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VIBRATIONAL ENERGY FLOWS BETWEEN COUPLED MEMBRANES

A. J. KEANE

Department of Engineering Science, University of Oxford, Parks Road, Oxford OX1 3PJ, England

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The vibrational energy flows occurring between a pair of unequal rectangular membranes coupled along a mutual edge are investigated. Since the membranes studied have different properties and form an L shape when coupled this calculation may not be carried out directly, instead requiring some form of approximation. Here, the line coupling is modelled by a series of point couplings with sufficient points to ensure convergence for the frequencies of interest. This L shaped composite membrane model is then perturbed by the addition of a number of random point masses. The effects caused by varying the number and degree of randomness of the attached masses on the ensemble statistics of the combined system are subsequently investigated. The statistics of this ensemble are considered for weak and strong coupling between the membranes and for point and "rain-on-the-roof" type forcing. The existence of energy flow vortices in some ensemble members at some frequencies is noted. Conclusions are drawn for the application of Statistical Energy Analysis (S.E.A.) to coupled two-dimensional subsystems.

1. INTRODUCTION

Statistical energy analysis (S.E.A) is concerned with the flows of energies between vibrating subsystems [1]. It offers the prospect of simple calculations that can be used to predict the behaviour of structures above those frequencies amenable to finite element analysis, yet below those where the statistical assumptions of acoustics become valid. However, it is by now well known, that S.E.A. is not without its problems, being reliable as a prediction tool only in rather special circumstances. Part of the problem with traditional S.E.A. (there are a number of related and more advanced alternatives; see for example reference [2] or [3]) lies in the fact that it is concerned with mean energy flow predictions. Such means are calculated either on an ensemble basis or, if an ensemble of similar structures is not available, across frequency intervals on a single structure. These mean values on their own can be misleading: to be of real use some form of higher order statistics are, of course, desirable. Most engineers tend immediately to think of Gaussian distributions when mean levels are quoted and thus ask for information on the standard deviation of the levels being described. In S.E.A., however the distributions tend to be highly skewed and often have means that lie well above the mode, even when dealing with logarithmic data. In such circumstances, 5 and 95% confidence levels become much more useful. Unfortunatley, such measures are extemely hard to predict without recourse to extensive testing or computational simulation. Nonetheless, some progress has been made in this direction; see, for example, references [4, 5]. This paper is concerned with a further extension of that work and deals with the behaviour of a pair of two-dimensional sybsystems coupled along a common boundary. Such cases are of significant practical

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importance since most problems of engineering interest contain areas of plated and stiffened structure, i.e., two-dimensional elements. They are, however, difficult to deal with computationally: first one needs accurate models of the two-dimensional structures, then a method for dealing with the line couplings that join them must be formulated and finally, some way found of perturbing the subsystems, if ensemble average statistics are to be created and analyzed.

Here, the subsystems are taken to be rectangular free-free membranes: i.e., they are governed by simple second order differentials equations with well understood eigensolutions. Such subsystems do suffer from some shortcomings when carrying out studies of the type considered here, since they are not used in practical stuctures and do not yield finite displacements at the points of loading when subjected to point forces. However, their mode-shapes do not differ greatly from those of plates at the frequencies of interest in S.E.A. and their modal convergence difficulties can be fairly easily overcome. Their Green functions are, in addition, very easy to set up and compute. Moreover, such second order problems are of wide interest since they occur in many other branches of physics, references [6–8].

The line coupling model to be adopted lies at the heart of this work and is here modelled via a discrete array of point couplings, leading to a relatively straightforward matrix analysis problem [9], rather than the integral equations required for a true line coupling. Since such integral relationships are often solved numerically via discretization schemes this is not thought to be a major shortcoming. Indeed, a subsequent section explores this aspect and shows that, given a frequency range of interest, it is possible to establish the number of point couplings required to adequately model the joint under study.

