COHOMOLOGICAL INVARIANTS FOR INFINITE GROUPS

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ABSTRACT

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The main objects of interest in this thesis are $\mathbf{H}_1\mathfrak{F}$-groups. These are groups that act on finite-dimensional contractible CW-spaces with finite stabilisers. Important examples of these are given by groups admitting a finite-dimensional classifying space for proper actions $E_{\mathfrak{F}}G$. A large part of the thesis is motivated by an old conjecture of Kropholler and Mislin claiming that every $\mathbf{H}_1\mathfrak{F}$-group G admits a finite-dimensional model for $E_{\mathfrak{F}}G$. The natural choice for studying algebraically $\mathbf{H}_1\mathfrak{F}$-groups is $\mathfrak{F}$-cohomology. This is a form of group cohomology relative to a G-set introduced by Nucinkis in 1999. In this theory there is a well-defined notion of $\mathfrak{F}$-cohomological dimension and we study its behaviour under taking group extensions. A conjecture of Nucinkis claims that every group G of finite $\mathfrak{F}$-cohomological dimension admits a finite-dimensional model for $E_{\mathfrak{F}}G$. Note that it is unknown whether the class $\mathbf{H}_1\mathfrak{F}$ is closed under taking extensions. It is also unknown whether the class of groups admitting a finite-dimensional classifying space for proper actions is closed under taking extensions.

In Chapter 3 we introduce and study the notion of $\mathfrak{F}$-homological dimension and give an upper bound on the homological length of non-uniform lattices on locally finite $\mathrm{CAT}(0)$ polyhedral complexes, giving an easier proof that generalises an important result for arithmetic groups over function fields, due to Bux and Wortman.

The first Grigorchuk group $\mathfrak{G}$ was introduced in 1980 and has been extensively studied since due to its extraordinary properties. The class $\mathbf{H}\mathfrak{F}$ of *hierarchically decomposable* groups was introduced by Kropholler in 1993. There are very few known examples of groups that lie outside $\mathbf{H}\mathfrak{F}$. We answer the question regarding the $\mathbf{H}\mathfrak{F}$-membership of $\mathfrak{G}$ by showing that $\mathfrak{G}$ lies outside $\mathbf{H}\mathfrak{F}$.

In the final chapter we introduce a new class of groups $\mathfrak{U}$, and show that the Kropholler-Mislin conjecture holds for a subclass of $\mathfrak{U}$ and discuss its validity in general.

Contents

Declaration of Authorship

I, Giovanni Gandini, declare that the thesis entitled *"Cohomological Invariants for Infinite Groups"* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly while in candidature for a research degree at this university;
- no part of this thesis has previously been submitted for a degree or any other qualification at this university or any other institution;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work will be published as: "*Cohomological Invariants and the Classifying Space for Proper Actions*" in Groups, Geometry and Dynamics.

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Introduction

Let G be a group and let $\mathfrak{F}$ be the class of finite groups. A G-CW-complex is *proper* if all its cell stabilisers are in $\mathfrak{F}$. If a proper G-CW-complex X has the property that for each $\mathfrak{F}$-subgroup K of G the fixed-point subcomplex X^K is contractible, then X is called a *classifying space for proper actions of* G (or a model for $E_{\mathfrak{F}}G$). The interest in this object is twofold: from a geometric group theory perspective the spaces (or small modifications of these) on which "interesting" groups act are often classifying spaces for proper actions satisfying some finiteness condition. Outside the vast world of group theory, the equivariant K-homology of the classifying space for proper actions forms the left-hand side of the celebrated Baum-Connes conjecture. This conjecture is an important step for Connes' non-commutative geometry programme.

Generalisations of constructions due to Milnor [**Mil56**] and Segal [**Seg68**] show that every group G admits a model for $E_{\mathfrak{F}}G$. The Bredon geometric dimension of G, denoted by $\operatorname{gd}_{\mathfrak{F}} G$, is the minimal dimension of a model for $E_{\mathfrak{F}}G$. The Bredon cohomological dimension $\operatorname{cd}_{\mathfrak{F}} G$ plays a role analogous to that of the integral cohomological dimension $\operatorname{cd} G$ in ordinary group cohomology and is an algebraic counterpart of $\operatorname{gd}_{\mathfrak{F}} G$. In particular, $\operatorname{cd}_{\mathfrak{F}} G$ is finite if and only if $\operatorname{gd}_{\mathfrak{F}} G$ is finite [**Lüc89**]. However, both invariants are often very difficult to compute. Hence several possible "easy" geometric and algebraic invariants that guarantee their finiteness have been proposed by various authors [**Gui08, BDT09, Nuc00**].

On the geometric side, Kropholler introduced the class of $\mathrm{H}_1\mathfrak{F}$-groups [**Kro93**]. A group belongs to $\mathrm{H}_1\mathfrak{F}$ if there is a finite-dimensional contractible G-CW-complex X with cell stabilisers in $\mathfrak{F}$. The following problem has been open for almost 20 years.

Conjecture 1 (Kropholler-Mislin, [**Gui08, Mis01**]). *Every* $\mathrm{H}_1\mathfrak{F}$*-group* G *admits a finite-dimensional model for* $E_{\mathfrak{F}}G$.

A result proved independently by Bouc [**Bou99**] and Kropholler and Wall [**KW11**] implies that the augmented cellular chain complex $C_*(X)$ of any finite-dimensional contractible proper G-CW-complex splits when restricted to the $\mathfrak{F}$-subgroups of G. Nucinkis introduced a cohomology theory relative to a G-set Δ in order to algebraically mimic the behaviour of $\mathbf{H}_1\mathfrak{F}$-groups [**Nuc99**]. This theory can be regarded as a cohomology relative to a class of proper short exact sequences as in IX & XII [**ML95**] or as cohomology relative to the $\mathbb{Z}\,G$-module $\mathbb{Z}\,\Delta$. It generalises cohomology relative to a subgroup to cohomology relative to a family of subgroups. When dealing with the family of $\mathfrak{F}$-subgroups, we will refer to this as $\mathfrak{F}$-cohomology. In this setup there is a well-defined $\mathfrak{F}$-cohomological dimension $\mathfrak{F}\operatorname{cd} G$ [**Nuc00**]. It is an open question whether every group of finite $\mathfrak{F}$-cohomological dimension lies in $\mathbf{H}_1\mathfrak{F}$. The converse holds by the result of Kropholler and Wall mentioned above. Of course it is also unknown whether every group of finite $\mathfrak{F}$-cohomological dimension admits a finite-dimensional model for $E_{\mathfrak{F}}G$, and this is conjectured in [**Nuc00**]. It is well-known that for any group Γ, $\operatorname{cd}_{\mathbb{Q}}\Gamma \leqslant \mathfrak{F}\operatorname{cd}\Gamma \leqslant \operatorname{cd}_{\mathfrak{F}}\Gamma \leqslant \operatorname{gd}_{\mathfrak{F}}\Gamma$, see for example [**BLN01**].

Remaining on the algebraic side, it is important to mention that Bahlekeh, Dembegioti and Talelli conjecture in [**BDT09**] that every group of finite Gorenstein cohomological dimension $\operatorname{Gcd} G$ has finite Bredon geometric dimension. Most of these conjectures have positive answers in two important cases. The length $l(H)$ of an $\mathfrak{F}$-subgroup H of G is the supremum over all n for which there is a chain $H_0 < H_1 < \cdots < H_n = H$. Firstly, by applications of a result of Lück [**Lüc00**] if G has a bound on the lengths of its $\mathfrak{F}$-subgroups then the finiteness of $\operatorname{cd}_{\mathfrak{F}} G$ is equivalent to the finiteness of $\operatorname{Gcd} G$ and $\mathfrak{F}\operatorname{cd} G$. Secondly, if G is a countable elementary amenable group then $\operatorname{cd}_{\mathbb{Q}} G < \infty$, $\mathfrak{F}\operatorname{cd} G < \infty$ and $\operatorname{cd}_{\mathfrak{F}} G < \infty$ are equivalent by a theorem of Flores and Nucinkis [**FN05**].

We start this thesis by discussing the general theory of $\mathfrak{F}$-cohomology and by introducing the notion of $\mathfrak{F}$-injective module. Next we show that the $\mathfrak{F}$-cohomology can be calculated either with $\mathfrak{F}$-injective resolutions in the second variable or with $\mathfrak{F}$-projective resolutions in the first variable and therefore the $\mathfrak{F}$-cohomological dimension is independent of the chosen resolution. Complete $\mathfrak{F}$-cohomology via $\mathfrak{F}$-projective modules was defined in [**Nuc99**]. We develop complete $\mathfrak{F}$-cohomology via $\mathfrak{F}$-injective modules and prove some standard facts.

Let $\dim$ be a function from the class of all groups to $\mathbb{N} \cup \{\infty\}$. We say that $\dim$ is *subadditive* if for every group extension $N \hookrightarrow G \twoheadrightarrow Q$ we have $\dim G \leqslant \dim N + \dim Q$. The good behaviour of $\mathfrak{F}$-cohomological dimension with respect to several group operations is known [**Nuc00**] but its behaviour with respect to taking group extensions remains unclear. In Chapter 2 we investigate the subadditivity of $\mathfrak{F}$-cohomological dimension and we prove that it is subadditive if and only if it is preserved under taking extensions by groups of prime order. Leary and Nucinkis [**LN03**] build a group extension such that $\mathrm{cd}_{\mathfrak{F}} N = \mathfrak{F}\,\mathrm{cd}\, N = 2n$ and $\mathrm{cd}_{\mathfrak{F}} Q = \mathfrak{F}\,\mathrm{cd}\, Q = 0$ but $\mathrm{cd}_{\mathfrak{F}} G = 3n$ and $\mathfrak{F}\,\mathrm{cd}\, G = 2n$. Under extra conditions the behaviour of $\mathrm{gd}_{\mathfrak{F}} G$ under taking group extensions is known [**Mis01, MP02, MP07**]. It is still unknown whether there exists a group of infinite Bredon geometric dimension that is an extension of two groups of finite Bredon geometric dimension. On the other hand it is known that the Gorenstein cohomological dimension is subadditive [**BDT09**]. The precise connections between the Gorenstein cohomological dimension and the $\mathfrak{F}$-cohomological and rational cohomological dimensions are unclear. We show that $\mathrm{Gcd}\, G \leqslant \mathfrak{F}\,\mathrm{cd}\, G$, but it is unknown whether the finiteness of $\mathrm{Gcd}\, G$ implies the finiteness of $\mathfrak{F}\,\mathrm{cd}\, G$. If there exists a group G that has $\mathrm{Gcd}\, G < \infty$ or $\mathfrak{F}\,\mathrm{cd}\, G < \infty$ but admits no finite-dimensional model for $E_{\mathfrak{F}} G$, then by the theorem of Lück, G cannot have a bound on the lengths of its $\mathfrak{F}$-subgroups.

In Chapter 3 we look at $(\mathfrak{F})$-cohomological conditions of finite type. We introduce the notion of $\mathfrak{F}$-homological dimension and we prove some standard results. An interesting source of groups with no bound on the orders of their $\mathfrak{F}$-subgroups is given by non-uniform lattices on locally finite $\mathrm{CAT}(0)$ polyhedral complexes; we close the chapter by bounding their homological length.

Branch groups are certain subgroups of the full automorphism groups of spherically homogeneous rooted trees. Several examples of finitely-generated periodic non-elementary amenable groups with no bound on the lengths of their $\mathfrak{F}$-subgroups lie in this class. Here we show that finitely-generated regular branch groups have infinite rational cohomological dimension, which implies that the $\mathfrak{F}$-cohomological dimension and the Bredon cohomological dimension are infinite as well. Let $\mathbf{H}\mathfrak{F}$ be Kropholler's class of *hierarchically decomposable groups* [**Kro93**]. The class $\mathbf{H}\mathfrak{F}$ is defined as the smallest class of groups containing the class $\mathfrak{F}$ and which contains a group G whenever there is an admissible action of G on a finite-dimensional contractible cell complex for which all isotropy groups already belong to $\mathbf{H}\mathfrak{F}$. An important question in this area is to determine which branch groups belong to the class $\mathbf{H}\mathfrak{F}$. Until the recent work [**ABJ$^+$09**], where groups with a strong global fixed-point property are constructed, the only way to show that a group G does not belong to $\mathbf{H}\mathfrak{F}$ was to find a subgroup of G isomorphic to Thompson's group $\mathbf{F}$. In Chapter 4 we show that the first Grigorchuk group $\mathfrak{G}$ is not contained in the class $\mathbf{H}\mathfrak{F}$. Furthermore, $\mathfrak{G}$ is a counterexample to a conjecture of Petrosyan [**Pet07**] and answers negatively a question of Jo-Nucinkis [**JN08**].

We introduce a new class of groups $\mathfrak{U}$ defined in terms of actions on trees and taking extensions starting from the class of groups admitting a finite-dimensional $E_{\mathfrak{F}}G$ with a bound on the orders of their $\mathfrak{F}$-subgroups. The final chapter is dedicated to the study of the class of $\mathfrak{U}$-groups. In particular we show that the Kropholler-Mislin conjecture holds for a subclass of $\mathfrak{U}$.

Notation. We use standard notation for classes of groups. A collection $\mathfrak{Y}$ of groups is a *class of groups* if it contains the trivial group and it is closed under taking isomorphisms. We write $\mathfrak{I}$ for the class consisting of the trivial group, $\mathfrak{F}$ for the class of finite groups, $\mathfrak{A}$ for the class of abelian groups and $\mathbf{Fr}$ for the class of free groups. For a class of groups $\mathfrak{X}$, $\mathbf{L}\mathfrak{X}$ denotes the class of locally $\mathfrak{X}$-groups, i.e. G lies in $\mathbf{L}\mathfrak{X}$ if and only if every finitely-generated subgroup of G lies in $\mathfrak{X}$. Less common group operations will be defined later in the thesis.

CHAPTER 1

$\mathfrak{F}$-Cohomology

We study cohomology of groups relative to a G-set Δ which first appeared in [**Nuc99**]. This theory can be regarded either as cohomology relative to a class of proper short exact sequences as developed by Mac Lane in Chapter IX & XII in [**ML95**] or as cohomology relative to a permutation G-module $k\,\Delta$. This theory generalises cohomology relative to a subgroup to a family of subgroups. In the first section we discuss the fundamental concepts. Next we introduce the notion of a Δ-injective module which we use later in complete relative cohomology. We repeat two constructions of complete cohomology groups in the context of relative cohomology using Δ-injective modules. The first complete relative cohomology groups that we construct, $\Delta\widetilde{\mathrm{Ext}}^n_{kG}(-,\,N)$, are built similarly to Mislin's completion via satellites [**Mis94**]. The second construction follows Benson and Carlson's approach [**BC92**] and this leads to the groups $\Delta\,\widetilde{\mathrm{BC}}^n_{kG}(-,N)$.

We show that as for ordinary cohomology [**Nuc98**], these constructions are equivalent.

Theorem. *Let N be a kG-module, then there is a natural equivalence of functors*

$$\Theta^n : \Delta\widetilde{\mathrm{Ext}}^n_{kG}(-,\,N) \to \Delta\,\widetilde{\mathrm{BC}}^n_{kG}(-,N), \forall n \in \mathbb{Z}.$$

1. Cohomology relative to a G-set

Let G be a group, k a commutative ring of coefficients and Δ an arbitrary G-set. We write $k\Delta$ for the free k-module on the set Δ. The abelian group $k\Delta$ can be regarded as a kG-module by extending the G-action on Δ to a linear k-action of G on $k\,\Delta$. The kG-module $k\,\Delta$ is called the *permutation module* on the G-set Δ. We write $\otimes$ for the tensor product over k, and if M and N are kG-modules, the G-action on $M \otimes N$ is defined as $(m \otimes n)g := mg \otimes ng$ for $g \in G$, $m \in M$ and $n \in N$.

Definition 1.1. [**Nuc99**, 2.1] A short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ of kG-modules is called Δ-*split* if

$$A \otimes k\Delta \rightarrowtail B \otimes k\Delta \twoheadrightarrow C \otimes k\Delta$$

is a kG-split sequence.

A kG-module is Δ-*projective* if it is a direct summand of a module of the form $M \otimes k\,\Delta$ for some kG-module M. It is easy to characterise Δ-projective kG-modules in a slightly different manner. In fact, the next lemma clarifies the definition of Δ-projectivity.

Lemma 1.2. *For a kG-module P the following are equivalent:*

(1) P is Δ-projective,
(2) the functor $\mathrm{Hom}_{kG}(P, -)$ *is exact on Δ-split sequences,*
(3) every Δ-split epimorphism $\alpha : N \twoheadrightarrow P$ splits over kG.

PROOF. This can be proved analogously to the case of ordinary projective modules. A proof for projective modules can be found in any homological algebra book. □

The category of right kG-modules, denoted by $\mathfrak{Mod}_{kG}$ has enough Δ-projectives since the obvious surjection $M \otimes k\,\Delta \twoheadrightarrow M$ given by $m \otimes \delta \mapsto m$ is Δ-split. To see this, let $M \otimes k\,\Delta \otimes k\,\Delta \twoheadrightarrow M \otimes k\,\Delta$ given by $m \otimes \delta_1 \otimes \delta_2 \mapsto m \otimes \delta_2$ the split is given by $m \otimes \delta \mapsto m \otimes \delta \otimes \delta$. Note that even if $k\,\Delta$ and M are finitely-generated kG-modules this construction does not provide a finitely-generated Δ-projective mapping onto M via a Δ-split surjection. This issue is briefly discussed in Chapter 3.

While it was clear what was the appropriate definition of a Δ-projective module, there is in the literature some uncertainty on the correct definition of a Δ-free module. The functor $- \otimes k\,\Delta$ is the left adjoint of the functor $\mathrm{Hom}(k\,\Delta, -)$, so it is reasonable to define Δ-*free modules* as direct sums of modules of the form $M_i \otimes k\,\Delta$, where M_i's are kG-modules. Moreover, with this definition there is a relative version of the Eilenberg Swindle Lemma:

Lemma 1.3. [**Nuc00**, 4.1] *If P is a Δ-projective then there exist a Δ-free module F such that $F \cong P \oplus F$.*

We want to have a theory that does not depend on the G-set Δ up to G-maps; more precisely:

Lemma 1.4. [**Nuc99**, 2.5] *Let Δ_1 and Δ_2 be two G-sets. If there exists a G-map $\phi : \Delta_1 \to \Delta_2$, then every Δ_2-split short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ of kG-modules is Δ_1-split.*

Lemma 1.5. [**Nuc99**, 2.3] *Let Δ_1 and Δ_2 be two G-sets. If there exists a G-map $\phi : \Delta_1 \to \Delta_2$, then every Δ_1-projective kG-module is Δ_2-projective.*

Since the class of Δ-projective modules is closed under taking direct summands and direct sums, it follows that the building blocks for the Δ-projectives are the direct summands of modules of the form $M \uparrow_H^G$ where H is a stabiliser of some orbit representative for Δ. Moreover, by Lemma 1.5 if M is any abelian group the module $M \otimes k\,\Delta$ with action on the right side is Δ-projective.

Let M and N be two kG-modules. The cohomology functors relative to Δ of M with coefficient in N are defined by

$$\Delta\,\mathrm{Ext}^n(M, N) := \mathrm{H}^n(\mathrm{Hom}_{kG}(\mathbf{P_M}, N)),$$

that is the *n-th right derived (relative) functor* of $\mathrm{Hom}_{\mathbb{Z}G}(-, N)$ as [**ML95**, pg. 389]. A Δ-projective dimension is well-defined, and we say that a group G has *Δ-cohomological dimension* over kG equal to n if k regarded as a kG-module with trivial action has Δ-projective dimension over kG equal to n.

There is an obvious way to write the *"standard Δ-projective resolution"* of the trivial kG-module k [**Nuc00**]. Let $P_i = k(\Delta^i)$ and $K_n = \ker d_{n-1}$, where the maps $d_i : P_{i+1} \to P_i$ are defined as

$$d_i(\delta_0, \delta_1, \ldots, \delta_i) = \sum_{k=0}^{i} (-1)^k (\delta_0, \delta_1, \ldots, \hat{\delta}_k, \ldots, \delta_i)$$

where $\hat{\delta}_k$ means δ_k is omitted.

The fact that any G-set Δ admits a decomposition as a disjoint union of sets of cosets suggests that we can use this cohomology to have a theory relative to a family of subgroups closed under conjugation. In this work we are mostly interested in the family of $\mathfrak{F}$-subgroups. More concretely, let Δ_0 be a set of orbit representatives for Δ and let G_δ be the stabiliser of $\delta \in \Delta_0$. Then we have:

$$\Delta = \bigsqcup_{\delta \in \Delta_0} \delta G \cong \bigsqcup_{\delta \in \Delta_0} G_\delta \backslash G.$$

Note that if one of the G_δ is non-trivial, then $\mathbb{Z}\,\Delta$ is not $\mathbb{Z}\,G$-projective. To see this is enough to recall that the module $\mathbb{Z}\,\Delta$ admits the decomposition $\mathbb{Z}\,\Delta \cong \bigoplus_\delta \mathbb{Z} \otimes_{\mathbb{Z}\,G_\delta} \mathbb{Z}\,G$ and that by [**CK96**, Lemma 6.1] the module $A \otimes_{\mathbb{Z}\,G_\delta} \mathbb{Z}\,G$ is $\mathbb{Z}\,G$-projective if and only if A is $\mathbb{Z}\,G_\delta$-projective.

Lemma 1.6. [**Nuc99**, 6.1] *For any pair Δ_1, Δ_2 of G-sets that satisfy the following condition*

$$(*) \qquad \Delta^H \neq \varnothing \Leftrightarrow H \leqslant G,\ H \in \mathfrak{F},$$

there exist G-maps: $\phi : \Delta_1 \to \Delta_2$ and $\rho : \Delta_2 \to \Delta_1$.

An example of a such G-set is $\Phi = \bigsqcup_{H \leqslant G, H \in \mathfrak{F}} H\backslash G$, but Lemma 1.5 implies that every G-set satisfying condition $(*)$ generates the same cohomology theory. For any such G-set we replace the letter Δ with $\mathfrak{F}$ to designate this case in cohomology. The following clarifies the concept of Δ-split short exact sequence.

Lemma 1.7. [**Nuc99**, 2.6, 6.2] *A short exact sequence of kG-modules $A \rightarrowtail B \twoheadrightarrow C$ is Δ-split if and only if it splits restricted to each stabiliser G_δ. In particular, $A \rightarrowtail B \twoheadrightarrow C$ is $\mathfrak{F}$-split if and only if it splits restricted to each $\mathfrak{F}$-subgroup of G.*

Concretely to compute group cohomology we replace kG-projective resolutions used in (ordinary) cohomology of groups with $\mathfrak{F}$-split resolutions made of direct summand of sums of induced modules from $\mathfrak{F}$-subgroups of G. Note that any $\mathfrak{F}$-split acyclic complex of kG-modules is k-split; and in particular every $\mathfrak{F}$-projective resolution of k is k-split. Recently it was noticed that even if $\mathfrak{F}$-cohomology is defined with respect to the family of $\mathfrak{F}$-subgroups it takes into consideration only the finite $\mathfrak{P}$-subgroups,. We write $\mathfrak{P}_{\mathfrak{F}}$ for the class $\mathfrak{F} \cap \mathfrak{P}$.

Theorem 1.8. [**LN10**]

- *A short exact sequence of kG-modules is $\mathfrak{F}$-split if and only if it is $\mathfrak{P}_{\mathfrak{F}}$-split.*
- *A kG-module is $\mathfrak{F}$-projective if and only if it is $\mathfrak{P}_{\mathfrak{F}}$-projective.*
- $\mathfrak{F}\,\mathrm{H}^*(G; -) \cong \mathfrak{P}_{\mathfrak{F}}\,\mathrm{H}^*(G; -)$.

Lemma 1.4 and Theorem 1.8 show that if G has a bound on the order of its $\mathfrak{F}$-subgroups, then when we build a G-set Φ satisfying $(*)$ we can just consider one subgroup per conjugacy class of maximal $\mathfrak{P}_{\mathfrak{F}}$-subgroups.

Lemma 1.9. *Let* $\mathrm{A} \rightarrowtail \mathrm{B} \twoheadrightarrow \mathrm{C}$ *be a short exact sequence of* $\mathbb{Z}$*-free* $\mathbb{Z}\,G$*-modules. Suppose that the short exact sequence*

$$A\downarrow_P \otimes \mathbb{Z}/|P|\,\mathbb{Z} \rightarrowtail B\downarrow_P \otimes \mathbb{Z}/|P|\,\mathbb{Z} \twoheadrightarrow C\downarrow_P \otimes \mathbb{Z}/|P|\,\mathbb{Z}$$

splits over $\mathbb{Z}\,P$ *for every* $\mathfrak{P}_{\mathfrak{F}}$*-subgroup* P *of* G*. Then* $\mathrm{A} \rightarrowtail \mathrm{B} \twoheadrightarrow \mathrm{C}$ *is* $\mathfrak{F}$*-split.*

PROOF. It follows from [**KW11**, Lemma 3.4] and Theorem 1.8. □

Remark 1.10. In view of Chouinard's Theorem [**Cho76**] it is natural to ask if $\mathfrak{F}$-cohomology can be reduced to a cohomological theory relative to the family $\mathfrak{E}$ of finite elementary abelian subgroups. This is not the case; to see this let P be a non-elementary abelian $\mathfrak{P}_{\mathfrak{F}}$-group and let $\{H_i\}$ be the family of conjugacy classes of its elementary abelian subgroups. The short exact sequence $K \rightarrowtail \oplus_i \mathbb{Z}\,H_i\backslash P \overset{\pi}{\twoheadrightarrow} \mathbb{Z}$ is $\mathfrak{E}$-split but does not split over $\mathbb{Z}\,P$. Let $\sigma_i \in \mathbb{Z}[H_i\backslash P]$ denote the sum of the cosets of H_i in P, that is $\sigma_i = \sum_{H_ip \in H_i\backslash P} H_ip$. Since P/H_i is a transitive P-set the only well-defined $\mathbb{Z}\,P$-map from $\mathbb{Z}$ to $\mathbb{Z}\,H_i\backslash P$ is the map $1 \mapsto m_i\sigma_i$ where m_i is a non-zero integer. Any $\mathbb{Z}\,P$-map $\iota : \mathbb{Z} \to \oplus_i \mathbb{Z}\,H_i\backslash P \cong \mathbb{Z}\,H_1\backslash P \oplus \mathbb{Z}\,H_2\backslash P \oplus \cdots \oplus \mathbb{Z}\,H_n\backslash P$ is defined by $1 \mapsto (m_1\sigma_1, \ldots, m_n\sigma_n)$ for some choice of $\{m_1, m_2, \ldots, m_n\}$. Since $\pi \circ \iota(1) = \sum_{i=1}^n m_i[P : H_i] = \sum_{i=1}^n m_i p^{n_i} \neq 1$ ($n_i \neq 0$ for all i's), π does not split over $\mathbb{Z}\,P$.

Lemma 1.11 (Shapiro's Lemma)**.** *Let* H *be a subgroup of* G *and* N *be a* $\mathbb{Z}H$*-module. Then*

$$\mathfrak{F}\,\mathrm{H}^n(H; N) \cong \mathfrak{F}\,\mathrm{H}^n(G; \mathrm{Coind}_H^G N),$$

where $\mathfrak{F}\,\mathrm{H}^*(G; -) := \mathfrak{F}\,\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, -)$.

PROOF. At first we recall that any $\mathfrak{F}$-projective resolution $\mathbf{P}$ of $\mathbb{Z}G$-modules can be regarded as an $\mathfrak{F}$-projective resolution of $\mathbb{Z}H$-modules. From the isomorphism $\mathrm{Hom}_{\mathbb{Z}H}(M, N) \cong \mathrm{Hom}_{\mathbb{Z}G}(M, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, N))$, it follows that

$$\mathrm{Hom}_H(\mathbf{P}, N) \cong \mathrm{Hom}_G(\mathbf{P}, \mathrm{Coind}_H^G N)$$

which completes the proof. □

There is also a Shapiro's Lemma for $\mathfrak{F}$-homology; this is presented in Chapter 3. There are a few immediate consequences of Shapiro's Lemma that we mention explicitly:

Corollary 1.12. *The group* $\mathfrak{F}\,\mathrm{H}^n(G;\mathrm{Coind}_{\{e\}}^G(A)) = 0$ *for every abelian group* A *and for every* $n > 0$.

Lemma 1.13. *If* $[G : H] < \infty$*, then*

$$\mathfrak{F}\,\mathrm{H}^*(H;\mathbb{Z}\,H) \cong \mathfrak{F}\,\mathrm{H}^*(G;\mathbb{Z}\,G).$$

PROOF. Since $[G : H] < \infty$ we have for any $\mathbb{Z}\,H$-module M, $\mathrm{Coind}_H^G M \cong M \uparrow_H^G$ [**Bro82**, Proposition 5.9, III]. $\square$

Lemma 1.14 (Transfer Maps)**.** *Let* M *be a* $\mathbb{Z}\,G$*-module and* $[G : H] < \infty$*. Then there exist the following maps*

$$\mathfrak{F}\,\mathrm{cor}_H^G : \mathfrak{F}\,\mathrm{H}^*(H;M) \to \mathfrak{F}\,\mathrm{H}^*(G;M),$$

$$\mathfrak{F}\,\mathrm{res}_H^G : \mathfrak{F}\,\mathrm{H}_*(G;M) \to \mathfrak{F}\,\mathrm{H}_*(H;M).$$

PROOF. Apply $\mathfrak{F}\,\mathrm{H}^*(G;-)$ to the canonical injection $M \rightarrowtail \mathrm{Hom}_{\mathbb{Z}\,H}(\mathbb{Z}\,G, M)$. From Shapiro's Lemma for $\mathfrak{F}$-cohomology there is a map $\alpha^* : \mathfrak{F}\,\mathrm{H}^*(G;M) \to \mathfrak{F}\,\mathrm{H}^*(H;M)$. Again, since $[G : H] < \infty$ we have $\mathrm{Coind}_H^G M \cong M \uparrow_H^G$ and we can apply $\mathfrak{F}\,\mathrm{H}^*(G;-)$ to the canonical surjection $\mathbb{Z}\,G \otimes_{\mathbb{Z}\,H} M \twoheadrightarrow M$ to obtain $\mathfrak{F}\,\mathrm{cor}_H^G : \mathfrak{F}\,\mathrm{H}^*(H;M) \to \mathfrak{F}\,\mathrm{H}^*(G;M)$.

Analogously, we apply the functor $\mathfrak{F}\,\mathrm{H}^*(G;-)$ to the canonical surjection $\mathbb{Z}\,G \otimes_{\mathbb{Z}\,H} M \twoheadrightarrow M$ and by Shapiro's Lemma for $\mathfrak{F}$-homology we obtain $\alpha_* : \mathfrak{F}\,\mathrm{H}_*(H;M) \to \mathfrak{F}\,\mathrm{H}_*(G;M)$. Again, since $[G : H] < \infty$ we have $\mathrm{Coind}_H^G M \cong M \uparrow_H^G$ and we can apply $\mathfrak{F}\,\mathrm{H}_*(G;-)$ to the canonical injection $M \rightarrowtail \mathrm{Hom}_{\mathbb{Z}\,H}(\mathbb{Z}\,G, M)$ to obtain $\mathfrak{F}\,\mathrm{res}_H^G : \mathfrak{F}\,\mathrm{H}_*(G;M) \to \mathfrak{F}\,\mathrm{H}_*(H;M)$. $\square$

2. Complete Δ-cohomology via Δ-injective modules

Since the augmented cellular complex of a n-dimension contractible G-CW-complex with $\mathfrak{F}$-stabilisers gives rise to an $\mathfrak{F}$-projective resolution of the trivial module $\mathbb{Z}$ over $\mathbb{Z}\,G$, the natural choice of objects to work in algebra are $\mathfrak{F}$-projective modules. In fact, Nucinkis developed the general theory using Δ-projective modules. However, even if there is no geometric counterpart we show in this section that the category of right K-modules $\mathfrak{Mod}_{kG}$ has enough Δ-*injective modules*. In order to state and prove standard results in group cohomology via $\mathfrak{F}$-injectives we need several dual results included in Chapters IX & XII [**ML95**] for relative injective objects in some suitable category, and for this we avoid including the proofs that are completely analogous to the ones for projectives.

Remark 1.15. Let N be a kG-module and let $\mathrm{Hom}_k(k\Delta, N)$ be the kG-module with the G-action defined as $(\varphi g)\delta = [\varphi(\delta g^{-1})]g$, for $\varphi \in \mathrm{Hom}_k(k\Delta, N)$. For every kG-module N we have a G-map $N \overset{\iota}{\rightarrowtail} \mathrm{Hom}_k(k\Delta, N)$ given by $\iota(n) = \varphi_n$, where $\varphi_n(\delta) = n$ for every $\delta \in \Delta$. Moreover, ι is Δ-split, in fact the injection $\iota_N : N \otimes k\Delta \to \mathrm{Hom}_k(k\Delta, N) \otimes k\Delta$ defined by $\iota_N(n \otimes \delta) = \varphi_n \otimes \delta$ splits. The splitting is given by $\varphi \otimes \delta \mapsto \varphi(\delta) \otimes \delta$. From now on we write Hom for the functor Hom_k.

