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STATISTICAL ENERGY ANALYSIS OF MULTIPLE, NON-CONSERVATIVELY COUPLED SYSTEMS

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In what follows, the analysis of Keane [1] for the energy flows around an arbitrary configuration of coupled multi-modal subsystems is extended to the case of non-conservative coupling. Here, the joints between any two subsystems are modelled by a spring *and* a damper, thus allowing for dissipation of energy to occur at the joint. The aim of this study is to give greater insight into the problem of energy flows through non-conservative couplings which has not been extensively discussed in the literature. Interest is focused on the effect of damping in the joints on the magnitudes of energy flows between, and energy levels in, each subsystem. The model derived is used to demonstrate the well known fact that selecting the correct level of damping in the joints surrounding a driven subsystem may cause a large percentage of the power input to the subsystem to be dissipated in these joints. This minimizes the overall power dissipated within the subsystems and thus the system total energy level. A Statistical Energy Analysis (SEA) model for non-conservatively coupled systems is then suggested, in which coupling damping loss factors are introduced into the main SEA energy balance equations to account for the energy dissipation in the joints. This model is shown to be exactly correct for the limiting case of weak coupling. Numerical examples which illustrate these various ideas are presented, with the use of parameters adopted in previous studies.

INTRODUCTION

Statistical Energy Analysis (SEA) is a method that has been developed to deal with the dynamic analysis of structures at high frequencies and where other methods, such as finite element analysis, become less practical. SEA deals with a complex system as being built up of a number of subsystems coupled together, where the interaction between these subsystems is defined in terms of time-averaged energy flows through the joints, while the vibrational response is characterized by the time-averaged vibrational energies of the various subsystems. The relationships between the energy flows and levels is expressed in terms of so-called "coupling loss factors", while the energy dissipation of each subsystem is related to its vibrational energy through viscous internal loss factors. Since the vibrational power input to each subsystem is in balance with the energy dissipated via its internal damping and the energy flowing through the joints directly connected to it, a set of algebraic linear simultaneous energy balance equations can be set up and inverted to

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obtain the subsystem vibrational energy levels, given suitable loss factors; see, for example, the book by Lyon [2].

SEA theory normally models the joints between subsystems as weak and conservative, but in real structures joints are not accurately modelled in this way. Experience shows that joints, as an unavoidable source of hysteresis, commonly dissipate more energy than structural damping; see, for example, the paper by Gaul [3]. As yet, only a few studies have been carried out on non-conservatively coupled subsystems. An early paper that considered this case is by Gersch [4], who extended the analysis of Scharton and Lyon [5] to cases involving non-conservative coupling. Later, Fahy and Yao [6] considered the energy flow between two non-conservatively coupled oscillators, and showed that the energy flows depend on the absolute energy values as well as on the difference between these energies, and that coupling damping of the same order as the internal damping is effective in controlling the energy of the indirectly driven subsystem. Sun *et al.* [7] extended this analysis to continuous structures coupled together. They showed that coupling damping increases the effective internal loss factors of the substructures and suggested modifications to the SEA energy balance equations. Chen and Soong [8] studied the case of non-conservatively coupled oscillators and showed that coupling damping may, in fact, increase the energy flow from the driven to the undriven oscillators, especially when the natural frequencies of the two oscillators are different.

In a more recent study, Beshara *et al.* [9] investigated the case of non-conservative coupling between two rods deterministically, using a modal approach. Exact expressions for the various energy flows were derived in terms of Green functions of the uncoupled rods. In that study the effects of the dissipative nature of the coupling were studied and the relationship between energy flows and energy levels recovered.

The main purpose of the present work is to extend the study of non-conservative coupling to the case of multiple subsystems coupled together. SEA of multiple subsystems has been studied before, but very few studies have considered the case of non-conservative couplings. Langley [10] considered the response of a multi-coupled dynamic system subject to random excitation using a Green function approach which allowed for a derivation of the main SEA equations in the most general case, provided that the coupling was conservative and weak. In another study, Langley [11] obtained exact expressions for the coupling loss factors in terms of the frequency- and space-averaged Green functions of the coupled systems, and showed that the conditions under which the exact theory works can be reduced to the standard SEA equations by introducing the assumption of weak coupling. In a later study, Keane [1] considered the case of an arbitrary configuration of rods coupled together, but in this analysis the expressions of the energy receptances were made in terms of the Green functions of the *uncoupled* systems, although only for the case of conservative coupling. Recently, Shankar and Keane [12] used a similar approach to predict the energy flows in a network of beams rigidly joined together, but again only considered conservative couplings.

In the present work, a network of rods joined together via non-conservative couplings is considered, following an approach similar to that adopted by Keane [1]: i.e., using the Green functions of the uncoupled subsystems to derive the various energy receptances of the system. This study demonstrates the effect of coupling damping on the various energy flows and energy levels. The results obtained are in agreement with the well known observation that selecting the correct level of damping in the joints directly surrounding a driven subsystem may cause the bulk of energy to be dissipated within these joints, thus bringing the total energy levels of the various subsystems down to minimal levels.

A SEA model is also suggested for two non-conservatively coupled subsystems, which accounts for the dissipation of energy at the joint. A modification to the main SEA energy balance equation is then proposed through the introduction of a set of "coupling damping loss factors" which describe the relationship between the energy dissipated at the joints and the subsystem energy levels. Finally, the results presented here show that the presence of coupling damping will affect the *in situ* values for loss factors obtained using the power injection method, leading to overestimated results, which is in agreement with the results obtained by using the approach presented by Cuschieri and Sun [13]. Furthermore, when coupling damping is ignored, the energy levels obtained by inverting the main SEA equations will be overestimated when the coupling damping is of the same order as the internal damping of the subsystems, which agrees well with the results of Fahy and Yao [6]. Therefore the inclusion of coupling damping loss factors would seem necessary in order to obtain reasonably accurate results in such cases.

2. THEORY

The basic approach used here is based on energy flow receptances, the aim being to write the response of a complex built-up system in terms of the responses of each individual subsystem and the properties of the joints. Hence, the expressions for the displacements of the joints are related to the displacements of the uncoupled elements through the Green functions of each individual subsystem and the properties of the joints. These expressions are then used to establish exact expressions for the energy flows through the various joints, the input power to each subsystem due to loading, the dissipated power due to coupling damping and the energy level of each individual subsystem.

Consider N subsystems and M point couplings, each of which comprises a spring and a damper, and the ends of which are connected together at the points labelled A and B respectively, as shown in Figure 1. Let $\{Y\}_A$ and $\{Y\}_B$be the displacements of the ends A and B from their mean positions due to forcing, and let $\{Y\}_{A0}$ and $\{Y\}_{B0}$ be the displacements of the ends A and B that would arise due to forcing alone in the absence of the couplings. Furthermore, let $\{Y\}_{AU}$ and $\{Y\}_{BU}$ be the displacements of the ends A and B due to the relative displacements between the ends A and B of the system while coupled, but excluding the effects of external forcing. These quantities are given by

$$\{Y\}_{AU} = [A]_A(\{Y\}_{BU} - \{Y\}_{AU}) \qquad \text{and} \qquad \{Y\}_{BU} = [A]_B(\{Y\}_{AU} - \{Y\}_{BU}), \quad (1, 2)$$

where $[A]_A$is a matrix of uncoupled subsystem Green functions and complex coupling strength products such as $\Omega_i g_a(x_1, x)$ where the Ω_i of each coupling is given $\Omega_i = K_i + i\gamma_i\,\omega$, and where the K_i and γ_i are, respectively, the spring and damper strength of the ith

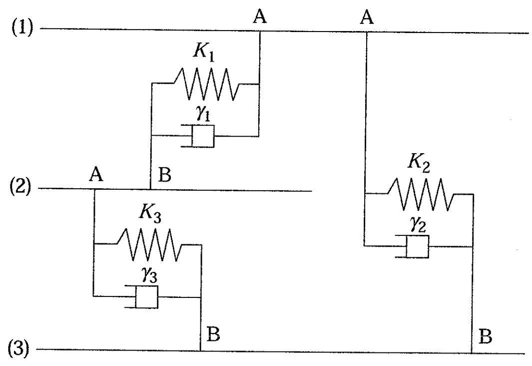

Figure 1. Three, damper and spring coupled, multi-modal subsystems.

coupling. Here $g_a(x_i, x)$ is the Green function that relates the displacement of subsystem a at x_i due to a unit harmonic force applied at the point x when uncoupled from all other subsystems.

When the external forces and the relative displacement between the ends are superposed, then the overall displacements of the ends A and B of the system are given by

$$\{Y\}_A = \{Y\}_{A0} - [A]_A(\{Y\}_A - \{Y\}_B) \qquad \text{and} \qquad \{Y\}_B = \{Y\}_{B0} + [A]_B(\{Y\}_A - \{Y\}_B). \tag{3, 4}$$

Solving these two equations simultaneously gives the total displacements of the ends A and B of the couplings in terms of the displacements of the uncoupled systems and their properties,

$$\{Y\}_A = [([I] + [A]_B) | [A]_A] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix} \begin{Bmatrix} \{Y\}_{A0} \\ \{Y\}_{B0} \end{Bmatrix} \tag{5}$$

and

$$\{Y\}_B = [[A]_B | ([I] + [A]_A)] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix} \begin{Bmatrix} \{Y\}_{A0} \\ \{Y\}_{B0} \end{Bmatrix}, \tag{6}$$

where

$$[D] = [I] + [A]_A + [A]_B. \tag{7}$$

These expressions for the total displacements are the same as those derived by Keane [1], but here a complex coupling strength is used which includes contributions for both the coupling springs and dampers. Notice that in the case of weak coupling the elements of $[A]_A$ and $[A]_B$ will be small compared to unity. In this case the vector of displacements $\{Y\}_A$ is approximately equal to the vector of displacements in the uncoupled situation, $\{Y\}_{A0}$, since in this case the coupling has very little effect on the total displacements. The condition $|D| \approx 1$ can therefore be used as a criterion for weak coupling.

The analysis next proceeds by finding expressions for the energy flows and energy levels of each subsystem when the system is subject to random forcing.