Lastly, to perturb the subsystems being studied, a number of point masses are attached to the surfaces of the membranes via very stiff springs. The sizes of these masses are then varied in a random fashion to generate the desired ensembles. A subsequent section investigates the number of masses and the degree of randomness required to ensure the desired behaviour of the overall system. It turns out that relatively few, quite small masses are needed to generate significant variations in the overall response variables being studied.

2. THE BASIC MODEL

Figure 1 illustrates the basic model to be studied. It consists of a membrane 1m by 1·618 m (a golden section) joined to a second membrane 0·8 m by 0·9 m (an arbitrary, not quite square, aspect ratio) by a line of springs (which act at right angles to the plane of the membranes). Both membranes are taken to be free at their edges, except where they meet each other. This, rather curious arrangement, of free-free but tensioned membranes is, of course, not physically realizable, but it does permit the two subsystems to be readily coupled along their edges. In addition, as has already been noted, it is convenient to deal with mathematically, since the eigensolutions of membranes are both easy to compute and well behaved. Here, the larger membrane is chosen to have unit total mass and a tension such that its first non-zero uncoupled natural frequency is 1 rad/s: i.e., a mass per unit area of 0·618 kg/m^2 and a tension of 0·1639 N/m. The smaller membranes has unit mass per unit area and a tension the same as the larger membrane, which gives a slightly different wave speed in the two subsystems. The frequency range to be studied is from 0·8 to 1·2 Hz or 5·023 to 7·540 rad/s, spanning modes 17–34 of larger membrane and modes 13–26 of the smaller one and leading to a mean modal spacing in the larger membrane of 0.136 rad/s. The viscous damping factor for this membrane is then set as 0·0272 rad/s so that its average modal overlap factor becomes 0·2. The same damping level is also used

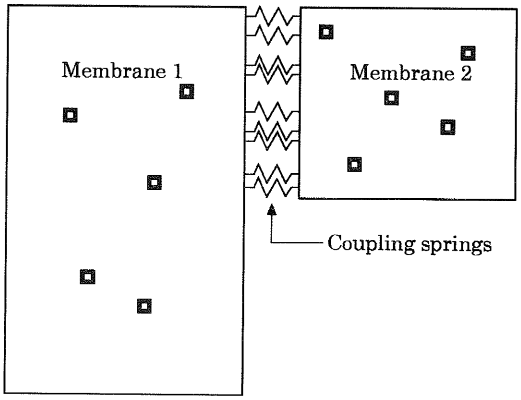

Figure 1. Two membranes coupled along a common boundary to form an L shape (with attached point masses marked □).

for the other subsystem resulting in a slighly lower modal overlap factor. The range of frequencies studied, the damping values chosen, and, by implication, the resulting modal overlap factors, have been adopted since it is known that S.E.A. tends to have difficulties in accurately predicting the behaviour of systems with low model overlaps at moderate frequency levels, a central interest of the current study.

The two membranes are joined so that the shorter edge of the smaller meets the longer of the larger and so that one corner of each is aligned. This leads to an overall system consisting of an L shape with different material parameters in the two parts. As has already been mentioned, here this joint is simulated by a row of springs. These are placed at random intervals along the joint so that no obvious filtering process can take place. Their number and strengths are discussed in a subsequent section where it is shown that using 30 such springs with stiffnesses of 0·01 N/m leads to an adequate model for a line coupling that is strong but not completely rigid.

To randomize this model a number of masses are attached to the membranes by very stiff springs set at randomly chosen positions. This aspect is also discussed in a subsequent section and it is shown that using five per membrane, with Gaussian distributed masses of mean 0·01 kg and normalized standard deviations of 20%, gives rise to suitable perturbations of the system behaviour. These are coupled to the membranes by springs of strengh 1 MN/m so that their individual natural frequencies are far above those being studied yet their behaviour is still amenable to the analysis process being adopted [9]. They are also lightly damped to limit their motions while causing some power to be dissipated at their attachment points.