Definition 1.16. [**ML95**] Let $\mathcal{A}$ be an abelian category and let $\mathcal{P}$ be a class of short exact sequences. A monomorphism $\phi : A \rightarrowtail B \in \mathcal{P}_m$ if there is an object $C \in \mathcal{A}$ and an epimorphism $B \twoheadrightarrow C$ such that A $\rightarrowtail$ B $\twoheadrightarrow$ C $\in \mathcal{P}$. Analogously, an epimorphism $\rho : B \twoheadrightarrow C \in \mathcal{P}_e$ if there is an object $A \in \mathcal{A}$ and a monomorphism $A \rightarrowtail B$ such that A $\rightarrowtail$ B $\twoheadrightarrow$ C $\in \mathcal{P}$.

The class $\mathcal{P}$ is a *proper class of short exact sequences* if the following hold:

- $\mathcal{P}$ is closed under taking isomorphisms,
- $A \rightarrowtail A \oplus B \twoheadrightarrow B \in \mathcal{P}$,
- if $A \overset{\phi}{\rightarrowtail} B$ and $B \overset{\rho}{\rightarrowtail} C \in \mathcal{P}_m$ then $A \overset{\rho\phi}{\rightarrowtail} C \in \mathcal{P}_m$,
- if $B \overset{\phi}{\twoheadrightarrow} C$ and $C \overset{\rho}{\twoheadrightarrow} D \in \mathcal{P}_e$ then $B \overset{\rho\phi}{\twoheadrightarrow} D \in \mathcal{P}_e$,
- if $A \overset{\phi}{\rightarrowtail} B$ and $B \overset{\rho}{\rightarrowtail} C$ are two monomorphisms such that $A \overset{\rho\phi}{\rightarrowtail} C \in \mathcal{P}_m$, then $A \overset{\phi}{\rightarrowtail} B \in \mathcal{P}_m$,
- if $B \overset{\phi}{\twoheadrightarrow} C$ and $C \overset{\rho}{\twoheadrightarrow} D$ are two epimorphisms such that $B \overset{\rho\phi}{\twoheadrightarrow} D \in \mathcal{P}_e$, then $C \overset{\rho}{\twoheadrightarrow} D \in \mathcal{P}_e$.

Lemma 1.17. [**Nuc99**, 3.1] *The set of Δ-split short exact sequences forms a proper class of short exact sequences.*

A kG-module I is called Δ-*injective* if for every Δ-split $\iota : A \rightarrowtail B$ and every $\alpha : A \to I$ there exists $\beta : B \to I$ such that $\beta\iota = \alpha$. Since for any $\mathbb{Z}G$-projective P, the surjection $\mathbb{Z}\Delta \otimes P \twoheadrightarrow P$ splits any $\mathbb{Z}G$-projective module is $\mathfrak{F}$-projective. Analogously, any $\mathbb{Z}G$-injective module is $\mathfrak{F}$-injective. We write $\mathcal{I}_\Delta(kG)$ for the class of Δ-injectives kG-modules. From basic properties of the functor Hom, we have that $\mathcal{I}_\Delta(kG)$ is closed under taking arbitrary direct product and finite direct sums.

Lemma 1.18. *A kG-module I is Δ-injective if and only if the functor* $\mathrm{Hom}_{kG}(-, I)$ *is exact on Δ-split short exact sequences.*

PROOF. Let I be a Δ-injective kG-module and $A \overset{\iota}{\rightarrowtail} B \overset{\tau}{\twoheadrightarrow} C$ be a Δ-split short exact sequence, by the left exactness of the contravariant functor $\mathrm{Hom}_{kG}(-, I)$ we only need to prove that $\gamma : \mathrm{Hom}_{kG}(B, I) \to \mathrm{Hom}_{kG}(A, I)$ is an epimorphism. Since ι is a Δ-split monomorphism and I is a Δ-injective module, every $\alpha \in \mathrm{Hom}_{kG}(A, I)$ factors through ι.

Conversely, assume that $\mathrm{Hom}_{kG}(-,\, I)$ is exact on Δ-split short exact sequences and let $\iota : A \rightarrowtail B$ be a Δ-split monomorphism. Since $\alpha_* : \mathrm{Hom}_{kG}(B, I) \to \mathrm{Hom}_{kG}(A, I)$ is an epimorphism for every $\alpha \in \mathrm{Hom}_{kG}(A, I)$ there is a $\beta \in \mathrm{Hom}_{kG}(B, I)$ with $\alpha = \alpha_*(\beta) = \iota\beta$. $\square$

From Lemma 1.4 the next result follows immediately.

Corollary 1.19. *Let Δ_1 and Δ_2 be two G-sets. If there exists a G-map $\phi : \Delta_1 \to \Delta_2$, then every Δ_1-injective kG-module is Δ_2-injective.*

Lemma 1.20. *For any kG-module N, the kG-module* $\mathrm{Hom}(k\Delta, N)$ *is Δ-injective. In particular, the category $\mathfrak{Mod}_{kG}$ has enough Δ-injectives.*

PROOF. Let $A \rightarrowtail B \twoheadrightarrow C$ be a Δ-split short exact sequence. Since $\mathrm{Hom}_{kG}(-, N)$ is a left exact functor, is additive and so preserves splitting, i.e. if $A \otimes k\Delta \rightarrowtail B \otimes k\Delta \twoheadrightarrow C \otimes k\Delta$ splits then $\mathrm{Hom}_{kG}(C \otimes k\Delta,\, N) \rightarrowtail \mathrm{Hom}_{kG}(B \otimes k\Delta,\, N) \twoheadrightarrow \mathrm{Hom}_{kG}(A \otimes k\Delta,\, N)$ splits. By applying the natural isomorphism [**ML95**, Ex. 4 pg. 272] $\mathrm{Hom}_{kG}(U,\, \mathrm{Hom}(V,\, T)) \cong \mathrm{Hom}_{kG}(U \otimes V,\, T)$ to every term in the short exact sequence we obtain $\mathrm{Hom}_{kG}(C,\, \mathrm{Hom}(k\,\Delta,\, N)) \rightarrowtail \mathrm{Hom}_{kG}(B,\, \mathrm{Hom}(k\,\Delta,\, N)) \twoheadrightarrow \mathrm{Hom}_{kG}(A,\, \mathrm{Hom}(k\,\Delta,\, N))$. Hence by Lemma 1.18 $\mathrm{Hom}(k\,\Delta, N)$ is Δ-injective. In Remark 1.15 we showed that the G-map $N \overset{\iota}{\rightarrowtail} \mathrm{Hom}_k(k\Delta,\, N)$ is Δ-split and therefore the category $\mathfrak{Mod}_{kG}$ has enough Δ-injectives. $\square$

Lemma 1.21. *Let I be a kG-module. Then the following are equivalent:*

(1) I is a direct summand of $\mathrm{Hom}(k\Delta, N)$ *for some kG-module N,*
(2) I is Δ-injective,
(3) every Δ-split monomorphism $\alpha : I \rightarrowtail J$ splits.

PROOF. The implication $(2) \Rightarrow (3)$ is obvious, just take the identity map $\alpha : I \to I$. For $(3) \Rightarrow (1)$, by Remark 1.15 $\iota : I \rightarrowtail \mathrm{Hom}(k\Delta, I)$ is Δ-split, and so splits by hypothesis, i.e. $I \oplus L \cong \mathrm{Hom}(k\Delta, I)$ for some kG-module L. For the remaining implication $(1) \Rightarrow (2)$, if I is a direct summand of a module of the form $\mathrm{Hom}(k\Delta, J)$ then by Lemma 1.20 and the splitting injection, $I \rightarrowtail \mathrm{Hom}(k\Delta, J)$, I is Δ-injective. $\square$

Lemma 1.22. *For any G-module L and any $\mathfrak{F}$-subgroup H of G, the kG-module $\mathrm{Coind}_H^G L$ is $\mathfrak{F}$-injective. For any G-set Δ and any $\delta \in \Delta$, $\mathrm{Coind}_{G_\delta}^G L$ is Δ-injective.*

PROOF. We first show that $\mathrm{Coind}_H^G L$ is $H\backslash G$-injective, that is the functor $\mathrm{Hom}_{kG}(-, \mathrm{Coind}_H^G L)$ is exact on H-split short exact sequences. Consider an H-split short exact sequence $\mathrm{A} \rightarrowtail \mathrm{B} \twoheadrightarrow \mathrm{C}$ of kG-modules. The short exact sequence $A \otimes_{kH} kG \rightarrowtail B \otimes_{kH} kG \twoheadrightarrow C \otimes_{kH} kG$ kG-splits by definition. For any kG-module M the functor $\mathrm{Hom}_{kG}(-, M)$ is exact on kG-split sequences. From $\mathrm{Hom}_{kG}(-,\, \mathrm{Hom}_{kH}(kG,\, L)) \cong \mathrm{Hom}_{kG}(- \otimes_{kH} kG,\, L)$ [**Bro82**, Proposition 5.6 III] and the isomorphism $- \otimes_{kH} kG \cong - \downarrow_H^G \otimes_{kH} kG$ it follows that $\mathrm{Coind}_H^G L$ is $H\backslash G$-injective. Let $\Phi = \bigsqcup_{K \leqslant G, K \in \mathfrak{F}} K\backslash G$. Each element in $H\backslash G$ has an $\mathfrak{F}$-stabiliser and so there exists a G-map $H\backslash G \to \Phi$. Hence by Corollary 1.19 $\mathrm{Coind}_H^G L$ is $\mathfrak{F}$-injective.

The previous argument applies by replacing H with G_δ, and there is an obvious G-map $G/G_\delta \to \Delta$. $\square$

Let $\mathcal{A}$ be an abelian category. A complex $C \rightarrowtail X_0 \to X_1 \to \cdots \to X_n \to \cdots$ over an object C in $\mathcal{A}$ is an *allowable injective resolution* of C if it is an allowable long exact sequence with X_i allowable injective for every i. Now we state the analogue of the Theorem [**ML95**, IX, 4.3] for injective complexes.

Theorem 1.23 (Comparison Theorem)**.** *Let $\mathcal{E}$ be an allowable class of short exact sequences in the abelian category $\mathcal{A}$. If $\gamma : A \to A'$ is a morphism of $\mathcal{A}$, $\epsilon' : A' \to \mathbf{Y}'$ an allowable injective complex over A' and $\epsilon : A \to \mathbf{Y}$ an allowable resolution of A, then there is a chain transformation $f : \mathbf{Y} \to \mathbf{Y}'$ of morphisms of $\mathcal{A}$ with $\epsilon'\gamma = f\epsilon$. Moreover, any two such chain transformations are chain homotopic.*

Since any two proper injective resolutions of an object A are chain homotopic by the previous theorem, the cohomology groups are well-defined.

Definition 1.24. Let $\mathcal{P}$ be a class of proper short exact sequences in an abelian category $\mathcal{A}$, and $C \rightarrowtail \mathbf{I}$ a proper injective resolution of an object C. Then we define, for an object A,

$$\underline{\mathrm{Ext}}^n_{\mathcal{P}}(A,\, C) := \mathrm{H}^n(\mathrm{Hom}_{\mathcal{A}}(A,\, \mathbf{I})).$$

Let $\mathcal{A}$ be an abelian category with enough proper injectives, $\mathcal{R}$ be a selective category and $T : \mathcal{A} \to \mathcal{R}$ be an additive covariant functor. Each object $A \in \mathcal{A}$ has a proper injective resolution $\epsilon : A \to \mathbf{Y}$. By Theorem 1.23 it follows that $R^n(A) = \mathrm{H}^n(T(\mathbf{Y}))$ is independent of the choice of the resolution $\mathbf{Y}$, and $R^n(\alpha) = \mathrm{H}^n(T(f)) : R^n(A) \to R^n(A')$ makes each R^n a covariant functor from $\mathcal{A}$ to $\mathcal{R}$. Hence R^n is the *n-th right derived functor of T* [**ML95**, XII, pg. 389]. The same holds for the right derived functors associated to a proper projective resolution and a contravariant functor T. Hence, we have the following *dimension shifting* corollaries.

Corollary 1.25. *Let $A \overset{\epsilon}{\rightarrowtail} E^0 \xrightarrow{d^1} E^1 \xrightarrow{d^2} E^2 \to \cdots$ be a proper injective resolution of an object A, and define $L^0 = \mathrm{im}(\epsilon)$ and $L^n = \mathrm{im}(d^n)$ for $n \geqslant 1$. Then if T is covariant, we have*

$$(R^{n+1}T)(A) \cong (R^nT)(L^0) \cong (R^{n-1}T)(L^1) \cong \cdots \cong (R^1T)(L^{n-1}).$$

In particular,

$$\underline{\mathrm{Ext}}^{n+1}_{\mathcal{P}}(C,\, A) \cong \underline{\mathrm{Ext}}^n_{\mathcal{P}}(C,\, L^0) \cong \cdots \cong \underline{\mathrm{Ext}}^1_{\mathcal{P}}(C,\, L^{n-1}).$$

Corollary 1.26. *Let $\cdots \to P_2 \xrightarrow{d^2} P_1 \xrightarrow{d^1} P_0 \overset{\epsilon}{\twoheadrightarrow} C$ be a proper projective resolution of an object C, and define $K_0 = \ker(\epsilon)$ and $K_n = \ker(d_n)$ for $n \geqslant 1$. Then if T is contravariant, we have*

$$(R^{n+1}T)(C) \cong (R^nT)(K_0) \cong (R^{n-1}T)(K_1) \cong \cdots \cong (R^1T)(K^{n-1}).$$

In particular, for the functors $\mathrm{Ext}^n_{\mathcal{P}}$, introduced in [**Nuc99**] *we have*

$$\mathrm{Ext}^{n+1}_{\mathcal{P}}(C,\, A) \cong \mathrm{Ext}^n_{\mathcal{P}}(K_0,\, A) \cong \cdots \cong \mathrm{Ext}^1_{\mathcal{P}}(K_{n-1},\, A).$$

Now we are able to compare $\underline{\mathrm{Ext}}^n_{\mathcal{P}}$ with $\mathrm{Ext}^n_{\mathcal{P}}$.

Theorem 1.27. *Let $A \rightarrowtail E^0 \to E^1 \to \cdots$ be a proper injective resolution of an object A and $\cdots \to P_1 \to P_0 \twoheadrightarrow C$ be a proper projective resolution of an object C. Then for all $n \geqslant 0$*

$$\mathrm{H}^n(\mathrm{Hom}(\mathbf{P_C},\, A)) \cong \mathrm{H}^n(\mathrm{Hom}(C,\, \mathbf{E_A})).$$

Thus the two definitions $\operatorname{Ext}^n_{\mathcal{P}}$ *and* $\underline{\operatorname{Ext}}^n_{\mathcal{P}}$ *have the same value on* $(C,\, A)$.

PROOF. This is just a repetition of the ordinary case that can be found, for example in [**Rot79**]. □

There is an obvious notion of *proper injective dimension*. Note that Lemma 1.21 holds in the generality of proper injectivity.

Theorem 1.28. *The following properties for an object* J *are equivalent:*

(1) J *is proper injective;*
(2) for each proper monomorphism $x : A \rightarrowtail B$, $x^* : \operatorname{Hom}(B,\, J) \to \operatorname{Hom}(A,\, J)$ *is an epimorphism;*
(3) every proper short exact sequence $J \rightarrowtail B \twoheadrightarrow C$ *splits;*
(4) for every module C, $\underline{\operatorname{Ext}}^1_{\mathcal{P}}(C,\, J)$ *vanishes.*

PROOF. This is analogous to [**ML95**, III, 7.1]. □

Complete Δ-Cohomology via Satellites Using Δ-injectives. The purpose of this subsection is to define the complete Δ-cohomology groups via satellites as in [**Nuc99**]. In contrast to Nucinkis' work, we use Δ-injective objects instead of Δ-projective ones.

Notation. When considering cohomology relative to a G-set Δ as above we denote $\underline{\operatorname{Ext}}^*_{\mathcal{P}}(A,\, C)$ by $\Delta\mathrm{I}\operatorname{Ext}^*_{kG}(A,\, C)$. The Δ*-injective dimension* of a kG-module M is the length of the shortest injective resolution of M. This is denoted by $\Delta\operatorname{id}_{kG} M$.

In order to establish an analogue of [**ML95**, XII, 7.3] we need to rewrite some notation. Since we assume that the category $\mathcal{A}$ has enough proper injective objects, there is for each object C of $\mathcal{A}$ a proper monomorphism $\sigma : C \rightarrowtail I$ with I proper injective; this gives a proper short exact sequence

$$E^C : C \rightarrowtail I \overset{x}{\twoheadrightarrow} J,$$

We call it, with abuse of language, a *short proper injective resolution* of C. Note that J is not proper injective in general.

Regard each proper short exact sequence $E : C \rightarrowtail B \twoheadrightarrow A$ as a complex in $\mathcal{A}$, say in dimensions 1, 0 and -1. Then $T(E) : T(A) \to T(B) \to T(C)$ is a complex in

$\mathcal{R}$; its one dimensional homology $H_1(T(E))$ is the object of $\mathcal{R}$ which makes

$$H_1(T(E)) \overset{\mu}{\rightarrowtail} T(A) \to T(B), \qquad \mathcal{R},$$

exact. Each morphism $\Gamma = (\gamma, \beta, \alpha) : E \to E'$ of proper short exact sequences in $\mathcal{A}$ gives a chain transformation $T(\Gamma) : T(E') \to T(E)$ and hence induces a morphism

$$H_1(\Gamma) : H_1(T(E')) \to H_1(T(E)), \qquad \mathcal{R},$$

which is characterized by $\mu H_1(\Gamma) = T(\alpha)\mu'$. Moreover, $H_1(\Gamma)$ depends only on γ, E and E' and not on α or β. To see this, let $\Gamma_0 = (\gamma, \beta_0, \alpha_0) : E \to E'$ be any other morphism (of proper short exact sequences) with γ as the first component. In the $\mathcal{A}$-diagram

$$\begin{tikzcd} C \arrow[r, hook, "\sigma"] \arrow[d, "0"'] & B \arrow[r, "x"] \arrow[d, "\beta-\beta_0"'] & A \arrow[d, "\alpha-\alpha_0"] \arrow[dl, dotted, "s"'] \\ C' \arrow[r, hook, "\sigma'"] & B' \arrow[r, "x'"] & A', \end{tikzcd}$$

$(\beta - \beta_0)\sigma = 0$, so $\beta - \beta_0 = sx$ for some s. Also $(\alpha - \alpha_0)x = x'(\beta - \beta_0) = x'sx$, so $x's = \alpha - \alpha_0$, $sx = \beta - \beta_0$. Hence s ia a homotopy $\Gamma \cong \Gamma_0$. By the additivity of T we have that $T(s)$ is a homotopy $T(\Gamma) \cong T(\Gamma_0) : T(E) \to T(E')$, so $H_1(\Gamma) = H_1(\Gamma_0)$. Now there exists:

- to each object C of $\mathcal{A}$ a short injective resolution E^C,
- to each $\gamma : C \to C'$ in $\mathcal{A}$ a morphism $\Gamma^\gamma = (\gamma, -, -) : E^C \to E^{C'}$,
- to each proper short exact sequence E in $\mathcal{A}$ a morphism $\Lambda^E = (1, -, -) : E^C \to E$.

Lemma 1.29. *Given $T : \mathcal{A} \to \mathcal{R}$ contravariant additive and the data above,*

$$S(C) = H_1(T(E^C)) \quad \textit{and} \quad S(\gamma) = H_1(\Gamma^\gamma) : S(C') \to S(C)$$

define a contravariant additive functor $S : \mathcal{A} \to \mathcal{R}$, while for μ,

$$E^* = \mu H_1(\Lambda^E) : S(C) \to T(A)$$

defines a natural transformation which makes (S, E^*, T) a $\mathcal{P}$-connected contravariant pair such that $S(C) \rightarrowtail T(K) \to T(J)$ is a left exact $\mathcal{R}$-sequence whenever J is proper injective and $C \rightarrowtail J \twoheadrightarrow K$ is a proper short exact sequence.

PROOF. The proof is analogous to the proof of [**ML95**, XII, 7.3]. $\square$

Theorem 1.30. *Let $\mathcal{A}$ be an abelian category with enough proper injectives. Then the contravariant pair $(T,\, E^*,\, S)$ is left $\mathcal{P}$-couniversal if and only if each proper short exact $\mathcal{A}$-sequence*

$$C \rightarrowtail J \twoheadrightarrow K$$

with J proper injective induces an exact $\mathcal{R}$-sequence $S(C) \rightarrowtail T(K) \to T(J)$. Then S is a left satellite of T.

PROOF. The proof is analogous to the proof of [**ML95**, XII, 7.7]. $\square$

Let $C \rightarrowtail J \twoheadrightarrow K$ be a proper short exact sequence with J proper injective. We define

$$S(C) := \ker(T(K) \to T(J)).$$

The functor $S(C)$ is additive and from above it is independent of the choice of the proper injective J. We call S the *left satellite of* T and we can define the iterated (left) satellites as: $S^0T(C) = T(C)$, $S^{-1}T(C) = S(C)$ and $S^{-n}T(C) = S^{-1}(S^{-n+1}T(C))$. If J is proper injective, then $S^{-n}T(J) = 0$ for every $n \geqslant 1$ by construction (as $J \rightarrowtail J \twoheadrightarrow 0$ is a proper short exact sequence and the basic properties of the functor T).

Lemma 1.31. *If T is a contravariant half proper-exact functor (i.e. $T(C) \to T(B) \to T(A)$ is exact for every proper short exact sequence $A \rightarrowtail B \twoheadrightarrow C$) and $\mathcal{A}$ has enough proper injectives, then for each proper short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ there is an associated long exact sequence*

$$\cdots \to S^{-n}T(C) \to S^{-n}T(B) \to S^{-n}T(A) \to S^{-n+1}T(C) \to \cdots$$
$$\cdots \to S^{-1}T(A) \to T(C) \to T(B) \to T(A).$$

PROOF. Lemma 1.29 tells us that $S(C) \to S(B) \to S(A) \to T(C) \to T(B) \to T(A)$ forms a complex. By iterating this argument we have that the sequence is a chain complex. The exactness can be proved as in [**CE56**, III, 3.1]. $\square$

Definition 1.32. A $\mathcal{P}$-*connected* sequence of contravariant functors is a sequence of functors $T^* = \{T^n,\, n \in \mathbb{Z}\} : \mathcal{A} \to \mathcal{R}$, which assigns to each proper short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ of a $\mathcal{A}$ a complex

$$\cdots \to T^{-n-1}(A) \to T^{-n}(C) \to T^{-n}(B) \to T^{-n}(A) \to T^{-n+1}(C) \to \cdots .$$

We say that the nonpositive part $T^{\leqslant 0} = \{T^{-n},\, n \geqslant 0\}$ of a $\mathcal{P}$-connected sequence of contravariant functors is of *proper cohomological type* if for each proper short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ we have a long exact sequence

$$\cdots \to T^{-n}(C) \to T^{-n}(B) \to T^{-n}(A) \to T^{-n+1}(C) \to \cdots \to T^{0}(A).$$

In order to prove the proper I-completion analogue of Mislin's completion [**Mis94**] we need the following result.

Proposition 1.33. *Let $T^{\leqslant 0}$ and $U^{\leqslant 0}$ be the nonpositive parts of $\mathcal{P}$-connected sequences of additive contravariant functors, and let $f^0 : T^0 \to U^0$ be a natural transformation. If, in addition, $U^{\leqslant 0}$ is of proper cohomological type and $U^{-n}(I) = 0$ for all $n \geqslant 0$ and all proper injectives I, then:*

(1) f^0 extends uniquely to $f^{\leqslant 0} : T^{\leqslant 0} \to U^{\leqslant 0}$ and $f^{\leqslant 0}$ factors uniquely through the canonical transformation $T^{\leqslant 0} \to S^{\leqslant 0}T^0$,

(2) if T^0 is half $\mathcal{P}$-exact and f^0 is an equivalence then the induced transformation $S^{\leqslant 0}T^0 \to U^{\leqslant 0}$ is an equivalence.

Before we can prove this proposition we need one more result.

Lemma 1.34. *Let $\mathcal{A}$ be an abelian category with enough proper injectives. Then the following are equivalent for a $\mathcal{P}$-connected sequence of contravariant functors $\{T^*, E^*\}$:*

(1) for each proper short exact $\mathcal{A}$-sequence $K \rightarrowtail I \twoheadrightarrow J$ with I proper injective, the sequence

$$T^{-n}(K) \rightarrowtail T^{-n+1}(J) \to T^{-n+1}(I)$$

is exact for every $n > 0$;

(2) for each $\mathcal{P}$-connected sequence of contravariant functors $\{V^*, E'^*\}$ and each natural transformation $f^0 : V^0 \to T^0$ there exist a unique natural transformation $f^n : V^{-n} \to T^{-n}$, extending f^0.

PROOF. Suppose that we have (1). Let $\{V^*, E'^*\}$ be a $\mathcal{P}$-connected sequence of contravariant functors and let $f^0 : V^0 \to T^0$ be a natural transformation. We construct by recursion on n the necessary natural transformation $f^n : V^{-n} \to T^{-n}$. Without loss of generality we can assume that f^i commutes with the connecting homomorphism for $0 \leqslant i < n$; now we apply point (1) of Theorem 1.30 to show that $(T^n,\, E^n,\, T_{n-1})$ is left $\mathcal{P}$-couniversal, and so we can construct a unique $f^n : V^{-n} \to T^{-n}$ with $E^n f^n = f^{n-1} E'^n$.

Suppose now, that we have (2). From T^0 we construct the left satellite S^1, and we iterate the construction to obtain $S^n : \mathcal{A} \to \mathcal{R}$ from S^{n-1} for every n. Since the resulting $\mathcal{P}$-connected sequence satisfies $S^n(K) \rightarrowtail S^{n-1}(J) \to S^{n-1}(I)$, it is couniversal and therefore it must coincide with the unique $\mathcal{P}$-connected sequence of contravariant functors $\{T^*, E^*\}$ for the given T^0. $\square$

Now we are ready to reformulate [**ML95**, XII, 8.4].

Lemma 1.35. *Let $\mathcal{A}$ be an abelian category with enough proper injectives. Then each contravariant functor T^0 is the 0-component for a $\mathcal{P}$-connected sequence of contravariant functors as in the previous lemma, where T^{-n} is the n-th iterated left satellite $S^{-n}T^0$.*

PROOF OF PROPOSITION 1.33. This is analogous of [**Nuc98**, 2.3]; $T^{\leqslant 0}$ satisfies (1) of Lemma 1.34 and so there exists an unique transformation $f^{\leqslant 0} : U^{\leqslant 0} \to T^{\leqslant 0}$. The canonical transformation $U^{\leqslant 0} \to S^{\leqslant 0}T^0$ can be deduced from Lemma 1.35. Applying Lemma 1.34 to $U^0 \to T^0$ without viewing V^0 as the 0-th component of the $\mathcal{P}$-connected sequence of satellites, gives us a unique transformation $S^{\leqslant 0}V^0 \to T^{\leqslant 0}$.

By Lemma 1.31 it follows that $S^{\leqslant 0}V^0$ is of proper cohomological type. Since in addition, $S^{\leqslant 0}U^0(I) = 0$ for all proper injective I, we can apply the first part to $T^0 \to U^0$ and therefore $S^{\leqslant 0}U^0 \to T^{\leqslant 0}$ is an equivalence. $\square$

A $\mathcal{P}$-connected sequence of contravariant functors $T^* = \{T^n,\, n \in \mathbb{Z}\}$ is called a *proper contravariant* $(-\infty,\, \infty)$*-cohomological functor*, if for each proper short exact sequence $A \rightarrowtail B \twoheadrightarrow C$, we have a long exact sequence $\cdots \to T^n(C) \to T^n(B) \to T^n(A) \to T^{n+1}(C) \to \cdots$. Following [**Nuc99**, 3.7] we define:

Definition 1.36. A proper contravariant $(-\infty,\, \infty)$-cohomological functor $T^* = \{T^n,\, n \in \mathbb{Z}\}$ is called *proper I-complete* if $T^n(I) = 0$ for all n and every proper injective module I. A morphism $V^* \to T^*$ of proper contravariant $(-\infty,\, \infty)$-cohomological functors is called a *proper I-completion* of V^* if T^* is proper I-complete and every morphism $V^* \to W^*$ into a proper contravariant I-complete functor W^* factors uniquely through $V^* \to T^*$.

Theorem 1.37. *Every proper contravariant $(-\infty, \infty)$-cohomological functor $T^* = \{T^n, \ n \in \mathbb{Z}\}$ admits a unique proper I-completion $\tau^* : T^* \to \check{T}^*$. This completion is obtained as:*

$$\check{T}^n(A) = \varinjlim_{k \geqslant 0} S^{-k}T^{n+k}(A).$$

PROOF. We follow the proof of [**Nuc98**, 2.5]. For every $n \in \mathbb{Z}$ we obtain a proper contravariant $(-\infty, \infty)$-cohomological functor by defining

$$T^j\langle n\rangle = \begin{cases} S^{j-n}T^n & \text{if } j < n \\ T^j & \text{if } j \geqslant n. \end{cases}$$

By (1) of Proposition 1.33, the identity morphism $T^n \to T^n$ extends uniquely to a canonical morphism, defined for all integers $j \leqslant n$, as $\iota_n^j : T^j \to S^{j-n}T^n$. This induces a unique morphism $\iota_n^* : T^* \to T^*\langle n\rangle$, where $\iota_n^j = id_{T^j}$ for all $j > n$. In the same fashion, we extend for all $m > n$, the identity on T^m to a unique $\iota_{n,m}^* : T^*\langle n\rangle \to T^*\langle m\rangle$. Since for all integers $m \geqslant n$ the morphisms $\iota_{n,m}^*$ are uniquely determined, we have obtained a direct system $\{T^*\langle n\rangle, \iota_{n,m}^*\}$. Thus, we can now define

$$\check{T}^* = \varinjlim T^*\langle n\rangle.$$

For all $m \geqslant n$ we have the equality $\iota_{n,m}^*\iota_n^* = \iota_m^*$, which enables us to define

$$\check{T}^* = \varinjlim \iota_n^* : T^* \to \check{T}^*,$$

which satisfies the universal property of Definition 1.36. □

Lemmas 2.6 and 2.7 in [**Nuc98**] or Lemmas 3.9 and 3.10 in [**Nuc99**] have proper injective versions.

Lemma 1.38. *Let T^* be a proper contravariant $(-\infty, \infty)$-cohomological functor and $n_0 \in \mathbb{Z}$ such that $T^n(I) = 0$ for all proper injective I and all $n \geqslant n_0$. Then $\tau^n(A) : T^n(A) \to \check{T}^n(A)$ is an isomorphism for all $n \geqslant n_0$.*

Lemma 1.39. *If $f^* : T^* \to V^*$ is a morphism of proper contravariant $(-\infty, \infty)$-cohomological functors where V^* is proper I-complete and if $f^n : T^n \to V^n$ is an equivalence for all $n \geqslant n_0$ then the induced morphism $\check{T}^* \to V^*$ is an equivalence.*

Cohomology relative to a G-set Δ has a proper I-completion defined as follows:

$$\Sigma_\Delta M = \operatorname{coker}(M \rightarrowtail \operatorname{Hom}(k\Delta, M))$$

and inductively,

$$\Sigma_\Delta^i M = \Sigma_\Delta \Sigma_\Delta^{i-1} M.$$

Since the definition of the left satellite is independent of the choice of the proper injectives, we define:

$$S^{-1}\,\Delta\mathrm{I}\,\mathrm{Ext}^n_{kG}(M,\,N) = \ker(\Delta\mathrm{I}\,\mathrm{Ext}^n_{kG}(\Sigma_\Delta M, N) \to \Delta\mathrm{I}\,\mathrm{Ext}^n_{kG}(\mathrm{Hom}(k\Delta,\,M), N))$$

and the proper ΔI-completion as:

$$\Delta\widetilde{\mathrm{Ext}}^n_{kG}(M,\,N) = \varinjlim_{k\geqslant|n|} S^{-k}\,\Delta\mathrm{I}\,\mathrm{Ext}^{n+k}_{kG}(M,\,N).$$

A Different Approach to Complete Relative Cohomology. In this subsection we present a Δ-relative version of Benson-Carlson's approach to complete cohomology via injectives that appears in [**Nuc98**].