3. ENERGY FLOWS AND LEVELS

Energy flows can be determined from the product of force and velocity at the ends of the couplings. Since the forcing considered is random in nature, the analysis is carried out in the frequency domain. Hence, the energy flow spectra will be derived in terms of the spectral densities of the forcing, which are assumed to be ergodic random processes with specified spectral densities.

Let $\Pi_{COUP\,Ai}$ be the power that enters coupling i at end A and let $\Pi_{COUP\,Bi}$ be the power that enters coupling i at end B. Obviously, those two quantities are not equal in magnitude because of the energy dissipation within the coupling. If the energy dissipated within the ith coupling damper is denoted by Π_{DCi} then, from the energy balance of the coupling, it can easily be concluded that

$$\Pi_{DCi} = \Pi_{COUP\,Ai} + \Pi_{COUP\,B_i}. \tag{8}$$

3.1. ENERGY FLOWS THROUGH THE COUPLING SYSTEMS

It has been shown by Beshara *et al.* [9] that the expression for the energy flow $\Pi_{\mathrm{COUP\,A}i}$ from the end A of coupling system i, which comprises a spring and a viscous damper, can be given in the frequency domain in terms of the cross-spectral densities of the displacements at the two ends of the coupling by

$$\Pi_{\mathrm{COUP\,A}i} = \mathrm{Re}\{\mathrm{i}\Omega_i \omega S_{Y_{\mathrm{A}i}Y_{\mathrm{B}i}}(\omega)\} + \gamma_i \omega^2 S_{Y_{\mathrm{A}i}Y_{\mathrm{A}i}}(\omega). \tag{9}$$

It is also well known that the cross-spectral density $S_{Y_{\mathrm{A}i}Y_{\mathrm{B}i}}(\omega)$ is given by $\lim_{T\to\infty}[Y^*_{\mathrm{A}i}(\omega)Y_{\mathrm{B}i}(\omega)]\ 2\pi/T$, and this means that in order to obtain the vector of energy flows $\{\Pi_{\mathrm{COUP\,A}i}\}$, only the diagonal elements of the product $\{Y\}^*_{\mathrm{A}}\{Y\}^{\mathrm{T}}_{\mathrm{B}}$ and the product $\{Y\}^*_{\mathrm{A}}\{Y\}^{\mathrm{T}}_{\mathrm{A}}$ need be considered. The first is written in terms of the displacements of the uncoupled system, i.e., due to external forcing alone, as

$$\{Y\}^*_{\mathrm{A}}\{Y\}^{\mathrm{T}}_{\mathrm{B}} = [([I] + [A]_{\mathrm{B}})^* | [A]^*_{\mathrm{A}}] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^* \begin{Bmatrix} \{Y\}^*_{\mathrm{A}0} \\ \{Y\}^*_{\mathrm{B}0} \end{Bmatrix}$$

$$\times \{\{Y\}^{\mathrm{T}}_{\mathrm{A}0} | \{Y\}^{\mathrm{T}}_{\mathrm{B}0}\} \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^{\mathrm{T}} [[A]_{\mathrm{B}} | ([I] + [A]_{\mathrm{A}})]^{\mathrm{T}}. \tag{10}$$

A similar expression can be produced for the product $\{Y\}^*_{\mathrm{A}}\{Y\}^{\mathrm{T}}_{\mathrm{A}}$. As has been mentioned earlier, the aim of this analysis is to find expressions for the energy flows in terms of the responses of the uncoupled problem, $\{Y\}_{\mathrm{A}0}$ and $\{Y\}_{\mathrm{B}0}$, which can then be written in terms of the Green functions of the individual subsystems and the forcing functions acting on these subsystems. If end A of coupling i is connected to subsystem a at x_i, then the displacement at the point of coupling due to a forcing function $F_a(x, t)$ which acts on this subsystem while uncoupled is given by

$$Y_{\mathrm{A}0i} = \int_a g_a(x_i, x) F_a(x, t)\, \mathrm{d}x, \tag{11}$$

where $F_a(x, t)$ gives the spatial distribution of the forcing acting on this subsystem (which is assumed to be a random function of time). If the forcing is separable in space and time, such that $F_a(x, t) = F_a(t) f_a(x)$, this relation can be written as

$$Y_{\mathrm{A}0i} = F_a(t) \int_a g_a(x_i, x) f_a(x)\, \mathrm{d}x. \tag{12}$$

$Y_{\mathrm{B}0i}$ can be defined similarly. Hence, the vector of the displacements of the uncoupled system, in terms of the Green functions of the individual subsystems and the external forcing, is given by

$$\begin{Bmatrix} \{Y\}_{\mathrm{A}0} \\ \{Y\}_{\mathrm{B}0} \end{Bmatrix} = [gf]\{F\}, \tag{13}$$

where the matrix $[gf]$ contains the integrals for the Green functions and forces of the individual subsystems taken over the subsystems. Each column corresponds to one subsystem, with the upper half corresponding to the ends A of the coupling systems and the lower half to the ends B. Hence,

$$\left\{\frac{\{Y\}^*_{A0}}{\{Y\}^*_{B0}}\right\}\{\{Y\}^T_{A0} | \{Y\}^T_{B0}\} = [gf]^*[S_{FF}][gf]^T, \tag{14}$$

where $[S_{FF}]$ is the matrix of the various cross-spectra of the forces acting on the various subsystems. The product $\{Y\}^*_A\{Y\}^T_B$ can then be written in terms of the spectral densities of the forcing as

$$\{Y\}^*_A\{Y\}^T_B = [([I] + [A]_B)^* | [A]^*_A] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^*$$

$$\times [gf]^*[S_{FF}][gf]^T \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^T [[A]_B | ([I] + [A]_A)]^T. \tag{15}$$

Similarly,

$$\{Y\}^*_A\{Y\}^T_A = [([I] + [A]_B)^* | [A]^*_A] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^*$$

$$\times [gf]^*[S_{FF}][gf]^T \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^T [([I] + [A]_B) | [A]_A]^T. \tag{16}$$

When the forces on the different subsystems are incoherent, the matrix $[S_{FF}]$ is diagonal and the following equation is then valid:

$$[gf]^*[S_{FF}][gf]^T = \sum_{a=1}^{N} [gf]^*_a [gf]^T_a S_{F_aF_a}. \tag{17}$$

Hence, the energy flow from end A of the ith coupling can be written in terms of the spectral densities of the forcing on each subsystem as

$$\Pi_{\mathrm{COUP}\,Ai} = \sum_{a=1}^{N} H_{Aia} S_{FaF_a}, \tag{18}$$

where

$$H_{\mathrm{A}ia} = \mathrm{Re}\left\{ \mathrm{i}\omega\Omega_i [([I] + [A]_{\mathrm{B}})^* | [A]^*_{\mathrm{A}}] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^*_i \right.$$

$$\times\ [gf]^*_a [gf]^{\mathrm{T}}_a \left[\begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^{\mathrm{T}} [[A]_{\mathrm{B}} | ([I] + [A]_{\mathrm{A}})]^{\mathrm{T}} \right]_i$$

$$+ \gamma_i \omega^2 \left[[([I] + [A]_{\mathrm{B}})^* | [A]^*_{\mathrm{A}}] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^* \right]_i$$

$$\left. \times\ [gf]^*_a [gf]^{\mathrm{T}}_a \left[\begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^{\mathrm{T}} [([I] + [A]_{\mathrm{B}}) | [A]_{\mathrm{A}}]^{\mathrm{T}} \right]_i \right\}. \quad (19)$$

Here H_{Aia} is the energy flow receptance from end A of the ith coupling due to forcing on the ath subsystem. If the end A of the ith coupling is not directly connected to the ath subsystem, the energy flow will be indirectly caused by the couplings between the various different subsystems.

In cases in which the forcing is not spatially separable within the individual subsystem but is still incoherent between subsystems, then

$$[gf]^*_a [gf]^{\mathrm{T}}_a = \iint_a g^*_a(x_l, x) g_a(x_m, y) f_{aa}(x, y)\, \mathrm{d}x\, \mathrm{d}y, \quad (20)$$

where $f_{aa}(x, y)$ is then the spatial distribution of the forcing function over subsystem a. In the case of "rain-on-the-roof" driving, $f_{aa}(x, y)$ is given by $f_{aa}(x, y) = \delta(x - y)(\rho/M)$ (ρ is the density and M is the mass); while in the case of point driving at x_0, $f_{aa}(x, y) = \delta(x - x_0)\delta(y - x_0)$. If the previous double integral is denoted by $[Q]_a$, then the receptance $H_{\mathrm{A}ia}$ may be written as,

$$H_{\mathrm{A}ia} = \mathrm{Re}\left\{ \mathrm{i}\omega\Omega_i \left[[([I] + [A]_{\mathrm{B}})^* | [A]^*_{\mathrm{A}}] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^* \right]_i \right.$$

$$\times [Q]_a \left[\begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^{\mathrm{T}} [[A]_{\mathrm{B}} | ([I] + [A]_{\mathrm{A}})]^{\mathrm{T}} \right]_i + \gamma_i \omega^2 \left[[([I] + [A]_{\mathrm{B}})^* | [A]^*_{\mathrm{A}}] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^* \right]_i$$

$$\left. \times [Q]_a \left[\begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^{\mathrm{T}} [([I] + [A]_{\mathrm{B}}) [A]_{\mathrm{A}}]^{\mathrm{T}} \right]_i \right\}. \quad (21)$$

Similarly, the energy flows from the ends B of the ith coupling system can be written as

$$\Pi_{\mathrm{COUP\,B}_i} = \sum_{a=1}^{N} H_{\mathrm{B}ia} S_{F_a F_a}, \quad (22)$$

where $H_{\mathrm{B}ia}$ is the energy flow receptance from end B of the ith coupling system due to forcing on subsystem a, and is given by the expression

$$H_{\mathrm{B}ia} = \mathrm{Re}\left\{ \mathrm{i}\omega\Omega_i \left[[[A]_{\mathrm{B}}^* \,|\, ([I] + [A]_{\mathrm{A}}^*)] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^* \right]_i \right.$$

$$\times [Q]_a \left[\begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^{\mathrm{T}} [([A]_{\mathrm{B}} + [I]) \,|\, [A]_{\mathrm{A}}]^{\mathrm{T}} \right]_i + \gamma_i \omega^2 \left[[[A]_{\mathrm{B}}^* \,|\, ([I] + [A]_{\mathrm{A}}^*)] \begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^* \right]_i$$

$$\left. \times [Q]_a \left[\begin{bmatrix} [D]^{-1} & [0] \\ [0] & [D]^{-1} \end{bmatrix}^{\mathrm{T}} [[A]_{\mathrm{B}}^* \,|\, ([I] + [A]_{\mathrm{A}})]^{\mathrm{T}} \right]_i \right\}. \tag{23}$$