It should be noted, however, that a tensioned membrane theoretically cannot support motions in an attached point mass, since membranes have zero bending stiffness and could therefore not generate the required forces. In such circumstances the point of attachment of the mass would remain stationary irrespective of the size of the mass, defeating the desired object of this study. This problem has been investigated by Langley and Keane [10] who showed that, when using a modal summation based Green function to model a membrane, as here, the inevitable requirement to truncate the summation at some point is, in essence, equivalent to assuming that the attached mass occupies a finite region of the membrane which removes the theoretical difficulties. Additionally, the rate of convergence of the modal sum to the value where the mass has no effect is very slow. It therefore turns out that such attachments form a simple way of perturbing the systems under study, as will be seen later. Moreover, since no such problems occur when dealing with bending plates and, indeed such methods are often used experimentally when studying plates, it is sensible to adopt a similar approach here. Similar remarks also apply to point forcing either along the coupling or at the point of excitation: in all cases the truncation of the modal summations implies a slight smearing out of the point force.

3. THEORY

The Green function based theory required to model a number of dynamical systems coupled by springs has been set out before [9], where it was shown that the problem of calculating the energy flow through any of the springs can be set up as a matrix inversion of terms based on the mode shapes and frequencies of the uncoupled subsystems, here two membranes and a number of point masses. The relevant shapes and frequencies are well known and will not be recapitulated. Only two difficulties need to be overcome when using this approach. First, sufficient numbers of terms must be used in the modal summations to ensure resonable convergence and secondly, the coupling springs used must not be so strong as to cause the matrix inversion process to become unstable. Both requirements are met here.

4. MODELLING THE LINE COUPLING

To ensure that the line coupling model proposed will be valid for the range of frequencies studied a test case has been investigated. This consists of two membranes with the same tension and mass per unit area coupled by very stiff springs and without any added masses; see Figure 2. Here the longer dimension of the previous larger subsystem has been reduced to 0·8 m and the mass per unit area of the smaller reduced to 0·618 kg/m^2, so that when joined the two become a single uniform, free-free membrane of 1·9 m by 0.8 m. If the line coupling model proposed is adequate, it should then be the case that the point receptances of the joined membranes should be the same as that for the single larger subsystem: i.e., plots of the energy flowing into the model from point, white noise excitation should be similar.

As has already been noted, the model used here requires the number of coupling springs to be specified together with their strengths and positions. To investigate the strength required one subsystem is forced and the energy levels of both subsystems studied at a single frequency of excitation as the coupling strength varies. Figure 3 illustrates such a calculation for 30 springs with the positions set out in Table 1 and point forcing at 0·16 m from the top and 0·17 m from the left-hand edge of the composite system. As can be seen, provided the coupling springs are over 10 N/m in strength, no further change in energy level takes place and the subsystems are effectively rigidly joined. Figures 4(a) and (b) then illustrate the input powers with coupling springs of 100 times this strength

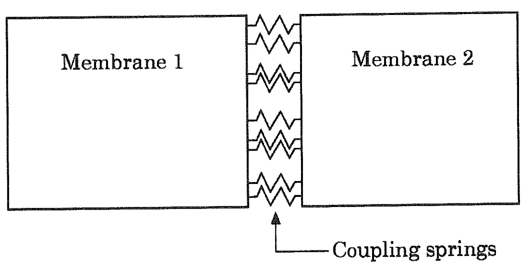

Figure 2. Two membranes coupled along a common boundary to form a rectangle.