Definition 1.40. Let $\Delta\mathrm{I}\,\mathrm{Hom}_{kG}(M,N)$ be the subgroup of $\mathrm{Hom}_{kG}(M,N)$, consisting of the homomorphisms in $\mathrm{Hom}_{kG}(M,N)$ factoring through a Δ-injective. We denote

$$[M,N]_{\Delta\mathrm{I}} = \mathrm{Hom}_{kG}(M,N)/\,\Delta\mathrm{I}\,\mathrm{Hom}_{kG}(M,N),$$

and by $\Delta\mathrm{I}\,\mathfrak{Mod}_{kG}$ the category having as objects the kG-modules and whose morphisms lay in $[-,-]_{\Delta\mathrm{I}}$. Note that there is an obvious surjection

$$\begin{aligned}\mathrm{Hom}_{kG}(M,N) &\twoheadrightarrow [M,N]_{\Delta\mathrm{I}}\\ \Psi &\mapsto [\Psi] = \Psi + \Delta\mathrm{I}\,\mathrm{Hom}_{kG}(M,N).\end{aligned}$$

Lemma 1.41. *Let N be a kG-module and $A \rightarrowtail B \twoheadrightarrow C$ be a Δ-split short exact sequence of kG-modules. Then every $[\phi] \in [A,N]_{\Delta\mathrm{I}}$ induces a unique $[\Psi] \in [C,\Sigma_\Delta N]_{\Delta\mathrm{I}}$. In particular Σ_Δ is a functor from $\mathcal{I}_\Delta(kG)$ to itself.*

PROOF. Take a representative ϕ of $[\phi]$ in $\mathrm{Hom}_{kG}(A,N)$. Since $\mathrm{Hom}(k\Delta,N)$ is a Δ-injective kG-module and τ is a Δ-split monomorphism, there exists a map $\bar\phi : B \to \mathrm{Hom}(k\Delta,N)$ making the left hand square of the diagram below commute. We define a map $\Psi : C \to \Sigma_\Delta N$ as $\Psi(c) = \pi\bar\phi(b)$ where $\sigma(b) = c$ which makes the right hand square commute (it is not unique, but this does not matter). Suppose that there is another pair $\bar\phi'$, Ψ' making the diagram commute:

$$\begin{tikzcd}
N \arrow[r, hook, "\iota"] & \mathrm{Hom}(k\Delta,N) \arrow[r, two heads, "\pi"] & \Sigma_\Delta N \\
A \arrow[u, "\phi"] \arrow[r, hook, "\tau"] & B \arrow[u, shift left, "\bar\phi"'] \arrow[u, shift right, dotted, "\bar\phi'"'] \arrow[r, two heads, "\sigma"] & C. \arrow[ul, "\Theta"'] \arrow[u, shift left, "\Psi"'] \arrow[u, shift right, dotted, "\Psi'"']
\end{tikzcd}$$

We want to show that $\Psi - \Psi'$ factors through $\mathrm{Hom}(k\Delta, N)$ implies $[\Psi] = [\Psi']$. We define the map $\Theta : C \to \mathrm{Hom}(k\Delta, N)$ by $\Theta(c) = (\bar{\phi} - \bar{\phi}')(b)$, where $\sigma(b) = c$. Hence we obtain $\pi\Theta\sigma = \pi(\bar{\phi} - \bar{\phi}') = (\Psi - \Psi')\sigma$, but σ is surjective and so $\pi\Theta = \Psi - \Psi'$. Therefore, $[\Psi] = [\pi\Theta + \Psi'] = [\Psi']$.

Suppose ϕ factors through a Δ-injective J, then $\Sigma_\Delta J$ is Δ-injective since the short exact sequence $J \rightarrowtail \mathrm{Hom}(k\Delta, J) \twoheadrightarrow \Sigma_\Delta J$ and since $\Psi^* : C \rightarrowtail \Sigma_\Delta J$ is a Δ-split monomorphism, Ψ factors through the relative injective $\Sigma_\Delta J$.

To show that Σ_Δ is a functor we can replace $A \rightarrowtail B \twoheadrightarrow C$ by $N \rightarrowtail \mathrm{Hom}(k\Delta, N) \twoheadrightarrow \Sigma_\Delta N$ for some kG-module N, and verify the following:

i) $\Sigma_\Delta N \in \Delta\mathrm{I}\,\mathfrak{Mod}_{kG}$: immediate by the definition.

ii) Let $M, N \in \Delta\mathrm{I}\,\mathfrak{Mod}_{kG}$ and $\phi \in [M, N]_{\Delta\mathrm{I}}$. Then $\Sigma_\Delta\phi \in [\Sigma_\Delta M, \Sigma_\Delta N]_{\Delta\mathrm{I}}$ follows from the way we have defined $\Sigma_\Delta\phi$.

iii) For $\phi \in [M, N]_{\Delta\mathrm{I}}$ and $\rho \in [N, O]_{\Delta\mathrm{I}}$ we have $\Sigma_\Delta\phi \circ \Sigma_\Delta\rho = \Sigma_\Delta(\rho\phi)$. We consider the diagram:

$$\begin{tikzcd}
M \arrow[r, hook, "\iota"] \arrow[d, "\phi"'] & \mathrm{Hom}(k\Delta, M) \arrow[r, two heads, "\pi"] \arrow[d, "\Delta\mathrm{I}\,\phi"'] & \Sigma_\Delta M \arrow[d, "\Sigma_\Delta\phi"'] \\
N \arrow[r, hook, "\tau"] \arrow[d, "\rho"'] & \mathrm{Hom}(k\Delta, N) \arrow[r, two heads] \arrow[d, "\Delta\mathrm{I}\,\rho"'] & \Sigma_\Delta N \arrow[d, "\Sigma_\Delta\rho"'] \\
O \arrow[r, hook, "\sigma"] & \mathrm{Hom}(k\Delta, O) \arrow[r, two heads] & \Sigma_\Delta O
\end{tikzcd}$$

We prove that $\Sigma_\Delta(\rho\phi) - \Sigma_\Delta\phi \circ \Sigma_\Delta\rho$ factors through a Δ-injective. We define

$$\begin{aligned}
\Psi \ : \ \Sigma_\Delta M \to & \quad \mathrm{Hom}(k\,\Delta, O) \\
\bar{m} \ \mapsto & \quad (\Delta\mathrm{I}(\rho\phi) - \Delta\mathrm{I}\,\rho \circ \Delta\mathrm{I}\,\phi)(m)
\end{aligned}$$

where $\pi m = \bar{m}$ for some $m \in \mathrm{Hom}(k\,\Delta, M)$. Suppose there is a morphism $m' \in \mathrm{Hom}(k\,\Delta, M)$ such that $\pi(m') = 0$, then by the exactness of the first row there is an $\tilde{m} \in M$ with $\iota\tilde{m} = m'$. Hence,

$$\begin{aligned}(\Delta\mathrm{I}(\rho\phi) - \Delta\mathrm{I}\,\phi \circ \Delta\mathrm{I}\,\rho)(\iota\tilde{m}) &= (\Delta\mathrm{I}\,\rho\phi)\iota(\tilde{m}) - \Delta\mathrm{I}\,\phi \circ \Delta\mathrm{I}\,\rho \circ \iota(\tilde{m}) \\ &= (\Delta\mathrm{I}\,\rho\phi)\iota(\tilde{m}) - (\Delta\mathrm{I}\,\phi)\tau\rho(\tilde{m}) \\ &= (\Delta\mathrm{I}\,\rho\phi)\iota(\tilde{m}) - (\sigma\phi)\rho(\tilde{m}) \\ &= \sigma(\rho\phi)(\tilde{m}) - (\sigma\phi)\rho(\tilde{m}) \\ &= 0,\end{aligned}$$

and so Ψ is an homomorphism.

iv) For every $M \in \Delta\mathrm{I}\,\mathfrak{Mod}_{kG}$, $\Sigma_\Delta(1_M) = 1_{\Sigma_\Delta M}$. Replace in the previous diagram the second row with a copy of the first row and erase the third one. The identities on $\mathrm{Hom}(k\,\Delta, M)$ and on Σ_Δ make the diagram commute. □

By Lemma 1.41 there is a well-defined chain:

$$[M, N]_{\Delta\mathrm{I}} \to [\Sigma_\Delta M, \Sigma_\Delta N]_{\Delta\mathrm{I}} \to [\Sigma^2_\Delta M, \Sigma^2_\Delta N]_{\Delta\mathrm{I}} \to \cdots$$

Definition 1.42. Let $M, N \in \mathfrak{Mod}_{kG}$. Then we define the 0-*th relative injective Benson-Carlson group* as:

$$\Delta\breve{\mathrm{BC}}^{0}_{kG}(M, N) = \varinjlim_{i \geqslant 0} [\Sigma^i_\Delta M, \Sigma^i_\Delta N]_{\Delta\mathrm{I}}.$$

We would like to show that the above defines a proper contravariant $(-\infty, \infty)$-cohomological functor which is proper I-complete. We begin with the classical Schanuel's Lemma.

Lemma 1.43. *Let $A \overset{\tau}{\rightarrowtail} I \twoheadrightarrow C$ and $A \overset{\iota}{\rightarrowtail} J \twoheadrightarrow D$ be short exact sequences in an abelian category $\mathcal{A}$. Suppose that there exist $\phi : I \to J$ and $\psi : J \to I$ such that $\psi\iota = \tau$ and $\phi\tau = \iota$. Then $J \oplus C \cong I \oplus D$. Suppose that there is an unique map $\phi : I \to J$ such that $\phi\tau = \iota$. Then there is a short exact sequence $I \rightarrowtail C \oplus J \twoheadrightarrow D$.*

PROOF. The proof is analogous to the proof of Lemma 2.7 in [**Nuc99**]. □

Corollary 1.44. *Let $A \rightarrowtail I \twoheadrightarrow C$ and $A \rightarrowtail J \twoheadrightarrow D$ be proper short exact sequences in $\mathcal{A}$. If J and I are proper injectives then*

$$J \oplus C \cong I \oplus D.$$

Furthermore, let $M \rightarrowtail \mathbf{J}$ and $M \rightarrowtail \mathbf{I}$ be two proper injective resolution of M and let D^n and C^n be the n-th cokernels respectively. Then $D^n \oplus \Delta$-inj $\cong C^n \oplus \Delta$-inj.

Lemma 1.45. *The functor* $[-, N]_{\Delta \mathrm{I}}$ *is half exact.*

PROOF. Let $A \overset{\iota}{\rightarrowtail} B \overset{\pi}{\twoheadrightarrow} C$ be a Δ-split exact sequence. We have to show that $[C, N]_{\Delta \mathrm{I}} \xrightarrow{\pi^*} [B, N]_{\Delta \mathrm{I}} \xrightarrow{\iota^*} [A, N]_{\Delta \mathrm{I}}$ is exact at $[B, N]_{\Delta \mathrm{I}}$.
From $\iota^* \pi^* [\phi] = [\pi \iota \phi] = [0]$ it follows that $\operatorname{im} \pi^* \subseteq \ker \iota^*$. Now we show that $\ker \iota^* \subseteq \operatorname{im} \pi^*$. Let $[\phi] \in \ker \iota^*$, then $\phi\iota$ factors through a Δ-injective I, and we have

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[d, "\alpha"'] & B \arrow[d, "\phi"] \arrow[dl, dotted, "\psi"'] \\ I \arrow[r, hook, "\beta"] & N \end{tikzcd}$$

where $\beta\alpha = \phi\iota$. Since I is Δ-injective and ι is a Δ-split monomorphism, there exists $\psi : B \to I$ such that $\psi\iota = \alpha$, and so $[\phi] = [\phi - \beta\psi]$. Since

$$\begin{aligned} \iota^*[\phi - \beta\psi] &= (\phi - \beta\psi)\iota \\ &= \beta\alpha - \beta\psi\iota \\ &= \beta\alpha - \beta\alpha = 0, \end{aligned}$$

then $\phi - \beta\psi \in \ker \iota$, but $\operatorname{Hom}(-, N)$ is a left exact functor and so $\phi - \beta\psi \in \operatorname{im} \pi^*$, hence $[\phi] = [\phi - \beta\psi] \in \operatorname{im} \pi^*$. $\square$

Lemma 1.46 (Horseshoe Lemma for proper injectives)**.** *Every proper short exact sequence* $\mathrm{A} \rightarrowtail \mathrm{B} \twoheadrightarrow \mathrm{C}$ *in an abelian category* $\mathcal{A}$ *with enough proper injectives can be embedded in a commutative diagram*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & B \arrow[r, two heads] \arrow[d, hook, dotted] & C \arrow[d, hook] \\ I \arrow[r, hook, dotted] \arrow[d] & J \arrow[r, two heads, dotted] \arrow[d, two heads, dotted] & L \arrow[d] \\ S \arrow[r, hook, dotted] & T \arrow[r, two heads, dotted] & V. \end{tikzcd}$$

in which all rows and columns are exact, the middle row splits and consists of proper injectives. Moreover, the sequences $A \rightarrowtail I \twoheadrightarrow S$ *and* $C \rightarrowtail L \twoheadrightarrow V$ *with* I *and* L *proper injective, may be given in advance.*

PROOF. Since $\mathcal{A}$ has enough proper injectives we can suppose without loss of generality that $A \rightarrowtail I \twoheadrightarrow S$ and $C \rightarrowtail L \twoheadrightarrow V$ with I and L proper injective are given. Now let $J = I \oplus L$. Clearly the middle row splits and since $A \rightarrowtail B$ is a proper monomorphism there exists $B \to I \oplus L$ that makes the upper squares commute. Let T be the coker of $B \rightarrowtail I \oplus L$ and we see that the lower squares commute. $\square$

Corollary 1.47. *Every Δ-split short exact sequence* $\mathrm{A} \rightarrowtail \mathrm{B} \twoheadrightarrow \mathrm{C}$ *gives rise to a Δ-split sequence of the form $\Sigma_\Delta A \rightarrowtail \underline{\Sigma}_\Delta B \twoheadrightarrow \Sigma_\Delta C$, where $\underline{\Sigma}_\Delta B := \mathrm{coker}(B \to \mathrm{Hom}(k\Delta, A) \oplus \mathrm{Hom}(k\Delta, C))$.*

PROOF. By the Horseshoe Lemma we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
\mathrm{Hom}(k\Delta, A) \arrow[r] \arrow[d] & \mathrm{Hom}(k\Delta, A) \oplus \mathrm{Hom}(k\Delta, C) \arrow[r] \arrow[d] & \mathrm{Hom}(k\Delta, C) \arrow[d] \\
\Sigma_\Delta A \arrow[r] & \underline{\Sigma}_\Delta B \arrow[r] & \Sigma_\Delta C.
\end{tikzcd}$$

By tensoring it by $k\Delta$ we obtain a diagram with the side columns and the first two rows split, and so the bottom row splits as well. $\square$

Lemma 1.48. *Every Δ-split short exact sequence induces a long exact sequence*

$$\cdots \to [\Sigma_\Delta B, N]_{\Delta \mathrm{I}} \to [\Sigma_\Delta A, N]_{\Delta \mathrm{I}} \to [C, N]_{\Delta \mathrm{I}} \to [B, N]_{\Delta \mathrm{I}} \to [A, N]_{\Delta \mathrm{I}}.$$

PROOF. By Lemma 1.45 we have exactness at $[B, N]_{\Delta \mathrm{I}}$. We first show exactness at $[\Sigma_\Delta B, N]_{\Delta \mathrm{I}}$. By Corollary 1.47 we obtain a short exact sequence

$$\Sigma_\Delta A \rightarrowtail \underline{\Sigma}_\Delta B \twoheadrightarrow \Sigma_\Delta C.$$

We apply Lemma 1.45 to have

$$[\Sigma_\Delta A, N]_{\Delta \mathrm{I}} \rightarrowtail [\underline{\Sigma}_\Delta B, N]_{\Delta \mathrm{I}} \twoheadrightarrow [\Sigma_\Delta C, N]_{\Delta \mathrm{I}}$$

which is exact at $[\underline{\Sigma}_\Delta B, N]_{\Delta \mathrm{I}}$. Schanuel's Lemma for proper injectives implies that $\underline{\Sigma}_\Delta B \oplus \Delta\text{-inj} \cong \Sigma_\Delta B \oplus \Delta\text{-inj}$, and so

$$\begin{aligned}
[\underline{\Sigma}_\Delta B, N]_{\Delta \mathrm{I}} &\cong [\underline{\Sigma}_\Delta B \oplus \Delta\text{-inj}, N]_{\Delta \mathrm{I}} \\
&\cong [\Sigma_\Delta B \oplus \Delta\text{-inj}, N]_{\Delta \mathrm{I}} \\
&\cong [\Sigma_\Delta B, N]_{\Delta \mathrm{I}},
\end{aligned}$$

and so we have exactness at $\Sigma_\Delta B$. Applying the Schanuel's Lemma to

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ A \arrow[r, hook] & \mathrm{Hom}(k\Delta, A) \arrow[r, two heads] & \Sigma_\Delta A \end{tikzcd}$$

we obtain a short exact sequence $B \rightarrowtail C \oplus \mathrm{Hom}(k\Delta, A) \twoheadrightarrow \Sigma_\Delta A$.

Since $[C \oplus \mathrm{Hom}(k\Delta, A), N]_{\Delta\mathrm{I}} \cong [C, N]_{\Delta\mathrm{I}}$, by applying Lemma 1.45 to the above short exact sequence we have exactness at $[C, N]_{\Delta\mathrm{I}}$.

To prove exactness at $[\Sigma_\Delta A, N]_{\Delta\mathrm{I}}$ we use a similar argument. We apply Schanuel's Lemma to

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, equal] & C \oplus \mathrm{Hom}(k\,\Delta, A) \arrow[r, two heads] \arrow[d] & \Sigma_\Delta A \arrow[d] \\ B \arrow[r, hook] & \mathrm{Hom}(k\,\Delta, B) \arrow[r, two heads] & \Sigma_\Delta B \end{tikzcd}$$

in order to obtain $C \oplus \mathrm{Hom}(k\,\Delta, A) \rightarrowtail \Sigma_\Delta A \oplus \mathrm{Hom}(k\,\Delta, B) \twoheadrightarrow \Sigma_\Delta B$. Hence

$$\begin{tikzcd} {[\Sigma_\Delta B, N]_{\Delta\mathrm{I}}} \arrow[r, hook] \arrow[d, equal] & {[\Sigma_\Delta A \oplus \mathrm{Hom}(k\,\Delta, B), N]_{\Delta\mathrm{I}}} \arrow[r, two heads] \arrow[d, equal] & {[C \oplus \mathrm{Hom}(k\,\Delta, A), N]_{\Delta\mathrm{I}}} \arrow[d, equal] \\ {[\Sigma_\Delta B, N]_{\Delta\mathrm{I}}} \arrow[r, hook] & {[\Sigma_\Delta A, N]_{\Delta\mathrm{I}}} \arrow[r, two heads] & {[C, N]_{\Delta\mathrm{I}}} \end{tikzcd}$$

is exact at the middle by Lemma 1.45. □

Now we define for every $n \in \mathbb{Z}$, the *n-th relative injective Benson-Carlson group*:

$$\Delta\,\widecheck{\mathrm{BC}}^n_{kG}(M, N) = \Delta\,\widecheck{\mathrm{BC}}^0_{kG}(M, \Sigma^n_\Delta N).$$

Even if $\Sigma^n_\Delta N$ is not defined for $n < 0$, the definition is reasonable since in the direct limit we omit only a finite number of initial terms. Since taking the direct limit respects exactness, we take the direct limit for each column to conclude the following.

Proposition 1.49. *The relative injective Benson-Carlson groups $\Delta\,\widecheck{\mathrm{BC}}^*_{kG}(-, N)$ defines a proper contravariant $(-\infty, \infty)$-cohomological functor.*

Theorem 1.50. *Let N be a kG-module. Then for all $n \in \mathbb{Z}$ there is a natural equivalence of functors*

$$\Theta^n : \Delta\widecheck{\mathrm{Ext}}^n_{kG}(-, N) \to \Delta\,\widecheck{\mathrm{BC}}^n_{kG}(-, N).$$

PROOF. Let $d : N \rightarrowtail I^*$ be a Δ-injective resolution of N such that for every $j \geqslant 1$ $\Sigma^j_\Delta N = \operatorname{coker}(I^j \to I^{j+1})$. From $\Sigma^{n-1}_\Delta N \rightarrowtail I^n \twoheadrightarrow \Sigma^n_\Delta N$ we have the commutative diagram:

$$\begin{tikzcd}
\operatorname{Hom}_{kG}(M, I^n) \arrow[r, hook, "\iota^*"] \arrow[d, "\pi"] & \operatorname{Hom}_{kG}(M, \Sigma^n_\Delta N) \arrow[r, two heads, "\delta"] \arrow[d, "\pi'"] & \Delta\mathrm{I}\operatorname{Ext}^1(N, \Sigma^{n-1}_\Delta N) \arrow[dl, dotted, "\theta^n"] \\
{[M, I^n]_{\Delta\mathrm{I}}} \arrow[r, hook, "{[\iota^*]}"] & {[M, \Sigma^n_\Delta N]_{\Delta\mathrm{I}}}. &
\end{tikzcd}$$

By the surjectivity of δ we can define the map $\theta^n : \Delta\mathrm{I}\operatorname{Ext}^1(N, \Sigma^{n-1}_\Delta N) \to [M, \Sigma^n_\Delta N]_{\Delta\mathrm{I}}$ by mapping every $\bar{a} \in \Delta\mathrm{I}\operatorname{Ext}^1(N, \Sigma^{n-1}_\Delta N)$ to $\pi'(a)$ where a is a δ-preimage of $\bar{a}$. Since $[M, I^n]_{\Delta\mathrm{I}} = 0$ the map θ^n is well-defined: choose $a' \neq a$ such that $\delta(a') = \bar{a}$ then $\delta(a - a') = 0$ and so $a - a' = \iota^*(\tilde{a})$ for some $\tilde{a} \in \operatorname{Hom}_{kG}(M, I^n)$ and $\pi'(a - a') = [\iota^*]\pi(\tilde{a}) = 0$.

We show that θ^n is surjective as π', since (dimension shifting)

$$\Delta\mathrm{I}\operatorname{Ext}^1(M, \Sigma^{n-1}_\Delta N) \cong \Delta\mathrm{I}\operatorname{Ext}^n(M, N)$$

we have for every $n \geqslant 1$ the surjection $\theta^n : \Delta\mathrm{I}\operatorname{Ext}^n(M, N) \twoheadrightarrow [M, \Sigma^n_\Delta N]_{\Delta\mathrm{I}}$. Now direct limits respect surjectivity, so we have for each integer n, the surjection:

$$\varinjlim_{k \geqslant |n|} \Delta\mathrm{I}\operatorname{Ext}^{n+k}(\Sigma^k_\Delta M, N) \twoheadrightarrow \varinjlim_{k \geqslant |n|} [\Sigma^k_\Delta M, \Sigma^{n+k}_\Delta N]_{\Delta\mathrm{I}} =: \Delta\,\breve{\mathrm{BC}}^n(M, N).$$

From the short exact sequence

$$\Sigma^k_\Delta(M) \rightarrowtail \operatorname{Hom}(k\,\Delta, \Sigma^k_\Delta M) \twoheadrightarrow {\Sigma_\Delta}^{k+1} M$$

we have

$$\zeta : \Delta\mathrm{I}\,\mathbb{E}^{n+k}_{kG}(\Sigma^k_\Delta M, N) \to \Delta\mathrm{I}\,\mathbb{E}^{n+k+1}_{kG}(\Sigma^{k+1}_\Delta M, N)$$

$$\begin{aligned}
\operatorname{im}\zeta = \ker(\Delta\mathrm{I}\operatorname{Ext}^{n+k+1}_{kG}(\Sigma^{k+1}_\Delta M, N) &\to \Delta\mathrm{I}\operatorname{Ext}^{n+k+1}_{kG}(\operatorname{Hom}(k\Delta, \Sigma^{k+1}_\Delta M), N)) \\
&= S^{-1}\,\Delta\mathrm{I}\operatorname{Ext}^{n+k+1}_{kG}(\Sigma^k_\Delta M, N) \\
&= S^{-k}\,\Delta\mathrm{I}\operatorname{Ext}^{n+k+1}_{kG}(\Sigma_\Delta M, N).
\end{aligned}$$

Hence,

$$\begin{aligned}
\varinjlim_{k \geqslant |n|} \Delta\mathrm{I}\operatorname{Ext}^{n+k}_{kG}(\Sigma^k_\Delta M, N) &= \varinjlim_{k \geqslant |n|} S^{-k}\,\Delta\mathrm{I}\operatorname{Ext}^{n+k+1}_{kG}(\Sigma_\Delta M, N) \\
&= \Delta\widetilde{\operatorname{Ext}}^{n+1}(\Sigma_\Delta M, N) \\
&= \Delta\widetilde{\operatorname{Ext}}^{n}(M, N).
\end{aligned}$$

Hence, for every $n \in \mathbb{Z}$ we have the surjection,

$$\Theta^n : \Delta\widetilde{\mathrm{Ext}}^n(M,N) \twoheadrightarrow \Delta\widetilde{\mathrm{BC}}^n(M,N).$$

We are left to deal with the injectivity of Θ^n, so we start by considering $\hat{x} \in \ker \Theta^n$. The element $\hat{x}$ can be represented by an element $\bar{x} \in \Delta\mathrm{I}\,\mathrm{Ext}^{n+k}(\Sigma^k_\Delta M, N)$ for some $k \geqslant |n|$ such that its image in $[\Sigma^k_\Delta M, \Sigma^{k+n}_\Delta N]_{\Delta\mathrm{I}}$ is zero, i.e. it factors through a Δ-injective. Using the Δ-injective resolution of N we can represent $\bar{x}$ by a cocycle $x : \Sigma^k_\Delta M \to I^{n+k+1}$, which factors through $\Sigma^{n+k}_\Delta N$. Thus we have obtained

$$\begin{tikzcd}
\Sigma^k_\Delta M \arrow[r, hook] \arrow[d, "x"'] \arrow[dr, "y"] & \mathrm{Hom}(k\,\Delta, \Sigma^k_\Delta M) \arrow[d, "\phi"] \\
I^{n+k+1} & \Sigma^{n+k}_\Delta N. \arrow[l, tail]
\end{tikzcd}$$

Since y is a representative of the image of $\bar{x}$ under Θ^n in $[\Sigma^k_\Delta M, \Sigma^{k+n}_\Delta N]_{\Delta\mathrm{I}}$ (that is zero) it factors through a Δ-injective. The injection $\Sigma^{n+k}_\Delta \rightarrowtail I^{n+k+1}N$ is Δ-split, and so we can assume that y factors through $\mathrm{Hom}(k\,\Delta, \Sigma^k_\Delta M)$ and the above diagram commutes.

Now we examine at

$$\begin{tikzcd}
\Sigma^k_\Delta M \arrow[r, hook, "\iota"] \arrow[d, "y"'] & I^{k+1}M \arrow[r, two heads, "\pi"] \arrow[d] \arrow[dl] & \Sigma^{k+1}_\Delta M \arrow[r, equal] \arrow[d, "\Sigma^y_\Delta"] \arrow[dl, "\Psi"] & \Sigma^{k+1}_\Delta M \arrow[d, "z"] \\
\Sigma^{n+k}_\Delta N \arrow[r, hook] & I^{n+k+1} \arrow[r, two heads] & \Sigma^{n+k+1}_\Delta N \arrow[r, hook] & I^{n+k+1}N
\end{tikzcd}$$

and we note that y factors through $I^{k+1}M$, a Δ-injective. Now z represents $\bar{x}$ in $\Delta\mathrm{I}\,\mathrm{Ext}^{n+k+1}_\Delta(\Sigma^{k+1}_\Delta M, N)$, z is a coboundary since $z = \tau'\sigma\Psi = d\Psi$, where $d : I^{n+k} \to I^{n+k+1}$. Hence $\delta\bar{x} = 0$ and $\hat{x} = 0$ in the direct limit. $\square$

Theorem 1.51. *Let M and N be kG-modules.*

(1) If M or N has finite Δ-injective dimension then $\Delta\,\widetilde{\mathrm{Ext}}^n_{kG}(M,N) = 0$, for all $n \in \mathbb{Z}$.
(2) $\Delta\,\widetilde{\mathrm{Ext}}^0_{kG}(M,M) = 0 \iff \Delta\,\mathrm{id}_{kG}\, M < \infty$.

PROOF. It is analogous to [**Nuc98**, Theorem 3.7]. $\square$

Complete Δ-injective Resolutions.

Definition 1.52. Let M be a kG-module. Then a *complete Δ-injective resolution* of M is a Δ-split exact sequence of Δ-injectives $\mathbf{I} = (I^*, d)$, indexed by the integers such that

(1) $\mathbf{I}$ coincides with a Δ-injective resolution of M in sufficiently high dimensions,

(2) the chain complex $\mathrm{Hom}_{kG}(J, I^*)$ is exact for any Δ-injective kG-module J.

Lemma 1.53. *Let M be a kG-module with a complete Δ-injective resolution $\mathbf{I} = (I^*, d)$. Then:*

(1) if $\mathbf{J} = (J^*, \delta)$ is a Δ-injective resolution of M with $J^i = 0$ for all $i < 0$, and coinciding with $\mathbf{I}$ in sufficiently high dimensions, then there is a chain map $\mathbf{J} \to \mathbf{I}$ unique up to homotopy;

(2) any two complete Δ-injective resolutions of a kG-module M are chain homotopy equivalent.

PROOF. The proof of (1) is analogous to [**Nuc98**, Lemma 7.2]. Part (2) follows from Theorem 1.23 and part (1). $\square$

Note that the functor $\mathrm{H}^*(\mathrm{Hom}_{kG}(-, I))$ is a proper (contravariant) complete $(-\infty, \infty)$-cohomological functor.

Theorem 1.54. *Let M be a kG-module with a complete Δ-injective resolution $\mathbf{I} = (I^*, d)$. Then for any kG-module N we have the equivalence*

$$\Delta\,\widetilde{\mathrm{Ext}}^*_{kG}(N, M) \cong \mathrm{H}^*(\mathrm{Hom}_{kG}(N, I^*)).$$

PROOF. Suppose $M \rightarrowtail \mathbf{J}$ is a Δ-injective resolution of the module M such that $\mathbf{I}$ coincides with $\mathbf{J}$ in sufficiently high dimensions. Then by Lemma 1.53 there is a natural transformation

$$\Delta\mathrm{I}\,\mathrm{Ext}^*_{kG}(-, M) \to \mathrm{H}^*(\mathrm{Hom}_{kG}(-, I^*)).$$

By Lemma 1.39 and by the uniqueness of the relative I-completion we have that this natural transformation is a natural equivalence. $\square$

For any ring R, the functors $\widetilde{\operatorname{Ext}}^*_R(M,N)$ and $\widehat{\operatorname{Ext}}^*_R(M,N)$ are naturally equivalent if and only if $\operatorname{silp} R$ and $\operatorname{spli} R$ are finite [**Nuc98**]. It would be interesting to know when $\Delta\,\widetilde{\operatorname{Ext}}^*_{kG}(M,N)$ and $\Delta\,\widehat{\operatorname{Ext}}^*_{kG}(M,N)$ are naturally equivalent. If k has finite self-injective dimension, Emmanouil answered an old open question by showing that $\operatorname{silp} kG < \infty \iff \operatorname{spli} kG < \infty$ [**Emm10**]. Maybe a similar result holds for $\Delta \operatorname{silp} kG$ and $\Delta \operatorname{spli} kG$, where these invariants are defined in the obvious way.

Corollary 1.55. *Let N be a kG-module with a complete Δ-injective resolution. Then,*

(1) for every Δ-injective module I in sufficiently high dimensions,

$$\Delta\mathrm{I}\operatorname{Ext}^i_{kG}(I,N) = 0,$$

(2) if Q is Δ-projective module,

$$\Delta\,\widetilde{\operatorname{Ext}}^*_{kG}(Q,N) = 0.$$

PROOF. Part (1) is immediate from Theorem 1.54. For part (2) let $N \rightarrowtail \mathbf{J}$ be a Δ-injective resolution of the module N and let $C_i = \operatorname{coker}(J_{i-1} \to J_i)$. By Theorem 1.54 and dimension shifting, it follows $\Delta\,\widetilde{\operatorname{Ext}}^n_{kG}(A,N) = \Delta\operatorname{Ext}^1_{kG}(A,C_{n-2})$ for any kG-module A. When A is Δ-projective the functor $\operatorname{Hom}_{kG}(A,-)$ is exact on Δ-split sequences, hence $\Delta\operatorname{Ext}^1_{kG}(A,C_{n-2}) = 0$ and so $\Delta\,\widetilde{\operatorname{Ext}}^n_{kG}(A,N) = 0$. □

Proposition 1.56. [**KT91**] *Let G be a group and $\mathrm{B}(G,\mathbb{Z})$ be the $\mathbb{Z}\,G$-module of bounded functions from G to $\mathbb{Z}$. Then $\mathrm{B}(G,\mathbb{Z})$ satisfies the following*

- *$\mathrm{B}(G,\mathbb{Z})$ is a free $\mathbb{Z}\,F$-module for each $\mathfrak{F}$-subgroup F of G,*
- *$\mathrm{H}^0(G;\mathrm{B}(G,\mathbb{Z})) \neq 0$.*

Moreover, we consider:

- $\mathrm{B}(\Delta,\mathbb{Z})$ for the set of bounded functions from Δ to $\mathbb{Z}$,
- $\mathrm{B}(\Delta,k) = \mathrm{B}(\Delta,\mathbb{Z}) \otimes_{\mathbb{Z}} k$ for the k-algebra of functions from Δ to k which takes only finitely many values.

As in [**Nuc99**] we define the G-action on $\mathrm{B}(\Delta,k)$ as $\phi^g(\delta) = \phi(\delta g^{-1})$. The module $\mathrm{B}(\Delta,\mathbb{Z})$ is in general not free over $\mathbb{Z}\,F$. To see this, first consider the $\mathfrak{F}$-split surjection $\mathrm{B}(\Delta,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}$. This implies that $\mathbb{Z}\,\Delta$ is a direct summand of $\mathrm{B}(\Delta,\mathbb{Z}) \otimes \mathbb{Z}\,\Delta$ but $\mathbb{Z}\,\Delta$ is not $\mathbb{Z}\,G$-free in general.