The dissipated power within the ith coupling system is then just

$$\Pi_{\mathrm{DC}i} = \sum_{a=1}^{N} (H_{\mathrm{A}ia} + H_{\mathrm{B}ia}) S_{F_a F_a}. \tag{24}$$

3.2. INPUT POWER

The input power due to forcing on subsystem a is calculated from the product of the force applied to the subsystem and velocity at the point of application and is given in the frequency domain by

$$\Pi_{\mathrm{IN}a} = \mathrm{Re}\left\{ \int_a -\mathrm{i}\omega Y_a^*(x) F_a(\omega) f_a(x)\, \mathrm{d}x \right\} = -\omega\, \mathrm{Im}\left\{ \int_a Y_a(x) F_a(\omega) f_a(x)\, \mathrm{d}x \right\}, \tag{25}$$

where $Y_a(x)$ is the total displacement at the point x on subsystem a, which is equal to the displacement due to the external forcing acting directly on subsystem a plus the displacement due to the forces acting on the subsystem through the couplings directly connected to it. The input power due to external forcing acting directly on subsystem a is given by

$$\Pi_{\mathrm{IN}a0} = -\omega\, \mathrm{Im}\left\{ \iint_a g_a(x_i, x_j) f_{aa}(x_i, x_j)\, \mathrm{d}x_i\, \mathrm{d}x_j \right\} S_{F_a F_a}, \tag{26}$$

where the forces acting on subsystem a are again assumed to be incoherent. The input power due to all the couplings with the other subsystems is given by

$$\Pi_{\mathrm{IN}ac} = -\omega\, \mathrm{Im}\left\{ \pm \sum_{l=1}^{M} \sum_{m=1}^{M} \Omega_l [D]_{lm}^{-1} \iint_a g_a(x, x_l) g_a(x_m, y) f_{aa}(x, y)\, \mathrm{d}x\, \mathrm{d}y \right\} S_{F_a F_a}, \tag{27}$$

where the summations are preceded by " $-$ " if the two ends of the coupling at x_l and x_m are both A or both B ends, and preceded by " $+$ " if they are one A and one B (note that if a coupling l or m is not directly connected to subsystem a, the relevant terms in the summations are set to zero: i.e., $g_a(x, x_l) = 0$ if coupling l is not directly coupled to subsystem a, etc.). The input power into subsystem a is thus given by

$$\Pi_{\mathrm{IN}a} = H_{\mathrm{IN}a} S_{F_a F_a}, \tag{28}$$

where the input energy flow receptance is

$$H_{\mathrm{IN}a} = -\omega \,\mathrm{Im}\left\{\iint_a g_a(x_i, x_j) f_{ss}(x_i, x_j)\,\mathrm{d}x_i, \mathrm{d}x_j \right.$$

$$\left. \pm \sum_{l=1}^{M} \sum_{m=1}^{M} \Omega_l [D]^{-1}_{lm} \iint_a g_a(x, x_l) g_a(x_m, y) f_{aa}(x, y)\,\mathrm{d}x\,\mathrm{d}y \right\}. \tag{29}$$

3.3. ENERGY LEVELS

If the vector of powers leaving the subsystems is denoted by $\{\Pi_{\mathrm{OUT}}\}$, then this vector can be related to the vector of the energies flowing through the coupling systems at the ends A and B

$$\left\{\frac{\{\Pi_{\mathrm{COUP\,A}}\}}{\{\Pi_{\mathrm{COUP\,B}}\}}\right\},$$

by the relation

$$\{\Pi_{\mathrm{OUT}}\} = [CON]\left\{\frac{\{\Pi_{\mathrm{COUP\,A}}\}}{\{\Pi_{\mathrm{COUP\,B}}\}}\right\}. \tag{30}$$

The elements of the connectivity matrix $[CON]$ are either 1 or 0, and indicate the presence or absence of a direct coupling between two subsystems. Hence the energy dissipated within each subsystem due to damping can easily be deduced from the energy balance for each subsystem as

$$\{\Pi_{\mathrm{DISS}}\} = \{\Pi_{\mathrm{IN}}\} - \{\Pi_{\mathrm{OUT}}\}. \tag{31}$$

Finally, the energy levels for each subsystem can be related to the energy dissipated due to damping by the well known relation

$$\{E\} = [c]^{-1}\{\Pi_{\mathrm{DISS}}\}, \tag{32}$$

where $[c]$ is the diagonal matrix of damping constants for each subsystem, the damping of each subsystem being assumed to be viscous and mass proportional

4. STATISTICAL ENERGY ANALYSIS MODELS

According to the preceding deterministic analysis, all the energy flows in a coupled system can be written in terms of the spectral densities of the driving forces acting on each subsystem as

$$\{\Pi_{\mathrm{COUP\,A}}\} = [H_{\mathrm{A}}]\{S_{F_aF_a}\}, \qquad \{\Pi_{\mathrm{COUP\,B}}\} = [H_{\mathrm{B}}]\{S_{F_aF_a}\}, \tag{33, 34}$$

$$\{\Pi_{\mathrm{OUT}}\} = [CON]\left[\frac{[H_{\mathrm{A}}]}{[H_{\mathrm{B}}]}\right]\{S_{F_aF_a}\}, \tag{35}$$

or

$$\{\Pi_{\mathrm{OUT}}\} = [H_{\mathrm{OUT}}]\{S_{F_aF_a}\}. \tag{36}$$

Here H_{Aij} and H_{Bij} are the receptances describing the energies that leaves ends A and B, respectively, of coupling $i$due to forcing applied on subsystem j. Likewise, H_{OUTij} describes the energy that leaves subsystem i due to forcing on subsystem j. Also, the vector of power input to each subsystem can be related to the forcing spectra as

$$\{\Pi_{IN}\} = [H_{IN}]\{S_{F_aF_a}\}, \quad (37)$$

where $[H_{IN}]$ is a diagonal matrix of dimensions $N \times N$. Hence, the vector of dissipated power within each subsystem can be written as

$$\{\Pi_{DISS}\} = [[H_{IN}] - [H_{OUT}]]\{S_{F_aF_a}\}. \quad (38)$$

Clearly, the vector of forcing spectra can be related to the subsystem energy levels by

$$\{S_{F_aF_a}\} = [[H_{IN}] + [H_{OUT}]]^{-1}\{\Pi_{DISS}\} = [[H_{IN}] - H_{OUT}]]^{-1}[c]\{E\}, \quad (39)$$

and this allows the energy flow vectors and the input power vector to be written in terms of the energy levels vector, by

$$\{\Pi_{COUP\,A}\} = [H_A][[H_{IN}] - [H_{OUT}]]^{-1}[c]\{E\} \quad \text{and} \quad \{\Pi_{COUP\,B}\} = [H_B][[H_{IN}] - [H_{OUT}]]^{-1}[c]\{E\}, \quad (40, 41)$$

or

$$\{\Pi_{COUP\,A}\} = [\beta_A][c]\{E\} \quad \text{and} \quad \{\Pi_{COUP\,B}\} = [\beta_B][c]\{E\}. \quad (42)$$

Also

$$\{\Pi_{OUT}\} = [H_{OUT}][[H_{IN}] + [H_{OUT}]]^{-1}[c]\{E\} \quad \text{or} \quad \{\Pi_{OUT}\} = [\alpha][c]\{E\}. \quad (43)$$

Now, obviously,

$$[\alpha] = [CON]\begin{bmatrix}[\beta_A]\\ [\beta_B]\end{bmatrix}, \quad (44)$$

or, alternatively,

$$\alpha_{ij} = \sum_{k=1}^{M} CON_{ik}\beta_{Akj} + \sum_{k=1}^{M} CON_{(i,k+M)}\beta_{Bkj}, \quad (45)$$

so that, finally,

$$\{\Pi_{IN}\} = [[I] + [\alpha]][c]\{E\}. \quad (46)$$

Here $[\beta_A]$ and $[\beta_B]$ are matrices of dimension $M \times N$, where $\beta_{Aij}c_j$ denotes the constant of proportionality between the energy flowing from end A of coupling i and the energy level of subsystem j. Also, $\beta_{Bij}c_j$ denotes the constant of proportionality between the energy flowing from end B of coupling i and the energy level of subsystem j. Notice that, in general, the energy leaving coupling system i at either end is related to the energy level of all subsystems comprising the complete system, and not only those two to which it is directly connected. Similarly, $\alpha_{ij}c_j$ denotes the constant of proportionality between the power leaving subsystem i and the energy level of subsystem j. In general, the energy leaving any subsystem i is related to the energy levels of *all* subsystems—not only the subsystems to which the coupling is directly connected.

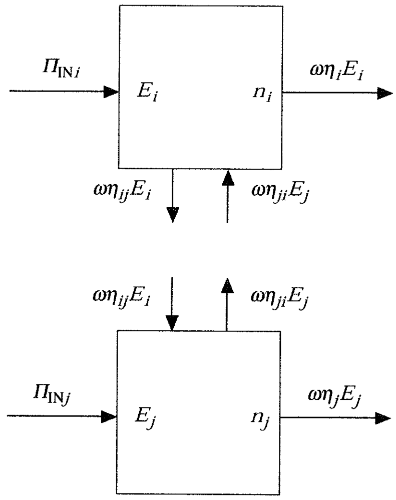

Figure 2. An energy flow model for two spring coupled subsystems.

