thanks to all of my family members: Parents, wife, son, daughters, sisters, brothers and also to other family members. Without their encouragement and understanding it would have been impossible for me to finish this work.

Contents

To my beloved Mother

Late Mrs Kankeyanathan Vimalaranee $\cdots$

CHAPTER 1

Introduction

1.1. Motivation and statement of the results

The study of C^*- algebra consists of two parts; one is concerned with the intrinsic structure of algebras and the other deals with the representations of a C^*- algebra. We lay the foundations for later discussion, giving elementary results on Banach algebra [**2**] and C^*-algebra [**13**]. Von Neumann introduced the theory of operator algebra and in 1943 [**31**], the work of Israel Gelfand, Mark Naimark and Irving Segal proposed an abstract characterization of C^*- algebra making no reference to Hilbert space. It is generally believed that C^*- algebra were first considered primarily for their use in quantum mechanics as model algebras of physical observables. In the case of the reduced C^*- algebra that space is a space of representations of the group.

The purpose of this thesis is to provide an illustration of an interesting and nontrivial interaction between analytic and geometric properties of a group. We provide an approximation property of operator algebras associated with discrete groups. There are various notions of finite dimensional approximation properties for C^*- algebra and more generally operator algebras. Among these are the completely bounded approximation property (CBAP), the strong invariant approximation property (SIAP), the approximation property (AP), the operator space approximation property (OAP), the strong operator space approximation property (SOAP) and exactness, and the reader is referred to [**7**], [**19**] and [**36**] for these a interesting concepts. A first result in this direction was Haagerup's [**18**] discovery that that the reduced C^*- algebra $\mathbb{F}_n$, $n \geq 2$ has the metric approximation property. Another important theorem, due to Lance [**27**], states that a group is amenable if and only if reduced C^*- algebra is nuclear.

For the reduced C^*- algebra of a discrete group most of these approximation properties have a number of equivalent reformulations in term of the discrete group: Haagerup and Kraus have proved in [**19**], that approximation property (AP) of G is equivalent to the SOAP of $C_r^*(G)$ and to the OAP of $C_r^*(G)$, Haagerup proved in [**19**], that the CBAP for $C_r^*(G)$ is equivalent to the weak amenability of G. Approximation properties of group C^*- algebra are now important tools in group theory.

Let G be a discrete group, then the characteristic function $\delta_g(s)$ of $g, s \in G$ is defined as follows [**13**]:

$$\delta_g(s) = \begin{cases} 1 & \text{if } g = s, \\ 0 & \text{if } g \neq s. \end{cases}$$

If we assume that the G is a discrete group then the functions δ_g form a basis for the Hilbert space $\ell^2(G)$ of square summable functions on G.

$$\ell^2(G) = \left\{ f : G \longrightarrow \mathbb{C} \text{ such that} \sum_{n \in G} |f(n)|^2 < \infty \right\}$$

One defines a scalar product as follows: For $f, g \in \ell^2(G)$

$$\langle f, g \rangle = \sum_{n \in G} f(n)\overline{g(n)}$$

Any element of $\ell^2(G)$ can be expressed as an infinite linear combination of δ_g with square summable sequence. Let $\ell^2(G)$ be Hilbert space, the algebra $B(\ell^2(G))$ of bounded linear maps from $\ell^2(G)$ to itelf is a C^*- algebra for the operator norm.

The group ring $\mathbb{C}[G]$ consists of all finitely supported complex-valued functions on G, that is of all finite combinations

$$f = \sum_{s\in G} a_s s$$

with complex coefficients.

Let us recall the left and right regular representation and reduced C^*- algebra of a discrete group. The left regular representation

$$\lambda : \mathbb{C}[G] \to B(\ell^2(G))$$

is defined by

$$\lambda(s)\delta_t(r) = \delta_t(s^{-1}r) = \delta_{st}(r) \text{ for } s, r \in G.$$

The right regular representation

$$\rho : \mathbb{C}[G] \to B(\ell^2(G))$$

is defined by

$$\rho(s)\delta_t(r) = \delta_t(rs) = \delta_{ts^{-1}}(r) \text{ for } s, r \in G.$$

The reduced C^*- algebra $C^*_\lambda(G)$ of a group G (which we shall assume to be discrete) arises from the study of the left regular representation λ of the group ring $\mathbb{C}[G]$ on the Hilbert space of square-summable functions on the group. The reduced group C^*- algebra G, denoted by $C^*_\lambda(G)$ is the completion of $\mathbb{C}[G]$ in the norm given, for $c \in \mathbb{C}[G]$, by

$$\|c\|_\lambda = \|\lambda(c)\| \,.$$

In the context of coarse geometry introduced by Roe [**30**]. There is a natural way to associate a C^*- algebra with a discrete metric space X. We shall denote the algebra of bounded operators associated with finite propagation kernels on X by $A^\infty(X)$. The uniform Roe algebra of a metric space X is the closure of $A^\infty(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on X.

The reduced C^*- algebra $C^*_\lambda(G)$ is naturally contained in $C^*_u(G)$ [**30**]. According to Roe [**30**] G has the invariant approximation property (IAP) if and only if

$$C^*_\lambda(G) = C^*_u(G)^G.$$

It is an interesting problem to determine which groups have this property.

An important ingredient in our study is the property RD of P. Jolissaint's results [**21**]. Let G be a discrete group. A length function on G is a map $\ell : G \longrightarrow \mathbb{R}$ taking values in the non-negative reals which satisfies the following conditions:

(1) $\ell(1) = 0$ where 1 is the identity element of the group;
(2) For every $g \in G$, $\ell(g) = \ell(g^{-1})$;
(3) For every $g, h \in G$, $\ell(gh) \leq \ell(g) + \ell(h)$.

For any length function ℓ and positive real number$s$, we define a Sobolev norm on the group ring $\mathbb{C}[G]$ by[**21**]:

$$\|f\|_{\ell,s} = \sqrt{\sum_{\gamma\in G} |f(\gamma)|^2 \,(1+\ell(\gamma))^{2s}}.$$

Following Jolissaint [**21**], we say that G has the Rapid Decay property (property RD) with respect to the length function ℓ if there exist $C \geq 0$ and $s > 0$ such that, for all $f \in \mathbb{C}[G]$,

$$\|f\|_* \leq C \|f\|_{\ell,s} ,$$

where $\|f\|_*$ denotes the operator norm of f acting by left convolution on $\ell^2(G)$ The rapid decay property for groups, generalizes Haagerup's [**18**] inequality for free groups and so for example of free groups have property RD

This property RD for groups has deep implications for the analytical, topological and geometric aspects of groups. Jolissaint proved in his thesis that groups of polynomial growth and classical hyperbolic groups have property RD, and the only amenable discrete groups that have property RD are groups of polynomial growth. He also showed that many groups, for instance $SL_3(\mathbb{Z})$, do not have the Rapid Decay property [**21**].

Examples of RD groups include group acting on CAT(0)-cube complexes [**12**], hyperbolic groups of Gromov [**16**], Coxeter groups [**12**], and torus knot groups [**22**].

Having introduced the basic notations, we study the interaction between property RD and the invariant approximation property, and in particular we show that the invariant approximation property for groups equipped with conditionally negative length function. We use the proof requires working familiarity with elements of von Neumann algebra theory, C^*- algebra, property RD, and key features of the uniform Roe algebra.

Our main result in this direction is the following (see Theorem 4.3.3).

THEOREM 1.1.1. *Let G be a discrete group satisfying the rapid decay property with respect to a conditionally negative length function ℓ. Then the group G has the invariant approximation property.*

We then use this to show the following groups have invariant approximation property (see Examples 4.3.5, 4.3.6, 4.3.7, 4.3.9 and 4.3.11):

- The classical hyperbolic group
- Hyperbolic groups
- $CAT(0)$-cubical groups
- finitely generated Coxeter group
- Torus Knot group

We also that if G is a free product group satisfying the rapid decay property with respect to a conditionally negative length function ℓ, then the group G has the invariant approximation property (see Example 4.3.14).

We give a general exposition of approximation properties which were initiated by Grothendieck [**5**]. His fundamental ideas have been applied to the study of groups and these noncommutative approximation properties have played a crucial role in the study of von Neumann algebras and C^*- algebra. Some weaker conditions (i.e., weak amenability and the approximation property) for locally compact groups have been studied by Haagerup and Kraus [**19**]. We recall basic definitions of approximation properties. Let C^*- algebra A is said to have the completely bounded approximation property (CBAP) if there is a positive number C such that the identity map on A can be approximated in the point norm topology by a net $\{\phi_\alpha\}$ of finite rank completely bounded maps whose completely bounded norm are bounded by C, that is if there exists a net of finite-rank maps $\{\phi_\alpha\} : A \longrightarrow A$ such that $\|\phi_\alpha\|_{cb} \leq C$ for some constant C and $\phi_\alpha \longrightarrow id_A$ in the point-norm topology on A. The infimum of all values of C for which such constants exist is denoted by $\Lambda_{cb}(A)$ [**19**]. We say that discrete G is weakly amenable if there is an approximate identity (ϕ_n) such that $C := \sup \|M_{\phi_n}\| < \infty$.

There are many other interesting approximation properties for C^*- algebra. It is shown in [**19**] that a C^*- algebra A has the operator approximation property (OAP) if there exists a net of finite-rank maps $T_\alpha : A \longrightarrow A$ such that $T_\alpha \longrightarrow id_A$ in the stable point-norm topology. The discrete group G has the approximation property (AP) if there is a net $\{\phi_\alpha\}$ in $A(G)$ such that $M_{\phi_\alpha} \longrightarrow id_{A(G)}$ in the stable point-norm topology on $A(G)$ [**19**]. Haagerup and Kraus (see [**19**]) show that a discrete group G has the approximation property (AP) if and only if $C^*_r(G)$ has the operator space approximation property (OAP) [**19**].

In particular, there is the following implication for discrete groups:

$$\text{CBAP} \Rightarrow \text{AP} \Rightarrow \text{IAP}.$$

We then use this to show the following groups have invariant approximation property:

- Amenable groups
- Hyperbolic groups [**29**]
- $CAT(0)$-cubical groups [**32**]
- $SL_2(\mathbb{Q}_p)$ [**4**].

We give a general exposition of invariant approximation property(IAP), which was initiated by Roe [**36**]. We study certain stability properties of invariant approximation property, and we show that it passes to extensions with a finite quotient, passes to subgroups and holds for direct products with finite group. An important result of this thesis is the following (see Theorem 6.1.1).

THEOREM 1.1.2. *Any subgroup H of a discrete group G with the invariant approximation property has the invariant approximation property.*

Brodzki, Niblo and Wright [**6**] show that the uniform Roe algebra of metric space is a coarse invariant up to Morita equivalence. We translate this result to the case of a coarse equivalence of metric spaces $X \longrightarrow Y \times N$, where N is finite. We use this to show that the invariant approximation property passes to direct products with finite factor (see Theorem 6.3.2).

THEOREM 1.1.3. *Let H be a discrete group with the IAP and K a finite group, then the direct product $G = H \times K$ has IAP.*

Our final result is to show that the invariant approximation property passes to finite extensions (see Theorem 6.2.3).

THEOREM 1.1.4. *Let G is a discrete group and H is a finite index normal subgroup of G with IAP,*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

then G has the IAP.

1.2. Organisation of the thesis

The following is a rough synopsis; see the start of each chapter for a more detailed outline: In Chapter 2, we explain some basic facts about C^*- algebra (see section 2.1), the left and right regular representation (see section 2.4), weak topologies (see section 2.2) and Von Neumann algebras (see section 2.3) [**31**], [**13**]. In Chapter 2, we also explain tensor product of C^*- algebras (see section 2.6).

In Chapters 3 to 6 have we develop various aspects of invariant approximation property. In Chapter 3, we recall coarse geometry, uniform Roe algebras, and the invariant approximation property under sections 3.1, 3.2, and 3.3 respectively. In Section 3.3 contains a discussion of the role played by the left or right-invariance of the metric. In Chapter 3, we show that uniform Roe algebras can be expressed as crossed products.

In Chapter 4, we explain the basic notions related to property RD for discrete groups. In Section 4.2, we study positive and negative type kernels. In this chapter, we prove that for a discrete group G satisfying the rapid decay property with respect to a conditionally negative length function ℓ, the reduced C^*- algebra $C^*_\lambda(G)$ has the invariant approximation property (see Theorem 4.3.3). We also provide some examples of groups (see Corollory 4.3.5, 4.3.7, 4.3.11, and 4.3.14) that have invariant approximation property of Theorem 4.3.3.

Chapter 5 contains the basic definitions of various approximation properties. In Section 5.2, we study strong invariant approximation property. This chapter also shows relationship among the CBAP, AP and

IAP (see Proposition 5.2.11), and IAP and SIAP (see Proposition 5.3.3). We show that the strong approximation property pass to semi direct products (see Proposition 5.4.2), and extensions for discrete exact groups (see Proposition 5.3.6). Section 5.5 provides an example, which does not have SIAP.

In Chapter 6, we show the a relationship between uniform Roe algebra and coarsely equivalence metric space (see Theorem 6.2.3). In addition to this we show that the invariant approximation property passes to subgroups (see Theorem 6.1.1), direct products with a finite group (see Theorem 6.2.4), and finite extensions (see Theorem 6.3.2). There are the main results of this thesis.

CHAPTER 2

C^*- algebra theory

We assume that the reader is familiar with basic notions in operator algebras (the reader is referred to Takesaki [**31**], Blackadar [**2**], Effros [**15**], Davidson [**13**], Brown and Ozawa [**7**] and Wassermann [**33**]). The aim of this chapter is to introduce some of the important topological techniques in the study of operator algebras, and in particular C^*- algebra. For the most part, this chapter consists of basic definitions of C^*- algebra and related topics. In Sections 2.4, 2.5, we study the left and right regular representations and the reduced group C^*- algebra. In Sections 2.2, 2.3, 2.6 of this work, we explain some basic facts about weak topologies, Von Neumann algebras and tensor product of C^*- algebra.

We first recall some basic facts about C^*- algebra.

2.1. Basic definitions

Most of the definitions given in this section is taken from [**31**] and [**13**].

DEFINITION 2.1.1. A complex normed algebra $\mathfrak{U}$ which is complete (as a topological space) and satisfies the inequality

$$\|AB\| \leq \|A\| \, \|B\| \quad \text{for all} \quad A, B \in \mathfrak{U}$$

is called a *Banach algebra*.

DEFINITION 2.1.2. We say that $\mathfrak{U}$ is *Banach* $*-$ *algebra* if $\mathfrak{U}$ is a complex algebra with a conjugate linear involution $*$ (called the adjoint) which is an anti-isomorphism. It is endowed with a map

$$*:\mathfrak{U}\longrightarrow\mathfrak{U},$$

given by

$$*:A\longrightarrow A^*,$$

satisfying the following properties: For all A, B in $\mathfrak{U}$ and λ in $\mathbb{C}$,

$$(A+B)^*=A^*+B^*$$

$$(\lambda A)^*=\overline{\lambda}A$$

$$A^{**}=A$$

$$(AB)^*=B^*A^*.$$

We say that $\mathfrak{U}$ is C^*- *algebra* if it is a Banach $*-$ algebra with the additional norm condition

$$\|A^*A\|=\|A\|^2 \text{ for all } A\in\mathfrak{U}.$$

EXAMPLE 2.1.3. The algebra of all bounded operators on a Hilbert space $\mathcal{H}$ is a C^*- algebra. Denote $B(\mathcal{H})$ is the C^*- algebra with respect to the operator norm. When $\mathcal{H}=\mathbb{C}^n$, we get the algebra $M_n(\mathbb{C})$ of $n\times n$ matrices with complex entries.

REMARK 2.1.4. Every finite dimensional C^*- algebra is a finite product of full matrix algebras.

EXAMPLE 2.1.5. Given a locally compact space X, the C^*- algebra $C_0(X)$ of continuous functions X to $\mathbb{C}$, vanishing at infinity, is a commutative C^*-algebra when equipped the norm

$$\|f\| := \sup_{x \in X} |f(x)|,$$

and the adjoint

$$f^*(x) = \overline{f(x)}.$$

REMARK 2.1.6. A norm closed subalgebra of a C^*- algebra is also C^*- algebra. A norm closed subalgebra of $B(\mathcal{H})$ will be called a concrete C^*- algebra.

DEFINITION 2.1.7. [**3**] A bounded operator $T : \mathcal{H} \longrightarrow \mathcal{H}'$ is a *compact* if the image by TB_1 of the closed unit ball B_1 in $\mathcal{H}$ is *relatively compact* $\mathcal{H}'$. This is equivalent to saying that whenever $\{x_n\}$ is a bounded sequence in $\mathcal{H}$, we can select a subsequence $\{x_{n_k}\}$ such that the sequence $\{Tx_{n_k}\}$ converges.

The linear space of all compact operators from $\mathcal{H}$ to $\mathcal{H}'$ is denoted by $\mathcal{K}(\mathcal{H}, \mathcal{H}')$. We write $\mathcal{K}(\mathcal{H})$ instead of $\mathcal{K}(\mathcal{H}, \mathcal{H})$.

DEFINITION 2.1.8. We say that an element A of a C^*- algebra $\mathfrak{U}$ is *self-adjoint* if $A^* = A$; N is *normal* if $N^*N = NN^*$; N is an *isometry* if $NN^* = I$; and U is *unitary* if $U^*U = I = UU^*$.

DEFINITION 2.1.9. A linear functional $\phi : A \longrightarrow \mathbb{C}$ on a C^*- algebra A is *positive* if and only if

$$\phi(x^*x) \geq 0 \text{ for all } x \in A.$$

A positive linear functional of norm one is called a *state* [**2**].

If $\phi(x^*x) \neq 0$ for every nonzero $x \in A$, then ϕ is said to be *faithful.*

DEFINITION 2.1.10. A map between C^*- algebra which preserves sum, scalar multiplication, product, and adjoint, is called a $*-$ *homomorphism.*

Maps between C^*- algebras can be extended to maps of matrix algebras in the following way. Let A and B be two C^*- algebras and

$$\phi : A \longrightarrow B$$

be a linear map. Then

$$\phi \otimes id_{M_n} : M_n(A) \longrightarrow M_n(B),$$

$$(a_{i,j}) \longmapsto (\phi(a_{i,j}))$$

is a linear map, denoted by ϕ_n. If ϕ is a $*-$ homomorphism then ϕ_n is also $*-$ homomorphism. The completely bounded norm of ϕ is defined as

$$\|\phi\|_{cb} = \sup\left\{\|\phi_n\| : \ n \ \in \ \mathbb{N}\right\}.$$

We say ϕ is completely bounded if

$$\|\phi\|_{cb} = \sup\left\{\|\phi_n\| : \ n \ \in \ \mathbb{N}\right\} < \infty.$$

We define the $CB(A, B)$ as the space of completely bounded maps from A to B with completely bounded norm.