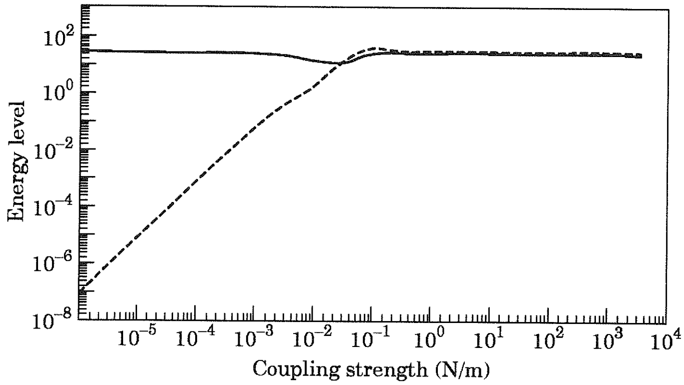

Figure 3. Energy levels at 6·2832 rad/s for the two coupled membranes of Figure 2 when using 30 coupling springs of varying strengths (membrane 1 solid line, membrane 2 dotted line).

for the cases of 20 and 30 coupling springs and a variety of modal summation bandwidths. These show that with 20 springs, significant differences remain between the two membrane and one membrane cases for all modal summation bandwidths while for 30 springs a reasonable match is obtained with a bandwidth of 100 rad/s. This summation bandwidth leads to the use of around 2000 modes in the summations and once again illustrates the high numbers of modes that can be needed to ensure reasonable convergence in such studies [11]. It should also be noted, as has been mentioned earlier, that point loadings applied to membranes give rise to some theoretical difficulties, but that truncating the Green function modal summations in effect smears out the point of action of the load. This feature explains why the curves for the two joined membranes in Figure 4(b) lie on either side of that for the single membrane: increasing the number of terms in the summation improves the convergence of the sums but also causes the individual springs to become more discrete in their action. The bandwidth of 100 rad/s thus represents a compromise in this respect. Nonetheless, the differences between the two membrane and single membrane curves in Figure 4(b) are always relatively slight.

Now the very high coupling spring strengths used for this test case ensure that the membranes are effectively rigidly joined where they meet, leading to the condition known as "equipartition of modal energies" in the subsystems. This case is not, however, the most interesting from an S.E.A. point of view, since it is not natural to divide a single uniform

TABLE 1

Spring positions as distances in m from the aligned corner of the membranes (those marked * *are not used in the* 20 *spring models)*

Spring no.	Position	Spring no.	Position	Spring no.	Position
1	0·0041	11	0·3200*	21	0·5067
2	0·0050	12	0·3300*	22	0·5105
3	0·0306	13	0·3564	23	0·5200*
4	0·0827	14	0·3637	24	0·5500*
5	0·0995	15	0·3972	25	0·6200*
6	0·1641	16	0·4200*	26	0·6403
7	0·1724	17	0·4244	27	0·6600*
8	0·1981	18	0·4400*	28	0·7186
9	0·2200*	19	0·4721	29	0·7200*
10	0·3059	20	0·4909	30	0·7300

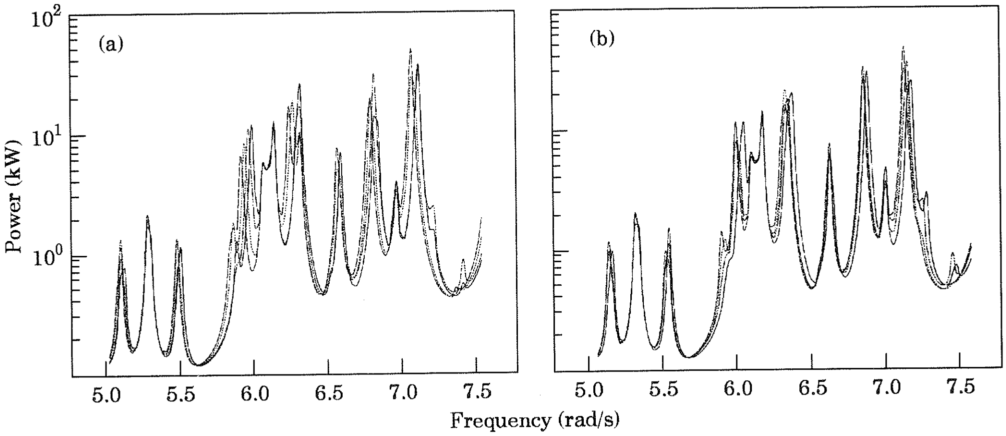

Figure 4. (a) Input powers, Π_{IN}, for the two coupled membranes of Figure 2 when using 20 coupling springs and various modal summation bandwidths: ——, 63 rad/s; ·····, 100 rad/s; — — —, 125 rad/s; – · · · – · · ·, single membrane. (b) As (a) but with 30 coupling springs.