5. CONSERVATIVE COUPLING AND SEA

For the case of conservative coupling, where all the γ_i in the previous analysis are zero, it is clear that the sum of the input powers to the whole system must be dissipated through the internal damping of each subsystem. This means that the sum of the vector $\{\Pi_{OUT}\}$ over the N subsystems must also be zero and so the sum of the elements of each column of the matrix $[\alpha]$ must be zero. This leads to

$$\alpha_{ii} = -\sum_{j \neq i}^{N} \alpha_{ji}, \qquad i = 1, \ldots, N. \tag{47}$$

When equation (47) is substituted into equation (46), an exact set of linear equations similar in structure to the well known SEA energy balance equations can be recovered, where the input power to subsystem i in terms of the energy levels of the various subsystems is given by

$$\Pi_{INi} = c_i E_i - \sum_{j \neq i}^{N} (\alpha_{ji} c_i E_i - \alpha_{ij} c_j E_j). \tag{48}$$

In the fundamental hypothesis of SEA, ensemble averages are taken over broadly similar systems which differ from each other in some minor way. Since all the processes involved are assumed to be ergodic these ensemble averages are usually assumed to be identical to the averages taken on one realization over a frequency bandwidth centred at the frequency of interest (here average quantities are denoted $\hat{}$). It may be recalled that, in SEA theory, two subsystems i and j coupled together are modelled as shown in Figure 2. The ensemble average of energy flow from subsystem i to subsystem j, $\hat{\Pi}_{ij}$, is assumed to be related only to the ensemble average of the energy levels of subsystems i and j throug h what are known as coupling loss factors by the relation

$$\hat{\Pi}_{ij} = \omega\eta_{ij}\hat{E}_i - \omega\eta_{ji}\hat{E}_j \tag{49}$$

and the main energy balance equation for conservative coupling then becomes

$$\hat{\Pi}_{\mathrm{IN}i} = \omega\eta_i\hat{E}_i + \sum_{j\neq i}(\omega\eta_{ij}\hat{E}_i - \omega\eta_{ji}\hat{E}_j). \tag{50}$$

It is clear that when an ensemble average is taken in equation (48), then it will be equivalent to the SEA equations if and only if

$$\omega\eta_{ji} = -c_j\hat{\alpha}_{ij}. \tag{51}$$

In general, this equation does not hold, since $c_j\hat{\alpha}_{ij}$ is the average constant of proportionality between the power leaving subsystem i and the energy level of subsystem j, and is related to the properties of *all* of the couplings directly connected with subsystem i rather than just the one that couples i and j, while $-\omega\eta_{ji}$ is the constant of proportionality between the energy flow through the coupling between i and j and the energy level of subsystem j only, and is identical, by definition, to $c_j\hat{\beta}_{\mathrm{A}kj}$ taking the coupling to be connected with i at the point A_k (or to $c_j\hat{\beta}_{\mathrm{B}kj}$ if the coupling is connected with i at the point B_k). This means that equation (51) is satisfied if

$$\hat{\beta}_{\mathrm{A}kj}(\text{or } \hat{\beta}_{\mathrm{B}kj}) = \hat{\alpha}_{ij} = \sum_{k=1}^{M} CON_{ik}\hat{\beta}_{\mathrm{A}kj} + \sum_{k=1}^{M} CON_{(i,k+M)}\hat{\beta}_{\mathrm{B}kj}; \tag{52}$$

cf., equations (45) and (46). For a system comprising just two subsystems coupled together, the terms $c_j\hat{\alpha}_{ij}$ and $\omega\eta_{ji}$wil l be identical, but problems arise for a system comprising more than two subsystems, such as that in Figure 1, and therefore SEA estimates will be in error even when the coupling loss factors are accurately estimated, either from the physical details of the couplings or by measurements on the subsystems. Moreover, such errors will arise whether or not the measurements are made *in situ* or on pairs of isolated subsystems. However, for the case of weak coupling, it can be shown that the energy flow through the coupling between i and j is mainly governed by the energy levels of the subsystems i and j, and equation (52) is then satisfied with little error, giving the well known result that SEA estimates tend to be in good agreement with exact results for cases of weak coupling.

6. NON-CONSERVATIVE COUPLING

When dealing with the case of non-conservative coupling, the general set of equations (42) and (43) is still applicable, but now there is an additional set of equations resulting from the energy balance at each coupling between the power dissipated in the coupling dampers and the energy flowing into the ends of the couplings.

In the previous section it was shown that

$$\{\Pi_{\mathrm{DC}}\} = [][H_{\mathrm{A}}] + [[H_{\mathrm{B}}]]\{S_{F_aF_a}\} = [H_{\mathrm{DC}}]\{S_{F_aF_a}\}, \tag{53}$$

and it follows that

$$\{\Pi_{\mathrm{DC}}\} = [H_{\mathrm{DC}}][[H_{\mathrm{IN}}] - [H_{\mathrm{OUT}}]]^{-1}[c]\{E\}, \tag{54}$$

or

$$\{\Pi_{\mathrm{DC}}\} = [\Gamma][\ ]\{E\}, \tag{55}$$

where $[\Gamma]$ is a matrix of dimension $M \times N$ and where $c_j\Gamma_{ij}$ is the constant of proportionality between the power dissipated in coupling i and the energy level of subsystem j, and is equal to $c_j(\beta_{\mathrm{A}ij} + \beta_{\mathrm{B}ij})$. Clearly, the power dissipated in any coupling is, in general, related to the

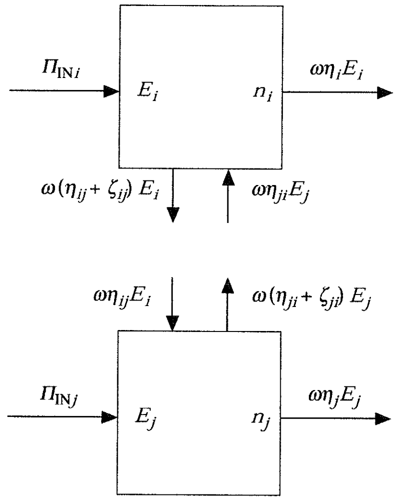

Figure 3. An energy flow model for two damper and spring coupled subsystems.

energy levels of all the subsystems comprising the system and not only to those two to which it is directly connected.

Energy balance for the overall system requires that the sum of the input powers to all the subsystems is equal to the sum of the energies dissipated in each subsystem plus the sum of the energies dissipated in the various couplings. This means that

$$\sum_{i=1}^{N} \{\Pi_{IN}\}_i = \sum_{i=1}^{N} \{\Pi_{DISS}\}_i + \sum_{k=1}^{M} \{\Pi_{DC}\}_k, \qquad (56)$$

TABLE 1

Parameter values for the example subsystems

Subsystem	Mass, M (kg)	Length, L (m)	Rigidity, AE (MN)	Damping, c (s^{-1})	Modal overlap factor
1	21·5346	5·182	17·85	88·95	0·07
2	17·9872	4·328	17·85	106·4	0·07
3	19·8698	4·781	17·85	93·75	0·07

TABLE 2

Parameter values for the example couplings

Coupling	Strength (N/m)	A end		B end	
		Subsystem	Position (m)	Subsystem	Position (m)
1	861·4	1	0·6094	2	3·047
2	5437·2	1	5·031	3	4·78
3	65·3	2	2·227	3	2·0

which leads to

$$\sum_{i=1}^{N} \{\Pi_{OUT}\}_i = \sum_{k=1}^{M} \{\Pi_{DC}\}_k, \qquad \text{or, alternatively,} \qquad \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_{ij} c_j E_j = \sum_{k=1}^{M}\sum_{j=1}^{N} \Gamma_{kj} c_j E_j, \tag{57, 58}$$

which in turn leads to

$$\sum_{i=1}^{N} \alpha_{ij} = \sum_{k=1}^{M} \Gamma_{kj} \quad \text{for } j = 1, 2, \ldots, N \qquad \text{and} \qquad \alpha_{jj} = \sum_{k=1}^{M} \Gamma_{kj} - \sum_{i \neq j}^{N} \alpha_{ij}, \tag{59, 60}$$

so the input power to subsystem i in terms of the energy levels of the subsystems then becomes

$$\Pi_{INi} = c_i E_i + \left(\sum_{k=1}^{M} \Gamma_{ki} - \sum_{j \neq i}^{N} \alpha_{ji} \right) c_i E_i + \sum_{j \neq i}^{N} \alpha_{ij} c_j E_j. \tag{61}$$

For the case of weak coupling, the power dissipated in any coupling is again mainly governed by the energy levels of the two subsystems to which it is directly connected: i.e., Γ_{ki} is approximately zero if coupling k is not directly connected to subsystem i. Hence, for weak coupling the following equation is valid with little error:

$$\Pi_{DCk} = \Gamma_{ki} c_i E_i + \Gamma_{kj} c_j E_j. \tag{62}$$

Here i and j are the two subsystems directly connected by coupling k. In this case, if M_i is the number of couplings connected to subsystem i, then

$$\alpha_{ii} = \sum_{k=1}^{M_i} \Gamma_{ki} - \sum_{j \neq i}^{N} \alpha_{ji}. \tag{63}$$

By assuming that $\omega\zeta_{ij} = c_i \hat{\Gamma}_{ki}$ (where the jth subsystem is connected to subsystem i via coupling k) and introducing equation (51), the energy flow through a non-conservative coupling system connecting i and j where the coupling is weak is approximately given by

$$\hat{\Pi}_{ij} = \omega(\eta_{ij} + \zeta_{ij})\hat{E}_i - \omega\eta_{ji}\hat{E}_j. \tag{64}$$

This result justifies modelling a non-conservative joint between two subsystems i and j which form part of a complex system in the way illustrated in Figure 3. Here ζ_{ij} are a set of additional loss factors representing the effects of coupling damping on energy transfer, which may be referred to as "coupling damping loss factors", being defined in a similar way to the normal coupling loss factors. The set of energy balance equations for the case of weak non-conservative coupling may then be written as

$$\hat{\Pi}_{INi} = \omega\eta_i \hat{E}_i + \sum_{j \neq i} (\omega(\eta_{ij} + \zeta_{ij})\hat{E}_i - \omega\eta_{ji}\hat{E}_j). \tag{65}$$

To summarize, for the case of weak coupling, the effect of coupling damping may be included in SEA through the introduction of a set of coupling damping loss factors which are defined in a similar manner to the traditional coupling loss factors. The preceding discussion also shows that, although the general form of the SEA equations may be recovered deterministically by using formulae for the "coupling loss factors" in terms of