2.2. Weak topologies

There are several important topologies on $B(\mathcal{H})$ that are weaker than the norm topology. The *weak operator topology* (WOT) on $B(\mathcal{H})$ is defined as the weakest topology such that the sets

$$\mathcal{W}\{T, x, y\} = \{U \in B(\mathcal{H}) : \langle (T-U)x, y\rangle \leq 1 \forall\ T \in B(\mathcal{H}) \text{ and } x, y \in \mathcal{H}\}$$

are open [**13**]. The sets

$$\mathcal{W}\{T_i, x, y; 1 \leq i \leq n\} := \bigcap_{i=1}^{n} \mathcal{W}\{T_i, x_i, y_i\}$$

form a base for the weak operator topology.

Definition 2.2.1. [**13**] A net T_α converges to T, for $\alpha \in \Lambda$ (index set) in the weak operator topology ($T_\alpha \overset{WOT}{\longrightarrow} T$) if and only if for all $x, y \in \mathcal{H}$, $\langle T_\alpha x, y\rangle \longrightarrow \langle Tx, y\rangle$ i.e, there exists a continuous linear functional $\phi : \mathcal{H} \longrightarrow \mathbb{C}$ such that $\phi(T_\alpha) \longrightarrow \phi(T)$.

The *strong operator topology* (SOT) is defined [**13**] by the open sets

$$\mathcal{W}\{T, x\} = \{U \in B(\mathcal{H}) : \|(T-U)x\| \leq 1\ \forall\ T \in B(\mathcal{H}) \text{ and } x \in\ \mathcal{H}\}.$$

Definition 2.2.2. [**13**] We say that T_α, $\alpha \in \Lambda$ (index set), a net T_α converges to T in the strong operator topology ($T_\alpha \overset{SOT}{\longrightarrow} T$) if and only if for all $x \in \mathcal{H}$, $\lim_\alpha T_\alpha x = Tx$.

Next we also explain the Von Neumann algebra. The following is taken from [**31**] and [**13**].

2.3. Von Neumann algebra

DEFINITION 2.3.1. Let M be a subset of $B(\mathcal{H})$, let the *commutant* of M be defined as

$$M^{'} = \{S \in B(\mathcal{H}) : \forall\, T \in M, \;\; ST = TS\}\,.$$

REMARK 2.3.2. If M is self-adjoint, then $M^{'}$ is a self-adjoint unital algebra. If $M^{'} = M$ then $(M^{'})^{*} = M^{'}$ and also $M^{''} := (M^{'})^{'}$, and $M^{'''} := (M^{''})^{'}$, etc. If $M^{''} \supset M$ and $M \subset T$. Then $M^{''} \subset T$.

DEFINITION 2.3.3. We say that weak operator topology is closed (WOT - *closed*) if $T_\alpha \in M^{'}$ and $T_\alpha \overset{WOT}{\longrightarrow} T$, then for every $S \in M$

$$ST = WOT_\alpha - \lim ST_\alpha = WOT_\alpha - \lim T_\alpha S = TS$$

DEFINITION 2.3.4. A $C^{*}-$ subalgebra of $B(\mathcal{H})$ which contains the identity operator and is closed in the weak operator topology is called a *von Neumann algebra.*

REMARK 2.3.5. The von Neumann algebra of G is the double commutant of $\mathbb{C}[G] \subseteq B(\mathcal{H})$.

2.4. Left and right regular representations

An important class of C^*- algebras arise in the study of groups. Let G be a discrete group, then the characteristic function $\delta_g(s)$ of $g, s \in G$ is defined as follows [**13**]:

$$\delta_g(s) = \begin{cases} 1 & \text{if } g = s, \\ 0 & \text{if } g \neq s. \end{cases}$$

If we assume that the G is a discrete group then the functions δ_g form a basis for the Hilbert space $\ell^2(G)$ of square summable functions on G.

The group ring $\mathbb{C}[G]$ consists of all finitely supported complex-valued functions on G, that is of all finite combinations

$$f = \sum_{s \in G} a_s s$$

with complex coefficients.

The convolution product and the adjoint are defined as follows:

$$\left(\sum_{s \in G} a_s s \right) \left(\sum_{t \in G} a_t t \right) = \sum_{s,t \in G} a_s a_t st$$

$$\left(\sum_{s \in G} a_s s \right)^* = \sum_{s \in G} \overline{a_s} s^{-1}.$$

Denote by $B(\ell^2(G))$ the C^*- algebra of all bounded linear operator on the Hilbert space $\ell^2(G)$. We may distinguish between the left regular representation, which is induced by the left multiplication action, and the right regular representation, which is comes from the multiplication on the right.

DEFINITION 2.4.1. [**13**] The *left regular representation*

$$\lambda : \mathbb{C}[G] \to B(\ell^2(G))$$

is defined by

$$\lambda(s)\delta_t(r) = \delta_t(s^{-1}r) = \delta_{st}(r) \text{ for } s, r \in G.$$

The *right regular representation* is given by

$$\rho(s)\delta_t(r) = \delta_t(rs) = \delta_{ts^{-1}}(r) \text{ for } s, r \in G.$$

The left regular representation is implemented using the familiar convolution formula

$$(\delta_g *_\lambda \delta_h)(s) = \sum_{t \in G} \delta_g(st^{-1}).\delta_h(t) = \delta_{gh}(s).$$

It follows that for any function $f \in \ell^2(G)$ the left action by δ_g is given by

$$(\delta_g *_\lambda f)(s) = \sum_{t \in G} \delta_g(st^{-1}).f(t) = f(g^{-1}s).$$

We can define the following right convolution:

$$(\delta_g *_\rho \delta_h)(s) = \sum_{t \in G} \delta_g(t^{-1}s).\delta_h(t) = \delta_{hg}(s),$$

which gives rise to the right regular representation:

$$(\delta_g *_\rho f)(s) = \sum_{t \in G} \delta_g(t^{-1}s).f(t) = f(sg^{-1}).$$

We note that:

$$\begin{aligned} \delta_g *_\rho \delta_h(s) &= \sum_{t \in G} \delta_g(t^{-1}s).\delta_h(t) \\ &= \sum_{t \in G} \delta_h(s((t^1)^{-1}).\delta_g(t^1) \\ &= \delta_h *_\lambda \delta_g(s), \end{aligned}$$

and hence:

$$\delta_g *_\rho \delta_h(s) = \delta_h *_\lambda \delta_g(s).$$

PROPOSITION 2.4.2. *The left and right representations commute, that is for all $s, t \in G$:*

$$\rho(s)\lambda(t) = \lambda(t)\rho(s).$$

PROOF. We have:

$$\begin{aligned}\rho(s)\lambda(t)\delta_r &= \rho(s)\delta_{tr} \\ &= \delta_{trs^{-1}} \\ &= \lambda(t)\delta_{rs^{-1}} \\ &= \lambda(t)\rho(s)\delta_r.\end{aligned}$$

Thus

$$\rho(s)\lambda(t) = \lambda(t)\rho(s).$$

□

REMARK 2.4.3. The left regular representation λ of the group ring $\mathbb{C}[G]$ assigns to each element $f \in \mathbb{C}[G]$ a bounded operator $\lambda(f)$ which acts on any $\zeta \in \ell^2(G)$ by convolution:

$$\lambda(f)(\zeta) = f * \zeta.$$

and

$$\lambda(f^*) = (\lambda(f))^*.$$

The image $\lambda(\mathbb{C}[G])$ of the group ring under the left regular representation is a $*-$ subalgebra of the algebra $B(\ell^2(G))$ of bounded operators on $\ell^2(G)$.

LEMMA 2.4.4. *The left and right regular representations λ and ρ are $*-$ homomorphisms.*

PROOF. Let $f, g \in \mathbb{C}[G]$,

$$\lambda(f)(\zeta) = f * \zeta \text{ and } \lambda(g)(\zeta) = g * \zeta.$$

Consider

$$\begin{aligned}
\lambda(f * g)(\zeta) &= (f * g) * \zeta \\
&= f * (g * \zeta) \\
&= f * (\lambda(g)\zeta) \\
&= (\lambda(f)\lambda(g)(\zeta).
\end{aligned}$$

Thus

$$\lambda(f * g) = \lambda(f)\lambda(g) \text{ for all } f, g \in \mathbb{C}[G].$$

Thus λ satisfies the product. Consider

$$\begin{aligned}
(\lambda(f) + \lambda(g)(\zeta) &= \lambda(f)(\zeta) + \lambda(g)(\zeta) \\
&= f * \zeta + g * \zeta \\
&= (f + g) * \zeta \\
&= \lambda(f + g)(\zeta)
\end{aligned}$$

Thus

$$\lambda(f + g) = \lambda(f) + \lambda(g) \text{ for all } f, g \in \mathbb{C}[G].$$

Thus λ satisfies the sum. It is easy to prove scalar multiplication, and adjoint. Therefore λ satisfies the properties of an $*-$ homomorphisms. The proof for ρ is similar. $\square$

LEMMA 2.4.5. *The left and right regular representations λ and ρ are unitary bounded representations.*

PROOF. Let us define an operator

$$\lambda_g : \ell^2(G) \longrightarrow \ell^2(G)$$

which for any function $\zeta \in \ell^2(G)$ is given by

$$\lambda_g\zeta(t) = (\delta_g * \zeta)(t) = \zeta(g^{-1}t).$$

We have

$$\begin{aligned} \langle \lambda_g\zeta, \eta \rangle &= \sum_{t\in G} \lambda_g\zeta(t)\overline{\eta(t)} \\ &= \sum_{t\in G} \zeta(g^{-1}t)\overline{\eta(t)} \\ &= \sum_{t'\in G} \zeta(t')\overline{\eta(gt')} \\ &= \langle \zeta, \lambda_{g^{-1}}\eta \rangle . \end{aligned}$$

This means that

$$\lambda_g^* = \lambda_{g^{-1}}.$$

We have for every $g \in G, \quad \zeta \in \ell^2(G)$:

$$\begin{aligned} \|\lambda_g\zeta\|^2 &= \sum_{t\in G} \left|\zeta(g^{-1}t)\right|^2 \\ &= \sum_{t\in G} |\zeta(t)|^2 \\ &= \|\zeta\|^2 \end{aligned}$$

therefore, λ_g is a unitary bounded representation. The proof for ρ is similar. □

LEMMA 2.4.6. *The left regular representation λ is a faithful representation.*

PROOF. Let us assume that, for some $f \in \mathbb{C}[G]$,

$$\lambda(f^* * f)(\delta_g)(s) = 0, \ \ \forall \ g, \ s \in G.$$

Then using the fact that λ is a $*-$ homomorphisms we have that

$$\begin{aligned} \lambda(f^* * f)(\delta_g)(s) &= \lambda(f^*).\lambda(f)\delta_g(s) \\ &= \sum_{t \in G} f^*(t)(\lambda(f)\delta_g)(t^{-1}s) \\ &= \sum_{t \in G} \sum_{t' \in G} f^*(t) f(t' \delta_g)(t')^{-1}(t^{-1}s) \\ &= \sum_{t \in G} \sum_{t' \in G} f(t^{-1}) f(t^{-1}sg^{-1}) \\ &= \|f\|_2^2 \end{aligned}$$

From this we deduce that $\|f\|_2 = 0$, and so $f = 0$, which implies that λ is faithful. ☐

The same argument can be used to show that ρ is a faithful representation as well.

2.5. The reduced group C^*- algebra

The reduced C^*- algebra $C^*_\lambda(G)$ of a group G (which we shall assume to be discrete) arises from the study of the left regular representation λ of the group ring $\mathbb{C}[G]$ on the Hilbert space of square-summable functions on the group.

Definition 2.5.1. [**13**] The *reduced group* C^*- *algebra* G, denoted by $C^*_\lambda(G)$ is the completion of $\mathbb{C}[G]$ in the norm given, for $c \in \mathbb{C}[G]$, by

$$\|c\|_\lambda = \|\lambda(c)\|$$

This means that the closure of $\mathbb{C}[G]$ for the operator norm as a subalgebra of $B(\ell^2(G))$ is called the reduced C^*- algebra $C^*_\lambda(G)$ of a group G. This is equivalently, it is the closure of $\mathbb{C}[G]$ is identified with its image under the left regular representation. i.e.

$$C^*_\lambda(G) := \overline{\lambda(\mathbb{C}[G])}.$$

The reduced C^*- algebra $C^*_\rho(G)$ of a group G (which we shall assume to be discrete) arises from the study of the right regular representation ρ of the group ring $\mathbb{C}[G]$ on the Hilbert space of square-summable functions on the group.

Definition 2.5.2. The *reduced group* C^*- *algebra* G, denoted by $C^*_\rho(G)$ is the completion of $\mathbb{C}[G]$ in the norm given, for $c \in \mathbb{C}[G]$, by

$$\|c\|_\rho = \|\rho(c)\|\,.$$

This means that the closure of $\mathbb{C}[G]$ for the operator norm as a subalgebra of $B(\ell^2(G))$ is called the reduced C^*- algebra $C^*_\rho(G)$ of a group

G. This is equivalently, it is the closure of $\mathbb{C}[G]$ is identified with its image under the right regular representation. i.e.

$$C^*_\rho(G) := \overline{\rho(\mathbb{C}[G])}$$

Next we also explain tensor product of C^*- algebra. The following is taken from [**31**] and Brown and Ozawa [**7**].

2.6. Tensor product of C^*- algebra

We now recall some basic facts about the tensor product of C^*- algebra. Let A and B be two C^*- algebras, and denote by $A \odot B$ their algebraic tensor product, which is a $*-$ algebra, such that $\|x^*x\| = \|x\|^2$ for all $x \in A \odot B$.

Definition 2.6.1. [**31**] The minimal tensor product is defined by taking $*-$ homomorphism,

$$\pi_1 \otimes \pi_2 : A \odot B \longrightarrow B(\mathcal{H}_1) \otimes B(\mathcal{H}_2) \subseteq B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

where π_1 is a representation of A in some Hilbert space $\mathcal{H}_1$ and π_2 is a representation of B in some Hilbert space $\mathcal{H}_2$. Thus we define

$$\|x\|_{\min} = \sup \|(\pi_1 \otimes \pi_2)(x)\|$$

where π_1, π_2 run over all representations of A and B respectively. The minimal tensor product is the completion $A \otimes_{\min} B$ of $A \odot B$ for this C^*- norm.

The minimal tensor product is the completion $A \otimes_{\min} B$ of $A \odot B$ for this C^*- norm.

DEFINITION 2.6.2. [**31**] The maximal C^*- norm of $x \in A \odot B$ is defined by

$$\|x\|_{\max} = \sup \|\pi(x)\|$$

where π run over all $*-$ homomorphisms from $A \odot B$ into some $B(\mathcal{H})$.

The maximal tensor product is the completion $A \otimes_{\max} B$ of $A \odot B$ for this C^*- norm.

CHAPTER 3

The Invariant Approximation Property

The uniform Roe C^*- algebra (also called uniform translation C^*- algebra) provides a link between coarse geometry and C^*- algebra theory. The uniform Roe algebra has a great importance in geometry, topology and analysis.

In Sections 3.1 and 3.2, we define what a coarse space is, and we study a number of ways of constructing a coarse structure on a set so as to make it into a coarse space. We also consider some of the elementary concepts associated with coarse spaces. A discrete group G has a natural coarse structure which allows us to define the the uniform Roe algebra, $C_u^*(G)$ [**30**]. We study in section 3.3, we recall the invariant approximation property.

In Section 3.4, we study Stone -Čech compactification and the crossed product of C^*- algebras. Our goal here is to characterise $C_u^*(G)$ as a crossed product algebra.

3.1. Coarse geometry

In this section we shall establish the basic definitions and notations for the category of coarse metric spaces. Coarse geometry is the study of the large scale properties of spaces. The notion of large scale is quantified by means of a coarse structure.

First we recall the following definitions:

DEFINITION 3.1.1. [**30**] Let $X,\ Y$ be metric spaces and $f : X \longrightarrow Y$ a not necessarily continuous map.

(1) The map f is called *coarsely proper* (or *metrically proper*), if the inverse image of a bounded set is bounded.
(2) The map f is called *coarsely uniform* (or *uniformly bornologous*), if for every $r > 0$ there is $s(r) > 0$ such that for all x_1, x_2 in X

$$d(x_1, x_2) \leq r \implies d(f(x_1), f(x_2)) \leq s(r).$$

(3) The map f is called a *coarse map*, if it is coarsely proper and coarsely uniform.
(4) Let S be a set. Two maps $f, g : S \longrightarrow X$ are called *close* if there is $C > 0$ such that for all s in S

$$d(f(s), g(s)) < C.$$

(5) A subset E of $X \times X$ is called *controlled* (or *entourage*), if the coordinate projection maps $\pi_1,\ \pi_2 : E \longrightarrow X$ are close.

DEFINITION 3.1.2. [**30**] A coarse structure on a set X is a collection of subsets of $X \times X$, called the *controlled* sets or *entourages* for the coarse structure, which contains the diagonal and is closed under the formation of subsets, inverses, products, and (finite) unions.

It is easy to see that the controlled sets associated to a metric space X have the following properties:

(1) Any subset of a controlled set is controlled;
(2) The transpose $E^t = \{(x, y) : (y, x) \in E\}$ of a controlled set E is controlled;
(3) The composition $E_1 \circ E_2$ of controlled sets E_1 and E_2 is controlled; where

$$E_1 \circ E_2 := \{(x, z) \in X \times X : \exists\, y \in X,\ (x, y) \in E_1 \,\text{and}\, (y, z) \in E_2\}\,;$$

(4) A finite union of controlled sets is controlled;
(5) The diagonal $\Delta_X := \{(x, x) : x \in X\}$ is controlled.

A set equipped with a coarse structure is called a *coarse space*. Coarse geometry is the study of metric spaces (or perhaps more general objects) from a 'large scale' point of view, so that two spaces which 'look the same from a great distance' are considered equivalent.

Definition 3.1.3. [**30**] Let X and Y be metric spaces. A (not necessarily continuous) map $f : X \longrightarrow Y$ is a *quasi-isometry equivalence* if there are constants C, A such that

$$d(x, y) \leq Cd(f(x), f(y)) + A$$

and

$$d(f(x), f(y)) \leq Cd(x, y) + A,$$

for all x and y in X.

DEFINITION 3.1.4. [**30**] Let X and Y be metric spaces. A coarse map $f : X \longrightarrow Y$ is a *coarse equivalence* if there exixst $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are coarsely uniform (or uniformly bornologous).

EXAMPLE 3.1.5. The following are some examples [**30**] of coarse spaces.

(1) The trivial coarse structure only consists of the diagonal and its subsets.
(2) The discrete coarse structure on a set X consists of the diagonal together with subsets E of $X \times X$ which contain only a finite number of points (x, y) of the diagonal.
(3) Any compact metric space is coarsely equivalent to a point. The set $\mathbb{R}$ is coarsely equivalent to $\mathbb{Z}$.
(4) The indiscrete coarse structure on a set X consists of the diagonal together with subsets E of $X \times X$.
(5) Let X be a coarse space and Y a subset of X. We can equip Y with a coarse structure declaring that the controlled subsets of $Y \times Y$ are those which are controlled when considered as subsets of $X \times X$.