Lemma 1.57. [**Nuc99**, 5.6]

(1) $\mathrm{B}(\Delta, k)$ *is a free k-module,*
(2) *there is a k-split inclusion* $k \rightarrowtail \mathrm{B}(\Delta, k)$ *of kG-modules,*
(3) *the inclusion* $k \rightarrowtail \mathrm{B}(\Delta, k)$ *is Δ-split.*

Lemma 1.58. *Let I be a Δ-injective kG-module. Then for any k-free kG-module L, the module* $\mathrm{Hom}(L, I)$ *is Δ-injective.*

PROOF. Define the action of G on $\phi \in \mathrm{Hom}(B, C)$ by $\phi^g(a) = \phi(ag^{-1})g$. Then by [**CK96**, 3.2] it follows that there is a natural isomorphism:

$$\mathrm{Hom}_{kG}(A \otimes B, C) \cong \mathrm{Hom}_{kG}(A, \mathrm{Hom}(B, C)).$$

Since the functor $- \otimes L$ is exact, and it takes split sequences to split sequences, the result follows now by the Δ-injectivity of I. Again $- \otimes -$ is always associative and since k is commutative, $- \otimes -$ is commutative and so $(A \otimes k\,\Delta) \otimes L \cong (A \otimes L) \otimes k\,\Delta$. □

Lemma 1.59. *Let I be a Δ-injective kG-module. Then it is a direct summand of* $I \otimes \mathrm{B}(\Delta, k)$.

PROOF. By tensoring $k \rightarrowtail \mathrm{B}(\Delta, k)$ we have a G-monomorphism $\iota : I \rightarrowtail I \otimes \mathrm{B}(\Delta, k)$. Then by (3) of Lemma 1.57 and by Lemma 1.17 we have that $k \rightarrowtail \mathrm{B}(\Delta, k)$ splits when restricted to each stabiliser G_δ, and so does ι, hence ι is Δ-split. By Lemma 1.21 we are done. □

We conclude the chapter with a result analogous to [**Nuc99**, 5.9].

Lemma 1.60 (A sufficient condition)**.** *Let L be a kG-module such that*

$$\Delta\,\mathrm{id}_{kG}(\mathrm{Hom}(\mathrm{B}(\Delta, k), L)) < \infty.$$

Then L has a complete Δ-injective resolution which splits under $\mathrm{Hom}(\mathrm{B}(\Delta, k), -)$.

PROOF. We write B for $\mathrm{B}(\Delta, k)$, $\bar{\mathrm{B}}$ for the cokernel of the injection $k \rightarrowtail B$, and $\mathrm{Cok}^r L$ for the r-th cokernel of a Δ-injective resolution of L. By Lemma 1.58 the module $\mathrm{Hom}(\mathrm{B}, \mathrm{Cok}^r L)$ is Δ-injective for every r and since $\Delta\,\mathrm{id}_{kG}(\mathrm{Hom}(\mathrm{B}, L)) < \infty$, we may assume $\mathrm{Hom}(\mathrm{B}, L)$ is Δ-injective by replacing L with a suitable cokernel. By part (2) of Lemma 1.57 we have a surjective G-map,

$$\pi : \mathrm{Hom}(\mathrm{B}, L) \twoheadrightarrow L,$$

that by 3 of Lemma 1.57 and by Lemma 1.17 we have that the inclusion $k \rightarrowtail \mathrm{B}$ splits for all finite subgroups of G, which implies that π is Δ-split. Now we write $\mathrm{Hom}^i(\bar{\mathrm{B}}, L)$ for $\mathrm{Hom}(\bar{\mathrm{B}}, \mathrm{Hom}(\bar{\mathrm{B}}, \ldots, \mathrm{Hom}(\bar{\mathrm{B}}, L \cdots)))$ i-times. By the above, $\mathrm{Hom}(\mathrm{B}, \mathrm{Hom}^i(\bar{\mathrm{B}}, L)$ is Δ-injective for every $i \geqslant 0$. Hence we have a Δ-injective exact complex:

$$\cdots \to \mathrm{Hom}(\mathrm{B}, \mathrm{Hom}^i(\bar{\mathrm{B}}, L)) \to \cdots \to \mathrm{Hom}(\mathrm{B}, \mathrm{Hom}(\bar{\mathrm{B}}, L)) \\ \to \mathrm{Hom}(B, L) \twoheadrightarrow L$$

that is Δ-split since the surjection to L is Δ-split.

Note that by Lemma 1.58 the module $\mathrm{Hom}(\mathrm{B}, \mathrm{Hom}^i(\bar{B}, L))$ is Δ-injective for every $i \geqslant 0$, and we have that the short exact Δ-split sequence

$$\mathrm{Hom}(\mathrm{B}, \mathrm{Hom}^{i+1}(\bar{B}, L)) \rightarrowtail \mathrm{Hom}(\mathrm{B}, \mathrm{Hom}(\mathrm{B}, \mathrm{Hom}^i(\bar{\mathrm{B}}, L))) \\ \twoheadrightarrow \mathrm{Hom}(\mathrm{B}, \mathrm{Hom}^i(\bar{B}, L))$$

by 3 of Lemma 1.57 splits. This clearly remains true even for a Δ-injective resolution of L. Splicing this together with a Δ-injective resolution $\mathbf{J}$ of L, we have a complete Δ-injective resolution $\mathbf{I}$ of L; of course we are left to check that $\mathrm{Hom}_{kG}(J, \mathbf{I})$ is exact for every Δ-injective J.

By Lemma 1.59 it is sufficient to show that $\mathrm{Hom}(J \otimes \mathrm{B}, \mathbf{I}) \cong \mathrm{Hom}(J, \mathrm{Hom}(\mathrm{B}, \mathbf{I}))$ is exact, but by the above $\mathrm{Hom}(\mathrm{B}, \mathbf{I})$ splits and so $\mathrm{Hom}(J, \mathrm{Hom}(\mathrm{B}, \mathbf{I}))$ is exact. $\square$

CHAPTER 2

$\mathfrak{F}$-cohomological dimension

In the context of cohomology relative to a G-set there is a well-defined notion of $\mathfrak{F}$-*cohomological dimension*. One of the main reasons why this dimension is important comes from the fact that every $\mathbf{H}_1\mathfrak{F}$-group has finite $\mathfrak{F}$-cohomological dimension. Nucinkis introduced $\mathfrak{F}$-cohomology in order to find an appropiate algebraic counterpart for the minimal dimension of a contractible G-CW-complex with $\mathfrak{F}$-stabilisers. This question remains unsolved after 12 years; in fact, it is also unknown if the finiteness of the $\mathfrak{F}$-cohomological dimension implies $\mathbf{H}_1\mathfrak{F}$-membership. There is hope that studying the behaviour of the $\mathfrak{F}$-cohomological dimension will shed some light on the question above. For a group G, $\mathfrak{F}\operatorname{cd} G$ denotes its $\mathfrak{F}$-cohomological dimension. In this chapter we focus our attention on its behaviour under forming group extensions and we prove:

Theorem. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a group extension with $\mathfrak{F}\operatorname{cd} N \leqslant n$. Moreover, assume that for any subgroup H of G with $\mathfrak{F}\operatorname{cd} H \leqslant n$ and any extension L of H by a group of prime order has $\mathfrak{F}\operatorname{cd} L \leqslant n$. Then $\mathfrak{F}\operatorname{cd} G \leqslant \mathfrak{F}\operatorname{cd} H + \mathfrak{F}\operatorname{cd} Q$.*

We make a brief comment about the result. The behaviour of $\mathbf{H}_1\mathfrak{F}$-groups and in particular the behaviour of the minimal dimension of a contractible G-CW-complex with $\mathfrak{F}$-stabilisers is not completely understood with respect to taking group extensions. There are several other algebraic dimensions defined for groups for which finiteness is conjectured to imply membership of $\mathbf{H}_1\mathfrak{F}$. It is important to recall that for these dimensions the exact and good behaviour with respect of taking group extensions is often well-known.

1. Basics, examples and motivation

We begin by recalling the main group invariant for discussion in this chapter. Let G be a group and Δ a G-set.

Definition 2.1. [**Nuc00**] The Δ-cohomological dimension of G is defined as

$$\begin{aligned}\Delta\operatorname{cd}_k G &:= \Delta\operatorname{pd}_{kG} k\\ &:= \inf\{n \mid k \text{ admits an } \Delta\text{-projective resolution of length } n\}\\ &= \sup\{n \mid \Delta\operatorname{H}^n(G;M) \neq 0, \text{ for some } \mathbb{Z}\,G\text{-module } M\},\end{aligned}$$

where $\Delta\operatorname{H}^*(G;-) := \Delta\operatorname{Ext}^*_{kG}(k,-)$. The last equality can be shown analogously to [**Bro82**, VIII, Lemma 2.1]. Since the augmented cellular chain complex of a finite-dimensional contractible G-CW complex with $\mathfrak{F}$-stabilisers is an $\mathfrak{F}$-projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\,G$, it is natural to work with $\mathbb{Z}$ as the ring of coefficients. We write $\Delta\operatorname{cd} G$ for $\Delta\operatorname{cd}_{\mathbb{Z}} G$ and $\mathfrak{F}\operatorname{cd} G$ when the G-set Δ has the property $(*)$.

The classifying space for proper actions. Let G be a group and X be a CW-complex. The cell complex X is a G-CW-complex if G acts admissibly on X by self-homeomorphisms. The action is *admissible* if the set-wise stabiliser of each cell coincides with its point-wise stabiliser. A G-CW-complex is *proper* if all its cell stabilisers are finite. If a proper G-CW-complex has the property that for each $\mathfrak{F}$-subgroup K of G the fixed-point subcomplex X^K is contractible, it is called a *classifying space for proper actions of* G and it is denoted by $E_{\mathfrak{F}}G$. A classifying space for proper actions of G can also be defined as a terminal object in the homotopy category of proper G-CW-complexes. Note that, since G does not act freely on X, usually it is not possible to recover G from X/G, contrary to what happens on free G-CW-complexes and there is no notion of proper Eilenberg-Mac Lane space for G.

Examples of models for $E_{\mathfrak{F}}G$ arise virtually everywhere in geometric group theory; many natural spaces on which infinite groups act are classifying spaces for proper actions. Forcing the action to be free leads to an infinite-dimensional space whenever G has torsion, while for any $\mathfrak{F}$-group F a point is a suitable $E_{\mathfrak{F}}F$. Of course, if G is torsion-free, then any model for EG is a model for $E_{\mathfrak{F}}G$. We will mention here some examples; the interested reader is advised to consult [**Lüc05**] for a more explicit and wide survey on the topic.

Examples 2.2.

- If $G = D_\infty := C_\infty \rtimes C_2$, then $\mathbb{R}$ with the usual D_∞-action is a model for $E_{\mathfrak{F}}G$.

- If G is a countable $\mathbf{L}\mathfrak{F}$-group, then G acts on a tree $\mathcal{T}$ with $\mathfrak{F}$-stabilisers and $\mathcal{T}$ is a 1-dimensional model for $E_{\mathfrak{F}}G$ [**Ser03**].
- If G is a word-hyperbolic group, then the second barycentric subdivision of the Rips complex $R_d(G)$ is a finite model for $E_{\mathfrak{F}}G$ for large enough d [**MS02**].
- The barycentric subdivision of the spline K_n' is a finite $(2n-3)$-dimensional model for $E_{\mathfrak{F}}\operatorname{Out}(F_n)$[**KV93, BV01**].

Remark 2.3. By generalisations of constructions in [**Mil56, Seg68**] every group G admits a model for $E_{\mathfrak{F}}G$. It is interesting to study groups admitting classifying spaces for proper actions satisfying some finiteness properties; the most popular properties are being finite-dimensional, being of finite type and being cocompact. Our work is mainly focused on the first property, and for a group G the minimal dimension of a model for an $E_{\mathfrak{F}}G$ is called *Bredon geometric dimension* and is denoted by $\operatorname{gd}_{\mathfrak{F}} G$.

The *orbit category* $\mathcal{O}_{\mathfrak{F}}G$ has as objects the G-sets $H\backslash G$ where H is an $\mathfrak{F}$-subgroup of G and the morphisms are the G-maps. An $\mathcal{O}_{\mathfrak{F}}G$*-module* is a contravariant functor from $\mathcal{O}_{\mathfrak{F}}G$ to the category of abelian groups. For any G-set X with $\mathfrak{F}$-stabilisers, $\mathbb{Z}[?, X]_G$ denotes the $\mathbb{Z}$-free module on the set of G-maps from ? to X. It turns out that the *free* $\mathcal{O}_{\mathfrak{F}}G$-modules are precisely the modules of the form $\mathbb{Z}[?, X]_G$ where X is a G-set with $\mathfrak{F}$-stabilisers, and their direct summands are the *projective* $\mathcal{O}_{\mathfrak{F}}G$-modules. Moreover, in the category of $\mathcal{O}_{\mathfrak{F}}G$-modules there are enough projective $\mathcal{O}_{\mathfrak{F}}G$-modules and there exists a well-defined notion of projective dimension. Let $\underline{\mathbb{Z}}$ be the $\mathcal{O}_{\mathfrak{F}}G$-module that sends every G-set to $\mathbb{Z}$. Then the *Bredon cohomological dimension* of G, denoted by $\operatorname{cd}_{\mathfrak{F}} G$, is the projective dimension of $\underline{\mathbb{Z}}$.

A group G that admits a model for $E_{\mathfrak{F}}G$ of dimension $n < \infty$ has finite Bredon cohomological dimension, finite $\mathfrak{F}$-cohomological dimension, finite rational cohomological dimension and the integral homology groups $\mathrm{H}_i(G)$ with arbitrary coefficients are torsion for $i > n$ [**KM98**]. There is another class of groups closely related to the class of groups having a finite-dimensional classifying space for proper actions, the class $\mathbf{H}_1\mathfrak{F}$. This is the first "non-trivial" step for building up Kropholler's class $\mathbf{H}\,\mathfrak{F}$. A group is an $\mathbf{H}_1\,\mathfrak{F}$-group if it admits a contractible finite-dimensional G-CW-complex with $\mathfrak{F}$-stabilisers. We introduce the notion of *Kropholler dimension*. This is, for a group G, the minimal dimension of a contractible G-CW-complex with $\mathfrak{F}$-stabilisers. We denote the Kropholler dimension of a group G by $\mathfrak{K}(G)$. Of course $G \in \mathbf{H}_1\mathfrak{F}$ if and only if $\mathfrak{K}(G) < \infty$.

Clearly every group that has a finite-dimensional model for $E_{\mathfrak{F}}$ is in $\mathbf{H}_1\,\mathfrak{F}$ and a conjecture of Kropholler and Mislin claims that this implication is reversible. If $G \in \mathbf{H}_1\,\mathfrak{F}$ then the augmented chain complex over $\mathbb{Z}$ is a resolution of finite length of $\mathbb{Z}$ over $\mathbb{Z}\,G$ made of permutation modules with $\mathfrak{F}$-stabilisers. A result proved independently by Bouc [**Bou99**] and Kropholler-Wall [**KW11**] implies that this resolution is $\mathfrak{F}$-split. Hence, every $\mathbf{H}_1\,\mathfrak{F}$-group has finite $\mathfrak{F}$-cohomological dimension and in particular $\mathfrak{F}\,\mathrm{cd}\,G \leqslant \mathfrak{K}(G)$. One of the main questions was how to algebraically characterise groups that admit finite-dimensional classifying spaces for proper actions. The problem was solved with Bredon cohomology. It turns out that every group G of finite Bredon cohomological dimension has a finite-dimensional $E_{\mathfrak{F}}G$ [**Lüc89**]; more precisely for any group G we have $\mathrm{gd}_{\mathfrak{F}}\,G \leqslant \max\{3, \mathrm{cd}_{\mathfrak{F}}\,G\}$. It is important to recall that in [**BLN01**] the authors show a family of counterexamples for the *proper Eilenberg-Ganea conjecture*; i.e. they exhibit examples of groups G such that $\mathrm{gd}_{\mathfrak{F}}\,G = 3$ and $\mathrm{cd}_{\mathfrak{F}}\,G = 2$. The Bredon cohomological dimension is often problematic to compute, and it would be interesting to find alternative algebraic invariants that guarantee its finiteness. With this in mind, our interest is to understand groups of finite $\mathfrak{F}$-cohomological dimension.

Another interesting unsolved question asks if the $\mathfrak{F}$-cohomological dimension mirrors the Kropholler dimension.

Recently Leary and Nucinkis looked at groups admitting a classifying space with stabilisers of prime power order $\mathfrak{P}_{\mathfrak{F}}$. They showed the following.

Lemma 2.4 ([**LN10**]). *A group G admits a finite-dimensional $E_{\mathfrak{P}_{\mathfrak{F}}}G$ if and only if every finite subgroup of G is of prime power order and G admits a finite dimensional $E_{\mathfrak{F}}G$.*

The class of groups admitting a finite-dimensional $E_{\mathfrak{P}_{\mathfrak{F}}}G$ is much smaller than the class of groups admitting a finite-dimensional $E_{\mathfrak{F}}G$. It does not even contain the class $\mathfrak{F}$. Does $\mathfrak{F}$-cohomology that is a cohomology theory relative to $\mathfrak{P}_{\mathfrak{F}}$ detect the finiteness of a model for an $E_{\mathfrak{F}}G$?

The Weyl-groups of a given group G are the groups $WH := N_G(H)/H$. When looking at the classifying space for proper actions or at cohomological theories relative to the family of $\mathfrak{F}$-subgroups the Weyl-groups play a relevant role. Assume that Δ is a G-set satisfying condition $(*)$ and that H is a finite subgroup of G. It is possible to show [**Nuc00**] that the fixed-point set Δ^H is a WH-set satisfying condition $(*)$. This leads to the strong relation that exists between $\mathfrak{F}$-cohomology and Bredon cohomology with respect to the family of finite subgroups. We assume the reader to be familiar with the basic theory of Bredon cohomology. The interested reader can consult [**Lüc89, MV03**]. Since we are exclusively interested in group cohomology with respect to the family of finite subgroups we write "Bredon cohomology" for "Bredon cohomology with respect to the family of $\mathfrak{F}$-subgroups". Evaluating an $\mathcal{O}_{\mathfrak{F}}G$-module at $H\backslash G$ gives a $\mathbb{Z}\,WH$-module and the next result links the Bredon cohomological dimension with the $\mathfrak{F}$-cohomological dimension.

Theorem 2.5. [**Nuc00**, 3.2] *Consider an $\mathcal{O}_{\mathfrak{F}}G$-projective resolution $\mathbf{P}(-)$ of the Bredon module $\underline{\mathbb{Z}}$. Evaluated at $H\backslash G$ for any finite subgroup H of G, it gives an $\mathfrak{F}$-projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\,WH$. In particular, we have for each finite subgroup H of G, $\mathfrak{F}\operatorname{cd} WH \leqslant \operatorname{cd}_{\mathfrak{F}} G$.*

So we can picture a Bredon projective resolution of $\mathbb{Z}$ as a sequence of $\mathfrak{F}$-projective resolutions over $\mathbb{Z}\,WH$ with various connecting homomorphisms, i.e. for every G-map from G/H to G/K there is a morphism of complexes from $\mathbf{P}(K\backslash G)$ to $\mathbf{P}(H\backslash G)$. An $\mathcal{O}_{\mathfrak{F}}G$-module is a *free* $\mathcal{O}_{\mathfrak{F}}G$-module if it is isomorphic to a direct sum of $\mathcal{O}_{\mathfrak{F}}G$-modules of the form $\mathbb{Z}[-, H\backslash G]$. At this point it is natural to ask if in the theorem above we can replace the $\mathcal{O}_{\mathfrak{F}}G$-module $\underline{\mathbb{Z}}$ with any $\mathbb{Z}$-free $\mathcal{O}_{\mathfrak{F}}G$-module. On the other hand we have:

Lemma 2.6. [**Nuc00**, 4.3] $\mathfrak{F}\operatorname{cd} WH \leqslant \mathfrak{F}\operatorname{cd} G$ *for every $\mathfrak{F}$-subgroup H of G.*

Remark 2.7. Since Leary and Nucinkis [**LN03**] showed the existence of groups G with $\mathfrak{F}\operatorname{cd} G < \operatorname{cd}_{\mathfrak{F}} G < \infty$ it is not possible in general to glue the resolutions of the Weyl-groups coming from the $\mathfrak{F}$-projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\,G$ to construct an $\mathcal{O}_{\mathfrak{F}}G$-projective resolution of $\underline{\mathbb{Z}}$ of the same length.

Lemma 2.8. *Let H be a subgroup of G, M be a kH-module and N be a kG-module. Then:*

- *$M \otimes_{kH} kG$ is kG-projective if and only if M is kH-projective;*
- $\operatorname{pd}_{kG} N \otimes_{kH} kG = \operatorname{pd}_{kH} N \leqslant \operatorname{pd}_{kG} N$.

PROOF. These hold in a more general context and proofs can be found in [**CK96**, Lemma 6.1, 6.3]. □

If G has finite $\mathfrak{F}$-cohomological dimension than this can be calculated by evaluating the $\mathfrak{F}$-cohomology groups in $\mathfrak{F}$-free modules.

Lemma 2.9. *If $\mathfrak{F}\,\mathrm{cd}\, G < \infty$, then*

$$\mathfrak{F}\,\mathrm{cd}\, G = \sup\{n \mid \mathfrak{F}\,\mathrm{H}^n(G; F) \neq 0,\ F\ \mathfrak{F}\text{-free}\},$$

where a module is $\mathfrak{F}$-free if it is of the form $\mathbb{Z}\,\Delta \otimes M$ for some $\mathbb{Z}\,G$-module M.

PROOF. It is analogous to [**Bro82**, 2.3, VIII]. □

Note that in the lemma above we are unable to take F as a permutation module with $\mathfrak{F}$-stabilisers. However, Proposition 2.18 tells us that if $\mathfrak{F}\,\mathrm{cd}\, G < \infty$ then there exists some G-set Ω with $\mathfrak{F}$-stabilisers and an $m \in \mathbb{N}$ such that $\mathfrak{F}\,\mathrm{H}^m(G; \mathbb{Z}\,\Omega) \neq 0$.

In very low dimension everything is known by Dunwoody's work on groups of rational cohomological dimension equal to 1 [**Dun79**].

Lemma 2.10. [**Nuc00**, 2.8, 2.9]

- $\mathfrak{F}\,\mathrm{cd}\, G = 0$ *if and only if* $G \in \mathfrak{F}$,
- $\mathfrak{F}\,\mathrm{cd}\, G \leqslant 1$ *if and only if* G *acts on a tree with* $\mathfrak{F}$*-stabilisers.*

Examples 2.11. Groups of $\mathfrak{F}$-cohomological dimension equal to 1 include virtually-free groups and infinite countable $\mathbf{L}\,\mathfrak{F}$-groups. Since the additive group of the rational numbers is torsion-free, we have

$$\mathfrak{F}\,\mathrm{cd}\, \mathbb{Q} = \mathrm{cd}\, \mathbb{Q} = 2,$$

where the second equality follows from Berstein's Theorem and the fact that $\mathbb{Q}$ is not free. Restricted wreath products of the form $F \wr \mathbb{Z}$ where F is a $\mathfrak{F}$-group are commonly called *lamplighter groups*. It is known that for a lamplighter group G there exists a 2-dimensional model for $E_{\mathfrak{F}}G$ by [**Lüc00**, Theorem 3.1]. We will see later in Proposition 2.30 a simple proof that for torsion-free groups the $\mathfrak{F}$-cohomological dimension agrees with the integral cohomological dimension. In particular, a free abelian group of rank n has $\mathfrak{F}$-cohomological dimension equal to n and the free abelian group of infinite countable rank has infinite $\mathfrak{F}$-cohomological dimension. This shows that the class of groups of finite $\mathfrak{F}$-cohomological dimension is not $\mathbf{L}$-closed.

Remark 2.12 (The augmentation module)**.** Let $I_\Delta = \ker \epsilon$, where $\epsilon : \mathbb{Z}\Delta \to \mathbb{Z}$ is defined as $(\delta)\epsilon = 1$, hence, $(\sum_{\delta\in\Delta} n_\delta \delta)\epsilon = \sum_{\delta\in\Delta} n_\delta$.

- $\mathfrak{F}\operatorname{cd} G \leqslant 1$ if and only if I_Δ is an $\mathfrak{F}$-projective module. This is equivalent to stating that the map $I_\Delta \otimes \mathbb{Z}\,\Delta \twoheadrightarrow I_\Delta$ splits. Note also that $I_\Delta \otimes \mathbb{Z}\,\Delta \cong \mathbb{Z}\,\underline{\Delta}$ for some G-set $\underline{\Delta}$ with $\mathfrak{F}$-stabilisers.
- The finiteness of the $\mathfrak{F}$-cohomological dimension does not this imply the finiteness of the *relative dimension* $\operatorname{rd} G := \operatorname{pd}_{\mathbb{Z}\,G} I_\Delta$. Any countable infinite periodic $\mathfrak{A}$-group G has infinite relative cohomological dimension but the $\mathfrak{F}$-cohomological dimension of G is equal to 1 [**Alo91**, Corollary 7].
- For any group G, $\mathfrak{F}\operatorname{cd} G \leqslant \operatorname{rd} G$. Let $\operatorname{rd} G \leqslant n$. Then by [**Alo91**, Theorem 3], there exists a n-dimensional acyclic G-CW-complex X such that G acts with $\mathfrak{F}$-stabilisers on the 0-skeleton and freely in positive dimension. The augmented cellular chain complex of X over $\mathbb{Z}$ gives the desired upper bound. It is important to remark that even in this extreme case we do not have an algebraic proof for $\mathfrak{F}\operatorname{cd} G \leqslant \operatorname{rd} G$.
- Let $\operatorname{Gcd} G$ be the *Gorenstein cohomological dimension of* G [**BDT09**]. Then $\operatorname{Gcd} G \leqslant \operatorname{rd} G$ since the $\mathbb{Z}\,G$-module $\mathbb{Z}\,\Delta$ is Gorenstein projective for any G-set Δ with $\mathfrak{F}$-stabilisers.
- Suppose $\operatorname{Gcd}\Gamma = n < \infty$. Consider the standard $\mathfrak{F}$-projective resolution of $\mathbb{Z}$. This is a Gorenstein projective resolution of $\mathbb{Z}$. The $(n-1)$th kernel K_n where $\iota : K_n \rightarrowtail \mathbb{Z}\,\Delta^{n-1}$ is Gorenstein projective. We ask the following. Is K_n $\mathfrak{F}$-projective? Is $\mathfrak{F}\operatorname{pd} K_n < \infty$? We know that ι is $\mathfrak{F}$-split. Note that by [**BDT09**] every cofibrant module is Gorenstein projective and by [**DT10**] the two concepts coincide over an $\mathbf{LH}\mathfrak{F}$-group. Does every cofibrant module over an $\mathbf{LH}\mathfrak{F}$-group have finite $\mathfrak{F}$-projective dimension?

Lemma 2.13 (Dimension shifting)**.**

- $\mathfrak{F}\operatorname{H}^n(G; A) \cong \mathfrak{F}\operatorname{H}^{n-1}(G; \operatorname{Hom}(I_\Delta, A))$*, for* $n \geqslant 2$.
- $\mathfrak{F}\operatorname{H}^1(G; A) \cong \operatorname{coker}\{\mathfrak{F}\operatorname{H}^0(G; \operatorname{Hom}(\mathbb{Z}\,\Delta, A)) \to \mathfrak{F}\operatorname{H}^0(G; \operatorname{Hom}(I_\Delta, A))\}$*, for* $n = 1$ *where* $\operatorname{Hom}(I_\Delta, A))$ *is a* $\mathbb{Z}\,G$*-module with the usual action.*

PROOF. It is analogous to VI, 12.1 in [**HS97**] or [**Bro82**, pg. 64]. Recall that since the sequence $I_\Delta \rightarrowtail \mathbb{Z}\,\Delta \twoheadrightarrow \mathbb{Z}$ is $\mathbb{Z}$-split, the functor Hom is exact and so we have that $\operatorname{coker}\{A \rightarrowtail \operatorname{Hom}(\mathbb{Z}\,\Delta, A)\} \cong \operatorname{Hom}(I_\Delta, A)$. □

Proposition 2.14. *Let G be an infinite group with finitely many conjugacy classes of $\mathfrak{P}_{\mathfrak{F}}$-subgroups and M be any $\mathbb{Z}\,G$-module. Consider a finitely-generated $\mathbb{Z}\,G$-module $\mathbb{Z}\,\Delta$ such that Δ satisfies $(*)$. Then $(M \otimes \mathbb{Z}\,\Delta)^G = 0$. In particular, $(P)^G = 0$ for any finitely-generated $\mathfrak{F}$-projective module P.*

PROOF. $(M \downarrow_H \uparrow_H^G)^G = 0$ for every $\mathfrak{F}$-subgroup H by [**Bro82**, Ex.4 pg.71]. Hence,

$$(M \otimes \mathbb{Z}\,\Delta)^G := \operatorname{Hom}_{\mathbb{Z}\,G}(\mathbb{Z}, M \otimes \mathbb{Z}\,\Delta) \cong \operatorname{Hom}_{\mathbb{Z}\,G}(\mathbb{Z}, M \otimes (\oplus_{\delta \in \Delta_0} \mathbb{Z}\,G_\delta \backslash G))$$
$$\cong \operatorname{Hom}_{\mathbb{Z}\,G}(\mathbb{Z}, \oplus_{\delta \in \Delta_0}(M \otimes \mathbb{Z}\,G_\delta \backslash G)) \cong \bigoplus_{\delta \in \Delta_0} \operatorname{Hom}_{\mathbb{Z}\,G}(\mathbb{Z}, M \otimes \mathbb{Z}\,G_\delta \backslash G) = 0,$$

where the last equality follows from the basic fact that $\operatorname{Hom}_{\mathbb{Z}\,G}(\mathbb{Z}, -)$ commutes with products, but since Δ_0 is finite, products and sums coincide. □

Theorem 2.15. [**Nuc00**, 2.7] *The property of having "finite $\mathfrak{F}\operatorname{cd}$" is closed under taking subgroups,* HNN*-extensions and free products with amalgamation.*

Lemma 2.16. *Let N be a $\mathbb{Z}\,G$-module, $n \in \mathbb{N}$ and*

$$M_n \rightarrowtail M_{n-1} \to \cdots \to M_1 \to M_0 \twoheadrightarrow M_{-1}$$

be an $\mathfrak{F}$-split exact sequence of $\mathbb{Z}\,G$-modules. Assume $\mathfrak{F}\operatorname{Ext}^i_{\mathbb{Z}\,G}(N, M_i) = 0$ for $0 \leqslant i \leqslant n$. Then $\mathfrak{F}\operatorname{Ext}^0_{\mathbb{Z}\,G}(N, M_{-1}) = 0$.

PROOF. It can be proved by induction similarly to [**DKLT02**, Lemma 7.3]. If $n = 0$ then $M_0 \cong M_{-1}$ and the statement is trivial.

Let $M'_{n-1} = \operatorname{coker}(M_n \rightarrowtail M_{n-1})$ then we have two exact $\mathfrak{F}$-split sequences:

(1) $M_n \rightarrowtail M_{n-1} \twoheadrightarrow M'_{n-1}$,
(2) $M'_{n-1} \rightarrowtail M_{n-2} \to \cdots \to M_0 \twoheadrightarrow M_{-1}$.

From the long exact sequence in $\mathfrak{F}$-cohomology that arise from (1) we have:

$$\cdots \to \mathfrak{F}\operatorname{Ext}^{n-1}(N, M_{n-1}) \to \mathfrak{F}\operatorname{Ext}^{n-1}(N, M'_{n-1}) \to \mathfrak{F}\operatorname{Ext}^n(N, M_r) \to \cdots$$

where $\mathfrak{F}\operatorname{Ext}^{n-1}(N, M_{n-1})$ and $\mathfrak{F}\operatorname{Ext}^n(N, M_n)$ are zero by assumption and so the group $\mathfrak{F}\operatorname{Ext}^{n-1}(N, M'_{n-1})$ is zero as well. Now by the induction hypothesis applied to (2) we can conclude $\mathfrak{F}\operatorname{Ext}^0_{\mathbb{Z}\,G}(N, M_{-1}) = 0$. □

Corollary 2.17. *If* $n \in \mathbb{N}$ *and*

$$M_n \rightarrowtail M_{n-1} \to \cdots \to M_0 \twoheadrightarrow \mathbb{Z}$$

is an $\mathfrak{F}$*-split resolution of the trivial* $\mathbb{Z}\,G$*-module* $\mathbb{Z}$*, then there exists* $0 \leqslant i \leqslant n$ *such that* $\mathfrak{F}\,\mathrm{H}^i(G; M_i) \neq 0$.

PROOF. Arguing by contradiction, we suppose that $\mathfrak{F}\,\mathrm{H}^i(G; M_i) = 0$ for all $0 \leqslant i \leqslant n$. By the lemma above follows $\mathfrak{F}\,\mathrm{H}^0(G; M_{-1}) = 0$ but $\mathfrak{F}\,\mathrm{H}^0(G; M_{-1}) \cong \mathrm{Hom}_{\mathbb{Z}\,G}(\mathbb{Z}, \mathbb{Z}) \neq 0$. $\square$

The next proposition is the relative version of [**DKLT02**, Proposition 7.7].

Proposition 2.18. *Let* G *be a group such that either the functors* $\mathrm{H}^*(G; \mathbb{Z}\,\Delta) = 0$ *or* $\mathfrak{F}\,\mathrm{H}^*(G; \mathbb{Z}\,\Delta) = 0$ *for every* G*-set* Δ *with* $\mathfrak{F}$*-stabilisers. Then* $\mathfrak{F}\,\mathrm{cd}\,G = \infty$.