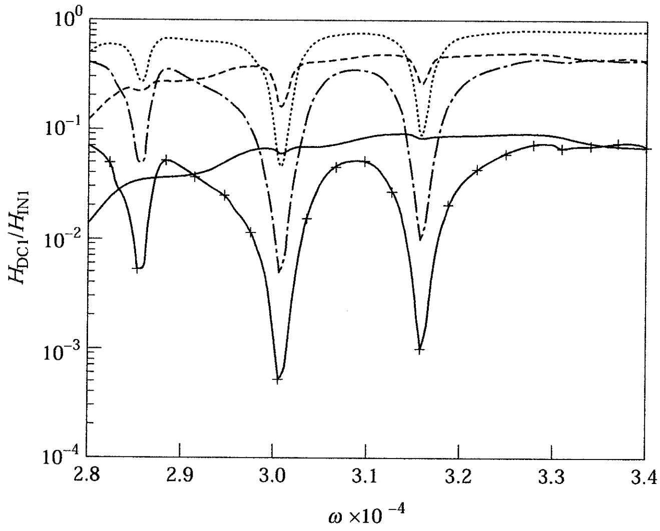

Figure 4. Variation in H_{DC1}/H_{IN1} with ωan d γ_1 for $\gamma_2 = \gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1. ——, $\gamma_1 = 100$ Ns/m; - - -, $\gamma_1 = 1000$ Ns/m; · · ·, $\gamma_1 = 10\,000$ Ns/m; — · — $\gamma_1 = 100\,000$ Ns/m; − + −, $\gamma_1 = 1\,000\,000$ Ns/m.

the various energy receptances, these factors are heavily dependent on the configuration of the system and the properties of all the couplings within it. However, in SEA theory, the coupling loss factor is simply a constant of propotionality between the energy flow in a coupling and the energy levels of the two subsystems on either side of the coupling: one of the main ideas behind SEA is to estimate these coupling loss factors for models consisting of only two subsystems coupled together and then to apply these factors to more complex sets of subsystems, so as to obtain the energy levels of the various subsystems based on simply the knowledge of the input powers and estimates of the internal damping

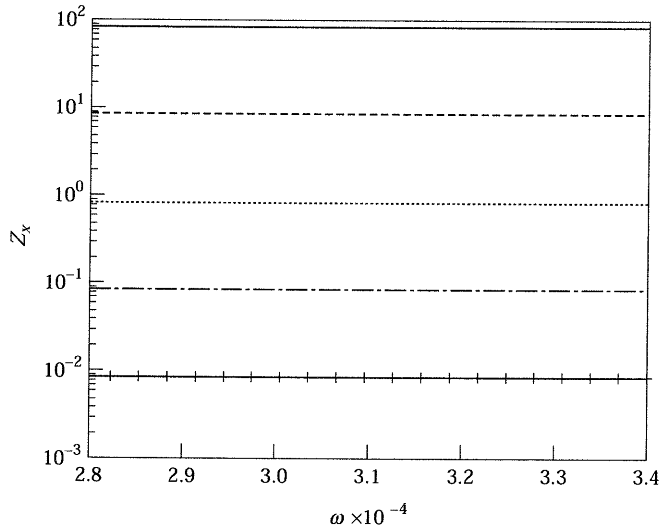

Figure 5. Variation in Z_x with ω and γ_1 for $\gamma_2 = \gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

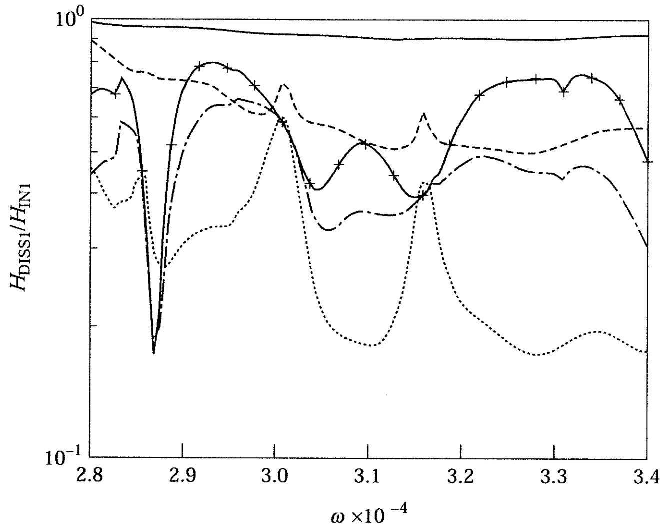

Figure 6. Variation in H_{DISS1}/H_{IN1} with ω and γ_1 for $\gamma_2 = \gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

of the subsystems; see for example, the book by Lyon [2] or that by Norton [14]. It is clear from the preceding analysis that, for strong coupling, this will definitely lead to significant errors no matter with what accuracy the coupling loss factors are determined. However, for cases of weak coupling an alternative formulation of the normal SEA equations may be proposed using "coupling damping loss factors" that enables the traditional form of the SEA equations to be preserved. Moreover, the analysis used here allows for an assessment of the error that may result from adopting these SEA assumptions, and this permits the introduction of coupling damping in a way that allows for an assessment of the magnitude of error that may arise from neglecting such complications in applications of SEA.

7. EXAMPLES

The theory presented in the previous sections is next illustrated through its application to two sets of numerical examples. The parameters values adopted here are the same as those used in a number of previous studies so as to aid comparison of the various results; seet Tables 1 and 2. The topology of the system is illustrated in Figure 1. In the first set of examples rod 1 is loaded by unit "rain-on-the-roof" forcing and the damper of coupling 1 is given increasing strength while the other two couplings remain conservative: i.e., $\gamma_2 = 0 = \gamma_3$.

In Figure 4 is shown the variation with driving frequency ω and coupling damping γ_1 of the ratio of the power dissipated in coupling 1, $\Pi_{DC1}(= H_{DC1}$ for unit forcing on just subsystem 1), to the overall input power to the driven system, $\Pi_{IN1}(= H_{IN1}$ for unit forcing). It may be seen that while γ_1 increases, the ratio H_{DC1}/H_{IN1} also increases until, at a certain level of γ_1, this ratio reaches maximum values in the range 0·6–0·8 for most frequencies (i.e., except for those which coincide with the natural frequencies of the undriven rod which is directly connected to the first, driven, rod). This indicates that, for this level of γ_1, a significant percentage of the overall power input to the system is dissipated

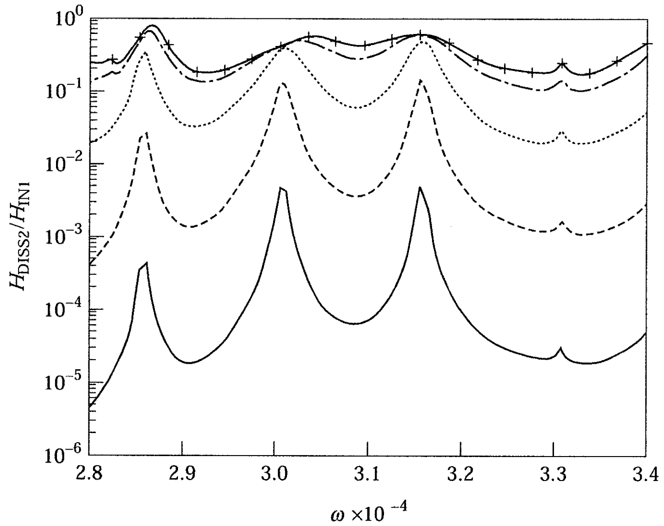

Figure 7. Variation in H_{DISS2}/H_{IN1} with ω and γ_1 for $\gamma_2 = \gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

in the coupling damper, thus minimizing the energy dissipated by the subsystems (although this does not mean that all the subsystems are their lowest energy levels). For greater values of γ_1 the ratio H_{DC1}/H_{IN1} decreases again, until it reaches values of the order 10^{-2} for large values of coupling damping (and where the coupling is then strong, i.e., $|D| \gg 1$).

This variation may be put into context by plotting the ratio of the longitudinal wave impedance of the first rod to the impedance of the first joint, Z_X. It is then seen that the value of the damper strength for which the ratio H_{DC1}/H_{IN1} is maximized arises for an

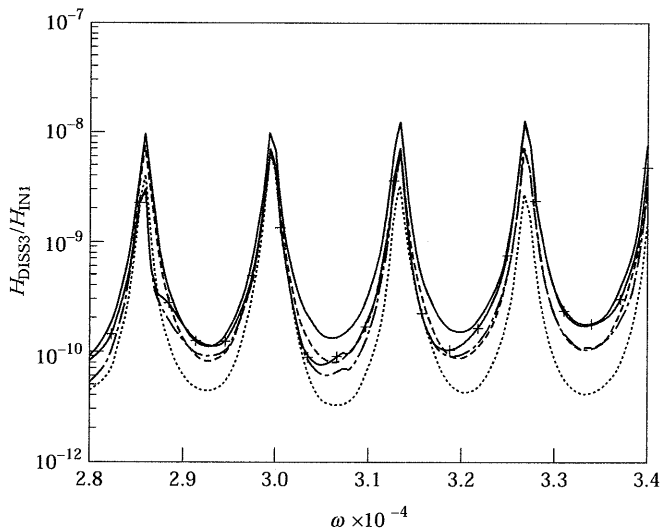

Figure 8. Variation in H_{DISS3}/H_{IN1} with ω and γ_1 for $\gamma_2 = \gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

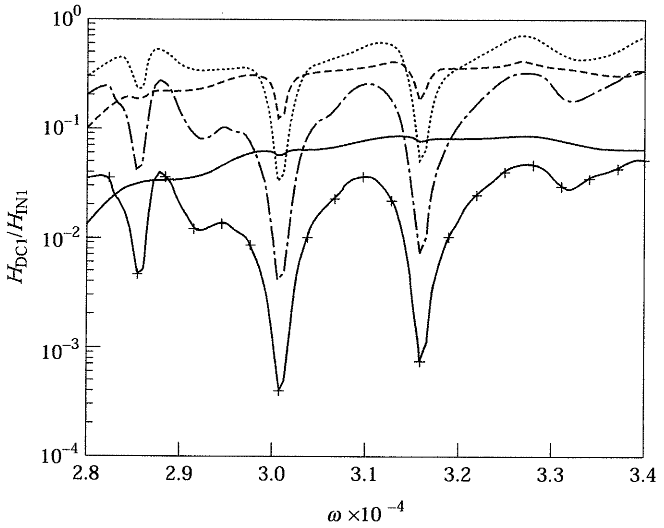

Figure 9. Variation in H_{DC1}/H_{IN1} with ω and γ_1 for $\gamma_1 = \gamma_2$, $\gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

impedance ratio around unity; see Figure 5. This result is perhaps not surprising, and clearly indicates how the distribution of power dissipation between the coupling dampers and the various subsystems and is significantly affected by the values of damper strength. Clearly, this distribution could be designed so that most of the input power would be dissipated in a given damper, thus reducing the power dissipated within the subsystems, which, being measures of subsystem energy levels, would then take minimum values.