DEFINITION 3.1.6. [**30**] A coarse structure on X is *connected* if each point of $X \times X$ belongs to some controlled set.

DEFINITION 3.1.7. [**30**] Let (X, d) be a metric space, we say the metric d induces a coarse structure on X, which is called a *bounded coarse structure*. More precisely, we can define the bounded coarse structure induced by the metric d as follows: Set

$$D_r := \{(x, y) \in X \times X : d(x, y) < r\}.$$

Then $E \subseteq X \times X$ is controlled, if $E \subseteq D_r$ for some $r > 0$.

EXAMPLE 3.1.8. The bounded coarse structure on a metric space (X, d) is the collection A of all subsets A of $X \times X$ such that

$$\sup\{d(x, y) : (x, y) \in A\}$$

is finite. With this structure, the integer lattice $\mathbb{Z}^n$ is coarsely equivalent to Euclidean space.

The following is an example of coarse structure.

EXAMPLE 3.1.9. [**30**] We say that a coarse structure is generated by the diagonals if it contains the diagonal and is closed under the formation of subsets, inverses, products, and (finite) unions. Let G be a finitely generated group. Then the bounded coarse structure associated to any word metric on G is generated by the diagonals

$$\Delta_g = \{(h, hg) : h \in G\}$$

as g runs over G.

DEFINITION 3.1.10. [**30**] Let X and Y be coarse spaces. A map $i : X \longrightarrow Y$ is a *coarse embedding* if it is a coarse equivalence between X and $i(X) \subseteq Y$.

We next recall some basic facts about the uniform Roe algebra and metric property of a discrete group.

3.2. The uniform Roe algebra

First we recall the following definitions; Let X be a discrete metric space.

DEFINITION 3.2.1. [**30**] We say that discrete metric space X has *bounded geometry* if for all R there exists N in $\mathbb{N}$ such that for all $x \in X$, $|B_R(x)| < N$, where

$$B_R(x) = \{y \in X : d(y, x) \leq R\}.$$

We will say that a kernel $\phi : X \times X \longrightarrow \mathbb{C}$

- is *bounded* if there, exists $M > 0$ such that

$$|\phi(s, t)| < M \text{ for all } s, t \in X$$

- has *finite propagation* if there exists $R > 0$ such that

$$\phi(s, t) = 0 \text{ if } d(s, t) > R.$$

Let $B(X)$ be a set of bounded finite propagation kernels on $X \times X$. Each such ϕ defines a bounded operator on $\ell^2(X)$ via the usual formula for matrix multiplication

$$\phi * \zeta(s) = \sum_{r \in X} \phi(s, r)\zeta(r) \text{ for } \zeta \in \ell^2(X).$$

Next, we show the operator associated with a bounded kernel is bounded.

LEMMA 3.2.2. *Let X be a bounded geometry discrete metric space. An operator associated with a bounded finite propagation kernel is bounded.*

PROOF. Let ϕ be a bounded propagation kernel on X and $\zeta \in \ell^2(X)$. Consider

$$\begin{aligned} \|\phi * \zeta\|_2^2 &= \sum_{x\in X} |\phi * \zeta(x)|^2 \\ &= \sum_{x\in X} \left| \sum_{y\in X} \phi(x,y)\zeta(y) \right|^2 . \end{aligned}$$

Given x, $\phi(x,y) \neq 0$ implies $y \in B_R(x)$, where R is the propagation of ϕ. The ball $B_R(x)$ is finite for all $x \in X$ and its size is bounded by $N \in \mathbb{N}$. Thus for every $x \in X$ the sum

$$\sum_{y\in X} \phi(x,y)\zeta(y)$$

has at most N nonzero terms and so

$$\begin{aligned} \left| \sum_{y\in X} \phi(x,y)\zeta(y) \right| &\leq \sum_{y\in X} |\phi(x,y)| \, |\zeta(y)| \\ &\leq \sum_{y\in X} M \, |\zeta(y)| \\ &\leq N_R M \, |\zeta(y)| \end{aligned}$$

where, by bounded geometry N_R is the upper bound on the number of elements in a ball $B_R(x)$. This is independent of $x \in X$, so

$$\|\phi * \zeta\|_2^2 \leq \sum_{x\in X} N_R^2 M^2 \, |\zeta(x)|^2 = N_R^2 M^2 \, \|\zeta\|_2^2$$

Therefore an operator associated with a bounded kernel is bounded. □

We shall denote the algebra of bounded operators associated with finite propagation kernels on X by $A^{\infty}(X)$.

Definition 3.2.3. The uniform Roe algebra of a metric space X is the closure of $A^{\infty}(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on X. This mean that the closure of $A^{\infty}(X)$ for the operator norm as a sub C^*- algebra of $B(\ell^2(X))$ of bounded operators on X is called the uniform Roe algebra of a metric space X.

If a discrete group G is equipped with its bounded coarse structure introduced in Example 3.1.9 then one can associated with it uniform Roe algebra $C^*_u(G)$ by repeating the above.

3.3. Invariant approximation property

A discrete group G has a natural coarse structure which allows us to define the uniform Roe algebra $C^*_u(G)$. A group G can be equipped with either the left or right-invariant the metric. Example of metric on a group G include the word metric, or a metric associated with a length function ℓ, defined in Definition 4.1.1. A choice of one of these determines whether $C^*_\lambda(G)$ or $C^*_\rho(G)$ is a subalgebra of the uniform Roe algebra $C^*_u(G)$ of G as we now explain. First we show that if the metric on G is right-invariant then

$$C^*_\lambda(G) \subset C^*_u(G).$$

Let d_1 be a right - invariant metric on G so that

$$d_1(x, y) = d_1(xg, yg) \ \forall \ g \in G.$$

For every $g \in G$, the operator $\lambda(g)$ is given by the following matrix:

$$A^\lambda_g(x, y) = \begin{cases} 1, & \text{if } x = gy, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, with this defintion we have

$$\begin{aligned} A^\lambda_g \delta_t(s) &= \sum_{y \in G} A^\lambda_g(s, y)\delta_t(y) \\ &= \delta_t(g^{-1}s) \\ &= \delta_{gt}(s) \\ &= \lambda_g(\delta_t)(s). \end{aligned}$$

Note that A^λ_g is right - invariant

$$A_g^\lambda(xt, yt) = \begin{cases} 1, & \text{if } xt = gyt \Longleftrightarrow x = gy, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore:

$$A_g^\lambda(x, y) = A_g^\lambda(xt, yt).$$

If the metric on G is right - invariant, A_g^λ is of finite propagation , because $A_g^\lambda(x, y)$ is non-zero when $g = xy^{-1}$ and so

$$d_1(x, y) = d_1(xy^{-1}, e) = d_1(g, e).$$

Hence any element of $\mathbb{C}[G]$ will give rise to a finite propagation kernel A_g^λ and this assignment extends to an inclusion

$$C_\lambda^*(G) \hookrightarrow C_u^*(G).$$

Next we show that if the metric on G is left - invariant then

$$C_\rho^*(G) \subset C_u^*(G).$$

Let d_2 be a left - invariant metric on G

$$d_2(x, y) = d_2(gx, gy) \;\; \forall \;\; g \in G.$$

For every $g \in G$, the operator $\rho(g)$ is given by the matrix.

$$A_g^\rho(x, y) = \begin{cases} 1, & \text{if } x = yg, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, with this defintion we have

$$\begin{aligned} A_g^\rho \delta_t(s) &= \sum_{y \in G} A_g^\rho(s, y)\delta_t(y) \\ &= \delta_t(sg^{-1}) \\ &= \delta_{tg}(s) \\ &= \rho_g(\delta_t)(s). \end{aligned}$$

Note that A_g^ρ is left - invariant

$$A_g^\rho(tx,ty) = \begin{cases} 1, & \text{if } tx = tyg \Longleftrightarrow x = yg. \\ 0, & \text{otherwise.} \end{cases}$$

Therefore:

$$A_g^\rho(x,y) = A_g^\rho(tx,ty).$$

If the metric on G is left - invariant, A_g^ρ is of finite propagation and $A_g^\rho \in C_u^*(G)$, because $A_g^\rho(x,y)$ is non-zero when $y^{-1}x = g$ and so

$$d_2(x,y) = d_2(y^{-1}x,e) = d_2(g,e).$$

Hence any element of $\mathbb{C}[G]$ will give rise to a finite propagation operators on $\ell^2(G)$ and this assignment extends to an inclusion

$$C_\rho^*(G) \hookrightarrow C_u^*(G).$$

Let us now choose a right invariant metric for G so that

$$C_\lambda^*(G) \hookrightarrow C_u^*(G).$$

The right regular representation ρ gives rise to the adjoint action on $C_u^*(G)$ defined by

$$Ad\rho(g)T = \rho(g)T\rho(g)^* = \rho(g)T\rho(g)^{-1}$$

for all $t \in G$, $T \in C_u^*(G)$. Our remarks above show that elements of $C_\lambda^*(G)$ are invariant with respect to this action and so $C_\lambda^*(G)$ is contained in the invariant subalgebra $C_u^*(G)^G$ of $C_u^*(G)$.

LEMMA 3.3.1. *If $T \in C_u^*(G)$, regarded as an operator on $\ell^2(G)$, arises from a kernel function $A(x,y)$, then for every $t \in G$, $Ad\rho(t)T$ is associated with the kernel function $A(xt,yt)$.*

PROOF. Using the definition of the adjoint action $Ad\rho(t)$. We compute that:

$$\begin{aligned}
(Ad\rho(t)T\zeta)(s) &= \rho(t)(T\rho(t)^*\zeta)(s) \\
&= T\left(\rho(t)^*\zeta\right)(st) \\
&= \sum_{x\in G} A(st,x)(\rho(t^{-1})\zeta)(x) \\
&= \sum_{x\in G} A(st,x)\zeta(xt^{-1}) \\
&= \sum_{y\in G} A(st,yt)\zeta(y).
\end{aligned}$$

Thus, $Ad\rho(t)T$ has kernel $A(xt,yt)$. □

In general, if $T \in C_u^*(X)$ then $\forall\ x, y \in G$:

$$\begin{aligned}
\langle Ad(\rho(t))T\delta_x, \delta_y\rangle &= \left\langle \rho(t)T\rho(t^{-1})\delta_x, \delta_y\right\rangle \\
&= \left\langle T\rho(t^{-1})\delta_x, \rho(t^{-1})\delta_y\right\rangle \\
&= \langle T\delta_{xt}, \delta_{yt}\rangle\,.
\end{aligned}$$

So the operator T is $Ad\rho-$ invariant if and only if

$$\forall\ x, y \in X\ \forall\ t \in G\ \ \langle T\delta_{xt}, \delta_{yt}\rangle = \langle T\delta_x, \delta_y\rangle\,.$$

We now define the invariant approximation property (IAP).

DEFINITION 3.3.2. [**30**] Let discrete group G equipped with a left - invariant metric. We say that G has the *invariant approximation property* (IAP) if and only if

$$C_u^*(G)^G = C_\lambda^*(G).$$

3.4. The uniform Roe algebra as crossed product

In this section, we will characterize $C_u^*(G)$ as a crossed product algebra.

We recall that the Stone - Čech compactification of a set X is a compact Hausdorff space βX, equipped with an inclusion of the discrete space X as an open dense subset and the following universal property: Every continuous function $f : X \longrightarrow Z$ extends uniquely to continuous function $\widetilde{f} : \beta X \longrightarrow Z$, where Z is a compact Hausdorff space. In particular, every bounded complex-valued function on X extends uniquely to a continuous function on βX.

The following is taken from Takesaki [**31**] and Brown and Ozawa [**7**].

Let G be a discrete group. Let $\alpha : G \curvearrowright H$ be an action of G on a C^*- algebra A: α is a homomorphism from group G into the group $Aut(A)$ of automorphisms of A. This means that for each $g \in G$ there is an automorphisms $\alpha(g)$ of A given by:

$$\alpha(g_1)\alpha(g_2) = \alpha(g_1 g_2).$$

The algebraic crossed product of A by G is the $*-$ algebra generated by A together with a unitary u_t corresponding to each $t \in G$, with the relation that

$$u_t a = \alpha(t)(a) u_t,$$

and

$$u_{t_1} u_{t_2} = u_{t_1 t_2}.$$

Any element of the algebraic crossed product of A by G is the formal sum

$$a = \sum a_t u_t,$$

where $a_t \in A$ and $t \in G$. We denote by $A[G]$ the $*-$ algebra of formal sums

$$a = \sum a_t u_t,$$

where

$$t \longmapsto a_t$$

is a map from G into A with finite support and where the operations are given by the following rules:

$$(a_t)(b_s) = a\alpha_t(b)ts,$$

$$(a_t)^* = \alpha_{t^{-1}}(a)t^{-1},$$

for $a, b \in A$ and $s, t \in G$.

DEFINITION 3.4.1. [**31**], [**7**] A *covariant representation* of $\alpha : G \curvearrowright A$, is a pair (π, ρ) where π and ρ are unitary representations of G and of A in the same Hilbert space $\mathcal{H}$ respectively, satisfying the covariance rule

$$\forall\ a\ \in\ G\ \forall\ t\ \in\ G,\ \ \rho(t)\pi(a)\rho(t)^* = \pi(\alpha_t(a)),$$

where $\pi : A \longrightarrow U(\ell^2(G))$ and $\rho : G \longrightarrow B(\ell^2(G))$, and $U(\ell^2(G))$ is the set of unitary bounded operators.

A covariant representation gives rise to a $*-$ homomorphism

$$\pi \times \rho : A[G] \longrightarrow B(\mathcal{H})$$

by

$$(\pi \times \rho)\left(\sum_{t\in G} a_t t\right) = \sum_{t\in G} \pi(a_t)\rho(t).$$

DEFINITION 3.4.2. [**31**], [**7**] The *full crossed product* of $A \rtimes G$ associated with $\alpha : G \curvearrowright A$ is the $*-$ algebra obtained as the completion of $A[G]$ in the norm

$$\|a\| = \sup_{\pi,\rho} \|(\pi \times \rho)(a)\|,$$

where (π, ρ) runs over all covariant representation of $\alpha : G \curvearrowright A$.

REMARK 3.4.3. [**31**], [**7**] By definition, every covariant representation (π, ρ) extends to a representation of $A \rtimes G$, denoted by $\pi \times \rho$.

DEFINITION 3.4.4. [**31**], [**7**] Let G be a discrete group. Let π be a representation of A on a Hilbert space $\mathcal{H}_0$ and

$$\mathcal{H} = \ell^2(G, \mathcal{H}_0) = \ell^2(G) \otimes \mathcal{H}_0.$$

We define a covariant representation $(\overline{\pi}, \overline{\lambda})$ of $\alpha : G \curvearrowright A$, acting on $\mathcal{H}$ by

$$\overline{\pi}(a)\zeta(t) = \pi(\alpha_{t^{-1}}(a))\zeta(t)$$

and

$$\overline{\lambda}(s)\xi(t) = \zeta(s^{-1}t)$$

for all $a \in A$, all $s, t \in G$ and all $\zeta \in \ell^2(G, \mathcal{H}_0)$. The covariant representation $(\overline{\pi}, \overline{\lambda})$ is said to be *induced* by π.

A induced covariant representation gives rise to a $*-$ homomorphism

$$\overline{\pi} \times \overline{\rho} : A[G] \longrightarrow B(\mathcal{H})$$

by

$$(\overline{\pi} \times \overline{\rho}) \left(\sum_{t \in G} a_t t \right) = \sum_{t \in G} \pi(\alpha_{t^{-1}}(a))\rho(t).$$

DEFINITION 3.4.5. [**31**], [**7**] The *reduced crossed product* of $A \rtimes_r G$ is the $*-$ algebra obtained as the completion of $A[G]$ in the norm

$$\|a\|_r = \sup \left\|(\overline{\pi} \times \overline{\lambda})(a)\right\|$$

for $a \in A[G]$, where π is a representation of A.

First we shall describe the following isomorphism:

$$C_u^*(G) \cong C(\beta G) \rtimes_r G \cong \ell^\infty(G) \rtimes_r G.$$

Any element of $f \in C(\beta G)[G] \subset C(\beta G) \rtimes_r G$, defined by

$$f = \sum_{t \in G} f_t t$$

where $f_t \in C(\beta G)$, $t \in G$, and only finitely many f_t are non-zero. We define a map

$$C(\beta G)[G] \longrightarrow C_c(\beta G \times G),$$

where $C_c(\beta G \times G)$ is the algebra of contionous compactly supported functions on $\beta G \times G$. For $f \in C(\beta G)[G]$, we put

$$\theta(f)(x,t) = f_t(x) \ \text{ for every } t \ \in \ G, \ \ x \ \in \ \beta G.$$

This map turns out to be an isomorphism:

$$C(\beta G)[G] \cong C_c(\beta G \times G).$$

The convolution product and the adjoint on the $*-$ algebra $C_c(\beta G \times G)$ are given by the following: Let $F, G \in C_c(\beta G \times G)$, then we define

$$(F * G)(x,s) = \sum_t F(x,t) G(t^{-1}x, t^{-1}s)$$

and

$$F^*(x,s) = \overline{F}(s^{-1}x, s^{-1}).$$

Let

$$J : G \times G \longrightarrow G \times G$$

be an involution defined by:

$$J : (s,t) \longrightarrow (s^{-1}, s^{-1}t).$$

The map

$$F \longmapsto F \circ J$$

establishes an isomorphism between $C_c(\beta G \times G)$ and $A^\infty(G)$.

The uniform Roe algebra $C_u^*(G)$ is the norm closure in $B(\ell^2(G))$ of the $*-$ subalgebra formed by the operators $Op(k)$, where k ranges over the bounded kernel with finite propagation, and $Op(k)$ is the bounded operator associated with k.

The following Theorem is from Roe [**30**] and Brown and Ozawa [**7**].