PROOF. It follows straight from [**DKLT02**, Corollary 7.6] that if $n \in \mathbb{N}$ and

$$M_n \rightarrowtail M_{n-1} \to \cdots M_0 \twoheadrightarrow \mathbb{Z}$$

is an exact sequence of $\mathbb{Z}\,G$-modules then there exists an $0 \leqslant i \leqslant n$ such that $\mathrm{H}^i(G; M_i) \neq 0$. Analogously to Corollary 2.17 it follows that $\mathfrak{F}\,\mathrm{H}^i(G; M_i) \neq 0$. Suppose $\mathfrak{F}\,\mathrm{cd}\,G = m$. Then by Lemma 2.28 there is an $\mathfrak{F}$-split resolution of $\mathbb{Z}$ of length m made of permutation modules with $\mathfrak{F}$-stabilisers, and Corollary 2.17 gives a contradiction. $\square$

As an immediate corollary we obtain a well-known result.

Corollary 2.19. *Let* G *be a non-trivial group such that* $\mathrm{H}^*(G; F) = 0$ *for every* $\mathbb{Z}\,G$*-free module* F*. Then* $\mathrm{cd}\,G = \infty$.

PROOF. We can assume G to be torsion-free and apply Proposition 2.18. $\square$

It is important to remember that if G is infinite and $\mathrm{H}^*(G; P) = 0$ for every $\mathbb{Z}\,G$-projective module P then $\mathrm{Gcd}\,G = \infty$.

Examples of groups with cohomology groups vanishing on all projective modules are given by the free abelian group of infinite countable rank, Thompson's group **F** and $\mathrm{GL}(n, F)$ where F is a subfield of the algebraic closure of $\mathbb{Q}$ [**CK97**, 5.3].

Proposition 2.20. [**Nuc00**, 2.5] *Suppose*

$$G = \varinjlim_{\lambda \in \Lambda} G_\lambda$$

where Λ is countable. Then $\mathfrak{F}\,\mathrm{cd}\,G \leqslant 1 + \sup\{\mathfrak{F}\,\mathrm{cd}\,G_\lambda \mid \lambda \in \Lambda\}$.

We state explicitly a few consequences of the proposition above.

Corollary 2.21.

- *A countable $\mathbf{L}\mathfrak{F}$-group has $\mathfrak{F}$-cohomological dimension at most 1.*
- *Let $1 = H_0 \leqslant H_1 \leqslant \cdots$ be a series of normal subgroups of G such that $\mathfrak{F}\,\mathrm{cd}\,G/H_i \leqslant k$ for all $i \geqslant 0$. Then $\mathfrak{F}\,\mathrm{cd}\,G/\bigcup H_i \leqslant 1 + k$.*
- *If G is countable and locally of $\mathfrak{F}\,\mathrm{cd} \leqslant k$, then $\mathfrak{F}\,\mathrm{cd}\,G \leqslant 1 + k$.*

Proposition 2.22. *Let A be a countable $\mathfrak{A}$-group. Then $\mathfrak{F}\,\mathrm{cd}\,A \leqslant r_0(A) + 1$, where $r_0(A)$ denotes the torsion-free rank of A.*

PROOF. Since $G \in \mathfrak{A}$ there exists $T \rightarrowtail A \twoheadrightarrow A/T$ where T is the torsion subgroup of A, A/T is a torsion-free quotient and $\mathfrak{F}\,\mathrm{cd}(T/A) = \mathrm{cd}(T/A) = r_0(T/A) = r_0(A)$. Since $T \in \mathbf{L}\mathfrak{F}$, then $\mathfrak{F}\,\mathrm{cd}(T) \leqslant 1$. The result follows from the spectral sequence in [**Nuc00**]. □

2. Group extensions

Suppose $N \rightarrowtail G \twoheadrightarrow Q$ is a group extension. In the context of ordinary cohomology it is an immediate consequence of the Lyndon-Hochschild-Serre spectral sequence that the property of having finite cohomological dimension is extension closed. Furthermore, if $\mathrm{cd}\,N = n$ and $\mathrm{cd}\,Q = m$ then $\mathrm{cd}\,G \leqslant n + m$. This property suggests the following question.

Question 2.23. Let $N \rightarrowtail G \twoheadrightarrow Q$ be a group extension, with $\mathfrak{F}\,\mathrm{cd}\,N = s$ and $\mathfrak{F}\,\mathrm{cd}\,Q = r$. Is $\mathfrak{F}\,\mathrm{cd}\,G \leqslant r + s$?

Remark 2.24. When Q is a torsion-free group, the above question has a positive answer by Proposition 2.4 [**Nuc00**].

In [**Nuc00**] it is shown that the class of groups of finite $\mathfrak{F}$-cohomological dimension is closed under taking extensions by groups of finite integral cohomological dimension.

We begin by recalling some results needed in the proof of the main theorem, Theorem 2.35.

Lemma 2.25. [**Nuc00**, 2.2] *Let $N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ be a group extension and let $\mathfrak{H}$ be a family of groups satisfying the following condition: if H is a subgroup of G and $H \in \mathfrak{H}$, then $\pi(H) \in \mathfrak{H}$. Then every $\mathfrak{H}$-split short exact sequence of $\mathbb{Z}\, Q$-modules is $\mathfrak{H}$-split when regarded as a sequence of $\mathbb{Z}\, G$-modules.*

For a $\mathbb{Z}\, H$-module M, we use the standard notation $M \uparrow_H^G := \mathbb{Z}\, G \otimes_{\mathbb{Z}\, H} M$.

Lemma 2.26. [**Nuc99**, 8.2] *Let H be a subgroup of G and let $A \rightarrowtail B \twoheadrightarrow C$ be an $\mathfrak{F}$-split short exact sequence of kH-modules. Then the sequence $A \uparrow_H^G \rightarrowtail B \uparrow_H^G \twoheadrightarrow C \uparrow_H^G$ is an $\mathfrak{F}$-split sequence of kG-modules.*

Any $\mathbb{Z}\, G$-module M induced up from an $\mathfrak{F}$-subgroup H of G is $\mathfrak{F}$-projective (Corollary 2.4, [**Nuc99**]). This is not true for arbitrary subgroups H, but holds if M is induced up from an $\mathfrak{F}$-projective $\mathbb{Z}\, H$-module.

Lemma 2.27. *Let H be a subgroup of G and P be an $\mathfrak{F}$-projective $\mathbb{Z}H$-module. Then $P \uparrow_H^G$ is an $\mathfrak{F}$-projective $\mathbb{Z}G$-module.*

PROOF. If $\Delta = \bigsqcup_{\delta \in \Delta_0} G_\delta \backslash G$ is a G-set that satisfies condition $(*)$ then Δ has an H-orbit decomposition of the form $\bigsqcup_{\delta \in \Delta_0} (\bigsqcup_{g \in \Omega_\delta} (H \cap G_\delta^g) \backslash H)$, where Ω_δ is a set of representatives of the double cosets $G_\delta g H$. Clearly Δ regarded as an H-set satisfies condition $(*)$. Let M be an $\mathfrak{F}$-projective $\mathbb{Z}\, H$-module. Then by definition M is a direct summand of $N \otimes \mathbb{Z}\, \Delta$ for some $\mathbb{Z}\, H$-module N. Since induction is an exact functor, $M \uparrow_H^G$ is a direct summand of $(N \otimes \mathbb{Z}\, \Delta) \uparrow_H^G$. The statement follows by the Frobenius Reciprocity $(N \otimes \mathbb{Z}\, \Delta) \uparrow_H^G \cong N \uparrow_H^G \otimes \mathbb{Z}\, \Delta$ (Exercise 2(a), 5, III [**Bro82**]). $\square$

Lemma 2.28. *Suppose G is a group of finite $\mathfrak{F}$-cohomological dimension equal to n. Then there is an $\mathfrak{F}$-projective resolution of $\mathbb{Z}$ of length n made of permutation modules with $\mathfrak{F}$-stabilisers.*

PROOF. Since $\mathfrak{F}\operatorname{cd} G = n$, the general relative Schanuel's Lemma implies that the kernel K_n of the standard $\mathfrak{F}$-projective resolution is $\mathfrak{F}$-projective and so $\mathbb{Z}(\Delta^n) \twoheadrightarrow K_n$ splits, i.e. $K_n \oplus P \cong \mathbb{Z}(\Delta^n)$. Let $\mathbb{Z}\, \hat{\Delta}$ be a module isomorphic to a direct sum of countably many copies of $\mathbb{Z}(\Delta^n)$. Then $K_n \oplus \mathbb{Z}\,\hat{\Delta} \cong \mathbb{Z}\,\hat{\Delta}$ and we have the required resolution:

$$\mathbb{Z}\,\hat{\Delta} \rightarrowtail \mathbb{Z}(\Delta^{n-1}) \oplus \mathbb{Z}\,\hat{\Delta} \to \cdots \to \mathbb{Z}\,\Delta \twoheadrightarrow \mathbb{Z}\,. \qquad \square$$

Note that in the proof above the relative Eilenberg swindle produces a permutation module; this does not hold for general $\mathfrak{F}$-projective modules. For further discussion consult Section 4, [**Nuc00**].

Corollary 2.29. *For any group G, $\operatorname{Gcd} G \leqslant \mathfrak{F}\operatorname{cd} G$.*

PROOF. Every permutation $\mathbb{Z}\,G$-module with $\mathfrak{F}$-stabilisers is a Gorenstein projective $\mathbb{Z}\,G$-module by Lemma 2.21 [**ABS09**]. The result now follows from Lemma 2.28. $\square$

Martinez-Pérez and Nucinkis prove using Mackey functors that for every virtually torsion-free group G the equality $\operatorname{vcd} G = \mathfrak{F}\operatorname{cd} G$ holds [**MPN06**]. We give a proof of a weaker result, sufficient for our purpose, using an elementary method.

Proposition 2.30. *Let G be torsion-free. Then G has finite $\mathfrak{F}$-cohomological dimension equal to n if and only if G has finite cohomological dimension equal to n.*

PROOF. If $\operatorname{cd} G = n$, then by Proposition 2.6 VIII in [**Bro82**] there is a $\mathbb{Z}G$-free resolution F_* of $\mathbb{Z}$ of length n. Since G is a torsion-free group, any $\mathbb{Z}G$-free module is $\mathfrak{F}$-projective and any acyclic $\mathbb{Z}$-split $\mathbb{Z}G$-complex is $\mathfrak{F}$-split. This shows that F_* is an $\mathfrak{F}$-projective resolution of $\mathbb{Z}$ of length n.

Now we consider the standard $\mathbb{Z}\,G$-free resolution of $\mathbb{Z}$:

$$\cdots \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \twoheadrightarrow \mathbb{Z},$$

where $F_i = \mathbb{Z}(G^{i+1})$. By the above this is an $\mathfrak{F}$-split sequence. By the relative general Schanuel's lemma applied to $K_n \rightarrowtail F_{n-1} \to F_{n-2} \to \cdots \to F_0 \twoheadrightarrow \mathbb{Z}$ it follows that K_n is $\mathfrak{F}$-projective. In particular K_n is a direct summand of F_n and so it is $\mathbb{Z}\,G$-projective. $\square$

Lemma 2.31 (Dimension shifting)**.** *Let $N_m \rightarrowtail N_{m-1} \to \cdots \to N_0 \twoheadrightarrow L$ be an $\mathfrak{F}$-split exact sequence of $\mathbb{Z}\,G$-modules such that $\mathfrak{F}\operatorname{pd} N_i \leqslant n$ for all $0 \leqslant i \leqslant m$. Then $\mathfrak{F}\operatorname{pd} L \leqslant m + n$.*

PROOF. We argue by induction on m. If $m = 0$ then $N_0 \cong L$ and $\mathfrak{F}\,\mathrm{pd}\,L \leqslant n$. Let $k \geqslant 1$ and assume that the statement holds for $m \leqslant k-1$. Consider the $\mathfrak{F}$-split short exact sequence $N_k \overset{\iota}{\rightarrowtail} N_{k-1} \twoheadrightarrow \mathrm{im}\,\iota$. By the induction hypothesis, $\mathfrak{F}\,\mathrm{pd}(\mathrm{im}\,\iota) \leqslant n+1$. We have an $\mathfrak{F}$-split resolution of L, $\mathrm{im}\,\iota \rightarrowtail N_{k-2} \to \cdots \to N_0 \twoheadrightarrow L$ of length $k-1$ made of modules of $\mathfrak{F}$-projective dimension at most n+1 and by the induction hypothesis we obtain $\mathfrak{F}\,\mathrm{pd}\,L \leqslant (k-1)+(n+1) = k+n$. $\square$

Proposition 2.32. *Let $N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ be a group extension with $\mathfrak{F}\,\mathrm{cd}\,Q \leqslant m$. Moreover, assume that any finite extension H of N has $\mathfrak{F}\,\mathrm{cd}\,H \leqslant n$. Then $\mathfrak{F}\,\mathrm{cd}\,G \leqslant n+m$.*

PROOF. For any finite extension H of N, let

$$P_n \rightarrowtail P_{n-1} \to \cdots \to P_0 \twoheadrightarrow \mathbb{Z}$$

be a $\mathfrak{F}$-projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. By Lemma 2.27 and Lemma 2.26, the resolution

$$P_n \uparrow_H^G \rightarrowtail P_{n-1} \uparrow_H^G \to \cdots \to P_0 \uparrow_H^G \twoheadrightarrow \mathbb{Z} \uparrow_H^G$$

is an $\mathfrak{F}$-projective resolution of $\mathbb{Z} \uparrow_H^G$ over $\mathbb{Z}G$.

Now, Lemma 2.28 implies that there is an $\mathfrak{F}$-projective resolution of $\mathbb{Z}$ over $\mathbb{Z}Q$ of the form

$$\mathbb{Z}\,\Delta \cong K \rightarrowtail \mathbb{Z}\,\Delta_{m-1} \to \cdots \to \mathbb{Z}\,\Delta_0 \twoheadrightarrow \mathbb{Z}.$$

By Lemma 2.25 the sequence above is $\mathfrak{F}$-split when regarded as a $\mathbb{Z}G$-sequence. Every permutation module $\mathbb{Z}\,\Delta_i$ and $\mathbb{Z}\,\Delta$ when regarded as a $\mathbb{Z}\,G$-module is isomorphic to some $\oplus_{j\in J}\mathbb{Z} \uparrow_{H_j}^G$ where $|H_j : N| < \infty$. To see this, consider the case of a homogeneous Q-set $\Omega = F\backslash G$, and regard Ω as a G-set via π. Then Ω is isomorphic to $\pi^{-1}(F)\backslash G$. If $|F| < \infty$ then $F \cong K/N$ where $[N : K] < \infty$ and $K \cong \pi^{-1}(F)$. By the above $\mathfrak{F}\,\mathrm{pd}(\mathbb{Z} \uparrow_{H_j}^G) < n$ and so the assertion follows from Lemma 2.31. $\square$

Corollary 2.33. *If $G = H \times K$, where $\mathfrak{F}\,\mathrm{cd}\,H \leqslant n$ and $\mathfrak{F}\,\mathrm{cd}\,K \leqslant m$, then $\mathfrak{F}\,\mathrm{cd}\,G \leqslant n+m$.*

PROOF. By Proposition 2.32 we can assume $|K| < \infty$ and we regard G as an extension of K by H. Any finite extension of K by a finite subgroup of H is finite and so it has $\mathfrak{F}$-cohomological dimension equal to 0. The result now follows by Proposition 2.32. $\square$

Proposition 2.32 is the relative analogue of Corollary 5.2 in [**MP02**] but in the context of $\mathfrak{F}$-cohomology we are able to strengthen the result, as we shall see in Theorem 2.35.

Since for virtually torsion-free groups the notion of $\mathfrak{F}$-cohomological dimension coincides with the notion of virtual cohomological dimension it is conceivable that taking finite extensions of groups of finite $\mathfrak{F}$-cohomological dimension does not raise the dimension. There are examples of non-virtually torsion-free groups that are extensions of two virtually torsion-free groups of finite virtual cohomological dimension [**Sch78**], but nonetheless these admit finite-dimensional classifying spaces for proper actions [**BLN01**].

In order to reduce the extension problem to extensions by groups of prime order we need the following observation.

Lemma 2.34. *Let $\mathfrak{P}$ be the class of p-groups. When considering the standard $\mathfrak{F}$-projective resolution $P_* \twoheadrightarrow Z$ we can replace the G-set Δ by $\Delta_{\mathfrak{P}}$, where $\Delta_{\mathfrak{P}} = \bigsqcup_{P \leqslant G,\, P \in \mathfrak{P} \cap \mathfrak{F}} G/P$.*

PROOF. The result is an immediate consequence of i) and ii) of Proposition 2.14 [**LN10**]. $\square$

Theorem 2.35. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a group extension with $\mathfrak{F}\operatorname{cd} N \leqslant n$. Moreover, assume that for any subgroup H of G with $\mathfrak{F}\operatorname{cd} H \leqslant n$ and any extension L of H by a group of prime order has $\mathfrak{F}\operatorname{cd} L \leqslant n$. Then $\mathfrak{F}\operatorname{cd} G \leqslant \mathfrak{F}\operatorname{cd} H + \mathfrak{F}\operatorname{cd} Q$.*

PROOF. Arguing as in Proposition 2.32 the problem can be reduced to extensions by groups of prime power order using Lemma 2.34. Now, if $N \rightarrowtail G \twoheadrightarrow P$ is such an extension then the quotient group P is nilpotent of prime power order. For any $p^n/|P|$ there exists by the correspondence theorem a normal subgroup S of G, $N \leqslant S \leqslant G$ such that S/N has order p^n and the result follows by induction. $\square$

Note that if $N \rightarrowtail G \twoheadrightarrow Q$ is a group extension such that $\operatorname{gd}_{\mathfrak{F}} N = n$ and $|Q| = k$ then $\operatorname{gd}_{\mathfrak{F}} G \leqslant n.k$ [**Mis01**]. It is unknown whether the finiteness of the $\mathfrak{F}$-cohomological dimension is preserved under taking (finite) extensions. However, this is the case for countable elementary amenable groups.

Proposition 2.36. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a group extension such that G is countable elementary amenable, $\mathfrak{F}\operatorname{cd} N \leqslant n$ and $\mathfrak{F}\operatorname{cd} Q \leqslant m$. Then $\mathfrak{F}\operatorname{cd} G \leqslant n + m + 1$.*

PROOF. The rational cohomological dimension of a group G $\operatorname{cd}_{\mathbb{Q}}$ is defined as the $\mathbb{Q}G$-projective dimension of the trivial $\mathbb{Q}G$-module $\mathbb{Q}$. If the trivial $\mathbb{Z}\,G$-module $\mathbb{Z}$ admits a resolution of length n made of permutation modules with $\mathfrak{F}$-stabilisers then tensoring it with $\mathbb{Q}$ over $\mathbb{Z}$ we obtain that $\operatorname{cd}_{\mathbb{Q}} G \leqslant n$. In particular, for every group G, $\operatorname{cd}_{\mathbb{Q}} G \leqslant \mathfrak{F}\operatorname{cd} G$. Corollary 3.3 [**Nuc00**] implies that for any group G, $\mathfrak{F}\operatorname{cd} G \leqslant \operatorname{cd}_{\mathfrak{F}} G$. Let $\operatorname{h} G$ be the Hirsch length of an elementary amenable group G. The inequality $\operatorname{h} G \leqslant \operatorname{cd}_{\mathbb{Q}} G$ holds by Lemma 2 in [**Hil91**]. Let $\operatorname{hd}_R G$ denote the homological dimension of G over RG. If G is any countable group G and R is a commutative ring of coefficients, then the following are well-known [**Bie81, Nuc04**]:

$$\operatorname{hd}_R G \leqslant \operatorname{cd}_R G \leqslant \operatorname{hd}_R G + 1,$$

$$\operatorname{hd}_{\mathfrak{F}} G \leqslant \operatorname{cd}_{\mathfrak{F}} G \leqslant \operatorname{hd}_{\mathfrak{F}} G + 1.$$

The class of elementary amenable groups is subgroup-closed and quotient-closed. By Theorem 1 in [**Hil91**] $\operatorname{h} G = \operatorname{h} N + \operatorname{h} Q$, and an immediate application of Theorem 1 in [**FN05**] gives the result. □

Furthermore, Serre's construction included in [**DD89**, V, 5.2] shows that, given a finite extension $N \rightarrowtail G \twoheadrightarrow Q$ with $\mathfrak{F}\operatorname{cd} N = n$ and $|Q| = k$, there exists an exact $\mathbb{Z}\,G$-resolution of $\mathbb{Z}$ made of permutation modules with stabilisers in $\mathfrak{F}$ of length $n.k$. However, it is unclear if this resolution is $\mathfrak{F}$-split and this suggests a more general question.

Question 2.37. Suppose G is a group that admits a resolution of finite length of the trivial $\mathbb{Z}\,G$-module $\mathbb{Z}$ made of permutation modules with stabilisers in $\mathfrak{F}$. Does G have finite $\mathfrak{F}$-cohomological dimension?

Note that Serre's construction can be made in topology; if N acts on a n-dimensional contractible cell-complex with $\mathfrak{F}$-stabilisers and Q has order k then G acts on a $(n.k)$-dimensional contractible cell-complex with $\mathfrak{F}$-stabilisers.

Remark 2.38. Arguing as in Corollary 2.29, every group admitting a resolution as in the question above has finite Gorenstein cohomological dimension. It is unknown if the converse holds.

Question 2.39. Can the theory of $\mathfrak{F}$-injective modules started in Chapter 1 be used for proving a relative version of the Grothendieck spectral sequence? Could this then be used to show a relative version of the Lyndon–Hochschild–Serre spectral sequence that would solve the extension problem?

It is natural to ask what happens if the ring of integers is replaced with the field of rationals as the coefficients ring.

The next result answer this and in some sense tells us that all these generalised theories end up, once evaluated in $\mathbb{Q}$, with the classic notion of rational cohomological dimension.

Proposition 2.40. *For any group G we have $\mathfrak{F}\operatorname{cd}_{\mathbb{Q}} G = \operatorname{cd}_{\mathbb{Q}} G$.*

PROOF. Arguing as in Lemma 2.28 we have that if $\mathfrak{F}\operatorname{cd}_{\mathbb{Q}} G = n$ then G admits a resolution of $\mathbb{Q}$ of length n made of permutation modules with $\mathfrak{F}$-stabilisers over $\mathbb{Q}G$. Every permutation module with $\mathfrak{F}$-stabilisers over $\mathbb{Q}G$ is $\mathbb{Q}G$-projective and so $\operatorname{cd}_{\mathbb{Q}} G \leqslant n$. On the other hand, suppose that $\operatorname{cd}_{\mathbb{Q}} G = n$. If F is a $\mathfrak{F}$-subgroup of G then every module over $\mathbb{Q}F$ is projective. From this it follows that any $\mathbb{Q}G$-projective resolution of $\mathbb{Q}$ is $\mathfrak{F}$-split and so $\mathfrak{F}\operatorname{cd}_{\mathbb{Q}} G \leqslant n$. □

Note that by Proposition 2.40 and [**Tal11**, 3.5] it follows that for any $\mathbf{LH}\,\mathfrak{F}$-group G, $\mathfrak{F}\operatorname{cd}_{\mathbb{Q}} G = \operatorname{Gcd}_{\mathbb{Q}} G = \operatorname{cd}_{\mathbb{Q}} G$.

An $\mathfrak{F}$-*resolution* of a $\mathbb{Z}\,G$-module M is a resolution made of permutation modules with $\mathfrak{F}$-stabilisers. For a group G, $\operatorname{cd}_{\mathfrak{M}_{\mathfrak{F}}} G$ denotes the *Mackey cohomological dimension of G* [**MPN06**]. We close the chapter with two diagrams.

The first illustrates the implications that hold in full generality.

$$\begin{tikzcd}
G \in \mathbf{H}_1\,\mathfrak{F} \arrow[d, Rightarrow] & \operatorname{gd}_{\mathfrak{F}} G < \infty \arrow[l, Rightarrow] \arrow[d, Leftrightarrow] \\
\mathfrak{F}\operatorname{cd} G < \infty \arrow[d, Rightarrow] & \operatorname{cd}_{\mathfrak{F}} G < \infty \arrow[d, Rightarrow] \\
\exists \text{ an } \mathfrak{F}\text{-resolution of } \mathbb{Z} \text{ of finite length} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \operatorname{cd}_{\mathfrak{M}_{\mathfrak{F}}} G < \infty \arrow[ul, Rightarrow] \\
\operatorname{Gcd} G < \infty & \operatorname{cd}_{\mathbb{Q}} G < \infty
\end{tikzcd}$$

In some sense all the implications in the diagram above are conjectured to be reversible as shown in the diagram below.

$$
\begin{tikzcd}
G \in \mathbf{H}_1\,\mathfrak{F} \arrow[rr, Rightarrow, "Kropholler-Mislin"] & & \mathrm{gd}_{\mathfrak{F}} G < \infty \\
\mathfrak{F}\,\mathrm{cd}\, G < \infty \arrow[urr, Rightarrow, "Nucinkis"] & & \\
\mathrm{Gcd}\, G < \infty \arrow[uurr, Rightarrow, "Bahlekeh-Dembegioti-Talelli"] & & \mathrm{cd}_{\mathbb{Q}}\, G < \infty \arrow[uu, Rightarrow]
\end{tikzcd}
$$

CHAPTER 3

$(\mathfrak{F})$-cohomological conditions of finite type

A cohomological finiteness condition is a group-theoretical property that is satisfied by any group admitting a finite $K(G,1)$. Since every non-trivial $\mathfrak{F}$-group does not admit a finite-dimensional $K(G,1)$, *being torsion-free* is a cohomological finiteness condition, but not a finiteness condition in the usual group-theoretical sense. On the other hand, the property of belonging to $\mathbf{L}\mathfrak{F}$ is a classical but not a cohomological finiteness condition. However there are finiteness conditions that agree, for example *being finitely-generated*, *being finitely presented*, etc.

A generalisation of these properties brings us to the concepts of cohomological conditions of finite type. More precisely, a group Γ is of type FP_n if the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a resolution of finitely-generated projective $\mathbb{Z}\Gamma$-modules up to dimension n. If Γ is of type FP_n for every $n \geqslant 0$, then Γ is said to be of type FP_∞. A group is of type F_n if it admits a $K(G,1)$ with finite n-skeleton; and Γ is of type F_∞ if it is of type F_n for every $n \geqslant 0$. For a group, being finitely-generated is equivalent to being of type FP_1. A group is finitely presented if and only if it is of type F_2. For $n \geqslant 2$, a group is of type F_n if and only if it is finitely presented and of type FP_n. Bestvina and Brady showed the existence of non-finitely presented groups of type FP_2 [**BB97**]. Relative versions of these are *$\mathfrak{F}$-cohomological conditions of finite type*. Informally, by this term we refer to the requirement of having an $\mathfrak{F}$-split resolution or a partial resolution of $\mathbb{Z}$ made of finitely-generated $\mathfrak{F}$-projective $\mathbb{Z}G$-modules.

In the first section we introduce the notion of a Δ-flat module. We consider the $\mathfrak{F}$-homological dimension and $\mathfrak{F}$-homological finiteness conditions for groups. After proving some of the usual properties we show the following:

Theorem. *Let M be a G-module of type $\mathfrak{F}\,\mathrm{FP}_\infty$. Then for any exact limit of kG-modules $\lim_{\lambda\in\Lambda} N_\lambda$ the natural map*

$$\mathfrak{F}\,\mathrm{Tor}_k(\lim_{\lambda\in\Lambda} N_\lambda, M) \to \lim_{\lambda\in\Lambda} \mathfrak{F}\,\mathrm{Tor}_k(N_\lambda, M)$$

is an isomorphism for any k.

The *homological finiteness length* of a group G, denoted by $\phi(G)$, is the supremum of the m such that G is of type FP_m. In the second section we consider non-uniform lattices on locally finite $\mathrm{CAT}(0)$ polyhedral complexes. Our main result is a bound on their homological finiteness lengths.

Theorem. *If Γ is a non-uniform lattice on a locally finite $\mathrm{CAT}(0)$ polyhedral complex of dimension n, then $\phi(\Gamma) < n$.*

1. $\mathfrak{F}$-homology and groups of type $\mathfrak{F}\mathbf{FP}_n$

Let G be a group and k a commutative ring of coefficients. Then G is of type FP_n over k if k admits a resolution of finitely-generated kG-projectives up to dimension n. A group G is of type FP_∞ over k if G is of type FP_n for every n. A group G is of type FP if k admits a finite resolution of finitely-generated kG-projectives. This is a particularly strong cohomological finiteness condition. Since every group of type FP has finite cohomological dimension, we have that $\mathfrak{F}$-groups are not of type FP, however they are of type FP_∞.

Clearly every group is of type FP_0 and by writing explicitly the augmentation ideal it easy to see that a group is finitely-generated if and only if it is of type FP_1 [**Bro82**]. A finitely presented group is of type FP_2, and the question of wheter this implication was reversible was open until Brady and Bestivina built infinitely presented groups of type FP_2 in their famous paper [**BB97**]. Briefly, to a finite flag complex L it is possible to associate a right angled Artin group G_L. If $L \neq \emptyset$, then there is a surjection to $\mathbb{Z}$ (given by mapping the generators to 1) and a short exact sequence $H_L \rightarrowtail G_L \twoheadrightarrow \mathbb{Z}$. They were able to determine the cohomology type of the kernel H_L by the homotopy type of the complex L.

One of the advantages of working with the type FP_∞ instead with the stronger type FP is that it allows torsion in the group. Examples of groups of type FP_∞ include $\mathfrak{F}$-groups (not of type FP), finitely-generated free groups (by Stallings-Swan Theorem they are of type FP), polycyclic groups and every finitely-generated one-relator group. The next theorem is used often as a crucial tool for proving the cohomological type of a group.

Theorem 3.1 ([**Bie81**], 1.3)**.** *For a group G the following are equivalent:*

- *G is of type FP_n;*
- *for any exact limit the natural map $\mathrm{H}_i(G, \lim M_*) \to \lim \mathrm{H}_i(G, M_*)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$;*
- *$\mathrm{H}^i(G; -)$ is continuous for $i < n$ and for any exact colimit, the natural map $\operatorname{colim} \mathrm{H}^n(G; M_*) \to \mathrm{H}^n(G; \operatorname{colim} M_*)$ is a monomorphism;*
- *$\mathrm{H}^i(G; -)$ is continuous at zero for $i \leqslant n$.*

Furthermore, it is enough to check the second condition on direct products.

A group G is of type $\mathfrak{F}\,\mathrm{FP}_n$ if $\mathbb{Z}$ admits a resolution of finitely-generated $\mathfrak{F}$-projectives up to dimension n. If G is of type $\mathfrak{F}\,\mathrm{FP}_n$ for every $n \geqslant 0$, then it said to be of type $\mathfrak{F}\,\mathrm{FP}_\infty$ and G is of type $\mathfrak{F}\,\mathrm{FP}$ if $\mathbb{Z}$ admits a finite-dimensional $\mathfrak{F}$-split resolution made of finitely-generated $\mathfrak{F}$-projective modules.

This section is partially motivated by the following question.

Question 3.2. Is there an analogous criterion for $\mathfrak{F}$-cohomology?

The answer is in general no. Since $\mathrm{H}^0(G; -) \cong \mathfrak{F}\,\mathrm{H}^0(G; -) \cong \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$, it follows that $\mathfrak{F}\,\mathrm{H}^0(G; -)$ is continuous at zero for every group G. But any group with infinitely many conjugacy classes of $\mathfrak{P}_{\mathfrak{F}}$-subgroups is not of type $\mathfrak{F}\,\mathrm{FP}_0$ by [**LN10**]. Nonetheless, a stronger question posed in [**Nuc99**] remains open.

Question 3.3. Is a group G of type $\mathfrak{F}\,\mathrm{FP}_\infty$ if and only if $\mathfrak{F}\,\mathrm{H}^n(G; -)$ is continuous for all $n \geqslant 0$?

Nucinkis proved that is G is a group of type $\mathfrak{F}\,\mathrm{FP}_\infty$ then $\mathfrak{F}\,\mathrm{H}^n(G; -)$ is continuous for all $n \geqslant 0$ [**Nuc99**], and we will prove an similar result for the $\mathfrak{F}$-homology groups.

There exist two more important results in the topic that are well worth mentioning here.

Proposition 3.4. [**LN10**] *A group G is of type $\mathfrak{F}\,\mathrm{FP}_0$ if and only if it has finitely many conjugacy classes of $\mathfrak{P}_{\mathfrak{F}}$-subgroups.*

Proposition 3.5. [**Nuc99**, 6.3, 7.2] *Every finitely-generated $\mathfrak{F}$-projective $\mathbb{Z}G$-module is of type FP_∞. Moreover, every $\mathbb{Z}G$-module of type $\mathfrak{F}\,\mathrm{FP}_\infty$ is of type FP_∞.*

Any finitely-generated $\mathfrak{F}$-projective module P is of the form $P \cong \bigoplus_{\lambda=1}^{n} Q_\lambda \uparrow_{G_\lambda}^{G}$ where the Q_λ are finitely-generated $\mathbb{Z}\, G_\lambda$-modules and the G_λ are $\mathfrak{F}$-subgroups of G. In particular Q_λ is a finitely-generated abelian group for every λ.

Definition 3.6. A kG-module M is Δ-*flat* if the functor $- \otimes_{kG} M$ is exact on Δ-split sequences. That is, whenever

$$A \overset{\iota}{\rightarrowtail} B \overset{\pi}{\twoheadrightarrow} C$$

is an exact Δ-split sequence of G-modules, then

$$A \otimes_{kG} M \overset{\iota\otimes 1_M}{\rightarrowtail} B \otimes_{kG} M \overset{\pi\otimes 1_M}{\twoheadrightarrow} C \otimes_{kG} M$$

is an exact sequence of abelian groups.