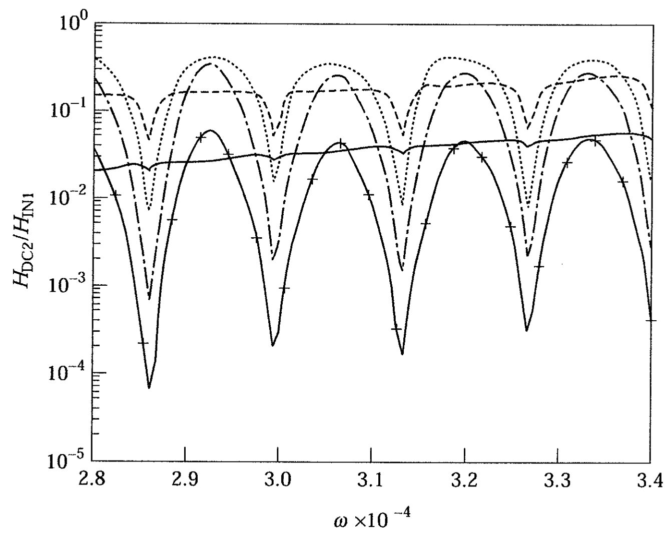

Figure 10. Variation in H_{DC2}/H_{IN1} with ω and γ_1 for $\gamma_1 = \gamma_2$, $\gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

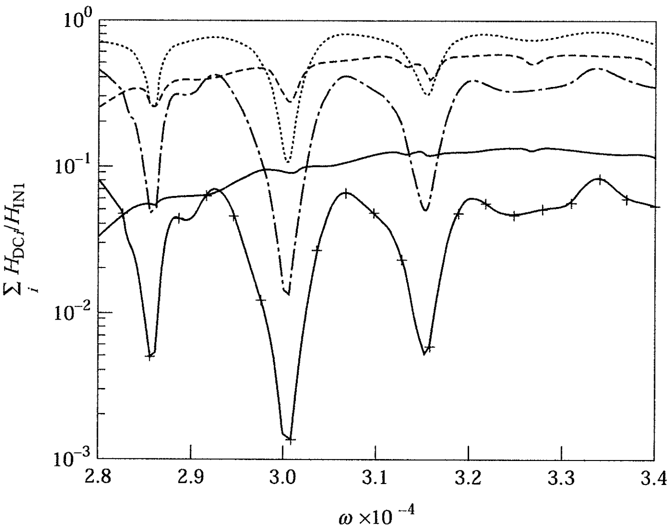

Figure 11. Variation in $\Sigma\, H_{DCi}/H_{IN1}$ with ω and γ_1 for $\gamma_1 = \gamma_2$, $\gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

The ratios of the power dissipated within subsystems 1, 2 and 3 to the power input into the driven subsystem are plotted versus driving frequency for increasing values of γ_1 in Figures 6–8, respectively. From these figures it may be seen that, for small γ_1, almost all of the power injected into subsystem 1 is dissipated within it. Since the coupling in this case is weak, the power transferred to subsystems 2 and 3 is very small and so are their resulting energy levels. As γ_1 increases to the point at which H_{DC1}/H_{IN1} is a maximum, the ratio of the power dissipated in subsystem 1 to the input power, H_{DISS1}/H_{IN}, reaches a

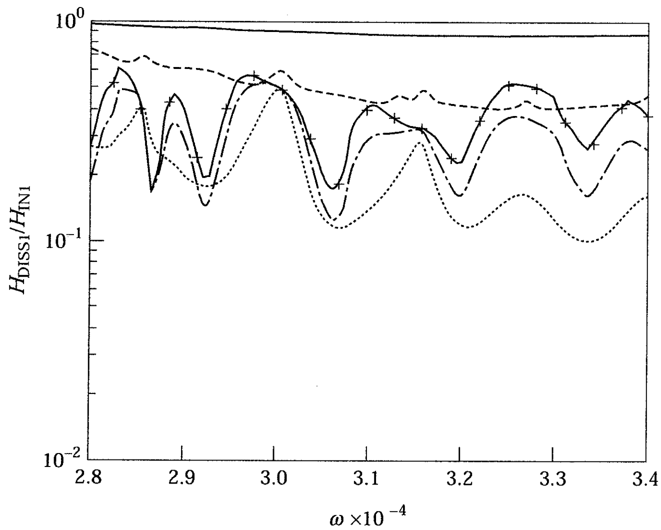

Figure 12. Variation in H_{DISS1}/H_{IN1} with ω and γ_1 for $\gamma_1 = \gamma_2$, $\gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

minimum level. The remaining input power is dissipated within the other subsystems. Although this is more than that for the case of weak γ_1, it should be noted that the power dissipated within subsystem 2 is still less than for strong γ_1. Thus, for this example 15–30% of the overall input power is dissipated in subsystem 1 and 2–20% in subsystem 2 when $\gamma_1 = 10\,000$ Ns/m, while when $\gamma_1 = 100$ Ns/m, more than 90% of the overall input power is dissipated in subsystem 1. Then, when $\gamma_1 = 1\,000\,000$ Ns/m, between 20% and 80% of the overall input power is dissipated in subsystem 1, with almost all of the remaining power being dissipated in subsystem 2: i.e., as much as 80% of the total (note that at this strength the coupling damper acts as an essentially solid link between the two subsystems, dissipating virtually no power). Finally, the power dissipated in subsystem 3 remains at very low levels throughout, because it is not strongly coupled with the other subsystems. These low levels show slight changes with the variation of γ_1, as might be expected.

In the second set of examples, the coupling damper between subsystems 1 and 3 is set to have the same magnitude as γ_1. In Figures 9 and 10 is shown the variation of the power dissipated in couplings 1 and 2 with frequency ω and coupling damping strength $\gamma_1 (= \gamma_2)$, respectively, as a ratio of the power input to the driven subsystem. It is seen that the variation of these ratios with the coupling damping strength has the same qualitative behaviour as in the previous examples. Furthermore, when these two quantities are added together to yield the ratio of the overall power dissipated in the dampers of couplings 1 and 2 to the input power, (see Figure 11) it is clear that for certain levels of γ_1 and γ_2 the ratio lies in the range 0.6–0.8 for most frequencies. Obviously, at this level of damping strength, the percentage of input power which is available to be dissipated in the subsystems via their internal damping will be a minimum and, again, this leads to minimum total energy levels in the subsystems; see Figures 12–14. It can further be seen from Figure 14 that the energy levels of subsystem 3 are higher than in Figure 8 because of the strong direct coupling with the driven subsystem in this case. The energy level of subsystem 2 is slightly reduced from that in the previous example, but has virtually identical behaviour with changes in driving frequency and damping levels. In other words, increasing damping in couplings 1 and 2 has changed the distribution of energy between subsystems 2 and 3, while the qualitative behaviour with variations in the coupling damping remains unchanged.

Finally, it is worth noting that adding damping only to the coupling between subsystems 2 and 3 has no discernible effect on the energy levels of the three subsystems (the variations being in the eighth decimal place). This obviously arises because the driven subsystem is then only weakly coupled to the other two despite the changes to γ_3.

To summarize, this brief study shows that in vibration control problems the distribution of input power which arises from external driving forces can readily be controlled by adding damping in the couplings directly connected to the driven system, as expected. This work further illustrates the well known method of tackling vibration transmission problems in complex built-up systems by adding damping in the joints near the driving points. Obviously, more detailed studies might reveal the most appropriate levels for such damping by looking at the problem as one in optimization, where the set of coupling damping levels in the various joints of the structure were selected so as to satisfy design requirements of minimal energy levels in any of the various subsystems.

8. SEA MODELLING

Next, the coupling damping loss factors which were introduced in equation (65) are evaluated deterministically for the case of weak coupling. Note, however, that the factors to be dealt with here are for just one realization and are more properly termed Energy

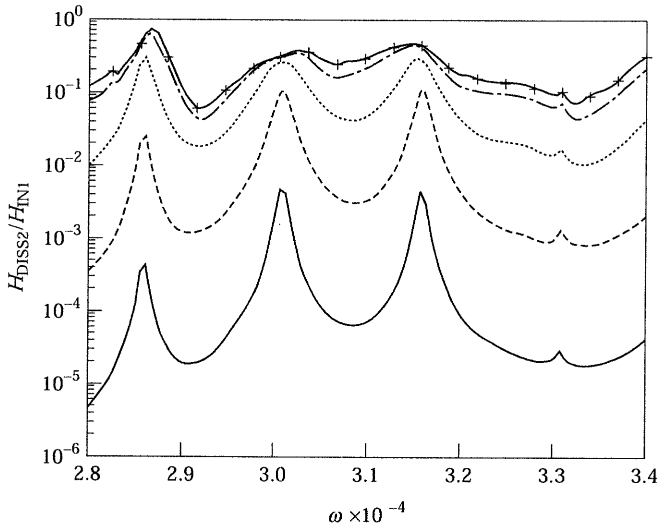

Figure 13. Variation in H_{DISS2}/H_{IN1} with ω and γ_1 for $\gamma_1 = \gamma_2$, $\gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

Flow Coefficients (EFCs), as defined by Fredö [15] (it was shown by Fredö that the EFCs tend towards the CLFs at high frequencies and therefore the results obtained give a reasonable estimate of the magnitude of the coupling loss factors). In Figures 15–17 are shown the values of η_{ij} and ζ_{ij} calculated for the system illustrated in Figure 1, where the parameters are chosen so that the coupling remains weak and equaion (65) is valid. It is clear from the figures that the coupling damping loss factors have high levels compared to the traditional coupling loss factors and therefore cannot simply be ignored.