THEOREM 3.4.6. *The map* $\Pi : f \longrightarrow Op(\theta(f) \circ J)$ *extends to an isomorphism between the* C^*- *algebras* $C_u^*(G)$ *and* $C(\beta G) \rtimes_r G$ *and*

$$C_u^*(G) \cong C(\beta G) \rtimes_r G \cong \ell^\infty(G) \rtimes_r G.$$

PROOF. To define $C(\beta G) \rtimes_r G$ we use representation π of $C(\beta G)$ in $\ell^2(G)$ given by

$$\pi(f)\xi(t) = f(t)\xi(t) \ \text{ for } \ f \in C(\beta G) \text{ and } \xi \in \ell^2(G).$$

Therefore $C(\beta G) \rtimes_r G$ is concretely represented on $B(\ell^2(G \times G))$. And also $f = \sum_{s \in G} f_s \in C(\beta G)[G]$ acts on $\ell^2(G \times G)$ by

$$(f.\xi)(x,t) = \sum_{s \in G} f_s(tx)\xi(x, s^{-1}t).$$

Let V be the unitary operator on $\ell^2(G \times G)$ defined by

$$V\xi(x,t) = \xi(x, t^{-1}x^{-1}).$$

We have

$$\begin{aligned} Vf\xi(x,t) &= V\sum_{s\in G} f_s(tx)\xi(x,s^{-1}t^{-1}) \\ &= \sum_{s\in G} f_s(tx)V\xi(x,s^{-1}t^{-1}) \\ &= \sum_{s\in G} f_s(tx)\xi(x,(s^{-1}t^{-1})^{-1}x^{-1}) \end{aligned}$$

It follows that

$$VfV^* = Id_{\ell^2(G)} \otimes Op(\theta(f)\circ J)$$

for every $f \in C(\beta G)[G] \subset C(\beta G) \rtimes_r G$. This means that Π is an isometry and so extends to a continous isomorphism.

$$C^*_u(G) \cong C(\beta G) \rtimes_r G.$$

It follows from the definition of βG that

$$C(\beta G) \cong \ell^\infty(G).$$

and the Theorem is proved. □

CHAPTER 4

Property RD and Invariant Approximation Property

In this chapter we discuss the rapid decay property (Property RD) for discrete group. This property was first considered by P.Jolissaint [**21**] and has generalised the work done by Haagerup on estimates of the regular representation for the free group [**18**].

Jolissaint proved in his thesis that groups of polynomial growth and classical hyperbolic groups have property RD, and the only amenable discrete groups that have property RD are groups of polynomial growth. He also showed that many groups, for instance $SL_3(\mathbb{Z})$, do not have the property RD [**21**].

De la Harpe improved Jolissaint's results and showed that the word hyperbolic groups of Gromov [**16**] have property RD as well, and this leads to the result of Connes and Moscovici that word hyperbolic groups satisfy the Novikov conjecture. Since then, many important works have been done on establishing the property RD, notably the works of Lafforgue [**25**], Chatterji [**8, 9, 10, 11, 12**] and Ruane, and Drutu and Sapir. Examples of RD groups include groups acting on CAT(0)-cube complexes [**12**].

In this chapter we study the relation between property RD and the invariant approximation property. Our main result (Theorem 4.3.3) states that groups which satisfy property RD with respect to a conditionally negative length function have the IAP.

4.1. Property RD and length functions

We begin with a description of property RD. Our discussion is based on Jolissaint's paper [**21**].

Definition 4.1.1. Let G be a discrete group. A *length function* on G is a map $\ell : G \longrightarrow \mathbb{R}$ taking values in the non-negative reals which satisfies the following conditions:

(1) $\ell(1) = 0$ where 1 is the identity element of the group;
(2) For every $g \in G$, $\ell(g) = \ell(g^{-1})$;
(3) For every $g, h \in G$, $\ell(gh) \leq \ell(g) + \ell(h)$.

A group equipped with a length function becomes a metric space with the left - invariant metric

$$d(\gamma, \mu) = \ell(\gamma^{-1}\mu).$$

Example 4.1.2. Let G be a discrete group with a finite generating set S. For convenience we will assume that S is symmetric, i.e. $S^{-1} = S$. For any $g \in G$, define

$$|g|_S = \min\{k : g = s_1 \dots s_k, s_i \in S\}.$$

This is the algebraic word length function of G induced by the generating set S.

Example 4.1.3. Consider $\mathbb{Z}^2$ with the symmetric generating set

$$S = \{(1,0), (0,1), (0,-1), (-1,0)\}.$$

For $(m, n) \in \mathbb{Z}^2$, we have the word length function

$$|(m, n)|_S = |m| + |n|,$$

where $|m|$ and $|n|$ are the absolute values of m and n respectively.

Let G be a countable, discrete group with symmetric finite generating sets S and S', yielding word-length functions $|.|_S$ and $|.|_{S'}$ respectively. As the generating sets are different, these length functions, and the metric functions they induce, are different.

EXAMPLE 4.1.4. Let X be a metric space with base point $x_0 \in X$ and let G be the group of isometries on X. For every $g \in G$, let

$$L_{x_0}(g) = d(x_0, g(x_0)).$$

Then L_{x_0} is a length function on G.

DEFINITION 4.1.5. Let ℓ be a length function on G. We define a *Sobolev* norm on the group ring of G as follows:

(1) For any length function ℓ and positive real number$s$, we define a Sobolev norm on the group ring $\mathbb{C}[G]$ by:

$$\|f\|_{\ell,s} = \sqrt{\sum_{\gamma\in G} |f(\gamma)|^2 (1 + \ell(\gamma))^{2s}}.$$

(2) If $s \in \mathbb{R}$, the Sobolev space of order s is the set $H^s_\ell(G)$ of functions ξ on G such that $\xi(1 + \ell)^s$ belongs to $\ell^2(G)$.

DEFINITION 4.1.6. Let $H < G$ be a subgroup of G and ℓ a length function on G. The restriction of ℓ to H induces a length function on H that we call the *induced length function.*

DEFINITION 4.1.7. If ℓ_1 and ℓ_2 are length functions on G, we say that ℓ_2 dominates ℓ_1 if there exist $a, b \in \mathbb{R}$ such that $\ell_1 \leq a\ell_2 + b$. If ℓ_1 dominates ℓ_2 and ℓ_2 dominates ℓ_1, then ℓ_1 and ℓ_2 are said to be equivalent.

LEMMA 4.1.8. *If ℓ_1 and ℓ_2 are equivalent then $\|f\|_{\ell_1,s}$ and $\|f\|_{\ell_2,s}$ are equivalent.*

PROOF. Since

$$\ell_1 \leq a\ell_2 + b,$$

we have

$$\begin{aligned} 1 + \ell_1 &\leq 1 + a\ell_2 + b \\ &\leq 1 + b + a(1+b)\,\ell_2 \\ &\leq c(1+b)\,(1+\ell_2) \end{aligned}$$

where $c = \max\{1, a\}$. Thus

$$\begin{aligned} \|f\|_{\ell_1,s} &= \left(\sum |f(x)|^2 \{1 + l_1(x)\}^{2s}\right)^{\frac{1}{2}} \\ &\leq \left(\sum |f(x)|^2 \left(c(1+b)\,(1+l_2(x))\right)^{2s}\right)^{\frac{1}{2}} \\ &\leq B^s \|f\|_{\ell_2,s} \end{aligned}$$

where $B^s = \{c(1+b)\}^s$.

Similarly

$$\|f\|_{\ell_2,s} \leq C\,\|f\|_{\ell_1,s}\,,$$

where C is a constant. Therefore $\|f\|_{\ell_2,s}$ and $\|f\|_{\ell_1,s}$ are equivalent. □

We are now ready to define property RD. The following definition is due to Jolissaint [**21**] (see also [**12**]).

DEFINITION 4.1.9. [**21**] Let ℓ be a length function on a discrete group G. We say that G has the *Rapid Decay property* (property (RD)) with respect to the length function ℓ if there exist $C \geq 0$ and $s > 0$ such that, for all $f \in \mathbb{C}[G]$,

$$\|f\|_* \leq C \|f\|_{\ell,s} ,$$

where $\|f\|_*$ denotes the operator norm of f acting by left convolution on $\ell^2(G)$.

We denote by $\mathbb{R}_+G$ the subset of $\mathbb{C}[G]$ consisting of functions with target in $\mathbb{R}_+$. The following proposition of Chatterji and Ruane [**12**] provides equivalent criteria for property RD and summaries various characterizations of property RD.

PROPOSITION 4.1.10. [**12**] *Let G be a discrete group endowed with a length function ℓ. Then the following are equivalent:*

(1) *The group G has property RD with respect to ℓ;*
(2) *There exists a polynomial P such that for any $r > 0$ and any $f \in \mathbb{R}_+G$ so that f vanishes on elements of length greater than r, we have*

$$\|f\|_* \leq P(r) \|f\|_2 ;$$

(3) *There exists a polynomial P such that, for any $r > 0$ and any two functions $f, g \in \mathbb{R}_+G$ so that f vanishes on elements of length greater than r, we have*

$$\|f * g\|_* \leq P(r) \|f\|_2 \|g\|_2 ;$$

(4) *There exists a polynomial P such that, for any $r > 0$ and any three functions $f, g, h \in \mathbb{R}_+G$ so that f vanishes on elements*

of length greater than r, we have

$$f * g * h(e) \leq P(r) \|f\|_2 \|g\|_2 \|h\|_2 ;$$

(5) *The space of rapidly decaying functions $H_\ell^\infty(G)$ is contained in the reduced C^*- algebra $C_\lambda^*(G)$;*

(6) *Any subgroup H in G has property RD with respect to the induced length function;*

PROOF. We sketch the proof given of the equivalence (1)$\Longleftrightarrow$ (2) by Chatterji and Ruane. We will be using the equivalence (1)$\Longleftrightarrow$ (2) later(section 4.3). We will If $f \in \mathbb{C}[G]$ is a function whose support is in $B(e,r)$ then

$$\begin{aligned} \|f\|_* &\leq C \|f\|_{l,s} \\ &= C\left(\sqrt{\sum_{\gamma \in B(e,r)} |f(\gamma)|^2 (1+l(\gamma))^{2s}}\right) \\ &\leq C\left(\sqrt{\sum_{\gamma \in B(e,r)} |f(\gamma)|^2 (1+r)^{2s}}\right) \\ &= C(1+r)^s \|f\|_2 . \end{aligned}$$

This is condition (2) where the polynomial is $P(r) = C(1+r)^s$.

To prove the converse, for $n \in \mathbb{N}$, let

$$S_n = \{\gamma \in G; n \leq l(\gamma)) < n+1\} ,$$

and denote by f_n the restriction of f to set S_n. We have

$$\begin{aligned}\|f\|_* &= \left\|\sum_{n=0}^{\infty} f_n\right\|_* \\ &\leq \sum_{n=0}^{\infty} \|f_n\|_* \\ &\leq \sum_{n=0}^{\infty} P(n+1)\,\|f_n\|_2\,,\end{aligned}$$

if we now replace the polynomial $P(n+1)$ by its highest order term we have the estimate

$$\|f\|_* \leq \sum_{n=0}^{\infty} C(n+1)^k\,\|f_n\|_2$$

for some constant C.

Dividing and multiplying by $(n+1)$ we can use the Cauchy -Schwartz inequality to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} C(n+1)^k\,\|f_n\|_2 &\leq \sum_{n=0}^{\infty} C(n+1)^{-1}(n+1)^{k+1}\,\|f_n\|_2 \\ &\leq C\left(\sum_{n=0}^{\infty}(n+1)^{-2}\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}(n+1)^{2k+2}\,\|f_n\|_2^2\right)^{\frac{1}{2}} \\ &= C\frac{\pi}{\sqrt{6}}\,\|f\|_{l,k+1}\,,\end{aligned}$$

where

$$\frac{\pi}{\sqrt{6}} = \left(\sum_{n=0}^{\infty}(n+1)^{-2}\right)^{\frac{1}{2}}.$$

□

Definition 4.1.11. [**21**] We say that a discrete group G has *polynomial growth* with respect to a length function ℓ if there exists a polynomial P such that the cardinality of the ball of radius r (denoted by $|B(e,r)|$) is bounded by $P(r)$.

EXAMPLE 4.1.12. [**21**] Let G be a discrete group endowed with a length function ℓ with respect to which G is of polynomial growth. Then G has property RD with respect to ℓ. Indeed, take $f \in \mathbb{C}[G]$ such that

$$\operatorname{supp}(f) = S_f \subseteq B(e,r),$$

then

$$\begin{aligned} \|f\|_* &\leq \|f\|_1 \\ &= \sum_{\gamma \in G} |f(\gamma)| \\ &= \sum_{\gamma \in S_f} |f(\gamma)| \\ &\leq \sqrt{|S_f|}\sqrt{\sum_{\gamma \in S_f} |f(\gamma)|^2} \\ &= \sqrt{|S_f|}\, \|f\|_2 \,, \end{aligned}$$

the last inequality being just the Cauchy-Schwartz inequality. If G is of polynomial growth, then

$$|S_f| \leq |B(e,r)| \leq P(r)$$

and thus

$$\|f\|_* \leq \sqrt{P(r)}\, \|f\|_2 \,.$$

We note that the following important result from [**21**].

THEOREM 4.1.13. *Let G be a discrete amenable group. Then G has property (RD) with respect to a length function ℓ if and only if G is of polynomial growth with respect to ℓ.*

4.2. Positive and negative type kernels

Let us briefly recall basic definitions and facts concerning positive and negative type kernels and functions.

DEFINITION 4.2.1. Let X be a set. A symmetric kernel on X is a function $f : X \times X \longrightarrow \mathbb{R}$ with $f(x, y) = f(y, x)$.

DEFINITION 4.2.2. [**30**] A kernel f has *conditionally positive* type if for all $m \in \mathbb{N}$, all $m-$tuples $x_1, x_2, \ldots, x_m$ of points of X and for all real scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$, one has

$$\sum_{i,j=1}^{m} \lambda_i \lambda_j f(x_i, x_j) \geq 0.$$

DEFINITION 4.2.3. [**30**] A kernel f has *conditionally negative* type if for all $m \in \mathbb{N}$, all $m-$tuples $x_1, x_2, \ldots, x_m$ of points of X, and for all real scalars $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\sum \lambda_i = 0$, one has

$$\sum_{i,j} \lambda_i \lambda_j f(x_i, x_j) \leq 0.$$

The following example is the connection between maps into Hilbert spaces and positive and negative type kernels.

EXAMPLE 4.2.4. [**30**]

(1) A constant function on $X \times X$ has conditionally negative type. A kernel of the form

$$f(x, y) = g(x)g(y),$$

where g is any real-valued function, has positive type.

(2) Suppose that $\mathcal{H}$ is a (real) Hilbert space. Then the kernel

$$f(x,y) = \langle x, y \rangle \,;\ \ x,\ \ y\ \ in\ \ \mathcal{H},$$

has positive type, and the kernel

$$f(x,y) = \|x - y\|^2 \,;\ \ x,\ \ y\ \ in\ \ \mathcal{H}$$

is of negative type.

The following result in [**30**], which relates positive and negative type kernels, is known as Schoenberg's Lemma.

LEMMA 4.2.5. [**30**] *Let f be a symmetric kernel on a space X. The following statements are equivalent.*

(1) *The kernel f is of negative type.*
(2) *For each $t > 0$ the kernel $exp(-tf)$ is of positive type.*

REMARK 4.2.6. [**30**] Let G be a group. A function of positive type on G is a function $\phi : G \longrightarrow \mathbb{R}$, $(x,y) \longmapsto \phi(x^{-1}y)$, is a kernel of positive type.

We recall some definitions:

DEFINITION 4.2.7. [**30**] We say that a kernel $f(x,y)$ on a coarse space X is *effective* if the sets

$$\{(x,y) :\ f(x,y) < R\}\,,\ \text{for}\ R > 0,$$

generate the coarse structure on X.

Let $C_E(X \times X)$ denote the algebra of bounded functions f on $X \times X$. which have the property that for each $\epsilon > 0$ the set

$$\{(x,y) \in X \times X :\ |f(x,y)|\ < \epsilon\}$$

is controlled. We assume that X is a uniformly discrete and of bounded geometry. It can be seen that $C_E(X\times X)$ is isomorphic to $C_0(G)$, where $C_0(G)$ is the algebra of functions vanishing at ∞.

These notations are brought together by the following result of Roe [**30**].

THEOREM 4.2.8. [**30**] *Let X be a coarse space. The following are equivalent:*

(1) *X can be coarsely embedded into a Hilbert space.*
(2) *There is an effective negative type kernel on X.*
(3) *The algebra $C_E(X \times X)$ has an approximate unit consisting of a sequence $\{u_n\}$ of normalized positive kernels.*

The normalized positive type kernels on X acting on $B(\ell^2(X))$ by Schur multiplication.

DEFINITION 4.2.9. [**30**] We say that f is a *normalized positive kernel* if $f(x, y) = 1$, for all $x, y \in X$.

LEMMA 4.2.10. [**30**] *Let f be a normalized positive type kernel on a set X. Then there is a unique unital completely positive map*

$$M_f : B(\ell^2(X)) \longrightarrow B(\ell^2(X))$$

such that

$$\langle (M_f T)\, \delta_x, \delta_y \rangle = f(x, y)\, \langle T\delta_x, \delta_y \rangle\,,$$

for all $T \in B(\mathcal{H})$.

4.3. RD and Invariant approximation property

In this section we show that the invariant approximation property for groups equipped with conditionally negative length function. The following proposition due to Roe shows a sufficient condition to invariant approximation property. We shall use this to prove the main Theorem of this chapter.

PROPOSITION 4.3.1. [**30**] *Suppose that there is an approximate unit for $C_0(G)$ comprised of a sequence of functions ϕ_n, such that*

(1) *each ϕ_n is of positive type and normalized,*
(2) *the operator M_{ϕ_n} of Schur multiplication by ϕ_n maps $\mathbb{L}(G)$ into $C^*_\lambda(G)$.*

Then G has the invariant approximation property.

We will need the following convergence result. Which is important proposition, which is used for the main result (see Theorem 4.3.3) of this Chapter.

PROPOSITION 4.3.2. [**30**] *Let G be a discrete group satisfying the rapid decay property with respect to a length function ℓ, for $s > 0$. Let $f \in H^s_l(G)$ be given by*

$$f = \sum_{g \in G} f_g \lambda(g), \quad f_g \in \mathbb{C}[G].$$

Then the series $\sum_{g\in G} f_g\lambda(g)$ converges in norm to an element of $C^*_\lambda(G)$.

PROOF. Let $\{F_n\}$ be a family of finite subsets of G such that

$$F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq F_n \cdots$$

and

$$G = \bigcup_{n\in\mathbb{N}} F_n.$$

Then for every $n \in \mathbb{N}$ we have by the rapid decay property

$$\begin{aligned} \|\lambda(f|_{F_n})\|_* &\leq C\left\{\sum_{g\in F_n} |f_g|^2 (1+\ell(g))^{2s}\right\}^{\frac{1}{2}} \\ &\leq C\,\|f\|_{l,s}. \end{aligned}$$

Letting n tend to infinity we see that the series on the left converges to an element $C^*_\lambda(G)$. □

We will use this Proposition 4.3.2 as follows. Let T in $\mathbb{L}(G)$ be represented by a series

$$\sum_{g\in G} b_g\lambda(g).$$

Then be complex coefficient b_g form a square-summable sequence. To see this, let $\{F_n\}$ be a family of finite subsets of G as in the previous Proposition 4.3.2. For every n, and $s \in G$.

$$\begin{aligned} \left\langle \sum_{g\in F_n} b_g\lambda(g)\delta_s, \sum_{g'\in F_n} b_{g'}\lambda(g')\delta_s \right\rangle &= \sum_{g\in F_n}\sum_{g'\in F_n} b_g\overline{b_{g'}}\left\langle \lambda(g)\delta_s, \lambda(g')\delta_s\right\rangle \\ &= \sum_{g\in F_n}\sum_{g'\in F_n} b_g\overline{b_{g'}}\left\langle \delta_{gs}, \delta_{g's}\right\rangle \\ &= \sum_{g\in F_n} |b_g|^2. \end{aligned}$$

But for every n,

$$\left\|\sum_{g\in F_n} b_g\lambda(g)\right\| \leq \|T\|$$

and so by letting n go to infinity we see that the sequence b_g is square summable.