Since $- \otimes_{kG} M$ is a right exact functor, a G-module M is Δ-flat if for any Δ-split monomorphism $\iota : A \rightarrowtail B$ the morphism $\iota \otimes 1_M : A \otimes_{kG} M \rightarrowtail B \otimes_{kG} M$ is a monomorphism.

Following from [**ML95**, 9, XII] we define for any G-module M the relative homological functors with coefficients in A as

$$\Delta \operatorname{Tor}_*(M, A) := \mathrm{H}_*(\mathbf{P} \otimes_{\mathbb{Z}\, G} A)$$

where $\mathbf{P}$ is a Δ-projective resolution of M. Note that $\Delta \operatorname{Tor}_0(M, A) \cong \operatorname{Tor}_0(M, A)$ $\cong M \otimes_{\mathbb{Z}\, G} A$, this can be either shown directly or it follows from [**ML95**, Theorem 9.1 XII]. We write $\Delta\, \mathrm{H}_*(G; A)$ for $\Delta \operatorname{Tor}_*(\mathbb{Z}, A)$ and when $A = \mathbb{Z}$, we will simply write $\Delta\, \mathrm{H}_*(G)$ for $\Delta \operatorname{Tor}_*(\mathbb{Z}, \mathbb{Z}) := \mathrm{H}_*(\mathbf{P} \otimes_{\mathbb{Z}\, G} \mathbb{Z}) \cong \mathrm{H}_*(\mathbf{P}^G)$.

The next result is an analogue to [**Ben98**, Corollary 3.6.7] and it is worth mentioning that [**Hoc56**] has the same spirit.

Lemma 3.7. *Let P be a Δ-projective and M any G-module. Then $P \otimes M$ is Δ-projective.*

PROOF. By tensoring the $\mathbb{Z}\, G$-split epimorphism $\phi : \mathbb{Z}\, \Delta \otimes P \twoheadrightarrow P$ with M we obtain the $\mathbb{Z}\, G$-split epimorphism $(\mathbb{Z}\, \Delta \otimes P) \otimes M \twoheadrightarrow P \otimes M$. Now the result is obvious by the associativity of the tensor product. $\square$

Corollary 3.8. *If $\mathfrak{F} \operatorname{cd} G = n$, then every $\mathbb{Z}\, G$-module M has $\mathfrak{F}$-projective dimension at most n.*

Lemma 3.9. *Let Δ_0 be a set of orbit representatives for Δ, $\delta \in \Delta_0$ and N be any $\mathbb{Z}\, G$-module. Then $N \uparrow_{G_\delta}^{G}$ is Δ-flat.*

PROOF. Let A $\rightarrowtail$ B $\twoheadrightarrow$ C be a Δ-split short exact sequence. For any G-module M we have by the Frobenius reciprocity and the associativity of the tensor product the isomorphism $M \otimes_{\mathbb{Z}\, G} N \uparrow_{G_\delta}^{G} \cong M \otimes_{\mathbb{Z}\, G} \mathbb{Z}\, G \otimes_{\mathbb{Z}\, G_\delta} N \cong M \otimes_{\mathbb{Z}\, G_\delta} N$. Since $A \otimes \mathbb{Z}\,\Delta \rightarrowtail B \otimes \mathbb{Z}\,\Delta \twoheadrightarrow C \otimes \mathbb{Z}\,\Delta$ splits, by Lemma 1.17 we have that $A \otimes \mathbb{Z}\, G_\delta \backslash G \rightarrowtail B \otimes \mathbb{Z}\, G_\delta \backslash G \twoheadrightarrow C \otimes \mathbb{Z}\, G_\delta \backslash G$ splits. Now tensoring this split short exact sequence with N over $\mathbb{Z}\, G$ gives the following exact sequence

$$A \otimes_{\mathbb{Z}\, G_\delta} \mathbb{Z}\, G \otimes_{\mathbb{Z}\, G} N \rightarrowtail B \otimes_{\mathbb{Z}\, G_\delta} \mathbb{Z}\, G \otimes_{\mathbb{Z}\, G} N \twoheadrightarrow C \otimes_{\mathbb{Z}\, G_\delta} \mathbb{Z}\, G \otimes_{\mathbb{Z}\, G} N.$$

That is, $A \otimes_{\mathbb{Z}\, G_\delta} N \rightarrowtail B \otimes_{\mathbb{Z}\, G_\delta} N \twoheadrightarrow C \otimes_{\mathbb{Z}\, G_\delta} N$. Hence, $N \uparrow_{G_\delta}^{G}$ is Δ-flat. □

Lemma 3.10. *A direct sum $\bigoplus_{i \in I} F_i$ of G-modules is Δ-flat if and only if each F_i is Δ-flat. In particular the module $\mathbb{Z}\,\Delta$ is Δ-flat.*

PROOF. This can be proved as [**Rot09**, 3.46, ii]. □

Corollary 3.11. *For any G-module N, the Δ-free module $\mathbb{Z}\,\Delta \otimes N$ is Δ-flat. Moreover every Δ-projective module is Δ-flat.*

PROOF. It follows immediately by the associativity of the tensor product and by Lemma 3.10. □

Theorem 3.12. *Let A and B be two $\mathbb{Z}\, G$-modules and let $\mathbf{P} \twoheadrightarrow A$ and $\mathbf{Q} \twoheadrightarrow B$ be two Δ-projective resolutions of A and B respectively. Then*

$$\mathrm{H}_*(\mathbf{P} \otimes_{\mathbb{Z}\, G} B) \cong \mathrm{H}_*(A \otimes_{\mathbb{Z}\, G} \mathbf{Q}).$$

PROOF. The proof is analogous to [**Rot09**, Theorem 6.32] and it relies only on the Δ-flatness of the Δ-projective modules. □

Lemma 3.13. *If M is the filtered colimit (for example the direct limit) of Δ-flat modules, then M is Δ-flat.*

PROOF. Let $M \cong \operatorname{colim} M_\lambda$. Since colimits commute with tensor products over $\mathbb{Z}\, G$ [**Bie81**, pg. 8] we have $\operatorname{colim} M_\lambda \otimes_{\mathbb{Z}\, G} - \cong \operatorname{colim}(M_\lambda \otimes_{\mathbb{Z}\, G} -)$. Now the result follows from the Δ-flatness of the M_λ and the fact that filtered colimits are exact in $\mathfrak{Mod}_{\mathbb{Z}\, G}$ [**Kro**, 7.4]. □

Remark 3.14. Let $\mathrm{A} \rightarrowtail \mathrm{B} \twoheadrightarrow \mathrm{C}$ be an $\mathfrak{F}$-split short exact sequence with B and C $\mathfrak{F}$-projective modules. Then A is obviously $\mathfrak{F}$-projective. If A and B are $\mathfrak{F}$-flats by the long exact sequence in $\mathfrak{F}$-homology we have that C is also $\mathfrak{F}$-flat.

We define $\mathfrak{F}\operatorname{hd} G$ as the length of the shortest $\mathfrak{F}$-flat resolution of $\mathbb{Z}$ over $\mathbb{Z}\,G$. Of course by Corollary 3.11 we have $\mathfrak{F}\operatorname{hd} G \leqslant \mathfrak{F}\operatorname{cd} G$ for every group G.

Theorem 3.15. *Let M be a G-module of type $\mathfrak{F}\,\mathrm{FP}_\infty$. Then for any exact limit of kG-modules $\lim_{\lambda\in\Lambda} N_\lambda$ the natural map*

$$\mathfrak{F}\operatorname{Tor}_k(\lim_{\lambda\in\Lambda} N_\lambda, M) \to \lim_{\lambda\in\Lambda} \mathfrak{F}\operatorname{Tor}_k(N_\lambda, M)$$

is an isomorphism for any k.

PROOF. Let $\mathbf{P} \twoheadrightarrow M$ be a $\mathfrak{F}$-projective resolution of M of finite type and let $K_i = \ker(P_i \to P_{i-1})$ and K_0 be M. Clearly K_i is finitely-generated for every i and for any exact limit we have the following projection:

$$\pi_i : \lim N_\lambda \otimes_{\mathbb{Z}\,G} K_i \twoheadrightarrow \lim(N_\lambda \otimes_{\mathbb{Z}\,G} K_i).$$

By Proposition 3.5 P_i is of type FP_∞ for every i (it would be enough to have finite presentability) and there is a natural isomorphism [**Bie81**, Theorem 1.3]

$$\lim N_\lambda \otimes_{\mathbb{Z}\,G} P_i \cong \lim(N_\lambda \otimes_{\mathbb{Z}\,G} P_i).$$

Now we will apply the 5-Lemma to the following commutative diagram

$$\begin{tikzcd}
\lim N_\lambda \otimes_{\mathbb{Z}\,G} K_{i-1} \arrow[r] \arrow[d, "\pi_{i-1}"] & \lim N_\lambda \otimes_{\mathbb{Z}\,G} P_i \arrow[r, two heads] \arrow[d, equal] & \lim N_\lambda \otimes_{\mathbb{Z}\,G} K_i \arrow[d, "\pi_i"] \\
\lim(N_\lambda \otimes_{\mathbb{Z}\,G} K_{i-1}) \arrow[r] & \lim(N_\lambda \otimes_{\mathbb{Z}\,G} P_i) \arrow[r, two heads] & \lim(N_\lambda \otimes_{\mathbb{Z}\,G} K_i)
\end{tikzcd}$$

and conclude that π_i is an isomorphism. By an application of [**Bie81**, Proposition 1.4] to the short exact sequence $K_{i-1} \rightarrowtail P_{i-1} \twoheadrightarrow K_i$, we have that K_i is finitely presented. Now by dimension shifting for every $i \geqslant 2$ we have $\mathfrak{F}\operatorname{Tor}_i(-, M) \cong \mathfrak{F}\operatorname{Tor}_1(-, K_{i-2})$. This last isomorphism gives for every $i \geqslant 1$ the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{F}\operatorname{Tor}_i(\lim N_\lambda, M) \arrow[r, tail] \arrow[d, "\phi_i"] & \lim N_\lambda \otimes_{\mathbb{Z}\,G} K_{i-1} \arrow[r] \arrow[d, equal] & \lim N_\lambda \otimes_{\mathbb{Z}\,G} P_{i-1} \arrow[d, equal] \\
\lim \mathfrak{F}\operatorname{Tor}_i(N_\lambda, M) \arrow[r, tail] & \lim(N_\lambda \otimes_{\mathbb{Z}\,G} K_{i-1}) \arrow[r] & \lim(N_\lambda \otimes_{\mathbb{Z}\,G} P_{i-1}).
\end{tikzcd}$$

By the 5-Lemma it follows that the natural morphism ϕ_i is an isomorphism for every $i \geqslant 0$, and this concludes the proof. $\square$

Lemma 3.16 (Shapiro's Lemma for $\mathfrak{F}$-homology)**.**

$$\mathfrak{F}\,\mathrm{H}_*(H; N) \cong \mathfrak{F}\,\mathrm{H}_*(G; \mathrm{Ind}_H^G N).$$

PROOF. This can be proved as Lemma 1.11. $\square$

Lemma 3.17. *Let T be a G-tree with edge set $E = \bigsqcup_{i \in I} L_i \backslash G$ and vertex set $V = \bigsqcup_{j \in J} N_j \backslash G$. Then there is a Mayer-Vietoris sequence:*

$$\cdots \to \mathfrak{F}\,\mathrm{H}_n(G, -) \to \bigoplus_{j \in J} \mathfrak{F}\,\mathrm{H}_n(L_j, -)$$
$$\to \bigoplus_{i \in I} \mathfrak{F}\,\mathrm{H}_n(N_i, -) \to \mathfrak{F}\,\mathrm{H}_{n-1}(G, -) \to \cdots$$

PROOF. From Bass-Serre theory there exists a short exact $\mathfrak{F}$-split sequence ϵ associate with T of the form $\oplus_{i \in I} \mathbb{Z}\, L_i \backslash G \rightarrowtail \oplus_{j \in J} \mathbb{Z}\, N_i \backslash G \twoheadrightarrow \mathbb{Z}$. The result follows from the long exact sequence in $\mathfrak{F}$-homology applied to ϵ. $\square$

We state explicitly two immediate corollaries.

Corollary 3.18.

- *Let $G = H *_L K$. Then there is a Mayer-Vietoris sequence in $\mathfrak{F}$-homology;*

$$\cdots \to \mathfrak{F}\,\mathrm{H}_k(L; -) \to \mathfrak{F}\,\mathrm{H}_k(H; -) \oplus \mathfrak{F}\,\mathrm{H}_k(K; -)$$
$$\to \mathfrak{F}\,\mathrm{H}_{k-1}(G; -) \to \mathfrak{F}\,\mathrm{H}_{k-1}(L; -) \to \cdots$$

- *Let $G = H *_{A,\,t}$. Then there is a Mayer-Vietoris sequence in $\mathfrak{F}$-homology;*

$$\cdots \to \mathfrak{F}\,\mathrm{H}_k(A; -) \to \mathfrak{F}\,\mathrm{H}_k(H; -)$$
$$\to \mathfrak{F}\,\mathrm{H}_k(G; -) \to \mathfrak{F}\,\mathrm{H}_{k-1}(A; -) \to \cdots$$

Corollary 3.19. *The class of groups of finite $\mathfrak{F}\,\mathrm{hd}$ is closed under taking subgroups,* HNN*-extensions and free products with amalgamation.*

Proposition 3.20 ([**Bie81**], 1.4)**.** *Let $\mathrm{A} \rightarrowtail \mathrm{B} \twoheadrightarrow \mathrm{C}$ be a short exact sequence of $\mathbb{Z}\,G$-modules. Then the following hold.*

(1) If A is of type FP_{n-1} and B is of type FP_n, then C is of type FP_n.
(2) If B is of type FP_{n-1} and C is of type FP_n, then A is of type FP_{n-1}.
(3) If A and C are of type FP_n then B is also of type FP_n.

Remark 3.21. A G-module is of type $\mathfrak{F}\,\mathrm{FP}_\infty$ if and only if it admits an $\mathfrak{F}$-projective resolution with finitely-generated $\mathfrak{F}$-syzygies (i.e. $\cdots \to P_1 \to P_0 \twoheadrightarrow M$ such that $\ker(P_i \to P_{i-1})$ is finitely-generated for every $i \geqslant 1$). One implication is trivial, and the other implication follows from part (3) of Proposition 3.20. Please note that for a module of type $\mathfrak{F}\,\mathrm{FP}_n$ we require finitely-generated $\mathfrak{F}$-syzygies up to dimension $n+1$; possibly it is not enough to have this condition up to dimension n.

We mention an immediate consequence of Remark 3.21 and Proposition 3.5.

Corollary 3.22. *Let M be a $\mathbb{Z}\,G$-module of type $\mathfrak{F}\,\mathrm{FP}_\infty$. Then every $\mathfrak{F}$-syzygy in an $\mathfrak{F}$-projective resolution of M of finite type is of type* FP_∞.

PROOF. Apply part (2) of Proposition 3.20.to an $\mathfrak{F}$-projective resolution of finite type of M. $\square$

Conjecture 2. [**LN10**] *A group G is of type $\mathfrak{F}\,\mathrm{FP}_\infty$ if and only if it is of type* FP_∞ *and has finitely many conjugacy classes of $\mathfrak{P}_{\mathfrak{F}}$-subgroups.*

The sufficient condition follows from [**Nuc99**, Proposition 7.2] and [**LN10**]. By Proposition 3.4, G is of type $\mathfrak{F}\,\mathrm{FP}_0$, i.e. there is $P_0 \twoheadrightarrow \mathbb{Z}$ such that P_0 is a finitely-generated $\mathfrak{F}$-projective (P_0 can be chosen as $\bigoplus_{i\in I} \mathbb{Z}\, P_i\backslash G$ where I is a set of representatives for the conjugacy classes of $\mathfrak{P}_{\mathfrak{F}}$-subgroups). Now if $\mathbb{Z}$ is of type FP_∞ and P_0 is of type FP_∞, by Proposition 3.5 it follows that the kernel K_0 of the surjection $P_0 \twoheadrightarrow \mathbb{Z}$ is of type FP_∞ by part (2) of Proposition 3.20. In particular K_0 is finitely-generated but we cannot say that $\mathbb{Z}\,\Delta \otimes K_0$ is a finitely-generated $\mathfrak{F}$-projective G-module. This banal observation suggests the next question.

Question 3.23. When is a $\mathbb{Z}\,G$-module of type FP_∞ also of type $\mathfrak{F}\,\mathrm{FP}_0$?

We know almost exclusively trivial examples of modules of type $\mathfrak{F}\,\mathrm{FP}_0$; these include finitely-generated $\mathfrak{F}$-projective modules, finitely-generated $\mathbb{Z}\,G$-projective modules (even if the group is not of type $\mathfrak{F}\,\mathrm{FP}_0$) and $\mathbb{Z}$-free modules that are finitely-generated as abelian groups.

Remark 3.24. Let G be a group with finitely many conjugacy classes of $\mathfrak{P}_{\mathfrak{F}}$-subgroups. By looking at the analogous condition in Bredon cohomology we might be tempted to formulate the following question. Is a $\mathbb{Z}\,G$-module M of type $\mathfrak{F}\,\mathrm{FP}_n$ if and only if $M\downarrow_P\uparrow^G$ is of type FP_n for every $\mathfrak{P}_{\mathfrak{F}}$-subgroup P? By [**Dyd82**, Proposition 2.1] it follows that $M\downarrow_P\uparrow^G$ is of type FP_0 (Proposition 3.4 implies it is of type FP_∞) if and only if $M\downarrow_P$ is finitely-generated over $\mathbb{Z}P$ and this is the case if and only if M is finitely-generated as an abelian group. Of course I_Δ is in general not finitely-generated as an abelian group and this approach is inconclusive.

Lemma 3.25. *Let M be a finitely-generated $\mathbb{Z}G$-module such that $M\downarrow_P$ is $\mathbb{Z}P$-projective for every $\mathfrak{P}_{\mathfrak{F}}$-subgroup P of G. Then M is of type $\mathfrak{F}\,\mathrm{FP}_0$. In particular, if $\mathrm{B}(G,\mathbb{Z})$ is finitely-generated then it is of type $\mathfrak{F}\,\mathrm{FP}_0$.*

PROOF. We can take any finitely-generated $\mathbb{Z}G$-projective module mapping onto M. Since $M\downarrow_F$ is $\mathbb{Z}F$-projective for every finite subgroup F of G, the surjection will be $\mathfrak{F}$-split. $\square$

Kropholler's class of hierarchically decomposable groups $\mathbf{H}\mathfrak{F}$. For a class of groups $\mathfrak{X}$ the closure operation $\mathbf{H}$ introduced in [**Kro93**] is defined as follows. A group G belongs to $\mathbf{H}_1\mathfrak{X}$ if there exists a finite-dimensional contractible G-CW-complex X with cell stabilisers in $\mathfrak{X}$. The hierarchy of classes $\mathbf{H}_\alpha\mathfrak{X}$ for each ordinal α is defined by transfinite recursion:

- $\mathbf{H}_0\mathfrak{X} = \mathfrak{X}$;
- if α is a successor ordinal then $\mathbf{H}_\alpha\mathfrak{X} = \mathbf{H}_1(\mathbf{H}_{\alpha-1}\mathfrak{X})$;
- if α is a limit ordinal then $\mathbf{H}_\alpha\mathfrak{X} = \bigcup_{\beta<\alpha}\mathbf{H}_\beta\mathfrak{X}$.

The operator $\mathbf{H}$ is defined by G belongs to $\mathbf{H}\mathfrak{X}$ if and only if G belongs to $\mathbf{H}_\alpha\mathfrak{X}$ for some ordinal α. The class $\mathbf{H}\,\mathfrak{F}$ is often called the class of *hierarchically decomposable groups*. It is important to recall that classes of groups with a hierarchical decomposition defined in terms of suitable actions on finite-dimensional complexes appeared previously in the literature, see for examples [**AS82, Ike84**].

We are primarily interested in the case $\mathfrak{X} = \mathfrak{F}$, but it is worth mentioning that recently Leary and Nucinkis considered the case $\mathfrak{X} = \mathfrak{P}_{\mathfrak{F}}$.

Theorem 3.26. [**LN10**] *Let $\mathfrak{P}_{\mathfrak{F}}$ be the class of groups of prime power order. The following relations hold:*

- $\mathfrak{F} \subsetneq \mathbf{H}_1\,\mathfrak{P}_{\mathfrak{F}}$,
- $\mathbf{H}_1\,\mathfrak{P}_{\mathfrak{F}} \subsetneq \mathbf{H}_1\,\mathfrak{F}$,
- $\mathbf{H}\,\mathfrak{P}_{\mathfrak{F}} = \mathbf{H}\,\mathfrak{F}$.

So if we are interested in studying the class $\mathbf{H}\,\mathfrak{F}$ we can consider the class $\mathfrak{P}_{\mathfrak{F}}$ but if we want to study the class $\mathbf{H}_1\,\mathfrak{F}$ we cannot restrict to the family $\mathfrak{P}_{\mathfrak{F}}$ since the $\mathbf{H}_1\,\mathfrak{F}$-group $\mathrm{SL}(\infty, \mathbb{F}_p)$ is not contained in $\mathbf{H}_1\,\mathfrak{P}_{\mathfrak{F}}$ [**LN10**]. Examples of groups that lie in $\mathbf{H}_1\mathfrak{F}$ are given by groups that admit finite-dimensional models for $E_{\mathfrak{F}}$. By applying the closure operation $\mathbf{L}$ to $\mathbf{H}\,\mathfrak{F}$ we obtain the class $\mathbf{LH}\,\mathfrak{F}$. This last class is very large; it contains all elementary amenable groups and all linear groups. There exist groups not contained in $\mathbf{LH}\,\mathfrak{F}$, for example Thompson's group $\mathbf{F}$ and the first Grigorchuk group $\mathfrak{G}$. It is an old, deep theorem of Kropholler that any torsion-free $\mathbf{H}\,\mathfrak{F}$-group of type FP_∞ has finite cohomological dimension. The group $\mathbf{F}$ is of type FP_∞ by [**BG84**], therefore $\mathbf{F} \notin \mathbf{H}\,\mathfrak{F}$. By Theorem 4.12, $\mathfrak{G} \notin \mathbf{H}\,\mathfrak{F}$ and since both groups are finitely-generated they are not contained in $\mathbf{LH}\,\mathfrak{F}$. Until recently it was unknown whether $\mathbf{H}_3\,\mathfrak{F}$ was distinct from $\mathbf{H}\,\mathfrak{F}$. In 2010 a major step towards understanding the hierarchy of $\mathbf{H}\,\mathfrak{F}$ was achieved.

Theorem 3.27. [**JKL10**]

- $\mathbf{H}_\alpha\,\mathfrak{F} < \mathbf{H}_{\alpha+1}\,\mathfrak{F}$ *for every* $\alpha \leqslant \omega_1$*,*
- $\mathbf{LH}_{\omega_1}\,\mathfrak{F} = \mathbf{LH}\,\mathfrak{F}$*,*
- $\mathbf{LH}_\alpha\,\mathfrak{F} < \mathbf{LH}\,\mathfrak{F}$ *for every* $\alpha < \omega_1$*.*

A reminder.

- $\mathfrak{F}\,\mathrm{FP}_\infty \Rightarrow \mathrm{FP}_\infty$ by Proposition 3.5.
- Torsion-free and $\mathrm{FP}_\infty \Rightarrow \mathfrak{F}\,\mathrm{FP}_\infty$. A $\mathbb{Z}\,G$-projective resolution of finite type of $\mathbb{Z}$ is an $\mathfrak{F}$-projective resolution of $\mathbb{Z}$, examples of these are limit groups.
- $\mathfrak{F}\,\mathrm{FP} \Leftrightarrow \mathfrak{F}\,\mathrm{FP}_\infty + \mathfrak{F}\,\mathrm{cd} < \infty$. It is an immediate consequence of the Generalised Schanuel's Lemma.
- $\mathfrak{F}\,\mathrm{FP} \nRightarrow \mathrm{FP}$. Any non-trivial $\mathfrak{F}$-group has infinite cohomological dimension.
- $\mathfrak{F}\,\mathrm{FP} \nRightarrow \underline{\mathrm{FP}}_0$. Examples in [**LN03**].
- $\mathrm{FP} \Rightarrow \mathfrak{F}\,\mathrm{FP}$. If a group G is of type FP then it is torsion-free. This implies that $\mathfrak{F}\,\mathrm{cd}\,G = \mathrm{cd}\,G < \infty$ and since G is of type FP_∞ is of type $\mathfrak{F}\,\mathrm{FP}_\infty$ as well.

- $\mathfrak{F}\,\mathrm{FP}_\infty \nRightarrow \mathfrak{F}\,\mathrm{FP}$. Thompson's group $\mathbf{F}$ is a torsion-free FP_∞ group of infinite cohomological dimension and so is a group of type $\mathfrak{F}\,\mathrm{FP}_\infty$ with infinite $\mathfrak{F}\,\mathrm{cd}$.
- If $G \in \mathbf{LH}\mathfrak{F}$ and it is of type FP_∞ then G admits a finite-dimensional model for $E_{\mathfrak{F}}G$ [**KM98**]. In particular, every $\mathbf{LH}\mathfrak{F}$-group of type $\mathfrak{F}\,\mathrm{FP}_\infty$ is of type $\mathfrak{F}\,\mathrm{FP}$.
- There exists a group $G \in \mathbf{H}_3\mathfrak{F}$ with finitely many conjugacy classes of $\mathfrak{P}_{\mathfrak{F}}$-subgroups and a $\mathbb{Z}\,G$-module of type FP_∞ but not $\mathfrak{F}$-projective by [**KLN**]. Let G be an $\mathbf{LH}\mathfrak{F}$-group and M be a $\mathbb{Z}\,G$-module of type FP_∞. Does M have finite $\mathfrak{F}$-projective dimension?
- $\mathrm{FP}_\infty \nRightarrow \mathfrak{F}\,\mathrm{FP}_\infty$. For this, Thompson's group $\mathbf{V}$ is of type F_∞ [**Bro87**] and contains every $\mathfrak{F}$-group. Now by Proposition 3.4, $\mathbf{V}$ is not of type $\mathfrak{F}\,\mathrm{FP}_0$.
- $H \leqslant G$ and G of type $\mathfrak{F}\,\mathrm{FP}_\infty \nRightarrow H$ of type $\mathfrak{F}\,\mathrm{FP}_\infty$. The free group on 2 generators contains the free group on countably many generators.

Since $\mathbf{F} \leqslant \mathbf{T} \leqslant \mathbf{V}$ and having finite $\mathfrak{F}\,\mathrm{cd}$ is a subgroup closed property we have $\mathfrak{F}\,\mathrm{cd}\,\mathbf{V} = \infty$. This leads to the following:

Question 3.28. Does there exist a group G such that $\mathfrak{F}\,\mathrm{cd}\,G < \infty$ and of type FP_∞ but not of type $\mathfrak{F}\,\mathrm{FP}_\infty$? In [**LN03**] the authors give examples of groups of type FP_∞ admitting a finite-dimensional classifying space for proper actions but not of type $\underline{\mathrm{FP}}_0$; nonetheless these groups have infinitely many conjugacy classes of $\mathfrak{F}$-subgroups but have finitely many conjugacy classes of $\mathfrak{P}_{\mathfrak{F}}$-groups and they are of type $\mathfrak{F}\,\mathrm{FP}_\infty$. These are examples of groups of type $\mathfrak{F}\,\mathrm{FP}$ which are virtually torsion-free but not torsion-free. Later we will show that the Houghton's groups give examples of groups of type FP_n and with finite $\mathfrak{F}\,\mathrm{cd}$ but not of type $\mathfrak{F}\,\mathrm{FP}_0$ for every $n \in \mathbb{N}$.

Let n be a non-negative integer; a group G satisfies condition $b(n)$ if every $\mathbb{Z}\,G$-module M, which is $\mathbb{Z}\,F$-projective for each $\mathfrak{F}$-subgroup F of G, has projective dimension at most n. G satisfies condition $B(n)$ if, for each $\mathfrak{F}$-subgroup F of G, the Weyl-group WF satisfies condition $b(n)$.

Theorem 3.29. [**Lüc00**] *Let G be a group and let $|\Lambda(G)|$ be the G-simplicial complex determined by the poset of non-trivial $\mathfrak{F}$-subgroups $\Lambda(G)$. Suppose*

$\dim|\Lambda(G)| < \infty$ *and suppose that* G *satisfies condition* $B(n)$ *for some non-negative integer* n. *Then* G *admits a model for* $E_{\mathfrak{F}}G$ *of dimension at most* $max\{3, n\}$ $+\lambda(n+1)$.

The next result, due to Kropholler, shows how cohomological conditions of finite type can give strong information about the group structure. This result is the key ingredient in the proof of the next Proposition and in Theorem 4.12.

Theorem 3.30 (Proposition, [**Kro93**]). *Every group of finite rational cohomological dimension and of type* FP_∞ *has a bound on the orders of its* $\mathfrak{F}$*-subgroups.*

Furthermore, Kropholler in [**Kro93**] shows that every $\mathbf{H}\mathfrak{F}$-group of type FP_∞ has a bound on the orders of its $\mathfrak{F}$-subgroups.

Proposition 3.31. *Let* G *be a group with finite* $\mathfrak{F}$*-cohomological dimension and of type* FP_∞. *Then* G *admits a finite-dimensional model for* $E_{\mathfrak{F}}G$.

PROOF. Let $\mathfrak{F}\,\mathrm{cd}\,G = n$. Then $\mathrm{cd}_{\mathbb{Q}}\,G \leqslant n$. Since G is of type FP_∞ we can apply Proposition 3.30 to conclude that G has a bound on the orders of $\mathfrak{F}$-subgroups. In particular $\dim|\Lambda(G)| < \infty$. By Theorem 4.4 in [**Nuc00**] G satisfies $B(n)$. Now the result follows from Theorem 3.29. The final step can be achieved also with [**Nuc00**, Corollary 4.5]. □

Corollary 3.32. *Every group of type* $\mathfrak{F}\,\mathrm{FP}$ *has a finite-dimensional model for the classifying space for proper actions.*

PROOF. By the above remark and Theorem 7.2 in [**Nuc99**] G is of type FP_∞ and we can apply Proposition 3.31. □

Question 3.33. Is every group of type VF (VFP) of type $\mathfrak{F}\,\mathrm{FP}_\infty$ and so of type $\mathfrak{F}\,\mathrm{FP}$?

By [**Bro82**, IX, 13.2] every group of type VF has only finitely many conjugacy classes of subgroups of prime power order (and so is of type $\mathfrak{F}\,\mathrm{FP}_0$). By a result of Serre [**Ser71**] any group G of type VF has a finite-dimensional $E_{\mathfrak{F}}G$ (and so $\mathfrak{F}\,\mathrm{cd}\,G < \infty$). Moreover, groups of type VF are finitely presented and of type FP_∞. There exist groups that act properly and cocompactly by isometries on $\mathrm{CAT}(0)$-spaces that are not virtually torsion-free [**BH99**, Example 7.10, III.Γ]; these have a finite model for $E_{\mathfrak{F}}$ and so groups of type $\mathfrak{F}\,\mathrm{FP}$ are not necessarily of type VF.

In Question 3.28 we recalled that there exist groups of type $\mathfrak{F}\,\mathrm{FP}$ with no model for $E_{\mathfrak{F}}G$ having finite 0-skeleton [**LN03**]. In other words the strongest relative finiteness condition cannot guarantee any condition of finite type for $E_{\mathfrak{F}}G$.

Lemma 3.34. *Every module M of type $\mathfrak{F}\,\mathrm{FP}_n$ is of type FP_n.*

PROOF. Choose an $\mathfrak{F}$-projective resolution with the first n-terms finitely-generated $\cdots \to P_{n+1} \to P_n \to \cdots \to P_0 \twoheadrightarrow M$. By Proposition 3.5, P_i is of type FP_∞ for $i \leqslant n$. Let K_i be the kernel of $P_i \to P_{i-1}$. Since K_{n-1} is a quotient of P_n, it is of type FP_0. Apply part (1) of Proposition 3.20 to conclude that K_{n-2} is of type FP_1. Iterate this process to obtain M of type FP_n. □

Remark 3.35. Let G be a group of type $\mathfrak{F}\,\mathrm{FP}_n$, M a finitely-generated $\mathbb{Z}\,G$-module. Then $\mathfrak{F}\,\mathrm{H}_k(G;M)$ and $\mathfrak{F}\,\mathrm{H}^k(G;M)$ are finitely-generated $\mathbb{Z}\,G$-modules for $0 \leqslant k \leqslant n$. If $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ is an $\mathfrak{F}$-projective resolution of $\mathbb{Z}$ of finite type, then $M \otimes_{\mathbb{Z}\,G} \mathbf{P}$ and $\mathrm{Hom}_{\mathbb{Z}\,G}(\mathbf{P}, M)$ are finitely-generated $\mathbb{Z}\,G$-modules and the claim is obvious.