structure to form two subsystems when building S.E.A. models; rather structures are subdivided along natural boundaries. In the main case of interest here this boundary occurs because of the change in membrane parameters at the joint. It is further reinforced by selecting coupling springs that are not sufficiently strong to lock the boundaries rigidly together. It is, however, equally the case, that overly weak couplings are to be avoided in such studies, since it is well known that S.E.A. tends to perform well with weak couplings almost irrespective of other features in the model. Consequently, a transitional value of 0·01 N/m is adopted for subsequent calculations, a value which lies just below the knee in the unforced membrane energy level curve presented in Figure 3: i.e., where the simple relationships between coupling strength and energy level begin to break down. Thus, 30 springs of strengths 0·01 N/m, positioned as per Table 1 and a summation bandwidth of 100 rad/s were adopted for most of the remainder of this work. When applied to the original L shaped model this leads to the typical input power spectrum for white noise forcing of Figure 5, the same excitation point being used as for the test case.

5. RANDOMIZING THE MODEL

S.E.A. is fundamentally interested in ensembles of similar but not identical systems. Thus, when carrying out computational studies it is desirable to focus attention on schemes for randomizing the models being studied. This aspect of S.E.A. research has been investigated recently [4, 5] and it has been shown that the safest course is to apply the randomization to the physical model and then to find the eigensolutions of the randomized model and hence its responses, leading to a realistic variation of natural frequencies and, perhaps more importantly, mode-shapes. This is not simply achieved when dealing with two-dimensional subsystems and here recourse is made to composite subsystems instead: i.e., each membrane is modified by the addition of a number of firmly attached point masses. The energies of the subsystems then become those of the membranes plus their attached masses; the energy flowing in the springs coupling the two membranes remains, of course, the energy flow between the subsystems.

To adopt this randomization scheme the number of masses to be used must be decided, as must the mass variations and attachment details adopted to form the ensemble of systems. Using an initial set of 10 masses with a mean of 0·01 kg and a normalized standard

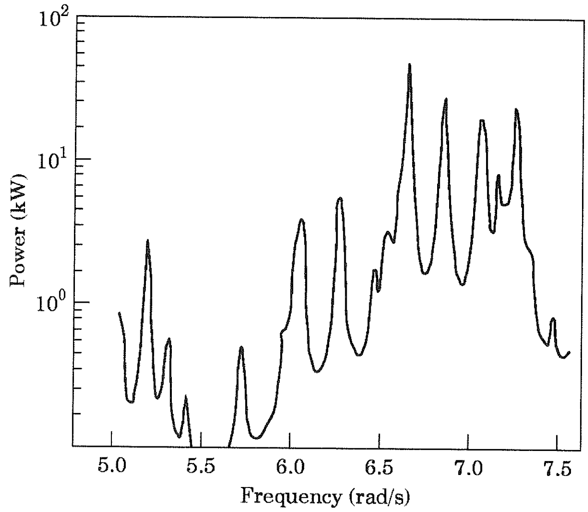

Figure 5. Input power Π_{IN}, for the L shaped membrane model of Figure 1 when using 30 coupling springs of strength 0·01 N/m, a modal summation bandwidth of 100 rad/s and no added masses.

deviation of 20% attached to the single 1·9 m by 0·8 m membrane studied in the previous section and excited at the same point leads to the input power spectrum of Figure 6 (which also shows the unloaded single membrane result from Figures 4(a) and (b)). Here the mass attachment points are randomly chosen and the attachment springs set with the very high value of 1 MN/m. The parameters used are detailed in Table 2. Clearly, a significant variation has been achieved between the initial case and the mass loaded membrane but it is still not obvious how a different set of random masses would affect this behaviour and this is important since the ensemble is to be created by modifying the attached masses.