We now prove the main result of this Chapter.

THEOREM 4.3.3. *Let G be a discrete group satisfying the rapid decay property with respect to a conditionally negative length function ℓ. Then the group G has the invariant approximation property.*

PROOF. By Schoenberg's Lemma (Lemma 4.2.5), for every n the function

$$\phi_n(\gamma) = exp(-\ell(\gamma)/n)$$

is of positive type and family $\{\phi_n\}$ forms an approximate unite for $C_0(G)$. The function ϕ_n are normalizrd as $\phi_n(e) = 1$ for all n. The Theorem 4.3.3 will be proved if we show that for every n, the map

$$T \longmapsto M_{\phi_n} T$$

sends

$$\mathbb{L}(G) \longrightarrow C^*_\lambda(G).$$

For this we use the rapid decay property and the method of Proposition 4.3.2.

If T is represented by the series

$$T = \sum_{g \in G} b_g \lambda(g)$$

then $M_{\phi_n} T$ is given by

$$M_{\phi_n} T = \sum_{g \in G} \phi_n b_g \lambda(g).$$

Then by the rapid decay property, for every $k \in \mathbb{N}$

$$\begin{aligned}
\left\|\sum_{g\in F_k} \phi_n(g)b_g\lambda(g)\right\| &= C\left(\sum_{g\in F_k} |\phi_n(g)b_g|^2 (1+\ell(g))^{2s}\right)^{\frac{1}{2}} \\
&= C\sup_{g\in G}\{|\phi_n(g)|(1+\ell(g))^s\}\left(\sum_{g\in F_k}|b_g|^2\right)^{\frac{1}{2}} \\
&= CK\sum_{g\in G}|b_g|^2 < \infty,
\end{aligned}$$

where $K = \sup_{g\in G}\{|\phi_n(g)|(1+\ell(g))^s\}$.

Then, letting $k \longrightarrow \infty$ we see that

$$\sum_{g\in F_k} \phi_n(g)b_g\lambda(g)$$

converges in norm to an element of $C^*_\lambda(G)$, and on the other hand it convegers to $M_{\phi_n}T$, proving the result. □

We now use this to show the following examples: First, we first recall the definitions of hyperbolicity for metric space.

Definition 4.3.4. A metric space (X, d) is said to be *hyperbolic* if there is a constant $\delta \geq 0$ such that for any points $w, x, y, z \in X$ we have that:

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + \delta.$$

Jolissaint showed that classical hyperbolic groups have property RD [**21**]. Faraut and Harzallah showed that the natural metrics on these hyperbolic spaces are conditionally negative and they give rise to conditionally negative length function on these group [**5**]. Hence we obtain:

Corollary 4.3.5. *Let G be a classical hyperbolic group. Then the group G has the invariant approximation property.*

We note that Ozawa has a more general result for hyperbolic groups [**29**].

THEOREM 4.3.6. [**29**] *Hyperbolic groups have the invariant approximation property.*

Let G be a CAT(0) cubical group, which means G acts properly and cocompactly on a CAT(0) cube complex [**28**]. Now according to Niblo and Reeves [**28**] given a group acting on a CAT(0) cube complex, they obtain a conditionally negative length kernel on the group which gives rise to a conditionally negative length function. Chatterji and Ruane [**12**] proved that CAT(0) cube complexes have property RD with respect to this length function provided that the action is properly discontinuous, stabilizers are uniformly bounded and the cube complexes have finite dimension. We deduce that:

COROLLARY 4.3.7. *CAT(0) cubical groups have the invariant approximation property.*

Recall the following:

DEFINITION 4.3.8. [**20**] A *Coxeter* group is a discrete group G given by the presentation with a finite set of generators

$$W = \{w_i, \dots, w_n\}$$

and a finite set of relations defined as follows:

$$w_i^2 = 1 = (w_i w_j)^{m_{i,j}}, \text{ where } m_{i,j} \text{ is either } \infty \text{ or an integer } \geq 2.$$

Chatterji proved that Coxeter groups have property RD [**10**]. Jolissaint showed that finitely generated Coxter groups have conditionally negative length function [**22**]. Hence we have

Example 4.3.9. Let G be a finitely generated Coxeter group. Then the group G has the invariant approximation property.

We recall the definition of torus knot groups:

Definition 4.3.10. [**22**][**21**] We define the *torus knot* groups by, for p and q positive integers such that $(p, q) = 1$,

$$G = \{x, y \mid x^p = y^q\},$$

Jolissaint showed that torus knot groups have conditionally negative length function [**22**] and proved that torus knot groups have property RD [**21**]. This gives us another example of invariant approximation property.

Corollary 4.3.11. *Let G be a torus knot group. Then the group G has the invariant approximation property.*

First we recall the definition of the free product $G_1 * G_2$ of two groups, G_1 and G_2.

Definition 4.3.12. [**22**] We say that the free product of $G_1 * G_2$ of two groups G_1 and G_2 is the set consisting of the empty word (denoted by e) together with all reduced words $w = a_1, a_2 \ldots a_n$, where the a_j's are elements of either G_1 or G_2 different from the identity and satisfy the condition:

$$a_j \in G_i, \text{ implies } a_{j+1} \in G_{3-i} \ (1 \leq j \leq n-1, \ i = 1, 2).$$

The following Theorem can be found in [**21**].

THEOREM 4.3.13. *If G_1 and G_2 have property (RD) then so does their free product $G = G_1 * G_2$.*

EXAMPLE 4.3.14. Let G be a free product two groups G_1 and G_2, which satisfying the rapid decay property with respect to a conditionally negative length function ℓ. By using Theorem 4.3.13, $G = G_1 * G_2$ have property RD. Jolissaint showed that, if G_1 and G_2 have conditionally negative length function then their free product $G_1 * G_2$ also has conditionally negative length function [**22**]. By using Theorem 4.3.3. We deduce that the G has the invariant approximation property.

CHAPTER 5

Strong Invariant Approximation Property

In this Chapter we will study the strong invariant approximation property in various contexts. In particular, we investigate its links to the completely bounded approximation property (CBAP), the strong invariant approximation property (SIAP), the approximation property (AP), the operator space approximation property (OAP), and exactness. The reader is referred to the book by Brown and Ozawa [**7**] for a beautiful exposition of these concepts. In this Chapter we describe and study the strong invariant approximation property for stability results.

In section 5.3, we show the following implications for discrete groups (see Proposition 5.2.11):

$$CBAP \Rightarrow AP \Rightarrow IAP.$$

Our interest in these properties comes from a link to the strong invariant approximation property (SIAP) of Zacharias, which implies the IAP (see Proposition 5.3.3). We shall use results of Haagerup and Kraus [**19**] on the AP to investigate some permanence properties of the IAP and the SIAP for discrete groups. This can be done most efficiently for exact groups.

5.1. Approximation properties

We begin with an outline of results of Haagerup and Kraus [**19**].

Definition 5.1.1. [**5**] A C^*- algebra A is *nuclear* if and only if it has the following *completely positive approximation property* (CPAP): The identity map on A can be approximated in the point norm topology by finite rank completely positive contractions. This means that there exist nets of operators $T_\alpha : A \longrightarrow M_{n_\alpha}(\mathbb{C})$ and $S_\alpha : M_{n_\alpha}(\mathbb{C}) \longrightarrow A$ such that for all $a \in A$

$$\lim_\alpha \|S_\alpha T_\alpha(a) - a\| = 0.$$

A C^*- algebra A has the metric approximation property (MAP) of Grothendieck if and only if the identity map on A can be approximated in the point-norm topology by a net of finite rank contractions.

Comparing the definitions we see that CPAP implies MAP (see for example [**5**]). Lance [**27**] has shown that G is a discrete group amenable if and only if its reduced C^*- algebra A has the CPAP which is equivalent to $C^*_r(G)$ being nuclear. Completely positive maps are in particular completely bounded, which suggest the following weakening of the CPAP.

Definition 5.1.2. [**5**] A C^*-algebra A is said to have the *completely bounded approximation property* (CBAP) if there is a positive number C such that the identity map on A can be approximated in the point norm topology by a net $\{\phi_\alpha\}$ of finite rank completely bounded maps whose completely bounded norms are bounded by C. This means that there exists a net of finite-rank maps $\{\phi_\alpha\} : A \longrightarrow A$ such that

$\|\phi_\alpha\|_{cb} \leq C$ for some constant C and $\phi_\alpha \longrightarrow id_A$ in the point-norm topology on A (i.e. $\|\phi_\alpha(x) - x\| \longrightarrow 0$ for all $x \in A$).

The infimum of all values of C for which such constants exist is denoted by $\Lambda_{cb}(A)$ and is called the Cowling - Haagerup constant. We set $\Lambda_{cb}(G) = \infty$ if the discrete group G does not have the CBAP. Obviously, a nuclear C^*- algebra has the metric approximation property. On the other hand, Haagerup [**18**] proved that the reduced C^*- algebra $\mathbb{F}_n,\ n \geq 2$ has the metric approximation property, a very remarkable result since $C^*_r(\mathbb{F}_n)$, $n \geq 2$, is not nuclear, $\mathbb{F}_n$ not being amenable.

We have the following definition of weak amenability.

Definition 5.1.3. [**5**] An approximate identity on G is a sequence (ϕ_n) of finitely supported functions such that ϕ_n uniformly converge to constant function 1. We say that discrete G is *weakly amenable* if there is an approximate identity (ϕ_n) such that

$$C := \sup_n \|M_{\phi_n}\|_{cb} < \infty.$$

We have the following important result by Haagerup [**19**].

Theorem 5.1.4. *Let G be a discrete group. The following are equivalent:*

(1) *G is weakly amenable,*
(2) *$C^*_r(G)$ has the CBAP.*

Definition 5.1.5. [**7**] We say that discrete group G is *amenable* if and only if there is an approximate identity consisting of positive definite functions.

LEMMA 5.1.6. *An amenable discrete group is weakly amenable.*

PROOF. We recall that G is an amenable discrete group if and only if there is an approximate identity on G consisting of positive definite functions (see definition 5.1.5). A sequence (ϕ_n) of finitely supported functions such that $\phi_n \longrightarrow 1$. Then M_{ϕ_n} completely positive on $C^*_\lambda(G)$ and also M_{ϕ_n} completely bounded and

$$\|M_{\phi_n}\|_{cb} = \phi_n(1).$$

Thus $\Lambda_{cb}(G) = 1$. Therefore G has CBAP. By Theorem 5.1.4, G is weakly ameable. □

Haagerup and Kraus have provided in [**19**] a detailed characterisation of AP.

First we recall the Fourier algebra

$$A(G) := \{f : f(t) = \langle \lambda(t)\xi \mid \eta\rangle \text{ for some } \xi, \eta \in \ell_2(G)\}$$

is the space of all coefficient function of the left regular representation λ. Given $f \in A(G)$, its norm is given by

$$\|f\| = \inf\{\|\xi\|\,\|\eta\| : f(t) = \langle \lambda(t)\xi \,|\, \eta\rangle\}.$$

With this norm, $A(G)$ is a Banach algebra with the pointwise multiplication [**19**].

A complex-valued function ϕ on G is a *multiplier* for $A(G)$ if the linear map

$$M_\phi(f) = \phi f$$

sends $A(G)$ to $A(G)$. If the map M_ϕ is completely bounded on $A(G)$, we call ϕ a completely bounded multiplier of $A(G)$. The set of multipliers of $A(G)$ is denoted by $M_0A(G)$. If $\phi \in A(G)$ then ϕ is a bounded continuous function and M_ϕ is a bounded operator on the space $A(G)$.

The discrete group G has the *approximation property* (AP) if there is a net $\{\phi_\alpha\}_{\alpha\in\wedge}$ in $A(G)$ such that $M_{\phi_\alpha} \longrightarrow id_{A(G)}$ in the stable point-norm topology on $A(G)$.

We say that the C^*- algebra A has the *strong operator approximation property* (SOAP) if there is a net T_α in $A(G)$ such that $T_\alpha \longrightarrow id_A$ in the stable point-norm topology.

If A is a C^*- algebra, and $\mathcal{H}$ is a separable infinite Hilbert space, a net T_α in $CB(A)$ is said to converge in the *stable point-norm topology* to T in $CB(A)$ if $T_\alpha \otimes id_{K(\mathcal{H})}(a) \longrightarrow T \otimes id_{\mathcal{K}(\mathcal{H})}(a)$ in norm for all $a \in A \otimes \mathcal{K}(\mathcal{H})$. Here $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators on $\mathcal{H}$.

We say that C^*- algebra, A has the *operator approximation property* (OAP) if there exists a net of finite - rank maps $T_\alpha : A \longrightarrow A$ such that $T_\alpha \longrightarrow id_A$ in the stable point-norm topology. This means that there exists a net of finite rank linear maps

$$T_\alpha : A \longrightarrow A$$

such that for all $x \in \mathcal{K}(\mathcal{H}) \otimes_{min} A$,

$$\|Id \otimes T_\alpha(x) - x\| \longrightarrow 0.$$

We have the following important result from Haagerup and Kraus [**19**]:

THEOREM 5.1.7. [**19**] *Let G be a discrete group. Then the following are equivalent:*

(1) *G has the AP,*

(2) $C_r^*(G)$ *has the operator approximation property (OAP),*

(3) $C_r^*(G)$ *has the strong operator approximation property (strong OAP).*

EXAMPLE 5.1.8. The following groups have AP [**19**]. This implies that these groups have the OAP, and thus also SOAP:

- $SL(2,\mathbb{Z})$[**19**]
- $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ [**19**]

Exactness of groups has been Kirchberg and Wassermann [**24**].

DEFINITION 5.1.9. [**33**] A C^*- algebra A is *exact* if, given any exact sequence

$$0 \longrightarrow J \longrightarrow B \longrightarrow C \longrightarrow 0$$

of C^*- algebras, the sequence

$$0 \longrightarrow A \otimes_{\min} J \longrightarrow A \otimes_{\min} B \longrightarrow A \otimes_{\min} C \longrightarrow 0$$

is again exact.

DEFINITION 5.1.10. [**33**] We say that a discrete group G is *exact* if and only if $C_r^*(G)$ is an exact C^*- algebra.

EXAMPLE 5.1.11. Kirchberg and Wassermann [**23**] show that if a $C_r^*(G)$ has the CBAP then G is exact. On the other hand a group G is weakly amenable if and only if it has the CBAP [**19**], and so all weakly ameanble groups are exact.

EXAMPLE 5.1.12. The following are examples of exact group:

- linear groups [**17**]

- Hyperbolic groups [**1**]
- Coxeter groups [**14**]
- countable subgroups of almost connected Lie groups [**17**]

5.2. Joachim Zacharias's IAP with coefficients

In this section we will give definition of the strong invariant approximation property. Let $S \subseteq B(\mathcal{H})$ be a closed subspace.

DEFINITION 5.2.1. [**36**] We define the operator space $C_u^*(G, S)$ as the closure of finite width matrices $[x_{s,t}]_{s,t \in G}$, where $x_{s,t} \in S$ and $\|x_{s,t}\|$ is uniformly bounded for all $s, t \in G$ acting on $\ell^2(G) \otimes \mathcal{H}$.

We have that

$$C_u^*(G) \otimes S \subseteq C_u^*(G, S).$$

In general $C_u^*(G, S)$ is an operator space and it is a C^*- algebra if S is C^*- algebra. $C_u^*(G, -)$ is a functor on the categery of C^*- algebras.

DEFINITION 5.2.2. [**36**] We say that $C_u^*(G, -)$ is an *exact functor* if the functor $A \longmapsto C_u^*(G, A)$ takes short exact sequences of C^*- algebras to short exact sequences of C^*- algebras, so that given a short exact sequence

$$0 \longrightarrow J \longrightarrow B \longrightarrow C \longrightarrow 0$$

of C^*- algebras the induced sequence

$$0 \longrightarrow C_u^*(G, J) \longrightarrow C_u^*(G, B) \longrightarrow C_u^*(G, C) \longrightarrow 0$$

is exact.

Let $S \subseteq B(\mathcal{H})$ be a closed subspace and let $\mathcal{H}$ be a Hilbert space. For any $x \in C_u^*(G,S)$: $x \in C_u^*(G,B(\mathcal{H}))$ such that $x_{s,t} \in S$ for all $s,t \in G$; $[x_{s,t}]_{s,t\in G}$ is the finite width matrices; $\|x_{s,t}\|$ is uniformly bounded for all $s,t \in G$ acting on $\ell^2(G) \otimes \mathcal{H}$. The following Theorem is proved in [**36**].

Theorem 5.2.3. [**36**] *For a discrete exact group the following conditions are equivalent:*

(1) *G is exact;*
(2) *$C_r^*(G)$ is exact;*
(3) *For all Hilbert spaces $\mathcal{H}$ and closed subspaces $S \subseteq B(\mathcal{H})$*

$$C_u^*(G,S) = \{x \in C_u^*(G,B(\mathcal{H}))\,; x_{s,t} \in S \textit{ for all } s,t \in G\}\,;$$

(4) *$C_u^*(G,-)$ is an exact functor.*

We describe an outline of proof of Theorem 5.2.3: We have

$$C_u^*(G,S) \subseteq \{x \in C_u^*(G,B(\mathcal{H}))\,; x_{s,t} \in S \text{ for all } s,t \in G\}\,;$$

For the reverse inclusion note that If $C_r^*(G)$ is exact, then there exists a net of finite width positive definite kernels

$$k_\alpha : G \times G \longrightarrow \mathbb{C}$$

(as in Theorem 2.1(3) [**36**]) such that for all $\epsilon > 0$ and every finite subset $F \subseteq G$ there is α_0 such that $|k_\alpha(s,t) - 1| < \epsilon$ whenever $st^{-1} \in F$ and $\alpha \geq \alpha_0$. The Schur multiplier M_{k_α} associated with k_α defines a completely positive map on $B(\ell^2(G) \otimes \mathcal{H})$ such that $M_{k_\alpha}(x) \longrightarrow x$ is norm for all $x \in C_u^*(G,B(\mathcal{H}))$. Moreover

$$M_{k_\alpha}\left(\{x \in C_u^*(G,B(\mathcal{H}))|\ x_{s,t} \in S \text{ for all } s,t \in G\}\right),$$

whose norms are uniformly bounded and is a subset of $C_u^*(G,S)$. This means that: (2) implies (3). (1) and (2) are equivalent by definition. (4) implies (1) follows from a characterisation of exactness of G given in Lemma 2.2 [**36**].