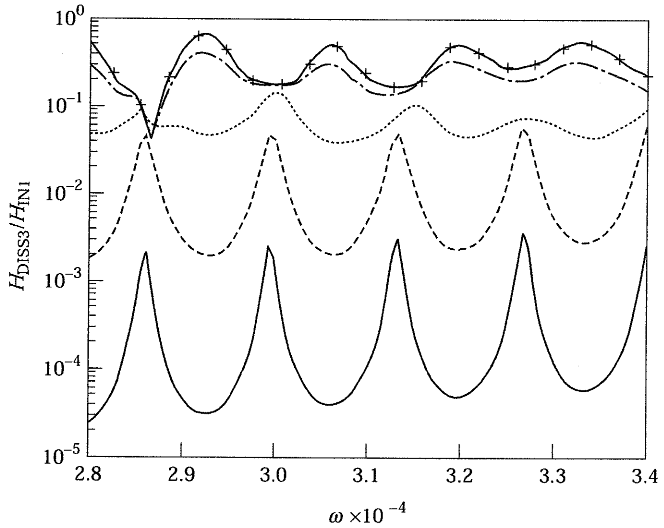

Figure 14. Variation in H_{DISS3}/H_{IN1} with ω and γ_1 for $\gamma_1 = \gamma_2$, $\gamma_3 = 0$ and "rain-on-the-roof" forcing of rod 1; key as Figure 4.

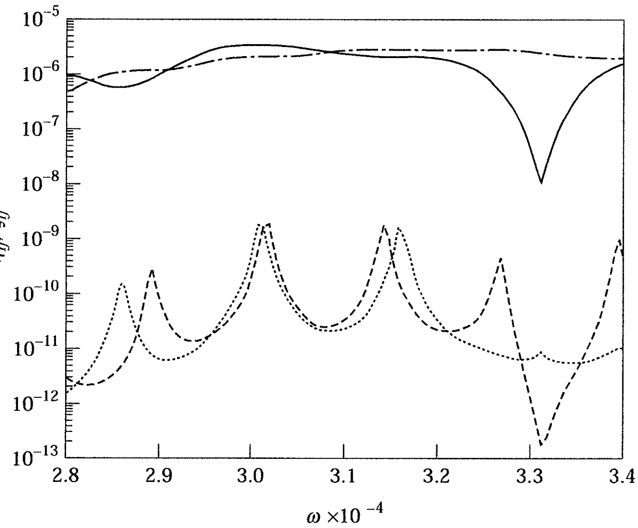

Figure 15. Variation in $\eta_{12}(\cdot\cdot\cdot)$, $\eta_{21}(- - -)$, $\zeta_{12}(-\cdot-)$ and $\zeta_{21}(—)$ with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1$ Ns/m.

The power injection method [16] has been suggested for measuring *in situ* the loss and coupling loss factors used in SEA. In this approach, power is injected to each subsystem in turn and energy levels are determined for each case. Energy balance equations may then be set up and inverted to determine the loss and coupling loss factors. Recently, Cuschieri and Sun [13] have suggested a variation of the power injection method in which the dissipation and coupling loss factors of a machine structure were estimated for both conservatively and non-conservatively coupled subsystems. However, for weakly coupled problems where the system involved contains dissipative joints, it is clear from comparing equations (50) and (65) that although the estimated coupling loss factors will be accurate,

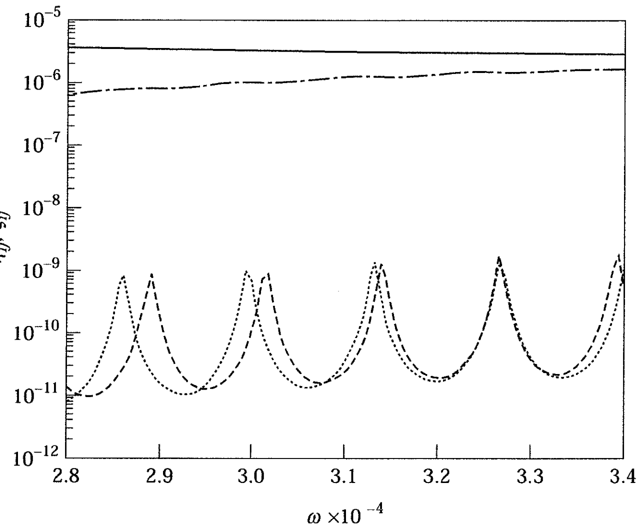

Figure 16. Variation in $\eta_{13}(\cdot\cdot\cdot)$, $\eta_{31}(- - -)$, $\zeta_{13}(-\cdot-)$ and $\zeta_{31}(—)$ with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1$ Ns/m.

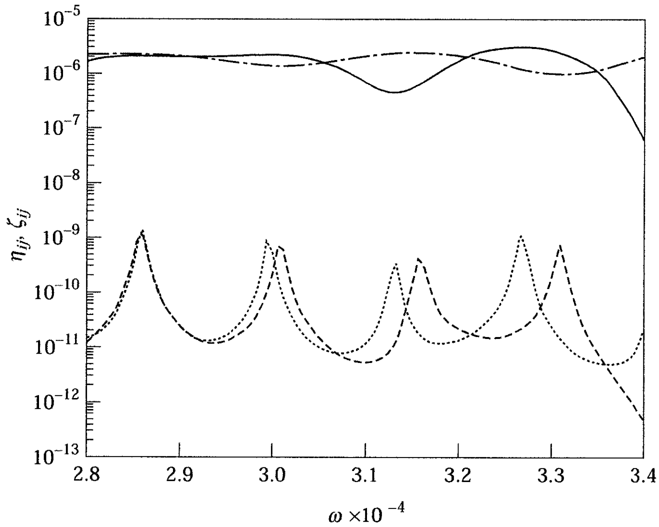

Figure 17. Variation in $\eta_{23}(\cdot\cdot\cdot)$, $\eta_{32}(- - -)$, $\zeta_{23}(- \cdot -)$ and $\zeta_{32}(—)$ with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1$ Ns/m.

the values of the damping loss factors will, in fact, be the total loss factors $\eta_i = \eta_i + \Sigma_{j \neq i} \zeta_{ij}$, which include the contribution of the dissipation at the joints [17].

Numerous techniques are available for the experimental measurement of damping loss factors, such as the reverberation decay technique and the steady-state energy flow technique; see, for example, the book by Norton [14] for more details. If the power injection method is applied to measure the coupling loss factors *in situ* while the loss factors are estimated independently by any of the aforementioned techniques, it is of interest to find out the effects of ignoring the coupling damping loss factors on the energy levels estimated by using the main energy balance equations. In Figures 18–20 are shown the energy levels of subsystem 1, 2 and 3, respectively, due to "rain-on-the-roof" forcing of subsystem 1 calculated directly and also by ignoring the coupling damping loss factors in the main energy balance equations (in both cases for small damping levels in the couplings (1 Ns/m) compared to the internal damping level of the subsystems). It is clear that the error in the estimated energy levels of the three subsystems is very small and can be ignored, which is perhaps as expected. However, for moderate levels of damping in the couplings (1000 Ns/m), for which the damping level is significant compared to the internal damping but for which the couplings are still weak, these errors increase significantly, and it then clear from Figures 21–23 that the estimated energy levels of the three subsystems are considerably higher than the actual levels, which suggests that coupling damping loss factors should be included in the analysis in order to obtain reasonably accurate results.

9. *IN SITU* DETERMINATION OF LOSS, COUPLING LOSS AND COUPLING DAMPING LOSS FACTORS

In a system of the type considered here, comprising N subsystems and M couplings, there are N loss factors, $N(N-1)$ coupling loss factors and $2M$ coupling damping loss factors which need to be determined. In the most general case in which all of the subsystems are interconnected, then $M = N(N-1)/2$, and in this case the total number of unknowns is

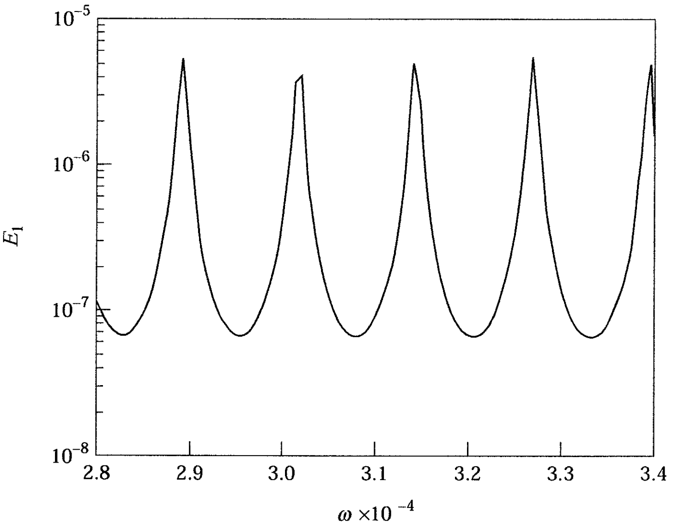

Figure 18. Variation in E_1 calculated exactly (——) and by ignoring coupling damping (- - - -) with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1$ Ns/m and "rain-on-the-roof" forcing of rod 1 (curves identical).