Proposition 3.36. *Let G be a group of type $\mathfrak{F}\,\mathrm{FP}_n$ ($\mathfrak{F}\,\mathrm{FP}$). If H is a subgroup of G of finite index then it is of type $\mathfrak{F}\,\mathrm{FP}_n$ ($\mathfrak{F}\,\mathrm{FP}$).*

PROOF. Let $\mathbf{P}$ be an $\mathfrak{F}$-resolution of $\mathbb{Z}$ over $\mathbb{Z}\,G$ with P_i finitely-generated for $i \leqslant n$. Restriction to H gives an $\mathfrak{F}$-resolution of $\mathbb{Z}$ over $\mathbb{Z}\,H$ (by Lemma 1.17 and the Mackey decomposition [**Bro82**, III, 5.6]) that is of finite type since $[G : H] < \infty$. The second part follows from the fact that restriction to H is an exact functor. □

Houghton's groups. The length $l(H)$ of an $\mathfrak{F}$-subgroup H of a group G is the supremum over all n for which there is a chain $H_0 < H_1 < \cdots < H_n = H$. It is easy to construct a group with no bound on the length of its $\mathfrak{F}$-subgroups that has finite rational cohomological dimension. By [**Ser03**] every countable infinite $\mathrm{L}\mathfrak{F}$-group has rational cohomological dimension equal to one. Taking finitely many HNN-extensions, free products with amalgamation or extensions by groups of finite cohomological dimension we obtain examples of groups with finite rational cohomological dimension that have no bound on the order of their $\mathfrak{F}$-subgroups.

Given a group it is often very difficult to establish its rational cohomological dimension. In this section we give examples of groups for which it is possible to determine the exact rational cohomological dimension. This can be achieved because they satisfy some strong cohomological finiteness conditions. Houghton's groups are examples of groups with no bound on the lengths of their $\mathfrak{F}$-subgroups. Here we determine their Bredon cohomological dimension.

Let n be a positive integer and $S = \mathbb{N} \times \{1, \dots, n\}$. Houghton's group $\mathfrak{H}_n$ is the group of permutations σ of S satisfying the following condition: there is an n-tuple $(m_1, \dots, m_n) \in \mathbb{Z}^n$ such that for each $i \in \{1, \dots, n\}$, $(x, i)\sigma = (x + m_i, i)$ for all sufficiently large $x \in \mathbb{N}$ [**Bro87**]. Brown proves the following:

Theorem 3.37. [**Bro87**, 5.1] *The group $\mathfrak{H}_n$ is of type FP_{n-1} but not of type FP_n. If n is at least 2 then $\mathfrak{H}_n$ is finitely presented.*

Proposition 3.38. [**LN01**] *Suppose that G is a group with $\operatorname{cd}_{\mathbb{Q}} G = n < \infty$ and G is of type FP_n over $\mathbb{Z}$. Then there is a bound on the orders of the $\mathfrak{F}$-subgroups of G.*

Proposition 3.39. *For every $n \geqslant 1$, $\operatorname{cd}_{\mathbb{Q}} \mathfrak{H}_n = \mathfrak{F}\operatorname{cd} \mathfrak{H}_n = \operatorname{cd}_{\mathfrak{F}} \mathfrak{H}_n = n$.*

PROOF. The group $\mathfrak{H}_n$ is isomorphic to an extension of the infinite finitary symmetric group (on a countable set) Θ by $\mathbb{Z}^{n-1}$. The group Θ is countable (Exercise 8.1.3 in [**DM96**]). Moreover it lies in $\mathbf{L}\mathfrak{F}$ and so $\mathfrak{F}\operatorname{cd}\Theta = 1$. The spectral sequence of Proposition 2.4 in [**Nuc00**] gives the bound $\mathfrak{F}\operatorname{cd}\mathfrak{H}_n \leqslant n$. Analogously by [**Lüc05**], $\operatorname{cd}_{\mathfrak{F}} \mathfrak{H}_n \leqslant n$. Every $\mathfrak{F}$-group embeds in $\mathfrak{H}_n$ and by Theorem 3.37, $\mathfrak{H}_n$ is of type FP_{n-1} but not of type FP_n. We apply Proposition 5.5 to conclude that $\mathfrak{F}\operatorname{cd}\mathfrak{H}_n$ and $\operatorname{cd}_{\mathfrak{F}} \mathfrak{H}_n$ cannot be strictly smaller than n, and so we have $\operatorname{cd}_{\mathbb{Q}} \mathfrak{H}_n = \mathfrak{F}\operatorname{cd}\mathfrak{H}_n = \operatorname{cd}_{\mathfrak{F}} \mathfrak{H}_n = n$. ∎

Examples 3.40. [**Bie81**, 2.14] Write $H_n = \times_{i=1}^{n} \langle h_i, k_i \rangle$, the n-direct power of free groups of rank 2. Let $F_\infty = \langle x_i \rangle_{i \in \mathbb{Z}}$ be the free group of infinite countable rank and $\mathbb{Q}_d$ be the additive subgroup of all rational numbers with denominator a power of the integer d greater than 1. Let H_n act on F_∞ as $x_j^{h_i} = x_j^{k_i} = x_{j+1}$ for all i, j. Let H_n act on $\mathbb{Q}_d$ as $q^{h_i} = q^{k_i} = dq$ for all i, q. Form the semidirect products: $A_n = F_\infty \rtimes H_n$ and $B_n = Q_d \rtimes H_n$. The groups A_n and B_n are of type FP_n but not of type FP_{n-1}. Since A_n and B_n are torsion-free they are of type $\mathfrak{F}\mathrm{FP}_n$ but not of type $\mathfrak{F}\mathrm{FP}_{n+1}$.

2. Finiteness properties and CAT(0) polyhedral complexes

The notion of a lattice in a locally compact group arises naturally in modern mathematics and has its roots in the study of Lie groups. A semisimple algebraic group over a local field can be realised as a group of automorphisms of its Bruhat–Tits building, and their lattices, called arithmetic lattices have been studied since the early 1970's. Other examples are given by tree lattices, which were introduced in the beginning of the 90's by Bass and Lubotzky. Tree lattices are lattices in the isometry group of a locally finite tree [**BL01**]. More recently, lattices in isometry groups of higher dimensional locally finite cell complexes have appeared in the literature [**Tho07, FT11**].

The homological finiteness length $\phi(G)$ of a group G is a generalisation of the concepts of finite generability and finite presentability. More precisely:

The *homological finiteness length* of Γ is defined as

$$\phi(\Gamma) = \sup\{m \mid \Gamma \text{ is of type } \mathrm{FP}_m\}.$$

It is worth mentioning that Abels and Tiemeyer generalise the above finiteness conditions for discrete groups to compactness properties of locally compact groups [**AT97**].

We begin by recalling the terminology and in doing so we follow closely [**Tho07**] and [**FT11**]. Let $\mathbf{X}^n$ be S^n, $\mathbb{R}^n$ or $\mathbb{H}^n$ with Riemannian metrics of constant curvature 1, 0 and -1 respectively. A finite-dimensional CW-complex X is a *polyhedral complex* if it satisfies the following:

- each open cell of dimension n is isometric to the interior of a compact convex polyhedron in $\mathbf{X}^n$;
- for each cell σ of X, the restriction of the attaching map to each open σ-face of codimension one is an isometry onto an open cell of X.

Let $\mathrm{Aut}(X)$ be the full group of cellular isometries of X. A subgroup $H \leqslant \mathrm{Aut}(X)$ acts *admissibly* on X if the set-wise stabiliser of each cell coincides with its point-wise stabiliser.

Remark 3.41. Every subgroup $G \leqslant \mathrm{Aut}(X)$ acts admissibly on the barycentric subdivision of X. Furthermore, if G acts admissibly on a $\mathrm{CAT}(0)$ polyhedral complex, then the fixed-point set X^G forms a subcomplex of X.

A subgroup Γ of a locally compact topological group G with left-invariant Haar measure μ is a *lattice* if:

- Γ is discrete, and

- $\mu(\Gamma\backslash G) < \infty$.

Moreover, $\mathrm{Aut}(X)$ is locally compact whenever X is locally finite and so it makes sense to talk about lattices on locally finite $\mathrm{CAT}(0)$ polyhedral complexes. A lattice Γ is said to be uniform if $\Gamma\backslash\mathrm{Aut}(X)$ is compact. Let G be a locally compact group with left-invariant Haar measure μ. Let Γ be a discrete subgroup of G and Δ be a G-set with compact and open stabilisers. The Δ-covolume, denoted by $\mathrm{Vol}(\Gamma\backslash\backslash\Delta)$, is defined to be $\sum_{\delta\in\Gamma\backslash\Delta}\frac{1}{|\Gamma_\delta|} \leqslant \infty$.

Lemma 3.42. [**BL01**, Chapter 1] *Let X be a locally finite $\mathrm{CAT}(0)$ polyhedral complex with vertex set $V(X)$. If Γ is a subgroup of $G = \mathrm{Aut}(X)$, then:*

- *Γ is discrete if and only if the stabiliser Γ_x is finite for each $x \in V(X)$;*
- *$\mu(\Gamma\backslash G) < \infty$ if and only if $\mathrm{Vol}(\Gamma\backslash\backslash X) < \infty$. Moreover, the Haar measure μ can be normalised in such a way that for every discrete $\Gamma \leqslant G$, $\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash X)$.*

Definition 3.43. The cohomological dimension of Γ over a ring R is defined by

$$\begin{aligned}\mathrm{cd}_R\,\Gamma &= \inf\{n \mid R \text{ admits an } R\Gamma\text{-projective resolution of length } n\}\\ &= \sup\{n \mid \mathrm{H}^n_{R\Gamma}(\Gamma; M) \neq 0, \text{ for some } R\Gamma\text{-module } M\}.\end{aligned}$$

Theorem 3.44 (Proposition 1, [**LN01**]). *Let G be a group with $\mathrm{cd}_{\mathbb{Q}}(G) = n < \infty$ and suppose that G is of type FP_n over $\mathbb{Z}$. Then there is a bound on the orders of the $\mathfrak{F}$-subgroups of G.*

Theorem 3.45. *If Γ is a non-uniform lattice on a locally finite $\mathrm{CAT}(0)$ polyhedral complex of dimension n, then $\phi(\Gamma) < n$.*

Proof. Let Γ be a non-uniform lattice on a locally finite $\mathrm{CAT}(0)$ polyhedral complex X of dimension n. By Lemma 3.42, $\mu(\Gamma\backslash\mathrm{Aut}(X)) = \sum_{\sigma\in\Gamma\backslash X}\frac{1}{|\Gamma_{\tilde\sigma}|}$, where $\sigma = [\tilde\sigma]$. Since Γ is non-uniform, the set $\Gamma\backslash X$ is infinite and so for any m there is some $\sigma \in \Gamma\backslash X$ such that $\frac{1}{|\Gamma_{\tilde\sigma}|} < \frac{1}{m}$. Therefore, there is no bound on the orders of the stabilisers (which are finite), and so there is no bound on the orders of the finite subgroups of Γ.

In view of Theorem 3.44, it only remains to argue that the rational cohomological dimension of Γ is at most n. Since every $\mathrm{CAT}(0)$ space is contractible [**BH99**], Γ acts on an n-dimensional contractible CW-complex with finite stabilisers. The augmented cellular chain complex of X is an exact sequence of the form:

$$\bigoplus_{i_n \in I_n} \mathbb{Z}[\Gamma_{i_n} \backslash \Gamma] \rightarrowtail \bigoplus_{i_{n-1} \in I_{n-1}} \mathbb{Z}[\Gamma_{i_{n-1}} \backslash \Gamma] \to \cdots \to \bigoplus_{i_0 \in I_0} \mathbb{Z}[\Gamma_{i_0} \backslash \Gamma] \twoheadrightarrow \mathbb{Z},$$

where Γ_{i_j} are finite subgroups of Γ for every $0 \leqslant j \leqslant n$. Since $\mathbb{Q}$ is flat over $\mathbb{Z}$ and $\mathbb{Q} \otimes \mathbb{Z}[H \backslash \Gamma] \cong \mathbb{Q}[H \backslash \Gamma]$ for any $H \leqslant \Gamma$, tensoring this sequence with $\mathbb{Q}$ over $\mathbb{Z}$ leads to the exact sequence:

$$\bigoplus_{i_n \in I_n} \mathbb{Q}[\Gamma_{i_n} \backslash \Gamma] \rightarrowtail \bigoplus_{i_{n-1} \in I_{n-1}} \mathbb{Q}[\Gamma_{i_{n-1}} \backslash \Gamma] \to \cdots \to \bigoplus_{i_0 \in I_0} \mathbb{Q}[\Gamma_{i_0} \backslash \Gamma] \twoheadrightarrow \mathbb{Q}.$$

Now, $\bigoplus_{i_j \in I_j} \mathbb{Q}[\Gamma_{i_j} \backslash \Gamma]$ is a $\mathbb{Q}\Gamma$-projective module for every $0 \leqslant j \leqslant n$, and so $\mathrm{cd}_{\mathbb{Q}} \Gamma \leqslant n$.

Hence, by Theorem 3.44, Γ is not of type FP_n. □

Remark 3.46. Note that if an $\mathfrak{F}$-group acts on a locally finite $\mathrm{CAT}(0)$ polyhedral complex, then it is contained in the stabiliser of some cell. Now, let F be a finite subgroup of a non-uniform lattice Γ acting admissibly on a locally finite $\mathrm{CAT}(0)$ polyhedral complex X. Since F acts admissibly on X, X^F is contractible [**BH99, BLN01**]. In particular, X is a model for $\underline{E}\,\Gamma$.

There are not many results that hold for all non-uniform lattices on $\mathrm{CAT}(0)$ polyhedral complexes. As a first immediate application we obtain a classical result.

Corollary 3.47. *If X is a tree, then every non-uniform lattice in $\mathrm{Aut}(X)$ is not finitely-generated. More generally, a non-uniform lattice on a product of n trees is not of type FP_n.*

Corollary 3.48. *Every non-uniform lattice on a locally finite 2-dimensional $\mathrm{CAT}(0)$ polyhedral complex is not finitely presented.*

Before the last corollary, we need to recall some more standard nomenclature. Let K be a global function field, and S be a finite non-empty set of pairwise inequivalent valuations on K. Let $\mathcal{O}_S \leqslant K$ be the ring of S-integers. Denote a reductive K-group by $\mathbf{G}$. Given a valuation v of K, K_v is the completion of K with respect to v. If L/K is a field extension, the L-rank of $\mathbf{G}$, $\mathrm{rank}_L\, \mathbf{G}$, is the dimension of a maximal L-split torus of $\mathbf{G}$. The K-group $\mathbf{G}$ is L-isotropic if $\mathrm{rank}_L\, \mathbf{G} \neq 0$. As in [**BW07**], to any K-group $\mathbf{G}$, there is associated a non-negative integer $k(\mathbf{G}, S) = \sum_{v \in S} \mathrm{rank}_{K_v} \mathbf{G}$. We are now ready to state and reprove the Theorem of Bux and Wortman.

Corollary 3.49 (Theorem 1.2, [**BW07**])**.** *Let* $\mathbf{H}$ *be a connected non-commutative absolutely almost simple* K*-isotropic* K*-group. Then* $\phi(\mathbf{H}(\mathcal{O}_S)) \leqslant k(\mathbf{H}, S) - 1$.

PROOF. Let $\mathbf{H}$ be a connected non-commutative absolutely almost simple K-isotropic K-group. Let H be $\prod_{v \in S} \mathbf{H}(K_v)$. There is a $k(\mathbf{H}, S)$-dimensional Euclidean building X associated to H. The space X is a locally finite $\mathrm{CAT}(0)$ polyhedral complex. The arithmetic group $\mathbf{H}(\mathcal{O}_S)$ becomes a lattice of H via the diagonal embedding. $\mathbf{H}$ is K-isotropic if and only if $\mathbf{H}(\mathcal{O}_S)$ is non-uniform by [**Har69**]. An application of Theorem 3.45 completes the proof. □

Remark 3.50. Theorem 3.45 gives the upper bound on the homological finiteness length of arithmetic groups over function fields; a historical overview can be found in [**BW07**]. In a recent remarkable paper [**BGW11**] Bux, Gramlich and Witzel showed that $\phi(\mathbf{H}(\mathcal{O}_S)) = k(\mathbf{H}, S) - 1$. Calculating the homological finiteness length of non-uniform lattices on $\mathrm{CAT}(0)$ polyhedral complexes is an ambitious open problem. We conclude by mentioning that Thomas and Wortman exhibit examples of non-finitely-generated non-uniform lattices on regular right-angled buildings [**TW11**]. This shows that the upper bound of Theorem 3.45 is not sharp and in particular, that the Theorem of Bux, Gramlich and Witzel does not hold for all non-uniform lattices on locally finite $\mathrm{CAT}(0)$ polyhedral complexes.

Theorem 3.51. *Let* $\mathbf{H}$ *be a connected non-commutative absolutely almost simple* K*-isotropic* K*-group. Then* $\mathrm{gd}_{\mathfrak{F}}(\mathbf{H}(\mathcal{O}_S)) = \mathfrak{K}(\mathbf{H}(\mathcal{O}_S)) = \mathrm{cd}_{\mathbb{Q}} \mathbf{H}(\mathcal{O}_S) = k(\mathbf{H}, S)$.

PROOF. Let H be $\prod_{v \in S} \mathbf{H}(K_v)$. There is a $k(\mathbf{H}, S)$-dimensional Euclidean building X associated to H. By Remark 3.46 we obtain the inequality $\mathrm{gd}_{\mathfrak{F}}(\mathbf{H}(\mathcal{O}_S)) \leqslant k(\mathbf{H}, S)$. By [**BGW11**] $\phi(\mathbf{H}(\mathcal{O}_S)) = k(\mathbf{H}, S) - 1$, and if $\mathrm{cd}_{\mathbb{Q}} \mathbf{H}(\mathcal{O}_S) < k(\mathbf{H}, S)$ an application of Proposition 1 in [**LN01**] gives that $\mathbf{H}(\mathcal{O}_S)$ has a bound on the orders of its $\mathfrak{F}$-subgroups, a contradiction. Therefore $\mathrm{gd}_{\mathfrak{F}}(\mathbf{H}(\mathcal{O}_S)) = \mathfrak{K}(\mathbf{H}(\mathcal{O}_S)) = k(\mathbf{H}, S)$. □

Remark 3.52. Since the groups in Corollary 3.49 have no bound on the orders of their $\mathfrak{F}$-subgroups they are not of type $\mathfrak{F}\,\mathrm{FP}_0$.

CHAPTER 4

Branch groups, rational cohomological dimension and $\mathbf{H}\mathfrak{F}$

A rooted tree $\mathcal{T}$ is *spherically homogeneous* if the valencies of the vertexes at a fixed level are finite and equal. First examples of such trees are given by n-ary regular rooted trees. Roughly speaking, a *branch group* is a subgroup of the full automorphism group of an infinite spherically homogeneous rooted tree $\mathcal{T}$ satisfying a number of conditions. This definition was introduced by Grigorchuk in 1997. It turns out that the class of branch groups is a very fruitful class of counterexamples in group theory. In 1980 Grigorchuk gave an example of a Burnside group $\mathfrak{G}$ and in 1984 he showed that $\mathfrak{G}$ had intermediate growth answering a question of Milnor [**Gri80, Gri84**]. Grigorchuck's group $\mathfrak{G}$ was first realised as a group of Lebesgue-measure-preserving transformations on the unit interval. Later it was noticed that $\mathfrak{G}$ can be realised as a subgroup of the automorphism group of a binary tree, and in modern terminology $\mathfrak{G}$ lies in the class of branch groups.

Until the recent work [**ABJ$^+$09**], where groups with a strong global fixed-point property are constructed, the only way to show that a group G did not belong to $\mathbf{H}\mathfrak{F}$ was to find a subgroup of G isomorphic to Thompson's group **F**. We show that the first Grigorchuk group $\mathfrak{G}$ has jump rational cohomology of height 1 and has infinite rational cohomological dimension. This is the first example of such a group and these properties imply the main result of the chapter.

Theorem. *The first Grigorchuk group $\mathfrak{G}$ is not in $\mathbf{H}\mathfrak{F}$.*

1. The rational cohomological dimension of some branch groups

In Remark 4.7 we will give a possible definition of a branch group, a geometric and an alternative algebraic definition of a branch group; can be found in [**BGŠ03**]. For now we prefer to give an explicit description of the first Grigorchuk group $\mathfrak{G}$. Let $\mathcal{T}$ be the binary rooted tree and let $a = ((1, 2))$ be the automorphism of $\mathcal{T}$ that permutes rigidly the two subtrees below the root. The group $\mathfrak{G}$ will be the group generated by the automorphisms a, b, c and d where the last three automorphisms are defined recursively as follows: $b = (a, c)$, $c = (a, d)$, and $d = (1, b)$. Each generator admits a *labeling* on $\mathcal{T}$ as shown below:

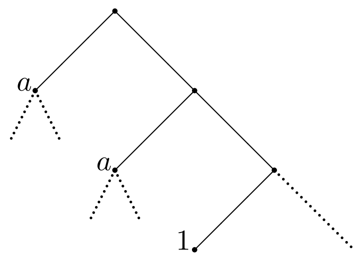

Fig.1. The automorphism b.

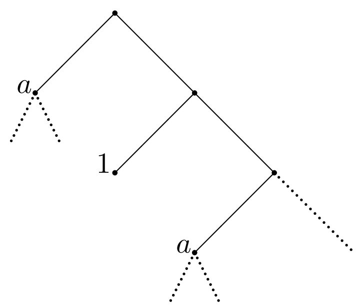

Fig.2. The automorphism c.

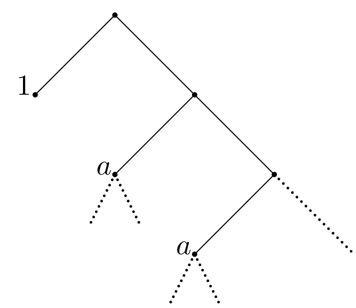

Fig.3. The automorphism d.

By Theorem 3.29, every group of finite $\mathfrak{F}$-cohomological dimension which has a bound on the lengths of its $\mathfrak{F}$-subgroups admits a finite-dimensional classifying space for proper actions.

In this section we calculate the rational cohomological dimension of some finitely-generated periodic groups with no such bound. Moreover, we look into the problem of determining which branch groups lie in the class $\mathbf{H}\mathfrak{F}$. In the next section we give a purely algebraic criterion, from which it follows that the first Grigorchuk group $\mathfrak{G}$ is not contained in $\mathbf{H}\mathfrak{F}$.

Usually if one wants to prove that a group G has finite $\operatorname{cd}_{\mathbb{Q}} G$ either one finds a suitable finite-dimensional G-space or decomposes the group G in order to control its rational cohomological dimension. On the other hand one usually proves that G has infinite $\operatorname{cd}_{\mathbb{Q}} G$ in the following way. Since having finite $\operatorname{cd}_{\mathbb{Q}} G$ is a subgroup-closed property it is enough to find an infinite chain of subgroups of strictly increasing rational cohomological dimension. For the groups we consider in this section there is no such chain, although we are able to establish their dimension because there is a chain of groups of strictly increasing cohomological dimension that uniformly embeds in them.

A group G is R-torsion-free if the order of every finite subgroup of G is invertible in the ring R.

Theorem 4.1. [**DD89**, V, 5.3] *Let G be a group and let H be a subgroup of G of finite index. If G is R-torsion-free, then $\operatorname{cd}_R H = \operatorname{cd}_R G$.*

Definition 4.2. [**Sau06**, 1.1] Let H and K be countable groups. A map $\phi : H \to K$ is called a *uniform embedding* if for every sequence of pairs $(\alpha_i, \beta_i) \in H \times H$ one has:

$$\alpha_i^{-1}\beta_i \to \infty \text{ in } H \iff \phi(\alpha_i)^{-1}\phi(\beta_i) \to \infty \text{ in } K,$$

where $\to \infty$ means eventually leaving every finite subset.

Note that this embedding is not necessarily a group homomorphism. Sauer proved the following remarkable result.

Theorem 4.3. [**Sau06**, 1.2] *Let G and H be countable groups and let R be a commutative ring. If $\operatorname{cd}_R H < \infty$ and H uniformly embeds in G, then $\operatorname{cd}_R H \leqslant \operatorname{cd}_R G$.*

Two groups H and G are said to be *commensurable* if there exist $H_1 \leqslant H, G_1 \leqslant G$ such that $[H : H_1] < \infty$, $[G : G_1] < \infty$ and $H_1 \cong G_1$. A group G is *multilateral* if it is infinite and commensurable to some proper direct power of itself.

Theorem 4.4. *Let G be a finitely-generated multilateral group. Then $\operatorname{cd}_{\mathbb{Q}} G = \infty$.*

PROOF. If A and B are two commensurable groups then by Theorem 4.1 it follows that $\operatorname{cd}_{\mathbb{Q}} A = \operatorname{cd}_{\mathbb{Q}} B$. Let G be a finitely-generated infinite group commensurable with G^k for some $k > 1$. First we show that G is commensurable to G^{k^n} for any $n \geqslant 1$. We proceed by induction on n. The base case $n = 1$ is obvious. Now $G^{k^{n+1}} \cong (G^{k^n})^k$; by the induction hypothesis G is commensurable to G^{k^n} and so G^k is commensurable to $(G^{k^n})^k$. Since G is commensurable to G^k and commensurability is transitive, we obtain that G is commensurable to $G^{k^{n+1}}$. By [**DlH00**, Exercise IV.A.12] there is an isometric embedding $\mathbb{Z} \hookrightarrow G$, from which it follows that there is an isometric embedding $\mathbb{Z}^{k^n} \hookrightarrow G^{k^n}$. An application of Theorem 4.3 gives $k^n = \operatorname{cd}_{\mathbb{Q}} \mathbb{Z}^{k^n} \leqslant \operatorname{cd}_{\mathbb{Q}} G^{k^n} = \operatorname{cd}_{\mathbb{Q}} G$. Since the last inequality holds for every non-negative integer n we have $\operatorname{cd}_{\mathbb{Q}} G = \infty$. $\square$

The converse of the theorem above does not hold. In fact the finitely-generated $\mathbf{H}_2\,\mathfrak{F}$-group of infinite cohomological dimension $\mathbb{Z} \wr \mathbb{Z}$ is not commensurable to any of its proper direct powers.

Tyrer Jones in [**Jon74**] constructs a finitely-generated non-trivial group G isomorphic to its own square; as an immediate application of Theorem 4.4 we obtain that $\operatorname{cd}_{\mathbb{Q}} G = \infty$.

Remark 4.5. If G is a finitely-generated multilateral group, then the proof of Theorem 3 [**Smi07**] extends verbatim by replacing G^n with G^{k^n} to conclude that $\operatorname{asdim} G = \infty$. For many groups the finiteness of the asymptotic dimension agrees with the finiteness of the rational cohomological dimension. However, Sapir in [**Sap11**] constructed a 4-dimensional closed aspherical manifold M such that the fundamental group $\pi_1(M)$ coarsely contains an expander, and so $\pi_1(M)$ has infinite asymptotic dimension but finite cohomological dimension.

Note that if G is a finitely-generated infinite group such that $G^n \hookrightarrow G$ with $n > 1$, then arguing as in Theorem 4.4 we obtain that G has infinite rational cohomological dimension. Of course if G is not periodic this shows that it contains a free abelian group of infinite countable rank. For example, it is well-known that for Thompson's group $\mathbf{F}$ we have the embedding $\mathbf{F} \times \mathbf{F} \hookrightarrow \mathbf{F}$.

Corollary 4.6. *Every finitely-generated regular branch group has infinite rational cohomological dimension.*

PROOF. For the precise definition of a regular branch group the reader is referred to [**BGŠ03**]. Let $\mathcal{T}$ be an m-ary regular rooted tree and G a finitely-generated regular branch group acting on $\mathcal{T}$. By definition, if G is branching over K then $[G : K] < \infty$ and $[\psi(K) : K^m] < \infty$, where ψ is the embedding of the stabiliser of the first level in the direct product G^m. Since $\psi(K) \cong K$ we have that K is commensurable with K^m. The group K is finitely-generated and so an application of Theorem 4.4 gives $\operatorname{cd}_{\mathbb{Q}} K = \infty$. The finiteness of the rational cohomological dimension is preserved under taking subgroups and so we have $\operatorname{cd}_{\mathbb{Q}} G = \infty$. □

Since the Gupta-Sidki group $\overline{\overline{\Gamma}}$ is a finitely-generated regular branch group [**BGŠ03**] we obtain as an application of Corollary 4.6 that $\operatorname{cd}_{\mathbb{Q}} \overline{\overline{\Gamma}} = \infty$. Note that since $\overline{\overline{\Gamma}}$ is a p-group with no bound on the orders of its elements it has no bound on the lengths of its $\mathfrak{F}$-subgroups. The hypothesis of finite generation in the corollary above is crucial since the $\mathbf{L}\mathfrak{F}$-group $\operatorname{Aut}_f(\mathcal{T})$ by Proposition 1.22 [**BGŠ03**] is a regular branch group.

Remark 4.7. We have proved Corollary 4.6 in the context of regular branch groups for convenience only. In fact it was pointed out to the author by Laurent Bartholdi, that also holds for the more general branch groups defined as follows.

A group G is *branch* if it admits a *branch structure*, i.e. there exists a sequence of groups $\{G_i\}_{i\in\mathbb{N}}$, a sequence of positive integers $\{n_i\}_{i\in\mathbb{N}}$ and a sequence of homomorphisms $\{\phi_i\}_{i\in\mathbb{N}}$ such that $G \cong G_0$, and for each i,

(1) $\phi_i : G_i \to G_{i+1} \wr \Sigma_{n_i}$ has finite kernel and finite cokernel, where Σ_{n_i} denotes the symmetric group on n_i-letters,
(2) the image of each ϕ_i acts transitively on Σ_{n_i}, and the stabiliser of any $j \in 1, \dots, n_i$ maps onto G_{i+1}.

The structure is non-trivial if all $n_i \geqslant 2$, and the ϕ_i are injective. It is possible to see that a branch group as above is a branch group in the geometric sense of

[**BGŠ03**]. Now, let G be a finitely-generated infinite group that admits a sequence of groups $\{G_i\}_{i\in\mathbb{N}}$ and a sequence of integers $\{n_i\}_{i\in\mathbb{N}}$, such that $G \cong G_0$ and for each i, G_i is commensurable with $G_{i+1}^{n_i}$. Arguing as in Theorem 4.4 it is easy to see that, if all $n_i \geqslant 2$, the rational cohomological dimension of G is infinite. Arguing as in Corollary 4.6 we deduce that every finitely-generated branch group has infinite rational cohomological dimension.

2. $\mathfrak{G} \notin \mathbf{H}\mathfrak{F}$

A group G is said to have *jump cohomology of height n over* R if there exists an integer $n \geqslant 0$ such that any subgroup H of finite cohomological dimension over R has $\mathrm{cd}_R(H) \leqslant n$.

Theorem 4.8. [**Pet07**, 3.2] *Let G be an R-torsion-free $\mathbf{H}\mathfrak{F}$-group with jump cohomology of height n over R. Then $\mathrm{cd}_R\, G \leqslant n$. In particular, any $\mathbf{H}\mathfrak{F}$-group G has jump rational cohomology of height n if and only if $\mathrm{cd}_{\mathbb{Q}}\, G \leqslant n$.*

Lemma 4.9. *Let G be a countable group with $\mathrm{cd}_{\mathbb{Q}}\, G < \infty$. Then there exists a finitely-generated subgroup H of G such that*

$$\mathrm{cd}_{\mathbb{Q}}\, H \leqslant \mathrm{cd}_{\mathbb{Q}}\, G \leqslant \mathrm{cd}_{\mathbb{Q}}\, H + 1.$$

Moreover, if $\mathfrak{F}\,\mathrm{cd}\, G < \infty$ then there exists a finitely-generated subgroup K such that

$$\mathfrak{F}\,\mathrm{cd}\, K \leqslant \mathfrak{F}\,\mathrm{cd}\, G \leqslant \mathfrak{F}\,\mathrm{cd}\, K + 1.$$

PROOF. The statement for the rational cohomological dimension follows from Theorem 4.3 in [**Bie81**] and for the $\mathfrak{F}$-cohomological dimension it follows from Proposition 2.5 in [**Nuc00**]. □

We say that a group G is *strongly multilateral* if it is multilateral and every finitely-generated subgroup of G is commensurable to some direct power of G.

Theorem 4.10. *Every finitely-generated strongly multilateral group has jump rational cohomology of height* 1.

PROOF. Let G be a finitely-generated strongly multilateral group. Then by Theorem 4.4 G has infinite rational cohomological dimension. Suppose H is a finitely-generated infinite subgroup of G, then by hypothesis H is commensurable with some direct power of G and so by Theorem 4.1 $\operatorname{cd}_{\mathbb{Q}} H = \infty$. Suppose now that H is an infinitely-generated subgroup of G of finite rational cohomological dimension. By Lemma 4.9 there exists $K \leqslant H$ such that K is finitely-generated and $\operatorname{cd}_{\mathbb{Q}} K \leqslant \operatorname{cd}_{\mathbb{Q}} H \leqslant \operatorname{cd}_{\mathbb{Q}} K + 1$. By the above K cannot be infinite and so $\operatorname{cd}_{\mathbb{Q}} H = 1$. □

Corollary 4.11. *If G is a finitely-generated strongly multilateral group, then G is not in $\mathbf{H}\mathfrak{F}$.*

PROOF. The group G has jump rational cohomology of height 1 but infinite rational cohomological dimension and so by Theorem 4.8 $G \notin \mathbf{H}\mathfrak{F}$. □

The first Grigorchuk group $\mathfrak{G}$ is an infinite periodic finitely-generated amenable group [**Gri80**]. $\mathfrak{G}$ can be obtained as a subgroup of the automorphism group of the rooted binary tree. Since $\mathfrak{G}$ has infinite $\mathbf{L}\mathfrak{F}$-subgroups [**Roz98**], it has no bound on the lengths of its $\mathfrak{F}$-subgroups. For the definition and further details the reader should consult [**BGŠ03**] or [**DlH00**].