Next, we define the set of fixed points of $C_u^*(G,S)^G$:

DEFINITION 5.2.4. We define

$$C_u^*(G,S)^G = \{T \in C_u^*(G,S)\,;\, Ad(\rho_t \otimes id)T = T \text{ for all } t \in G\}\,.$$

We now define Joachim Zacharias's IAP with coefficients (SIAP):

DEFINITION 5.2.5. [**36**] We say that a discrete group G has the *strong invariant approximation property* (SIAP) if for any closed subspace S of the compact operators $\mathcal{K}$ (on $\ell^2(\mathbb{N})$). We have an isomorphism

$$C_u^*(G,S)^G = C_\lambda^*(G) \otimes S \text{ holds.}$$

We have the following Lemma of Joachim Zacharias [**36**].

LEMMA 5.2.6. [**36**] *Suppose that G is exact and $S \subseteq B(\mathcal{H})$ is an arbitrary closed subspace, then*

$$C_u^*(G,S)^G = (C_u^*(G) \otimes S)^G.$$

Sketch of proof of Lemma 5.2.6: Given $A \in C_u^*(G,S)^G$ we have

$$A(x,y) = A(xt,yt) \in S \text{ for all } x,y,t \in G.$$

In particular

$$A(x,y) = A(xy^{-1},e) \in S \text{ for all } x,y \in G.$$

Let $F \subseteq G$ be finite and $A^F(x,y)$ the element obtained from A by replacing $A(x,y)$ by 0 when $xy^{-1} \notin F$. We have

$$A^F = \sum_{t \in F} \lambda_t \otimes A(t,e) \in \mathbb{C}[G] \odot S,$$

where $\mathbb{C}[G] \odot S$ is the algebraic tensor product and $\mathbb{C}[G]$ is the group ring. Since,

$$M_{k_\alpha}\left(A^F\right) = M_{k_\alpha}\left(A\right),$$

provided k_α has width F. (This means that there is $M \geq 0$ such that $|k_\alpha| \leq M$ for all $\alpha \in G$.) But

$$M_{k_\alpha}\left(A^F\right) = \sum_{t \in F} M_{k_\alpha}(\lambda_t) \otimes A(t,e) \in C_u^*(G) \odot S$$

thus

$$M_{k_\alpha}\left(C_u^*(G,S)^G\right) \subseteq C_u^*(G) \odot S \;\; \forall \; \alpha.$$

Since

$$M_{k_\alpha}(A) \longrightarrow A \;\; \forall \, A \;\; \in \;\; C_u^*(G,S).$$

it follows that

$$\begin{aligned} C_u^*(G,S)^G &= (C_u^*(G) \otimes S) \cap C_u^*(G,S)^G \\ &= (C_u^*(G) \otimes S)^G. \end{aligned}$$

The following Theorem can be found in [**36**].

Theorem 5.2.7. *For a discrete exact group G the following conditions are equivalent:*

(1) *G has the strong invariant approximation property (Joachim Zacharias's IAP with coefficients (SIAP));*
(2) *$C_u^*(G,S)^G = (C_u^*(G) \otimes S)^G = C_\lambda^*(G) \otimes S$ for any closed subspace $S \subseteq \mathcal{K}$;*

(3) *$C_u^*(G,S)^G = (C_u^*(G) \otimes S)^G = C_\lambda^*(G) \otimes S$ for any Hilbert space $\mathcal{H}$ and closed subspace $S \subseteq B(\mathcal{H})$;*

(4) *G has the AP.*

Comment on proof of Theorem 5.2.7: Given G exact with the strong invariant approximation property (SIAP), for any closed subspace S of the compact operator $\mathcal{K}$ (on $\ell^2(\mathbb{N})$) we have the equality

$$C_u^*(G,S)^G = C_\lambda^*(G) \otimes S.$$

By Lemma 5.2.6

$$C_u^*(G,S)^G = (C_u^*(G) \otimes S)^G$$

for any closed subspace S. Then

$$C_u^*(G,S)^G = (C_u^*(G) \otimes S)^G = C_\lambda^*(G) \otimes S$$

for any closed subspace $S \subseteq \mathcal{K}$. This implies that (1) $\Longleftrightarrow$ (2). Next we describe the (1) $\Longrightarrow$ (4): Given G exact with the strong invariant translation approximation property (SIAP), we need to show that G has AP. Haagerup and Kraus have shown that G has AP if and only if $C_r^*(G)$ has strong OAP. They also show that $C_r^*(G)$ has strong OAP if and only if $C_r^*(G)$ has the slice map property for closed subspace $S \subseteq \mathcal{K}$. Joachim Zacharias [**36**] shows that $C_r^*(G)$ has the slice map property for closed subspace $S \subseteq \mathcal{K}$. Let A and B be C^*- algebras and $X \subset B$ be a closed subspace. For arbitrary A the kernel of the map

$$A \otimes B \longrightarrow A \otimes C$$

is the Fubini product: We define

$$F(A,B,J) = \{x \in A \otimes B : (\phi \otimes id)(x) \in J \ \forall \phi \in A^*\}.$$

We say a triple (A, B, J) satisfies the slice map property if

$$F(A, J) = A \otimes J.$$

Next we describe the (4) $\Longrightarrow$ (3): G has the AP, by using definitions of AP. Let $\phi_\alpha \in A_c(G)$, which means that ϕ_α have finite support, such that

$$M_{\phi_\alpha}(x) \longrightarrow x, \ \ \forall x \in C_u^*(G, B(\mathcal{H})).$$

But

$$M_{\phi_\alpha} Ad\rho(t) = Ad\rho(t) M_{\phi_\alpha}, \ \ \forall \ t \ \in \ G.$$

Let us first note that completely bounded multiplier $\phi_\alpha \in M_0A(G)$ define $M_{\phi_\alpha} \in CB(C_u^*(G, S))$ for any operator space $S \subseteq B(\mathcal{H})$, given by

$$M_{\phi_\alpha} [x_{s,t}] = \left[\phi_\alpha(st^{-1}) x_{s,t}\right].$$

But a completely bounded operators $M_{\phi_\alpha} \in CB\left(C_u^*(G, S)\right)$ maps invariant elements to invariant elements. Moreover,

$$M_{\phi_\alpha}\left(C_u^*(G, S)^G\right) \subseteq \mathbb{C}[G] \odot S.$$

By similar proof of Lemma 5.2.6. Since

$$M_{\phi_\alpha}(x) \longrightarrow x \ \ \forall x \in C_u^*(G, S).$$

It follows that

$$C_u^*(G, S)^G \subseteq C_\lambda^*(G) \otimes S.$$

Therefore

$$C_u^*(G, S)^G = C_\lambda^*(G) \otimes S.$$

This implies that (4) $\Longrightarrow$ (3). Next we describe the (3) $\Longrightarrow$ (1). We assume that

$$C_u^*(G, S)^G = (C_u^*(G) \otimes S)^G = C_\lambda^*(G) \otimes S$$

for any Hilbert space $\mathcal{H}$ and closed subspace $S \subseteq \mathcal{H}$.

Therefore

$$C_u^*(G, S)^G = C_\lambda^*(G) \otimes S$$

for any Hilbert space $\mathcal{H}$ and closed subspace $S \subseteq \mathcal{H}$.

This implies that (3) $\Longrightarrow$ (1).

REMARK 5.2.8. For a discrete exact group G, G has the AP $\Longleftrightarrow$ G has SIAP (Zacharias's IAP with coefficients).

We note also the following results.

LEMMA 5.2.9. [**19**] *If G is weakly amenable, then G has the AP.*

LEMMA 5.2.10. *If G has the AP, then G has the IAP.*

PROOF. For group C^*- algebra of discrete groups the OAP if and only if G has AP [**19**]. Thus G has exactness, and so condition (4) in Thorem 5.2.7 implies the other condition for all discrete group G. Moreover (see Theorem 5.2.7), the strong invariant approximation property with coefficients implies the one without coefficients. This means:

$$C_u^*(G, \mathbb{C})^G = C_u^*(G)^G \otimes \mathbb{C} = C_\lambda^*(G) \otimes \mathbb{C}.$$

and therefore

$$C_u^*(G)^G = C_\lambda^*(G),$$

Thus G has IAP, $\square$

We are now ready to prove the following proposition.

PROPOSITION 5.2.11. *The following implications hold for a discrete group:*

$$CBAP \Rightarrow AP \Rightarrow IAP.$$

PROOF. By Theorem 5.1.4 if G is discrete group, then G is weakly amenable if and only if $C_r^*(G)$ has the CBAP. But G is weakly amenable implies that G has AP. As now we use Lemmas 5.2.9 and 5.2.10 to conclude

$$CBAP \Rightarrow AP \Rightarrow IAP.$$

□

REMARK 5.2.12. The converse of the first implication does not hold: a counter example is given by $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$: since AP is preserved by semi-direct products [**19**], this group has the AP. But Haagerup [**19**] proved that it does not have the CBAP.

The following groups are all weakly amenable. This implies that these groups have the AP, and thus also IAP:

- Amenable groups
- Hyperbolic groups [**29**]
- $CAT(0)$-cubical groups [**32**]
- $SL_2(\mathbb{Q}_p)$ [**4**]

REMARK 5.2.13. For discrete groups we have the following implications:

$$\text{Amenability} \implies \text{weak amenability} \implies \text{AP} \implies \text{exactness.}$$

The first implication is explained in Lemma 5.1.6. The first implication is not an equivalence: the non-abelian free groups are weakly amenable, but they are not amenable. The second implication is proved by Lemma 5.2.9 and also this implication is not an equivalence: a counter-example is given by $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$; this group has the AP [**19**]. But it was proved in [**19**] that it is not weakly amenable. The third implication is not an equivalence: Haagerup and Kraus showed in [**19**] that $SL_2(\mathbb{Z})$ is an exact group without AP.

5.3. Analytic properties of strong IAP

In this section, we study some of the analytic properties of the strong invariant approximation property for discrete exact groups.

REMARK 5.3.1. For a discrete exact group G the following are equivalent.

(1) G has the AP.
(2) $C_r^*(G)$ has the OAP.
(3) G has SIAP (Zacharias's IAP with coefficients)

The above remark means for a discrete exact group the following properties are actually equivalent:

$$AP \Longleftrightarrow OAP \Longleftrightarrow SIAP.$$

REMARK 5.3.2. For a discrete exact group G, by Remark 5.3.1 and Remark 5.2.8, G has the SIAP if for any closed subspace $S \subseteq B(\mathcal{H})$ the equality

$$C_u^*(G,S)^G = C_\lambda^*(G) \otimes S \text{ holds.}$$

But G has IAP, so

$$C_u^*(G)^G = C_\lambda^*(G)$$

and therefore

$$C_u^*(G, S)^G = C_u^*(G)^G \otimes S$$

for any closed subspace $S \subseteq B(\mathcal{H})$.

Next, we show the following:

PROPOSITION 5.3.3. *SIAP implies IAP for discrete exact groups.*

PROOF. Let G be a group with SIAP. By applying Theorem 5.2.7 to case $S = \mathbb{C}$, we have that

$$C_u^*(G, \mathbb{C})^G = C_u^*(G \otimes \mathbb{C})^G = C_\lambda^*(G) \otimes \mathbb{C}.$$

But

$$C_u^*(G \otimes \mathbb{C})^G = C_u^*(G)^G \otimes \mathbb{C},$$

so that

$$C_u^*(G)^G \otimes \mathbb{C} = C_\lambda^*(G) \otimes \mathbb{C}.$$

This implies

$$C_u^*(G)^G = C_\lambda^*(G).$$

□

We note also the following results.

THEOREM 5.3.4. [**34**] *Let G be a discrete group. Let*

$$1 \longrightarrow H \longrightarrow G \overset{\pi}{\longrightarrow} G/H \longrightarrow 1.$$

If H is a normal subgroup in G with H and G/H are exact, then G is exact.

We have the following important result of Haagerup and Kraus [**19**].

THEOREM 5.3.5. *Let G be locally compact group, and suppose that H is a closed normal subgroup of G. If H and G/H have the AP, then G has the AP.*

In the following Proposition, we show that the strong invariant approximation property (SIAP) passes to extensions for discrete exact groups. Note that $AP \iff SIAP$ for discrete exact groups (see Remark 5.3.1).

PROPOSITION 5.3.6. *Let G be a discrete group. Let*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} G/H \longrightarrow 1.$$

Let us assume that H is a normal subgroup in G, and that H and G/H are exact groups. If H and G/H have the SIAP, then G has SIAP.

PROOF. Let G be a discrete group and suppose that H is a normal subgroup of G. By Remark 5.3.1, if H has the SIAP then H has AP. If G/H has the SIAP then G/H has the AP. By Theorem 5.3.5, if H and G/H have the AP, then G has the AP. By Theorem 5.3.4, if H and G/H are exact groups, then so is G. Thus G is a discrete exact group with AP. By Remark 5.3.1, G has SIAP. □

5.4. The stability properties of strong IAP

In this section, we first show that the semidirect product of two discrete exact groups with the SIAP has the SIAP. We have the following important result in [**19**].

PROPOSITION 5.4.1. *The semidirect product of two discrete groups with the AP has the AP.*

From this we can deduce the following.

PROPOSITION 5.4.2. *The semidirect product of two discrete exact groups with the SIAP has the SIAP.*

PROOF. For an exact group G, G has AP $\Longleftrightarrow$ G has SIAP. The semidirect product of two discrete groups with the AP has the AP [**19**]. The semidirect product of two discrete exact groups is an exact group [**35**]. The semidirect product of two discrete groups with the AP has the AP (see Proposition 5.4.1). Therefore the semidirect product of two discrete exact groups with the SIAP has the SIAP. □

EXAMPLE 5.4.3. [**7**] We have the following short exact sequence of groups

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}) \longrightarrow SL(2,\mathbb{Z}) \longrightarrow 1.$$

Indeed, $\mathbb{Z}^2$ and $SL(2,\mathbb{Z})$ are weakly amenable groups [**19**]. The semidirect product of two discrete groups with the AP has the AP [**19**]. Thus, $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ has AP [**19**]. By using Proposition 5.2.11. Therefore $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ has IAP.

We also note the following.

PROPOSITION 5.4.4. *Let* $\{G_i, i \in I\}$ *be a family of amenable groups, and let* H *be an open compact subgroup of* G_i *for each* $i \in I$*. Then* $G = *_H G_i$ *has the invariant approximation property.*

PROOF. Amalgamated products of amenable groups are weakly amenable [**4**]. Then G is weakly amenable. By Proposition 5.2.11, G has the invariant approximation property. □

5.5. Examples of groups without the strong IAP

Lafforgue and de la Salle [**26**] have proved that $SL_3(\mathbb{Z})$, does not satisfy CBAP and OAP property.

THEOREM 5.5.1. [**26**] $SL_3(\mathbb{Z})$ *does not have the AP. Equivalently, the reduced* C^*- *algebra of* $SL_3(\mathbb{Z})$ *does not have the operator space approximation property (OAP), and hence does not have the completely bounded approximation property (CBAP).*

Linear groups are exact [**17**]. This provides an example of exact C^*-algebra without the OAP [**26**], by Remark 5.3.1. In particular we can conclude that $SL_3(\mathbb{Z})$ does not have the SIAP.

CHAPTER 6

Invariant Approximation Property for Subgroups and Extensions

In this chapter we will study the invariant approximation property in various contexts. First we shall show that it passes to subgroups (see Theorem 6.1.1). An interesting question, which we will address next is the behavior of this property with respect to group extensions. To prepare for that we first study a relationship of uniform Roe algebras attached to coarsely equivalent metric spaces in the following case. Let X be a bounded geometry metric space and assume that there is a bijective coarse equivalence

$$\phi : X \longrightarrow Y \times N,$$

where N is a finite metric space. Then there is an isomorphism

$$\begin{aligned} C_u^*(X) &\cong C_u^*(Y) \otimes C_u^*(N) \\ &\cong C_u^*(Y) \otimes M_n(\mathbb{C}), \end{aligned}$$

where $n = |N|$ (see Theorem 6.2.3).

In section 6.2, we shall use this result to prove that the invariant approximation property is preserved under taking direct product with a finite group : let H be a discrete group with the IAP and K a finite group. Then the direct product $G = H \times K$ has IAP (see Theorem 6.2.3).

We then study a generalization of this result to extensions with finite quotient: let G be a discrete group then, if H is a finite index normal subgroup of G with the IAP, G also has the IAP (see Theorem 6.3.2). An important technical tool in the proof is the fact that the left regular representation λ_G is equivalent to the left regular representation $\lambda_H \otimes \lambda_{G/H}$ (see Proposition 6.3.7). These are the main results of this thesis.

6.1. The IAP passes to subgroups

THEOREM 6.1.1. *Any subgroup H of a discrete group G with the invariant approximation property has the invariant approximation property.*

This proof is based on an idea of Joachim Zacharias. I am grateful to him for sharing this idea with me.