$(2N^2 - N)$. As has already been noted, for weakly coupled problems with coupling damping, the SEA energy balance equations may be written as

$$\hat{\Pi}_{INi} = \omega\eta_i\hat{E}_i + \sum_{j \neq i}(\omega(\eta_{ij} + \zeta_{ij})\hat{E}_i - \omega\eta_{ji}\hat{E}_j), \qquad i = 1, N, \tag{66}$$

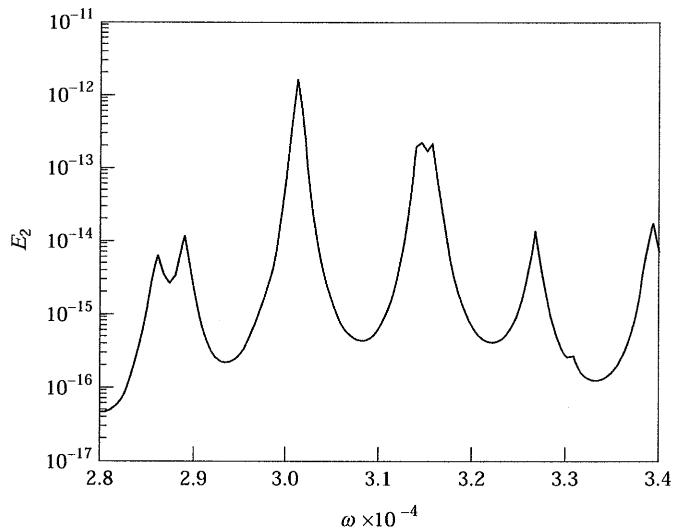

Figure 19. Variation in E_2 calculated exactly (——) and by ignoring coupling damping (- - - -) with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1$ Ns/m and "rain-on-the-roof" forcing of rod 1 (curves identical).

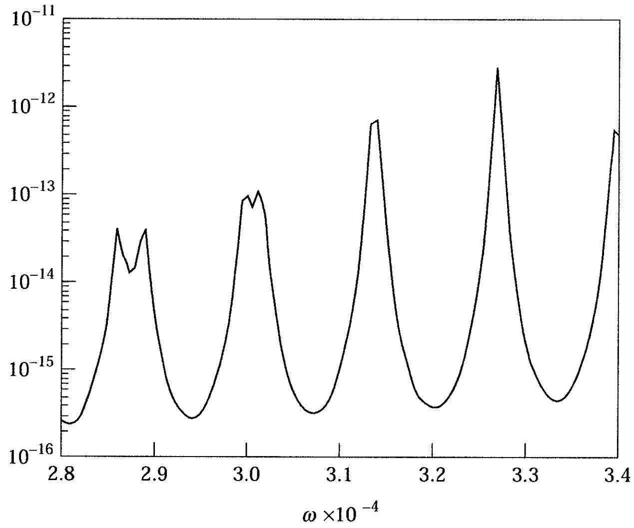

Figure 20. Variation in E_3 calculated exactly (——) and by ignoring coupling damping (- - - -) with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1$ Ns/m and "rain-on-the-roof" forcing of rod 1 (curves identical).

and

$$\hat{\Pi}_{DC_k} = \omega\zeta_{ij}\hat{E}_i + \omega\zeta_{ji}\hat{E}_j, \qquad k = 1, M, \tag{67}$$

where in this last equation i and j refer to the two subsystems directly connected through coupling k.

In order to determine the various loss factors *in situ*, the power injection method is often used, in which each subsystem is excited in turn and the energy levels of the various

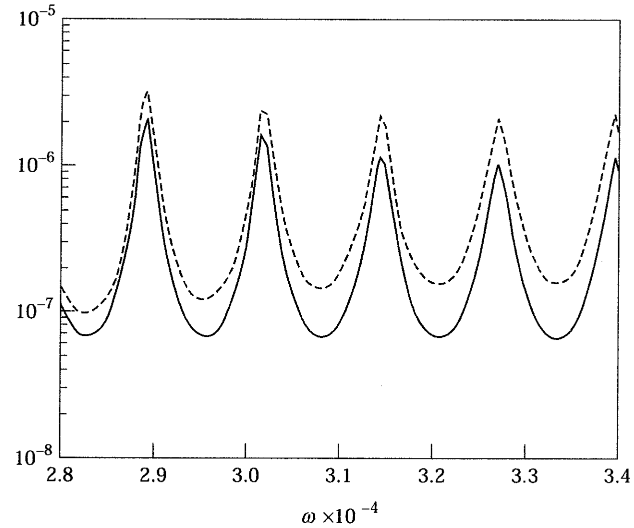

Figure 21. Variation in E_1 calculated exactly (——) and by ignoring coupling damping (- - - -) with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1000$ Ns/m and "rain-on-the-roof" forcing of rod 1.

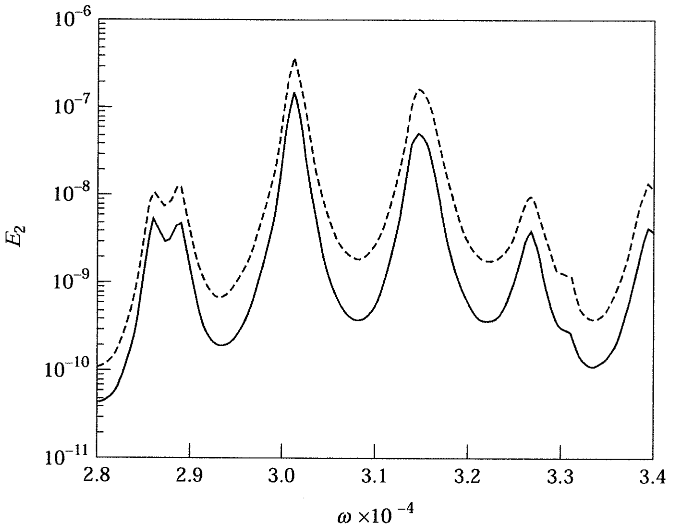

Figure 22. Variation in E_2 calculated exactly (——) and by ignoring coupling damping (- - - -) with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1000$ Ns/m and "rain-on-the-roof" forcing of rod 1.

subsystems are then measured. This allows N^2 equations to be set up describing the energy balances of the N subsystems. Obviously, for non-conservatively coupled subsystems, these equations are generally not sufficient to determine all of the $N^2 + 2M$ loss, coupling loss and coupling damping loss factors. The only way in which it is possible to find these additional quantities is by measuring the power dissipated at each joint for each case of excitation, so that an additional $M \times N$ equations can be constructed and the overall

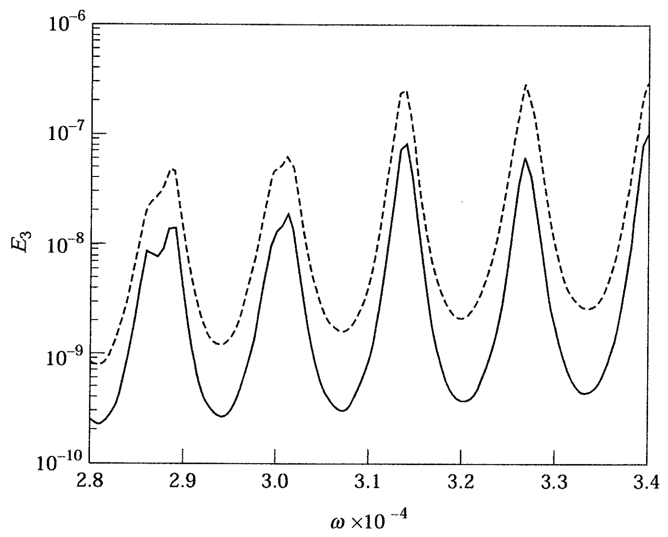

Figure 23. Variation in E_3 calculated exactly (——) and by ignoring coupling damping (- - - -) with ω for $\gamma_1 = \gamma_2 = \gamma_3 = 1000$ Ns/m and "rain-on-the-roof" forcing of rod 1.

number of equations ($N^2 + N \times M$) is then always sufficient to determine the unknowns using a standard least squares method.

In practice, it is not usually possible to measure the energy dissipated at each joint. However, if $\eta_i' = \eta_i + \Sigma_{j \neq i} \zeta_{ij}$ denotes the dissipation loss factor subsystem i, which includes the effects of dissipation at the joints directly connected to it in addition to the internal damping, then the power injection method can be used in a way similar to that suggested by Cuschieri and Sun [13] to determine all of the coupling loss factors η_{ij} and the dissipation loss factors η_i'. If the internal loss factors η_i are known from other methods, then it is straight forward to estimate the effect of the joint dissipations $\Sigma_{j \neq i} \zeta_{ij}$as compared to the internal damping η_i, which is dependent only on the subsystem material. However, it should once again be noted that the individual values of ζ_{ij} may not be found experimentally except for the case of two subsystems coupled together and where the internal loss factors are already known.

10. CONCLUSIONS

Energy flow relationships for multiple, non-conservatively coupled subsystems have been established by using a Green function approach. Deterministic expressions for the energy flows, energy levels and energy dissipations in the various parts of the overall system have been given in the frequency domain for both point and "rain-on-the-roof" driving. The effects of changes in coupling damping on the various power receptances have been illustrated through the use of numerical examples for "rain-on-the-roof" forcing.

It has been shown that the flow of energy around a complex built-up system can be controlled by adding damping in the joints directly connected to the driven subsystems, and that the level of damping can be chosen so as to minimize the subsystems' energy levels (as is well known in the field of practical vibration control). The model proposed here allows for the various power receptances to be utilized in such vibration control problems in a straight forward way.

Next, a SEA model for non-conservatively coupled systems has been suggested. It has been shown that the energy dissipation at the joints in weakly coupled problems can be accounted for through the introduction of a set of coupling damping loss factors which describe the relationship between the energy dissipated at the joints and the energy levels of the subsystems on either side of these joints. Expressions for these coupling damping loss factors have been given in terms of the various power receptances, along with the loss and coupling loss factors.

It has been shown that the power injection method can be used to obtain the coupling loss factors and the total subsystem internal loss factors. These latter quantities include the internal loss factors, which depend on the subsystem material and the sum of the coupling damping loss factors of all couplings directly connected to the subsystem. Moreover, if the values of the internal loss factors have already been determined, the power injection method then allows an estimate to be made of the significance of the coupling damping loss factors in a problem and of the errors which would result from ignoring damping in the joints when using the traditional SEA equations. It has also been shown that individual coupling damping loss factors cannot be determined experimentally unless the energy dissipated at each joint is measured separately, which may be difficult in practice.

Finally, it is worth emphasizing that the SEA model suggested in this work for non-conservatively coupled systems is valid only for weakly coupled problems. The use of SEA models for strongly coupled systems with coupling damping needs further study and lies outside the scope of this paper. Nevertheless, the results presented here may be of interest to those interested in the experimental side of SEA because they illustrate some of the possible effects of coupling damping on such work.

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