Theorem 4.12. *The first Grigorchuk group $\mathfrak{G}$ has jump rational cohomology of height 1, and has infinite rational cohomological dimension. Hence $\mathfrak{G}$ is not in $\mathbf{H}\mathfrak{F}$.*

PROOF. By VIII.14 and .15 [**DlH00**] $\mathfrak{G}$ is commensurable with its square, infinite and finitely-generated. Any finitely-generated infinite subgroup of $\mathfrak{G}$ is commensurable with $\mathfrak{G}$ [**GW03**] and so by Corollary 4.11 $\mathfrak{G} \notin \mathbf{H}\mathfrak{F}$. □

Other consequences.

Remark 4.13. Theorem 4.12 has two futher consequences.
Conjecture [**Pet07**]. For every group G without R-torsion the following are equivalent.

- G has jump cohomology of height n over R.
- G has periodic cohomology over R starting in dimension $n + 1$.
- $\operatorname{cd}_R G \leqslant n$.

Obviously from Theorem 4.12 it follows that $\mathfrak{G}$ is a counterexample to the above conjecture.

Jo-Nucinkis in [**JN08**] ask the following.
Question. Let G be a group such that every proper subgroup H of G of finite

Bredon cohomological dimension satisfies $\mathrm{cd}_{\mathfrak{F}} H \leqslant n$ for some positive integer n. Is $\mathrm{cd}_{\mathfrak{F}} G < \infty$?

Since a group G has rational cohomological dimension equal to 1 if and only if it has Bredon cohomological dimension equal to 1 [**Dun79**], Theorem 4.12 shows that $\mathfrak{G}$ provides a negative answer to their question.

Given Theorem 4.12, it is easy to see that for any $n \geqslant 1$, the group $\mathfrak{G} \times \mathbb{Z}^{n-1}$ has infinite rational cohomological dimension and jump rational cohomology of height n.

The question of Jo-Nucinkis is a "proper actions version" of an older question of Mislin-Talelli that asks whether there exists a torsion-free group with jump integral cohomology but infinite cohomological dimension. Note that every virtually torsion-free branch group G contains a free abelian group of infinite countable rank. To see this take a ray and an edge per level just hanging off it. Then there is a non-trivial element of infinite order a_n hanging off each edge since the rigid stabiliser of the nth-level $\mathrm{Rstt}_G(n)$ has finite index in G and G is spherically transitive. These elements generate distinct infinite cyclic subgroups of G that obviously commute since they act on distinct subtrees and so they generate $\bigoplus_{\mathbb{N}} \mathbb{Z}$. This implies that G has infinite rational cohomological dimension and does not have jump rational cohomology. Moreover, no torsion-free subgroup of finite index in G can answer Mislin-Talelli question. A more detailed study of the subgroup lattices of virtually torsion-free branch groups would be very interesting. In fact it is unknown whether there exists a torsion-free group $G \in \mathbf{H}\,\mathfrak{F} \setminus \mathbf{H}_3\,\mathfrak{F}$.

Question 4.14. Does every finitely-generated periodic regular branch group have a finitely-generated strongly multilateral subgroup?

Remark 4.15. Note that if G is an $\mathbf{H}\mathfrak{F}$-group, then $\mathrm{Gcd}\, G < \infty$ implies that $\mathrm{cd}_{\mathbb{Q}} G < \infty$. This can be shown in the following way. First we recall that $\mathrm{spli}(RG)$ is the supremum of the projective lengths of the injective RG-modules. The invariant $\kappa(RG)$ is the supremum of the projective dimensions of the RG-modules that have finite RF-projective dimension for all $\mathfrak{F}$-subgroups of G. For any group G, $\mathrm{Gcd}\, G < \infty$ if and only if $\mathrm{spli}(\mathbb{Z}G) < \infty$ by Remark 2.10 in [**ABS09**]. Assume now that G is an $\mathbf{H}\mathfrak{F}$-group of finite Gorenstein cohomological dimension. By Theorem C in [**CK98**] $\mathrm{spli}(\mathbb{Q}G) = \kappa(\mathbb{Q}G)$. By [**GG87**] $\mathrm{spli}(\mathbb{Q}G) \leqslant \mathrm{spli}(\mathbb{Z}G)$; in particular if $\operatorname{spli}(\mathbb{Q}G) < \infty$ then $\kappa(\mathbb{Q}G) < \infty$. Since $\mathbb{Q}$ is $\mathbb{Q}F$-projective for every $\mathfrak{F}$-subgroup F of G we have $\operatorname{cd}_{\mathbb{Q}} G < \infty$.

It is known from recent work of Dembegioti and Talelli [**DT10**] that the notions of a Gorenstein projective module and a cofibrant module coincide over $\mathbf{H}\mathfrak{F}$-groups. We suspect that the Gorenstein projective modules over an $\mathbf{H}\mathfrak{F}$-group G are exactly direct summands of $\mathbb{Z}\, G$-modules obtained as extensions of permutation modules with $\mathfrak{F}$-stabilisers. If this holds then the inequality $\operatorname{cd}_{\mathbb{Q}} G \leqslant \operatorname{Gcd} G$ would be immediate.

It would be very interesting to compute the Gorenstein cohomological dimension of $\mathfrak{G}$. In fact, $\mathfrak{G}$ could be a counterexample to the conjecture of Bahlekeh, Dembegioti and Talelli.

Corollary 4.16. *$\mathfrak{G}$ does not contain a group of finite $\mathfrak{F}$-cohomological dimension for which the extension property fails to be subadditive.*

PROOF. $\mathfrak{G}$ is just infinite and by Theorem 4.1 every normal subgroup N of $\mathfrak{G}$ has infinite rational cohomological dimension, so $\mathfrak{F}\operatorname{cd} N = \infty$. Assume L is a subgroup of $\mathfrak{G}$ such that $H \hookrightarrow L \twoheadrightarrow Q$, with $\mathfrak{F}\operatorname{cd} H = n$, $|Q| < \infty$ and $n < \mathfrak{F}\operatorname{cd} L < \infty$. Then, by Theorem 4.12 it follows that $\mathfrak{G}$ has jump rational cohomology of height 1 and L is not finitely-generated. From Lemma 4.9 $\operatorname{cd}_{\mathbb{Q}} L \leqslant 1$. By Dunwoody's theorem [**Dun79**] $\operatorname{cd}_{\mathbb{Q}} L \leqslant 1$ if and only if L acts on a tree T with $\mathfrak{F}$-stabilisers. We can assume $|L| = \infty$ and the tree T is a one dimensional model for $E_{\mathfrak{F}}L$, so $\operatorname{cd}_{\mathbb{Q}} L = \mathfrak{F}\operatorname{cd} L = \operatorname{cd}_{\mathfrak{F}} L = \operatorname{gd}_{\mathfrak{F}} L = 1$ and the result follows from Theorem 4.1. $\square$

CHAPTER 5

Some $\mathbf{H}_1\mathfrak{F}$-groups with unbounded torsion

Let $\mathcal{U}$ be the smallest class of groups containing all groups of finite $\mathfrak{F}$-cohomological dimension with a bound on the orders of their $\mathfrak{F}$-subgroups closed under taking extensions and fundamental groups of graphs of groups. This class contains all groups of finite virtual cohomological dimension, Gromov hyperbolic groups, Burnside groups of large odd exponent, more generally all groups of finite Bredon cohomological dimension with a bound on the order of their $\mathfrak{F}$-subgroups, all countable $\mathbf{L}\mathfrak{F}$-groups, lamplighter groups, Houghton's groups, every countable infinite free product of $\mathfrak{F}$-groups, Dunwoody's inaccessible group [**Dun93**], countable elementary amenable groups and many others.

We first show that the class $\mathcal{U}$ admits a natural hierarchical decomposition and establish some of its basic properties. Then, we prove that the Kropholler-Mislin conjecture holds for $\mathfrak{U}^{\star}_{\omega_0}$, a subclass of $\mathcal{U}$.

Theorem. *Every $\mathbf{H}_1\,\mathfrak{F}$-group contained in the class $\mathfrak{U}^{\star}_{\omega_0}$ admits a finite-dimensional classifying space for proper actions.*

1. The class $\mathfrak{U}$ and its hierarchy

Let $\mathfrak{X}$ be any class of groups, define the group operation $\mathbf{F_1}$ as $\mathbf{F_1}\mathfrak{X}$ consists of those groups which are isomorphic to a fundamental group of graph of $\mathfrak{X}$-groups. Note that if $\mathfrak{X} \subseteq \mathfrak{Y}$ then $\mathfrak{X} \subseteq \mathbf{F_1}\mathfrak{X} \subseteq \mathbf{F_1}\mathfrak{Y}$. Let $\mathfrak{B}$ be the class of groups of finite $\mathfrak{F}$-cohomological dimension with a bound on the orders of their $\mathfrak{F}$-subgroups. For each ordinal α we define the class $\mathfrak{U}_\alpha$ inductively

- $\mathfrak{U}_0 = \mathfrak{I}$,
- $\mathfrak{U}_\alpha = (\mathbf{F_1}\,\mathfrak{U}_{\alpha-1})\,\mathfrak{B}$ if α is a successor ordinal,
- $\mathfrak{U}_\alpha = \bigcup_{\beta<\alpha}\mathfrak{U}_\beta$ if α is a limit ordinal.

The class $\mathfrak{U}$ is defined as $\mathfrak{U} = \bigcup_{\alpha\geqslant 0}\mathfrak{U}_\alpha$.

Lemma 5.1. *The class $\mathfrak{U}$ coincides with the class $\mathcal{U}$.*

PROOF. Clearly $\mathfrak{U} \subseteq \mathcal{U}$ and $\mathfrak{U}$ is closed under taking fundamental groups of graphs of groups. In order to show $\mathcal{U} \subseteq \mathfrak{U}$ we need only to verify that the class $\mathfrak{U}$ is extension closed. By Bass-Serre theory it follows that if G/N acts on a tree T, then G has an action on T such that N fixes every vertex of T. Hence, if $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes of groups then $\mathfrak{X}(\mathbf{F_1}\,\mathfrak{Y}) \subseteq \mathbf{F_1}(\mathfrak{X}\,\mathfrak{Y})$.We argue by induction on β to show $\mathfrak{U}_\alpha\,\mathfrak{U}_\beta \subseteq \mathfrak{U}_{\alpha+\beta}$. If $\beta = 1$ then $\mathfrak{U}_\alpha\,\mathfrak{U}_1 = \mathfrak{U}_\alpha\,\mathfrak{B} \subseteq (F\,\mathfrak{U}_\alpha)\,\mathfrak{B} = \mathfrak{U}_{\alpha+1}$.

- Suppose β is a successor ordinal, $\beta = \gamma + 1$.

$$
\begin{aligned}
\mathfrak{U}_\alpha\,\mathfrak{U}_\beta &= \mathfrak{U}_\alpha((\mathbf{F_1}\,\mathfrak{U}_\gamma)\,\mathfrak{B}) \\
&\subseteq (\mathfrak{U}_\alpha(\mathbf{F_1}\,\mathfrak{U}_\gamma))\,\mathfrak{B} && \text{(by universality, [\textbf{Rob72}, pg. 2])} \\
&\subseteq (\mathbf{F_1}(\mathfrak{U}_\alpha\,\mathfrak{U}_\gamma))\,\mathfrak{B} && \text{(by the above)} \\
&\subseteq (\mathbf{F_1}(\mathfrak{U}_{\alpha+\gamma}))\,\mathfrak{B} && \text{(by induction)} \\
&= \mathfrak{U}_{\alpha+\beta}\,.
\end{aligned}
$$

- Suppose β be a limit ordinal, then $\mathfrak{U}_\beta = \bigcup_{\gamma<\beta}\mathfrak{U}_\gamma$.

$$
\begin{aligned}
\mathfrak{U}_\alpha\,\mathfrak{U}_\beta &= \mathfrak{U}_\alpha(\bigcup_{\gamma<\beta}\mathfrak{U}_\gamma) \\
&= \bigcup_{\gamma<\beta}\mathfrak{U}_\alpha\,\mathfrak{U}_\gamma \\
&\subseteq \bigcup_{\gamma<\beta}\mathfrak{U}_{\alpha+\gamma} && \text{(by induction)} \\
&= \mathfrak{U}_{\alpha+\beta}\,.
\end{aligned}
$$

□

Proposition 5.2. *The class $\mathfrak{U}$ is closed under taking free products with amalgamation,* HNN-*extensions, countable directed unions, extensions.*

PROOF. It is obvious that $\mathfrak{U}$ is closed under taking free products with amalgamation and HNN-extensions. If G is a countable directed union of $\mathfrak{U}$-groups then G acts on a tree with stabilisers conjugate to groups in the directed system and so $G \in \mathfrak{U}$. The class $\mathfrak{U}$ coincide with the class $\mathcal{U}$ that is closed under taking extensions by definition.

For any class of groups $\mathfrak{X}$ we write $G \in \mathbf{S}\mathfrak{X}$ if $G \leqslant K \in \mathfrak{X}$. Note that $\mathbf{SF_1}\mathfrak{X} \subseteq \mathbf{F_1S}\mathfrak{X}$; in fact if $G \in \mathbf{SF_1}\mathfrak{X}$, G is a subgroup of a group K that acts on a tree T with stabilisers in $\mathfrak{X}$ and so T is a G-tree with stabilisers that are subgroups of the stabilisers of the K-tree T. Clearly if a class $\mathfrak{X}$ is $\mathbf{S}$-closed then $\mathbf{SF_1}\mathfrak{X} \subseteq \mathbf{F_1}\mathfrak{X}$, and note that the class $\mathfrak{B}$ is $\mathbf{S}$-closed. Arguing as in Lemma 5.1 we have that $S\,\mathfrak{U} \subseteq \mathfrak{U}$. $\square$

Let $\mathbf{F}\,\mathfrak{B}$ be the smallest class of groups containing the class $\mathfrak{B}$ and which contains a group G whenever G can be realised as the fundamental group of a graph of groups already in $\mathbf{F}\,\mathfrak{B}$. The class $\mathbf{F}\,\mathfrak{F}$ was considered by Richard John Platten in his PhD thesis, however $\mathbf{F}\,\mathfrak{B}$ and $\mathbf{F}\,\mathfrak{F}$ differ. For example, any non-trivial group of finite cohomological dimension with Serre's property FA does not belong to $\mathbf{F}\,\mathfrak{F}$ but lies in $\mathfrak{B}$.

Lemma 5.3. *The class $\mathfrak{U}$ is contained in $\mathbf{H}\,\mathfrak{F}$, the class $\mathbf{F}\,\mathfrak{B}$ is properly contained in $\mathfrak{U}$ and $\mathfrak{U}$ is not closed under taking quotients.*

PROOF. By definition $\mathfrak{B} \subset \mathbf{H}_1\,\mathfrak{F}$, $\mathbf{H}\,\mathfrak{F}$ is obviously closed under taking groups acting on trees with stabilisers in $\mathbf{H}\,\mathfrak{F}$ and is extension closed by [**Kro93**, 2.3]. In particular $\mathfrak{U} \subseteq \mathbf{H}\,\mathfrak{F}$.

Let H be a non-trivial $\mathfrak{F}$-group and let P be Pride's group of cohomological dimension equal to two with Serre's property FA [**Pri83**]. Clearly the group $G = H \wr P$ has no bound on the orders of its $\mathfrak{F}$-subgroups and it lies in $\mathfrak{U}_2 \setminus \mathfrak{U}_1$. By [**CK11**] G has Serre's property FA, and does not lie in $\mathfrak{B}$ and so $G \notin \mathbf{F_1}\,\mathfrak{B}$. Note that $\mathrm{cd}_{\mathbb{Q}}\, G \leqslant 3$.

The first Grigorchuk group is a 3-generated group but by Theorem 4.12 it is not an $\mathbf{H}\,\mathfrak{F}$-group, therefore it is not a $\mathfrak{U}$-group and $\mathfrak{U}$ is not closed under taking quotients. $\square$

2. $\mathfrak{U}$-groups of finite $\mathfrak{F}$-cohomological dimension

Lemma 5.4. *Let T be a G-tree with edge set $E = \bigsqcup_{i \in I} L_i \backslash G$ and vertex set $V = \bigsqcup_{j \in J} N_j \backslash G$. Then there is a Mayer-Vietoris sequence:*

$$\cdots \to \mathrm{H}^n_{\mathfrak{F}}(G, -) \to \bigoplus_{j \in J} \mathrm{H}^n_{\mathfrak{F}}(N_j, \mathrm{res}_{I_{N_j}} -)$$
$$\to \bigoplus_{i \in I} \mathrm{H}^n_{\mathfrak{F}}(L_i, \mathrm{res}_{I_{L_j}} -) \to \mathrm{H}^{n+1}_{\mathfrak{F}}(G, -) \to \cdots$$

PROOF. By Corollary 3.4 in [**KMPN08**] the augmented Bredon cell complex

$$\mathbb{Z}[-, E] \rightarrowtail \mathbb{Z}[-, V] \twoheadrightarrow \underline{\mathbb{Z}}$$

is a short exact sequence of $\mathcal{O}G$-modules. Now applying the long exact sequence in Bredon cohomology we obtain

$$\cdots \to \mathrm{Ext}^n_{\mathfrak{F}}(\underline{\mathbb{Z}}, -) \to \bigoplus_{j\in J} \mathrm{Ext}^n_{\mathfrak{F}}(\mathbb{Z}[-, N_j\backslash G], -) \\ \to \bigoplus_{i\in I} \mathrm{Ext}^n_{\mathfrak{F}}(\mathbb{Z}[-, L_i\backslash G], -) \to \mathrm{Ext}^{n+1}_{\mathfrak{F}}(\underline{\mathbb{Z}}, -) \to \cdots$$

We show that $\mathrm{Ext}^n_{\mathfrak{F},G}(\mathbb{Z}[-, H\backslash G], -) \cong \mathrm{Ext}^n_{\mathfrak{F},H}(\underline{\mathbb{Z}}, -)$.

By [**Sym05**, Lemma 2.7] we have $\mathbb{Z}[-, H\backslash G] \cong \mathrm{ind}_{I_H} \underline{\mathbb{Z}}$. From the adjoint isomorphism it follows that induction with I_H is a left adjoint to restriction with I_H:

$$\mathrm{mor}_{\mathfrak{F},G}(\mathrm{ind}_{I_H} \underline{\mathbb{Z}}, -) \cong \mathrm{mor}_{\mathfrak{F},H}(\underline{\mathbb{Z}}, \mathrm{res}_{I_H} -)$$

and the result now follows. $\square$

We recall that a less direct proof of the lemma above, involving a spectral sequence appeared in [**FD11**, Corollary 4.7].

Corollary 5.5. *Let T be a G-tree with edge set $E = \bigsqcup_{i\in I} L_i\backslash G$ and vertex set $V = \bigsqcup_{j\in J} N_j\backslash G$. If there is a non-negative integer n such that $\mathrm{cd}_{\mathfrak{F}} L_i \leqslant n$ and $\mathrm{cd}_{\mathfrak{F}} N_i \leqslant n$ for all i, then $\mathrm{cd}_{\mathfrak{F}} G \leqslant n+1$.*

PROOF. It is an immediate consequence of Lemma 5.4. $\square$

Define $\mathbf{F_B}$ the group operation as $\mathbf{F_B}\mathfrak{X}$ consists of those groups which are isomorphic to a fundamental group of graph of $\mathfrak{X}$-groups such that there is a finite bound $\mathbf{B}$ on the differences between the $\mathfrak{F}\,\mathrm{cd}$ and $\mathrm{cd}_{\mathfrak{F}}$ over all the vertex and edge groups. For each ordinal α we define the class $\mathfrak{U}^\star_\alpha$ inductively

- $\mathfrak{U}^\star_0 = \mathfrak{I}$
- $\mathfrak{U}^\star_\alpha = (\mathbf{F_B}\,\mathfrak{U}^\star_{\alpha-1})\,\mathfrak{B}$ if $\alpha < \omega_0$,
- $\mathfrak{U}^\star_\alpha = \bigcup_{\beta<\omega_0} \mathfrak{U}^\star_\beta$ if $\alpha = \omega_0$.

Theorem 5.6. *Let G be a $\mathfrak{U}^\star_{\omega_0}$-group of finite $\mathfrak{F}$-cohomological dimension. Then G has finite Bredon cohomological dimension. In particular, every $\mathbf{H_1}\,\mathfrak{F}$-group contained in the class $\mathfrak{U}^\star_{\omega_0}$ has finite Bredon cohomological dimension.*

PROOF. Suppose $G \in \mathfrak{U}^\star_\alpha$ and $\mathfrak{F}\,\mathrm{cd}\, G = n$. If α is a successor ordinal, then G is an extension $N \rightarrowtail G \twoheadrightarrow Q$ with $N \in \mathbf{F_B}\,\mathfrak{U}^*_{\alpha-1}$ and $Q \in \mathfrak{B}$. By Corollary 5.5 we have that $\mathrm{cd}_{\mathfrak{F}} N \leqslant n+\mathbf{B}+1$. Since Q has a bound d on the orders of its $\mathfrak{F}$-subgroups we can apply [**Lüc00**, Theorem 3.1] to conclude that $\mathrm{cd}_{\mathfrak{F}} G \leqslant (n+\mathbf{B}+1)d+\mathrm{cd}_{\mathfrak{F}} Q$. Hence, since $\mathrm{cd}_{\mathfrak{F}} Q < \infty$ we have $\mathrm{cd}_{\mathfrak{F}} G < \infty$.

Let $\mathfrak{R}$ be the class of groups of finite $\mathfrak{F}$-cohomological dimension. If $\alpha = \omega_0$ then $G \in \mathfrak{U}^\star_{\omega_0} \cap \mathfrak{R} = (\bigcup_{\beta<\omega_0} \mathfrak{U}^\star_\beta) \cap \mathfrak{R} = \bigcup_{\beta<\omega_0}(\mathfrak{U}^\star_\beta \cap \mathfrak{R})$. By the above for $\beta < \omega_0$, every group in $\mathfrak{U}^\star_\beta \cap \mathfrak{R}$ has finite Bredon cohomological dimension, and so does G. □

Proposition 5.7. *Suppose that there exists a function $\rho : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{gd}_{\mathfrak{F}} G \leqslant \rho(\mathfrak{K}(G))$ for every group G of finite Bredon cohomological dimension. Then the Kropholler-Mislin conjecture holds inside $\mathfrak{U}_{\omega_0}$.*

PROOF. This can be proved as in Theorem 5.6. Notice that the bound does not have to be universal. □

Remark 5.8. Note that if there is a countable (periodic) $\mathbf{H}_1\,\mathfrak{F}$-group that does not belong in $\mathfrak{U}$ then there is a finitely-generated (periodic) $\mathbf{H}_1\,\mathfrak{F}$-group with no bound on the orders of its $\mathfrak{F}$-subgroups. To see this, suppose that G is a countable (periodic) $\mathbf{H}_1\,\mathfrak{F}$-group that does not belong to $\mathfrak{U}$. Then G is the directed union of its finitely-generated (periodic) subgroups that are $\mathbf{H}_1\,\mathfrak{F}$-groups. The group G acts on a tree with stabilisers conjugate to groups in the directed union. If every stabiliser were in $\mathfrak{U}$ so would G, giving a contradiction.

Proposition 5.9. *Examples of $\mathfrak{U}$-groups:*

(1) free groups $\mathbf{Fr} \subsetneq \mathfrak{U}_1$, countable $\mathbf{L}\mathbf{Fr}$-groups belong to $\mathfrak{U}_1$,
(2) free Burnside groups of large odd exponent $B(m,n) \in \mathfrak{U}_1 \setminus \mathfrak{U}_0$, in particular Petrosyan's class $\mathcal{N}^{cell}(\mathcal{P}_6) \neq \mathfrak{U}$,
(3) $\mathfrak{U}_2$ contains all countable $\mathfrak{A}$-groups, and $\mathfrak{U}$ contains all countable elementary amenable groups,
(4) Gromov hyperbolic groups, more generally every group G admitting a finite model for $E_{\mathfrak{F}}G$,
(5) let $\{F_i\}_{i\in I}$ be an infinite countable ordered family of finite subgroups such that $|F_i| < |F_{i+1}|$. If $G = \ast_{i\in I} F_i$, then $G \in \mathfrak{U}_2 \setminus \mathfrak{U}_1$,
(6) for every n, the Houghton's groups $\mathfrak{H}_n \in \mathfrak{U}_2 \setminus \mathfrak{U}_1$,
(7) Dunwoody's inaccessible group $\mathfrak{D} \in \mathfrak{U}_3$ [**Dun93**].

Note that every group mentioned above lies in $\mathfrak{U}^\star_{\omega_0}$.

PROOF. By the Stallings-Swan theorem every free group has integral cohomological dimension equal to 1 and so the assertion is obvious. If $G \in \mathbf{LFr}$, then $\operatorname{cd} G \leqslant 2$, therefore $G \in \mathfrak{B}$.

It is known by [**Adi79**] that $B(m,n)$ are infinite for large enough exponent and that they have a bound on the orders of their $\mathfrak{F}$-subgroups. By [**Iva91**] they admit an action on a contractible 2-dimensional CW-complex with cyclic stabilisers and so they are contained in $\mathfrak{B} \setminus \mathfrak{I}$ and so in $\mathfrak{U}_1 \setminus \mathfrak{U}_0$. If $G \in \mathcal{N}^{cell}(\mathcal{P}_6)$, then either it contains a free subgroup on two generators or it is countable elementary amenable [**Pet09**, Theorem 3.9]. A finitely-generated infinite periodic group cannot be elementary amenable, therefore free Burnside groups of large odd exponent are not contained in $\mathcal{N}^{cell}(\mathcal{P}_6)$ but they belong to $\mathfrak{B}$.

Every finitely-generated $\mathfrak{A}$-group lies in $\mathfrak{B}$ and so every countable $\mathfrak{A}$-group G can be realised as group acting on a tree with finitely-generated $\mathfrak{A}$-stabilisers and so $G \in \mathfrak{U}_2$. For example, the free abelian group of infinite countable rank lies in $\mathfrak{U}_2 \setminus \mathfrak{U}_1$. Clearly $\mathfrak{B}$ contains all $\mathfrak{F}$-groups and $\mathfrak{U}$ is closed under taking countable directed unions and so it contains all countable $\mathbf{L}\,\mathfrak{F}$-groups. By Proposition 5.2, $\mathfrak{U}$ is closed under taking extensions and so it contains all countable elementary amenable groups.

Note that since every $\mathfrak{F}$-subgroup of a group G fixes a point of $E_{\mathfrak{F}}G$, it is contained in a 0-cell stabiliser. Therefore, if G admits a finite model for $E_{\mathfrak{F}}G$, then it has finitely many conjugacy classes of $\mathfrak{F}$-subgroups and therefore G has a bound on the orders of its $\mathfrak{F}$-subgroups. Gromov hyperbolic groups admit a finite model for $E_{\mathfrak{F}}$ by [**MS02**].

Let $G = \ast_{i \in I} F_i$ as above. Then G has no bound on the orders of its $\mathfrak{F}$-subgroups but it is realised as the fundamental group of a graph of $\mathfrak{F}$-groups, and so $G \in \mathfrak{U}_2 \setminus \mathfrak{U}_1$.

The group $\mathfrak{H}_n$ is isomorphic to an extension of the infinite countable finitary symmetric group Θ by $\mathbb{Z}^{n-1}$. The group Θ is countable [**DM96**, Exercise 8.1.3], moreover it lies in $\mathbf{L}\,\mathfrak{F}$ and so $\Theta \in \mathbf{F_1}\,\mathfrak{U}_1$ and $\mathfrak{H}_n \in \mathfrak{U}_2$.

The group $\mathfrak{D}$ is the fundamental group of a graph X of groups. Every edge group is finite and the only non-finite vertex group is isomorphic to a free product with amalgamation $Q_n \ast_{H_\omega} H$. Where Q_n is the fundamental group of an infinite graph of groups with all finite edge and vertex groups, H_ω is an infinite countable $\mathbf{L}\,\mathfrak{F}$-group ($H_\omega \in \mathfrak{U}_2 \setminus \mathfrak{U}_1$) and H is isomorphic to a semidirect product of the infinite finitary symmetric group on a countable set by an infinite cyclic group ($H \in \mathfrak{U}_2 \setminus \mathfrak{U}_1$). From the construction it is clear that $\mathfrak{D} \in \mathfrak{U}_3$ and $\operatorname{cd}_{\mathbb{Q}} \mathfrak{D} \leqslant 4$. Note that $\mathfrak{D}$ has no bound on the orders of its $\mathfrak{F}$-subgroups by construction or by Linnell's theorem on inaccessible groups [**Lin83**]. $\square$

Let $\underline{G}$ and G denote the finitely-generated groups constructed respectively in [**DJ98**] and [**DJ99**]. These have the following remarkable decomposition properties: $\underline{G} = A *_{\mathbb{Z}} \underline{G}$ and $G = G *_{\mathbb{Z}} G$.

Proposition 5.10. *The group $\underline{G}$ has finite $\mathfrak{F}$-cohomological dimension. It has a bound on the orders of its $\mathfrak{F}$-subgroups and so it belongs to $\mathfrak{B}$.*

PROOF. In [**Dun11**] it is shown that the group $\underline{G}$ can be realised as the fundamental group of a graph of groups Y with two orbits of vertices VY and two orbits of edges EY. Each $\mathfrak{F}$-subgroup of $\underline{G}$ must lie in one of the conjugates of the A factors. Since $A = \langle a, b \,|\, b^3 = 1, a^{-1}ba = b^{-1} \rangle$ every $\mathfrak{F}$-subgroup has order bounded by 3. Moreover, since $\underline{G}$ is the fundamental group of a graph of groups with all virtually cyclic stabilisers, the Mayer-Vietoris sequence for $\mathfrak{F}$-cohomology gives that $\mathfrak{F}\operatorname{cd} \underline{G} < \infty$. Hence $\underline{G} \in \mathfrak{B}$ and $\underline{G}$ has finite Bredon cohomological dimension. $\square$

This is in contrast with finitely-generated infinite groups of the form $\underline{G} = A \times \underline{G}$ (with A non-finite) by an argument similar to Theorem 4.4. We see that these groups must have infinite rational cohomological dimension.

Question 5.11. Are there countable $\mathbf{H}_1$ $\mathfrak{F}$-groups not contained in the class $\mathfrak{U}$?

Arithmetic groups over global function fields are countable $\mathbf{H}_1$ $\mathfrak{F}$-groups and we expect to be a suitable source for answering positively the question above.

Proposition 5.12. *Let T be a G-tree with edge set $E = \bigsqcup_{i \in I} L_i \backslash G$ and vertex set $V = \bigsqcup_{j \in J} N_j \backslash G$. Then $\mathfrak{F}\operatorname{cd}(G) \leqslant \sup\{\mathfrak{F}\operatorname{cd} L_i, \mathfrak{F}\operatorname{cd} N_j$ with $i \in I, j \in J\} + 1$.*

PROOF. This is an immediate consequence of the Mayer-Vietoris sequence in $\mathfrak{F}$-cohomology associated to the short exact $\mathfrak{F}$-split sequence: $\mathbb{Z}E \rightarrowtail \mathbb{Z}V \twoheadrightarrow \mathbb{Z}$. $\square$

Corollary 5.13. *If Nucinkis' conjecture holds, then there exists a function $\phi : \mathbb{N} \to \mathbb{N}$ such that $\operatorname{gd}_{\mathfrak{F}} G \leqslant \phi(\mathfrak{F}\operatorname{cd}(G))$ for every group G of finite $\mathfrak{F}$-cohomological dimension.*

PROOF. Assume by contradiction that there is no such function. Then there exists some $n > 1$ and a family of group $\{G\}_{i\in\mathbb{N}}$ such that $\mathfrak{F}\,\mathrm{cd}\,G_i \leqslant n$ for every i and $\mathrm{gd}_{\mathfrak{F}}\,G_i \to \infty$ for $i \to \infty$.
The group $G = \ast_{i\in\mathbb{N}}\,G_i$ has $\mathfrak{F}\,\mathrm{cd}\,G \leqslant n+1$ but $\mathrm{gd}_{\mathfrak{F}}\,G = \infty$ giving a contradiction. The inequality $\mathfrak{F}\,\mathrm{cd}\,G \leqslant n+1$ follows from Proposition 5.12. Since G contains subgroups of arbitrarily large Bredon geometric dimension we have $\mathrm{gd}_{\mathfrak{F}}\,G = \infty$. $\square$

Theorem 5.14. *Let T be a G-tree with edge set $E = \bigsqcup_{i\in I} L_i\backslash G$ and vertex set $V = \bigsqcup_{j\in J} N_j\backslash G$. Then $\mathfrak{K}(G) \leqslant \sup\{\mathfrak{K}(L_i)_{i\in I}, \mathfrak{K}(N_j)_{j\in J}\} + 1$. In particular $G \in \mathbf{H}_1\,\mathfrak{F}$ if and only if there is a bound on the Kropholler dimensions of the edge and vertex groups.*

PROOF. Replace the edge and vertex groups with suitable $\mathbf{H}_1\,\mathfrak{F}$-spaces of minimal dimension and proceed as in [**Lüc05**] to obtain an $\mathbf{H}_1\,\mathfrak{F}$-space for G. $\square$

Corollary 5.15. *If Kropholler-Mislin conjecture holds, then there exists a function $\gamma : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{gd}_{\mathfrak{F}}\,G \leqslant \gamma(\mathfrak{K}(G))$ for every $\mathbf{H}_1\,\mathfrak{F}$-group G.*

PROOF. It follows from Theorem 5.14 and it can be proved as in Corollary 5.12. $\square$

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