PROOF. Let us fix a set of representatives R of the right cosets G/H so that for every element $g \in G$ there is a unique representation $g = h_g r_g$ where $h_g \in H$ and $r_g \in R$. We then have the isomorphism of Hilbert spaces:

$$\ell^2(G) \cong \ell^2(H) \otimes \ell^2(G/H),$$

given by

$$\delta_g \longmapsto \delta_{h_g} \otimes \delta_{r_g},$$

with the converse map given by

$$\delta_h \otimes \delta_r \longmapsto \delta_{hr}.$$

The uniform Roe algebra $C_u^*(H)$ acts on this space by $T \otimes 1$ for every $T \in C_u^*(H)$, which gives an embedding, i.e

$$C_u^*(H) \hookrightarrow C_u^*(G)$$

by

$$T \longmapsto T \otimes 1.$$

Using this inclusion, we shall show that

$$C_u^*(H)^H \cong C_u^*(H)^G.$$

First, it is clear that a $G-$ invariant operator in $C_u^*(H)$ is also $H-$ invariant operator, restricting the $Ad\rho$ action from G to H. To show the converse,

$$C_u^*(H)^H \subseteq C_u^*(H)^G,$$

we proceed as follows. We want to extend a kernel on $H \times H$ which is invariant with respect to the $Ad\rho_H$ action to a kernel on $G \times G$ which is invariant with respect to the $Ad\rho_G$ action. Given $a(h, h')$ we define

$$A : G \times G \longrightarrow \mathbb{C}$$

as follows: for every $s, t \in G$ and $h, h' \in H$

$$A(s,t) = \begin{cases} a(h,h'), & \text{if there exists } r \in R \text{ s.t } (s,t) = (hr, h'r), \\ 0, & \text{otherwise.} \end{cases}$$

Now we need to show that $A(s, t)$ is $Ad\rho_G-$invariant. If we write

$$rt = h_1 r_1 \text{ for } h_1 \in H, \; r, r_1 \in R$$

we get

$$\begin{aligned} Ad\rho_G(t)A(hr, h'r) &= A(hrt, h'rt) \\ &= A(hh_1r_1, h'h_1r_1) \\ &= a(hh_1, h'h_1) \\ &= a(h, h^{'}) \\ &= A(hr, h'r). \end{aligned}$$

Given that invariant Roe kernels form a dense subset of $C^*_u(H)^H$, it follows that

$$C^*_u(H)^H \subseteq C^*_u(H)^G,$$

and so we have an isomorphism,

$$C^*_u(H)^H \cong C^*_u(H)^G.$$

Let $T \in C^*_u(H)^G$. Then $T \in C^*_u(G)^G$ and $T \in C^*_u(H)$, and we have

$$C^*_u(H)^G \subseteq C^*_u(G)^G \cap C^*_u(H).$$

Since

$$C^*_u(G)^G \cap C^*_u(H) \subseteq C^*_u(H)^G,$$

we have

$$C^*_u(H)^G = C^*_u(G)^G \cap C^*_u(H).$$

We now want to show that a similar isomorphism holds for the regular C^*- algebras:

$$C^*_\lambda(H) \cong C^*_\lambda(G) \cap C^*_u(H).$$

First there is an inclusion

$$\mathbb{C}[G] \longrightarrow A^\infty(G),$$

$$g \longmapsto U_g(x, y),$$

where

$$U_g(x,y) = \begin{cases} 1, & \text{if} \quad x = gy, \\ 0, & \text{otherwise.} \end{cases}$$

This extends to a ring homomorphism so we have

$$\mathbb{C}[G] \hookrightarrow A^\infty(G) \hookrightarrow C_u^*(G),$$

where $A^\infty(G)$ is the uniform translation algebra. Since H is normal subgroup of G, we have an inclusion

$$\mathbb{C}[H] \hookrightarrow \mathbb{C}[G].$$

Then

$$\Phi : \mathbb{C}[H] \xrightarrow{\cong} \mathbb{C}[G] \cap A^\infty(H).$$

By taking completion of both sides, we have

$$C_\lambda^*(H) \cong C_\lambda^*(G) \cap C_u^*(H).$$

We now suppose that G has IAP. Then

$$C_u^*(G)^G = C_\lambda^*(G),$$

and using the above results we have that,

$$\begin{aligned} C_u^*(H)^H &\cong C_u^*(H)^G \\ &= C_u^*(G)^G \cap C_u^*(H) \\ &= C_\lambda^*(G) \cap C_u^*(H) \\ &\cong C_\lambda^*(H). \end{aligned}$$

Hence

$$C_u^*(H)^H \cong C_\lambda^*(H)$$

and so the IAP passes to subgroups. □

6.2. The IAP for direct products with finite group

In this section, we show that the invariant approximation property is preserved under taking direct product with a finite group. We first recall the definition of Morita equivalence:

DEFINITION 6.2.1. [**6**] We say that two unital C^*-algebras A and B are Morita equivalent if and only if they are stably isomorphic, which means that $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the algebra of compact operators.

The following Theorem can be found in [**6**].

THEOREM 6.2.2. [**6**] *If X and Y are uniformly discrete bounded geometry spaces, and X is coarsely equivalent to Y then, $C_u^*(X)$ is Morita equivalent to $C_u^*(Y)$.*

This statement can be made a little more precise in the following situation.

THEOREM 6.2.3. *Let X be a bounded geometry metric space and assume that there is a bijective coarse equivalence*

$$\phi : X \longrightarrow Y \times N,$$

where Y is a bounded geometry metric space and N is a finite metric space. Then there is an isomorphism

$$\begin{aligned} C_u^*(X) &\cong C_u^*(Y) \otimes C_u^*(N) \\ &\cong C_u^*(Y) \otimes M_n(\mathbb{C}). \end{aligned}$$

where $n = |N|$.

PROOF. We shall assume that the bijection ϕ is implemented by means of two maps

$$f : X \longrightarrow Y \text{ and } \pi : X \longrightarrow N$$

so that

$$\phi(x) = (f(x), \pi(x)) \text{ for all } x \in X.$$

The bijection ϕ gives rise to a unitary isomorphism

$$\ell^2 (X) \cong \ell^2 (Y) \otimes \ell^2 (N) .$$

This induces a continuous isomorphism

$$\Phi : B(\ell^2(X)) \xrightarrow{\cong} B(\ell^2(Y) \otimes \ell^2(N)) \cong B(\ell^2(Y)) \otimes M_n(\mathbb{C}),$$

where we use the fact that $\ell^2 (N) = \mathbb{C}^n$. We shall show that Φ restricts to an isomorphism

$$\Phi : C^*_u(X) \longrightarrow C^*_u(Y) \otimes M_n(\mathbb{C}).$$

First we need to show that, if T is a finite propagation operator on $\ell^2(X)$ then

$$\Phi(T) \in C^*_u(Y) \otimes M_n(\mathbb{C}).$$

For every $i = 1 \dots n$, let $X_i = \pi^{-1}(i)$ and note that the restriction of f to X_i gives a bijection

$$f|_{X_i} : X_i \xrightarrow{\cong} Y.$$

We shall denote by V_i the corresponding unitary isomorphism

$$V_i : \ell^2(X_i) \xrightarrow{\cong} \ell^2(Y),$$

and let P_i be the projection

$$P_i : \ell^2 (X) \longrightarrow \ell^2 (X_i) .$$

Then any operator $T \in C_u^*(X)$ admits a decomposition

$$T = \sum_{i,j=1}^{n} P_i T P_j$$

where $P_i T P_j$ is an operator from $\ell^2(X_j)$ to $\ell^2(X_i)$.

Let $S_{i,j} = P_i T P_j$. Then

$$V_i S_{i,j} V_j^* : \ell^2(Y) \longrightarrow \ell^2(Y)$$

is a unitary isomorphism and we have

$$\Phi(S_{i,j}) = V_i S_{i,j} V_j^* \otimes E_{ij},$$

where E_{ij} is the $(i,j)-$th elementary matrix. We want to show that $V_i S_{i,j} V_j^*$ is a finite propagation operator on Y. This will follows from the fact that

$$f : X \longrightarrow Y$$

is a coarse map. Let $y_1,\ y_2 \in Y$. Then

$$\begin{aligned} \left\langle V_i S_{i,j} V_j^* \delta_{y_1}, \delta_{y_2} \right\rangle &= \left\langle V_i P_i T P_j V_j^* \delta_{y_1}, \delta_{y_2} \right\rangle \\ &= \left\langle T P_j V_j^* \delta_{y_1}, P_i V_i^* \delta_{y_2} \right\rangle \\ &= \left\langle T \delta_{x_1}, \delta_{x_2} \right\rangle, \end{aligned}$$

where x_1 is the preimage of y_1 in X_j and x_2 is the preimage of y_2 in X_i. As T is a bounded propagation operator, there exists $R > 0$ so that

$$\langle T\delta_{x_1}, \delta_{x_2} \rangle = 0 \text{ when } d(x_1, x_2) > R.$$

Since f is a coarse map, $\exists\ S > 0$ such that

$$d_Y(f(x_1), f(x_2)) > S \Rightarrow d_X(x_1, x_2) > R.$$

As f is a surjection we now have that for all $y_1,\ y_2 \in Y$ such that $d_Y(y_1,y_2) > S$, there exist x_1 in X_j, x_2 in X_i such that $d_X(x_1,x_2) > R$ and

$$\begin{aligned}\left\langle V_i S_{i,j} V_j^* \delta_{y_1}, \delta_{y_2}\right\rangle &= \langle T\delta_{x_1}, \delta_{x_2}\rangle \\ &= 0\end{aligned}$$

So $V_i S_{i,j} V_j^* \in C_u^*(Y)$ has required. Next, we need to show that Φ is an isomorphism and for this we shall construct an inverse map

$$\Psi : C_u^*(Y) \otimes M_n(\mathbb{C}) \longrightarrow C_u^*(X).$$

If $T \otimes E_{ij} \in C_u^*(Y) \otimes M_n(\mathbb{C})$. Then define

$$\Psi(T \otimes E_{ij}) = P_i V_i^* T V_j P_j.$$

Using the same argument as before we prove that the operator $P_i V_i^* T V_j P_j$ is of finite propagation, since f is a coarse equivalence. We extend Ψ by linearity and continuity to a map

$$\Psi : C_u^*(Y) \otimes M_n(\mathbb{C}) \longrightarrow C_u^*(X).$$

We need to show that

$$\Psi \circ \Phi = \Phi \circ \Psi = Id.$$

First we have

$$\begin{aligned}\Phi \circ \Psi(T \otimes E_{i,j}) &= \Phi(P_i V_i^* T V_j P_j) \\ &= \sum_{l,k} V_k P_k (P_i V_i^* T V_j P_j) P_l V_l^* \otimes E_{k,l}.\end{aligned}$$

Note that for $1 \leq l, k \leq n$

$$P_k P_i = \begin{cases} 0 & \text{if } k \neq i, \\ P_i & \text{if } k = i. \end{cases}$$

and

$$P_jP_l = \begin{cases} 0 & \text{if } l \neq j, \\ P_j & \text{if } l = j. \end{cases}$$

Hence the above sum can be simplified as follows

$$\begin{aligned} \Phi \circ \Psi(T \otimes E_{i,j}) &= \sum_{k,l} V_kP_k(P_iV_i^*TV_jP_j)P_lV_l^* \otimes E_{k,l} \\ &= V_iP_iV_i^*TV_jP_jV_j^* \otimes E_{i,j}. \end{aligned}$$

Since $P_j|_{\ell^2(X_j)} = id_{X_j}$, we have

$$V_jP_jV_j^* = V_jV_j^* = Id_{X_j},$$

and

$$V_iP_iV_i^* = V_iV_i^* = Id_{X_j},$$

we have

$$\begin{aligned} \Phi \circ \Psi(T \otimes E_{i,j}) &= V_iP_iV_i^*TV_jP_jV_j^* \otimes E_{i,j} \\ &= T \otimes E_{i,j}. \end{aligned}$$

Moreover:

$$\begin{aligned} \Psi \circ \Phi(T) &= \Phi^{-1}\left\{\sum_{l,k} V_kP_kTP_lV_l^* \otimes E_{k,l}\right\} \\ &= \sum_{l,k} P_iV_i^*V_kP_kTP_lV_l^*V_jP_j \\ &= \sum_{i,j} P_iTP_j \\ &= T. \end{aligned}$$

Therefore

$$\Psi \circ \Phi = \Phi \circ \Psi = Id.$$

We conclude that

$$C_u^*(X) \cong C_u^*(Y) \otimes M_n(\mathbb{C}) \cong C_u^*(Y) \otimes C_u^*(N);$$

Hence the result. □

Next we prove that the invariant approximation property is preserved under taking direct product with a finite group.

THEOREM 6.2.4. *Let H be a discrete group with the IAP and K a finite group, then the direct product $G = H \times K$ has the IAP.*

PROOF. Let us denote the identification $G = H \times K$ by ϕ:

$$\phi : G \xrightarrow{\cong} H \times K.$$

Then

$$C_u^*(G) = C_u^*(H \times K).$$

The map ϕ is $G-$ equivariant we have

$$C_u^*(G)^G = C_u^*(H \times K)^{H \times K}.$$

By Theorem 6.2.3, we have,

$$C_u^*(H \times K) \cong C_u^*(H) \otimes C_u^*(K)$$

so that

$$C_u^*(H \times K)^{H \times K} \cong \left(C_u^*(H) \otimes C_u^*(K)\right)^{H \times K}.$$

Since the identification $G = H \times K$ is a group isomorphism, the unitary isomorphism

$$\ell^2(G) = \ell^2(H) \otimes \ell^2(K)$$

induces a unitary equivalence

$$\lambda_G \cong \lambda_H \otimes \lambda_K$$

and

$$\rho_G \cong \rho_H \otimes \rho_K.$$

This means $H \times K$ acts on $C_u^*(H) \otimes C_u^*(K)$ by $Ad\rho_H \otimes Ad\rho_K$ and so

$$C_u^*(H \times K)^{H \times K} \cong C_u^*(H)^H \otimes C_u^*(K)^K.$$

By the same remark,

$$C_\lambda^*(G) \cong C_\lambda^*(H) \otimes C_\lambda^*(K).$$

K is a finite group, so it amenable and so has the IAP, Roe [**30**]. Since H has the IAP by assumption

$$\begin{aligned} C_u^*(G)^G &= C_u^*(H)^H \otimes C_u^*(K)^K \\ &= C_\lambda^*(H) \otimes C_\lambda^*(K) \\ &= C_\lambda^*(H \times K) \\ &= C_\lambda^*(G). \end{aligned}$$

Therefore

$$C_u^*(G)^G = C_\lambda^*(G).$$

□

6.3. The IAP passes to extensions with a finite quotient

In this section, we show that the invariant approximation property passes to extensions with a finite quotient. Consider a group G with a finite index normal subgroup H, so that we have the following exact sequence.

$$1 \longrightarrow H \overset{i}{\hookrightarrow} G \overset{\pi}{\longrightarrow} G/H \longrightarrow 1.$$

We identify G with $H \times G/H$ as a set as follows. We choose a set-theoretic section of π in the above sequence which amounts to a choice of a finite set $R \subset G$ of coset representatives in G/H. Then any element $g \in G$ can be written uniquely as $g = h_g r_g$, where $h_g \in H$, $r_g \in R$. We then define

$$\phi : G \longrightarrow H \times G/H$$

$$g \longmapsto (h_g, r_g).$$

While G/H is a group, it is not true in general that

$$r_i r_j \in R \text{ when } r_i,\, r_j \in R.$$

The product on R, arising from the group structure on G/H, can be described in terms of the product in G as follows. For $r_1, r_2 \in R$, we denote by $r_1 * r_2$, the representative R of the coset of Hr_1r_2. We then note that

$$(Hr_1)(Hr_2) = Hr_1r_2 = H\left(r_1r_2\left(r_1 * r_2\right)^{-1}\right)\left(r_1 * r_2\right).$$

where $r_1r_2\left(r_1 * r_2\right)^{-1} \in H$. We will later need an explicit formula that will express the product $gg^{'}$ in the form

$$gg^{'} = h_{gg^{'}} r_{gg^{'}}.$$

This is done as follows. If $g = h_g r_g$, $g^{'} = h_{g'} r_{g'}$ then

$$\begin{aligned} gg^{'} &= (h_g r_g)\left(h_{g'} r_{g'}\right) \\ &= h_g \left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'} \\ &= h_g \left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'} \left(r_g * r_{g'}\right)^{-1} \left(r_g * r_{g'}\right). \end{aligned}$$

Since H is a normal subgroup of G, $r_g h_{g'} r_g^{-1} \in H$. By previous remark $r_g r_{g'} \left(r_g * r_{g'}\right)^{-1} \in H$ and so $h_g \left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'} \left(r_g * r_{g'}\right)^{-1} \in H$ and $r_g * r_{g'} \in R$, giving the required representation. We can now equip the set $H \times G/H$ with the product:

$$(h_g, r_g)(h_{g'}, r_{g'}) = \left(h_g \left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'} \left(r_g * r_{g'}\right)^{-1}, r_g * r_{g'}\right).$$

The identity of $R \cong G/H$ will be denoted by e_R, and identity elements of H will be denoted by $e_H \in H$:

$$\begin{aligned} (h_g, r_g)(e_H, e_R) &= \left(h_g \left(r_g e_H r_g^{-1}\right) r_g e_R \left(r_g * e_R\right)^{-1}, r_g * e_R\right) \\ &= \left(h_g \left(r_g r_g^{-1}\right) r_g r_g^{-1}, r_g\right) \\ &= \left(h_g \left(r_g r_g^{-1}\right) r_g r_g^{-1}, r_g\right) \\ &= (h_g, r_g) \end{aligned}$$

Next we need to find $(e, s)^{-1} \in H \times G/H$: The inverse of $s \in R \cong G/H$ will be denoted by $\overline{s}$. If (e, s) and $(h, \overline{s}) \in H \times G/H$: we have

$$(e, s)(h, \overline{s}) = \left((shs^{-1})(s\overline{s})(s * \overline{s})^{-1}, (s * \overline{s})\right)$$

If $(e, s)^{-1} = (h, \overline{s})$ then

$$\left((shs^{-1})(s\overline{s})(s * \overline{s})^{-1}, (s * \overline{s})\right) = (e, e),$$

If $s * \overline{s} = e = \overline{s} * s$. Then $\overline{s} = s^{-1}t$, for some $t \in H$ if and only $s\overline{s} = t$ and

$$(shs^{-1})t = e \text{ if and only } t = sh^{-1}s^{-1},$$

thus

$$\overline{s} = s^{-1}t = h^{-1}s^{-1} \text{ if and only } h = (\overline{s}s)^{-1}.$$

Thus,

$$(e,s)^{-1} = (h, \overline{s}) = ((\overline{s}s)^{-1}, \overline{s}).$$

Therefore $H \times G/H$ is a group structure. We record the following:

LEMMA 6.3.1. *Let the set $H \times G/H$ be equipped with the above product. Then the bijection*

$$\phi : G \longrightarrow H \times G/H,$$

is a group isomorphism.

PROOF.

$$\begin{aligned} \phi(gg') &= \left(h_g\left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'} \left(r_g * r_{g'}\right)^{-1}, \left(r_g * r_{g'}\right)\right) \\ &= \left(h_g, r_g\right)\left(h_{g'}, r_{g'}\right) \\ &= \phi(g)\phi(g'). \end{aligned}$$

□

From now on we shall denote by $H \overline{\times} G/H$ the set $H \times G/H$ equipped with the above product. As remarked above, $H \overline{\times} G/H$ is a group, but it is not the direct product of H and G/H.

We note that H is a subgroup of $H \overline{\times} G/H$, since for $h, h' \in H$

$$(h,e)(h',e) = (hh',e).$$

However, G/H is not in general a subgroup of $H\overline{\times}G/H$ as for any $r, r^{'} \in R$

$$(e,r)(e,r^{'}) = \left(rr^{'}\left(r * r^{'}\right)^{-1}, r * r^{'}\right).$$

G/H is a subgroup when the assignment:

$$[r] \longmapsto r \ \in R \ \subset \ G$$

is a a group homomorphism. i.e. when $r * r' = rr^{'}$, which happens when G is the direct product of H and G/H.

Our main goal in this chapter is to prove the following result.

THEOREM 6.3.2. *Let G be a discrete group. If H is a finite index normal subgroup of G with the IAP then G satisfies the IAP.*

The strategy of proof is as follows: First we will establish the isomorphism of C^*- algebras

$$C^*_u(G) \cong C^*_u(H) \otimes C^*_u(G/H).$$

Then it is natural to ask if a similar isomorphism can be obtained for regular C^*- algebras, i.e.

$$C^*_\lambda(G) \cong C^*_\lambda(H) \otimes C^*_\lambda(G/H). \tag{1}$$

We shall prove that this is true by constructing a unitary equivalence between λ_G and $\lambda_H \otimes \lambda_{G/H}$, where λ_G, λ_H and $\lambda_{G/H}$ are the left regular representations of the respective groups. We will also need to understand the action of G on $C^*_u(G)$ and of $H \times G/H$ on $C^*_u(H \times G/H)$ and we will prove that

$$C^*_u(G)^G \cong C^*_\lambda(H) \otimes C^*_\lambda(G/H). \tag{2}$$

Together with the isomorphism (1) this will show that

$$C_u^*(G)^G \cong C_\lambda^*(G),$$

so that G has the IAP.

The bijection

$$\phi : G \xrightarrow{\cong} H \times G/H,$$

defined above gives, by Theorem 6.2.3 an isomorphism of C^*- algebras

$$\Phi : C_u^*(G) \xrightarrow{\cong} C_u^*(H) \otimes C_u^*(G/H)$$

where we use the fact that G/H is finite, so there is no ambiguity concerning the choice of tensor product. In our present context, this isomorphism can be described as follows: Because H is a normal subgroup of G, its left and right cosets are the same because for every $r \in R$

$$rH = (rHr^{-1})r \cong Hr.$$

It follows that the group G can be given a disjoint union decomposition as either left or right cosets, which leads to isomorphisms of Hilbert spaces

$$\ell^2\left(G\right) = \bigoplus_{r \in R} \ell^2\left(Hr\right) = \bigoplus_{r \in R} \ell^2\left(rH\right).$$

This coset decomposition is preserved by both the left and the right multiplication by elements of H.

Using the right coset decomposition, we have the following isomorphism of Hilbert spaces

$$\ell^2\left(G\right) = \bigoplus_{r \in R} \ell^2\left(Hr\right).$$

We will denote by P_r the orthogonal projection

$$P_r : \ell^2(G) \longrightarrow \ell^2(Hr), \;\; r \in R$$

and by V_r the unitary isomorphism

$$\ell^2(Hr) \longrightarrow \ell^2(H)$$

given by

$$\delta_{hr} \longmapsto \delta_h, \text{ for all } h \in H.$$

We note that P_r commutes with the left regular representation and it also commutes with the right representation modulo the isomorphism

$$rH \cong Hr.$$

To see that this is true for the left cosets, we argue as follows: We can represent each function $\zeta \in \ell^2(G)$ as a linear combination

$$\zeta = \sum_{r \in R} \alpha_r \zeta_r,$$

where $\zeta_r \in \ell^2(rH)$ (this is understood as a subspace of $\ell^2(G)$ so that ζ_r is a function in $\ell^2(G)$ which vanishes outside rH) then

$$P_s(\zeta) = \zeta_s.$$

Then for every $t \in G$ and $h \in H$

$$\rho(h)\zeta(t) = \sum_{r \in R} \zeta_r(th).$$

We have

$$\begin{aligned} (P_s \rho(h)\zeta)(t) &= (\rho(h)\zeta_s)(t) \\ &= \zeta_s(th) \\ &= (\rho(h)P_s\zeta)(t). \end{aligned}$$

As in Theorem 6.2.3, every element $T \in C_u^*(G)$ can be represented as

$$T = \sum_{r, r' \in R} P_r T P_{r'}.$$

This decomposition is invariant with respect to $\rho(h)$, $h \in H$ (assuming left-coset decomposition for G). We also note that the unitary isomorphism

$$V_r : \ell^2(rH) \longrightarrow \ell^2(H),$$

commutes with ρ ; because

$$\begin{aligned} V_r\left(\rho(t)\delta_{rh}\right) &= V_r\left(\delta_{rht}\right) \\ &= \delta_{ht} \\ &= \rho(t)V_r\delta_{rh}. \end{aligned}$$

Now recall that the isomorphism

$$\Phi : C^*_u(G) \xrightarrow{\cong} C^*_u(H) \otimes C^*_u(G/H)$$

is given by

$$\Phi : T = \sum_{r,s\in R} P_r T P_s \longmapsto \sum_{r,s\in R} V_r P_r T P_s V_s^* \otimes E_{r,s}.$$

We obtain the following important Proposition:

Proposition 6.3.3. *The isomorphism Φ commutes with the adjoint action $Ad\rho$ of H.*

PROOF. We use the left coset decomposition so that ρ commutes the with P_r and V_r. $\forall\ h \in H$

$$\begin{aligned}
\Phi(Ad\rho(h)T) &= \Phi(\rho(h)T\rho(h)^*) \\
&= \Phi\left(\sum_{r,s\in R} P_r Ad\rho(h)(T)P_s\right) \\
&= \Phi\left(\sum_{r,s\in R} P_r \rho(h)T\rho(h)^* P_s\right) \\
&= \sum_{r,s\in R} (V_r P_r Ad\rho(h)TP_s V_s^* \otimes E_{r,s}) \\
&= \sum_{r,s\in R} Ad\rho(h)\,(V_r P_r T P_s V_s^*) \otimes E_{r,s} \\
&= Ad\rho(h)\left(\sum_{r,s\in R} V_r P_r T P_s V_s^* \otimes E_{r,s}\right) \\
&= Ad\rho(h)\Phi(T).
\end{aligned}$$

In the last equality we use the fact that H acts trivially on G/H. □

We can summarise these calculation as follows.

THEOREM 6.3.4. *We have*

$$C_u^*(G)^H \cong C_u^*(H)^H \otimes C_u^*(G/H).$$

If H has the IAP, then

$$C_u^*(G)^H \cong C_\lambda^*(H) \otimes C_u^*(G/H).$$

PROOF. By using Proposition 6.3.3 we have that

$$C_u^*(G)^H \cong C_u^*(H \times G/H)^H$$

which gives the isomorphism

$$C_u^*(G)^H \cong C_u^*(H)^H \otimes C_u^*(G/H)$$

since the action of H on G/H is trivial. When H has the IAP,

$$C_u^*(H)^H \cong C_\lambda^*(H)$$

and the result follows. $\square$

REMARK 6.3.5. Note that if $T \in C_u^*(G)^H$, then for every $s \in R$.

$$Ad\left(\rho(s)\right) T \in C_u^*(G)^H.$$

Indeed, take any $h \in H$, then

$$\begin{aligned} Ad\rho(h)Ad\rho(s)\left(T\right) &= Ad\rho(hs)\left(T\right) \\ &= Ad\left(\rho\left(s\left(s^{-1}hs\right)\right)\right)\left(T\right) \\ &= Ad\rho(s)Ad\rho\left(\left(s^{-1}hs\right)\right)\left(T\right) \\ &= Ad\rho(s)\left(T\right). \end{aligned}$$

Now take $r, t \in R$ and $T \in C_u^*(G)^H$. We have

$$\begin{aligned} Ad\rho(r)Ad\rho(t)\left(T\right) &= Ad\rho(rt)T \\ &= Ad\left(\rho\left((r*t)(r*t)^{-1}rt\right)\right)T \\ &= Ad\left(\rho(r*t)\right)\left(Ad(\rho\left((r*t)^{-1}rt\right)\right)T\Big) \\ &= Ad\left(\rho(r*t)\right)\left(T\right) \end{aligned}$$

because $(r*t)^{-1}rt \in H$ and T is $Ad\rho_H-$ invariant. We have that

$$Ad\left(\rho(rt)\right)\left(T\right) = Ad\left(\rho(r*t)\right)\left(T\right).$$

This proves (Remark 6.3.5) the first part of the following important Theorem.

THEOREM 6.3.6. *The group* $(R,*) \cong G/H$ *acts on* $C_u^*(G)^H$ *using the action induced by* $Ad\rho_G$*. We also have the following:*

$$C_u^*(G)^G \cong \left(C_u^*(G)^H\right)^{G/H}.$$

PROOF. First note if $T \in \left(C_u^*(G)^H\right)^{G/H}$ then $T \in C_u^*(G)^G$. Indeed, since for every $g \in G$, such that $g = h_g r_g$ and

$$\begin{aligned} Ad\rho(h_g r_g)T &= Ad\rho\left(r_g\left(r_g^{-1}h_g r_g\right)\right)(T) \\ &= Ad\rho(r_g)Ad\left(r_g^{-1}h_g r_g\right)(T) \\ &= T. \end{aligned}$$

So:

$$\left(C_u^*(G)^H\right)^{G/H} \subseteq C_u^*(G)^G.$$

We also have

$$C_u^*(G)^G \subseteq C_u^*(G)^H,$$

and

$$C_u^*(G)^G \cong \left(C_u^*(G)^H\right)^{G/H}.$$

□

Note also:

$$\begin{aligned} C_u^*(G)^G &\subseteq C_u^*(G)^H \\ &\subseteq C_u^*(H)^H \otimes C_u^*(G/H) \\ &\subseteq M_n(C_\lambda^*(H)). \end{aligned}$$

Thus

$$C_u^*(G)^G \cong M_n(C_\lambda^*(H))^{Ad\rho(G/H)}.$$

We now prove the following.

THEOREM 6.3.7. *Let H be normal finite index subgroup of G. Then*

$$\lambda_H \cong \lambda_H \otimes \lambda_{G/H}$$

and

$$\rho_H \cong \rho_H \otimes \rho_{G/H}.$$

PROOF. Let R be a set of coset representatives, which is the same for the left and the right cosets. We identify $(R,*)$ with G/H as before. Then if $g = h_g r_g$, $h_g \in H$, $r_g \in R$ then

$$\begin{aligned}
\lambda(g)\bigoplus_{r\in R}\ell^2(Hr) &= \bigoplus_{r\in R}\ell^2\left(g^{-1}Hr\right)\\
&= \bigoplus_{r\in R}\ell^2\left(r_g^{-1}h_g^{-1}Hr\right)\\
&= \bigoplus_{r\in R}\ell^2\left(r_g^{-1}h_g^{-1}(r_g^{-1})^{-1}\left(r_g^{-1}Hr_g\right)r_g^{-1}r\right)\\
&= \bigoplus_{r\in R}\ell^2\left(ad(r_g^{-1})(h_g^{-1})r_g^{-1}Hr_g r_g^{-1}r\right)\\
&= \bigoplus_{r\in R}\ell^2\left(ad(r_g^{-1})(h_g^{-1})Hr_g^{-1}r\left(r_g^{-1}*r\right)^{-1}\left(r_g^{-1}*r\right)\right).
\end{aligned}$$

Since $r_g^{-1}r\left(r_g^{-1}*r\right)^{-1} \in H$ we have that $Hr_g^{-1}r\left(r_g^{-1}*r\right)^{-1}$ is in bijection with H and so

$$\lambda(g)\bigoplus_{r\in R}\ell^2(Hr) = \bigoplus_{r\in R}\ell^2\left(ad(r_g^{-1})(h_g^{-1})H\left(r_g^{-1}*r\right)\right).$$

Now using the isomorphism Φ this is mapped to

$$\bigoplus_{r\in R}\ell^2\left(ad(r_g^{-1})(h_g^{-1})\right)\otimes \mathbb{C}\delta_{r_g^{-1}*r}$$

or in other words

$$\Phi\lambda_G(h_g r_g) \cong \lambda_H(ad(r_g^{-1})(h_g)\otimes \lambda_{G/H}(r_g).$$

Given that

$$ad(r_g^{-1}) : H \longrightarrow H$$

is a group isomorphism, $\lambda_H \circ ad$ is is a representation of H equivalent to λ_H.

Similarly, if if $g = r_g h_g$, $h_g \in H$, $r_g \in R$ then

$$\begin{aligned}
\rho(g) \bigoplus_{r \in R} \ell^2(rH) &= \bigoplus_{r \in R} \ell^2(rHr_g h_g) \\
&= \bigoplus_{r \in R} \ell^2\left(rr_g\left(r_g^{-1}Hr_g\right)h_g\right) \\
&= \bigoplus_{r \in R} \ell^2(rr_g H h_g) \\
&= \bigoplus_{r \in R} \ell^2\left((r * r_g)(r * r_g)^{-1} rr_g H h_g\right) \\
&= \bigoplus_{r \in R} \ell^2\left((r * r_g) H h_g\right)
\end{aligned}$$

where we use that $(r * r_g)^{-1} rr_g \in H$ and so $(r * r_g)^{-1} rr_g H$ is again in bijection with H and so induces a unitary isomorphism on $\ell^2(H)$. The isomorphism Φ will transport this to :

$$\Phi\rho_G(r_g h_g) = \rho_H(h_g) \otimes \rho_{G/H}(r_g).$$

□

We now finish the proof of Theorem 6.3.2 as follows. It follows from the Theorem 6.3.7 that

$$C_u^*(G)^G \cong C_u^*(H)^H \otimes C_u^*(G/H)^{G/H}$$

and that

$$C_\lambda^*(G) \cong C_\lambda^*(H) \otimes C_\lambda^*(G/H).$$

Therefore, if H has the invariant approximation property, and using the fact that finite groups satisfy this property we have

$$\begin{aligned} C_u^*(G)^G &\cong C_u^*(H)^H \otimes C_u^*(G/H)^{G/H} \\ &\cong C_\lambda^*(H) \otimes C_\lambda^*(G/H) \\ &\cong C_\lambda^*(G). \end{aligned}$$

This proves Theorem 6.3.2.

Bibliography

1. S. Adams, *Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups*, Topology **33** (1994), no. 4, 765–783. MR 1293309 (96g:58104)
2. B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.
3. B. Bollobas, *Linear Analysis-an introductory course*, Cambridge Mathematical Textbooks, 1999.
4. M. Bożejko and M. A. Picardello, *Weakly amenable groups and amalgamated products*, Proc. Amer. Math. Soc. **117** (1993), no. 4, 1039–1046. MR MR1119263 (93e:43005)
5. J. Brodzki and G. A. Niblo, *Approximation properties for discrete groups*, C^*-algebras and elliptic theory, Trends Math., Birkhäuser, Basel, 2006, pp. 23–35. MR MR2276913 (2007h:46089)
6. J. Brodzki, G. A. Niblo, and N. Wright, *Property A, partial translation structures, and uniform embeddings in groups*, J. Lond. Math. Soc. (2) **76** (2007), no. 2, 479–497. MR MR2363428 (2009h:46137)
7. N. P. Brown and N. Ozawa, *C^*- algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008. MR 2391387 (2009h:46101)
8. I. Chatterji, *Property (RD) for cocompact lattices in a finite product of rank one Lie groups with some rank two Lie groups*, Geom. Dedicata **96** (2003), 161–177. MR MR1956838 (2003k:22016)
9. ———, *Property of rapid decay*, Work shop on the Rapid Decay property available at http://www.aimath.org/WWN/rapiddecay/IntroRD.pdf, AIM Workshop, 2006.
10. ———, *On property (RD) for Certain Discrete Groups*, PhD thesis, ETH Zurich, September, 2001.
11. I. Chatterji, C. Pittet, and L. Saloff-Coste, *Connected Lie groups and property RD*, Duke Math. J. **137** (2007), no. 3, 511–536. MR MR2309152 (2008b:22005)
12. I. Chatterji and K. Ruane, *Some geometric groups with rapid decay*, Geom. Funct. Anal. **15** (2005), no. 2, 311–339. MR MR2153902 (2006d:20070)
13. K.R. Davidson, *C^*- algebra by Example*, vol. 6, Field Institute Monographs,American Mathematical Soc,, Providence, RI, 1986.

14. A. Dranishnikov and T. Januszkiewicz, *Every Coxeter group acts amenably on a compact space*, Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT), vol. 24, 1999, pp. 135–141. MR 1802681 (2001k:20082)
15. G. Effros, *Why the circle is connected: an introduction to quantized topology*, Math. Intelligencer **11** (1989), no. 1, 27–34. MR MR979021 (90d:46084)
16. M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73. MR MR623534 (83b:53041)
17. Erik Guentner, Nigel Higson, and Shmuel Weinberger, *The Novikov conjecture for linear groups*, Publ. Math. Inst. Hautes Études Sci. (2005), no. 101, 243–268. MR 2217050 (2007c:19007)
18. U. Haagerup, *An example of a non nuclear C^*- algebra, which has the metric approximation property*, Inventiones Mathematicae **50** (1978), no. 3, 279–293.
19. U. Haagerup and J. Kraus, *Approximation properties for group C^*- algebras and group von Neumann algebras*, Trans. Amer. Math. Soc. **344** (1994), no. 2, 667–699. MR MR1220905 (94k:22008)
20. T. Januszkiewicz, *For Coxeter groups $Z^{|g|}$ is a coeffcient of a uniformly bounded representation*, Preprint, 1999.
21. P. Jolissaint, *Rapidly decreasing functions in reduced C^*- algebras of groups*, Trans. Amer. Math. Soc. **317** (1990), no. 1, 167–196. MR MR943303 (90d:22006)
22. ______, *Borel cocycles, approximation properties and relative property T*, Ergodic Theory Dynam. Systems **20** (2000), no. 2, 483–499. MR MR1756981 (2001f:22015)
23. E. Kirchberg and S. Wassermann, *Exact groups and continuous bundles of C^*- algebras*, Math. Ann. **315** (1999), no. 2, 169–203. MR MR1721796 (2000i:46050)
24. Eberhard Kirchberg and Simon Wassermann, *Permanence properties of C^*-exact groups*, Doc. Math. **4** (1999), 513–558 (electronic). MR 1725812 (2001i:46089)
25. V. Lafforgue, *A proof of property (RD) for cocompact lattices of $SL(3,\mathbb{R})$ and $SL(3,\mathbb{C})$*, J. Lie Theory **10** (2000), no. 2, 255–267. MR MR1774859 (2001g:22020)
26. V. Lafforgue and d. Mikael, *Non commutative l^p spaces without the completely bounded approximation property*, (Preprint,http://arXiv:1004.2327v2,April 19, 2010).
27. E. C Lance, *On nuclear c^*- algebras.*, J. Funct. Anal. **12** (1973), no. 3, 157–176.
28. G. Niblo and L. Reeves, *Groups acting on* CAT(0) *cube complexes*, Geom. Topol. **1** (1997), approx. 7 pp. (electronic). MR MR1432323 (98d:57005)
29. N. Ozawa, *Weak amenability of hyperbolic groups*, Groups Geom. Dyn. **2** (2008), no. 2, 271–280. MR MR2393183 (2009f:20065)
30. J. Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003. MR MR2007488 (2004g:53050)

31. M. Takesaki, *Theory of operator algebras. I*, Encyclopaedia of Mathematical Sciences, vol. 124, Springer-Verlag, Berlin, 2002, Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5. MR MR1873025 (2002m:46083)
32. A. Valette, *Weak amenability of right-angled Coxeter groups*, Proc. Amer. Math. Soc. **119** (1993), no. 4, 1331–1334. MR MR1172955 (94a:46103)
33. S. Wassermann, *Exact* C^*- *algebra and related topics,lecture notes series*, 19. Seoul National University, Research Institute of Mathematics, 1994.
34. R. Willett, *Some notes on property A*, Limits of graphs in group theory and computer science, EPFL Press, Lausanne, 2009, pp. 191–281. MR 2562146 (2010i:22005)
35. G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139** (2000), no. 1, 201–240. MR MR1728880 (2000j:19005)
36. J. Zacharias, *On the Invariant Translation Approximation Property for Discrete Groups*, Proc. Amer. Math. Soc. **134** (2006), no. 7, 1909–1916 (electronic). MR MR2215118 (2007f:46056)