To quantify the degree of randomness in an S.E.A. model Keane and Manohar [4] suggested a term called the statistical overlap factor. This is defined as twice the standard deviation (s.d.) of the natural frequencies divided by the mean modal spacing. They showed that if this quantity was greater than two, an S.E.A. analysis tends to lead to well behaved and smooth mean response spectra (in some cases, a value of unity is sufficient, while in others three may be required). However, to adopt this formula the s.d. of the natural

TABLE 2

Initial added mass parameters, masses in kg and positions in m from the upper left corner of the membrane

Mass no.	Mass	Horz. posn.	Vert. posn.
1	0·009671	0·0995	0·3295
2	0·009260	0·8865	0·4954
3	0·010277	0·7244	0·6639
4	0·010035	0·4711	0·3411
5	0·007733	0·4562	0·7752
6	0·006885	1·8122	0·5841
7	0·008568	1·3048	0·6799
8	0·014238	1·1661	0·6904
9	0·012408	1·0893	0·6310
10	0·011061	1·2774	0·4917

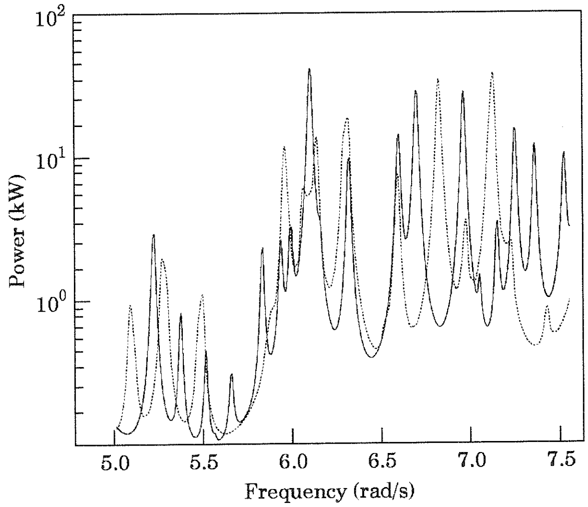

Figure 6. Input power, Π_{IN}, for a single rectangular membrane with 10 added masses, also showing response for membrane with no added masses taken from Figure 4 (dotted line).

frequencies must be known. Here, because of the use of composite subsystems, these may not be calculated directly and numerical methods must be adopted instead. To do this the peaks in the input power spectrum for point white noise forcing are used to represent the composite subsystem natural frequencies; see the solid line in Figure 7(a), which is part of the curve given in Figure 6. Figure 7(a) further shows the results of studying an ensemble of 1000 members formed by varying the existing random masses according to a Gaussian scheme where the mean of each individual mass is taken to be the initial value in Table 2 and the normalized standard deviation is 20% of these already random masses. The crosses in the diagram represent the positions of all peaks in the ensemble members and the dotted line the mean response of the ensemble (those crosses marked '+' are associated with the central peak while those marked '×' are for the others). The dashed line shows the probability density function (PDF) of natural frequency for the central peak. Figures 7(b) and (c) show similar information when the added masses are varied with normalized standard deviations of 10% and 50%, respectively (n.b., the mass values were always constrained to be positive during these studies). As can be seen from the figures, the peaks, and therefore the natural frequencies in the subsystem, are quite readily randomized in this way. Moreover, the natural frequency PDF's show no clearly recognizable shape, although their slight asymmetry is perhaps reminiscent of the Gamma distribution sometimes cited in this context [1, 5]. The complex nature of these distributions gives some idea of the difficulties inherent in using simple mathematical forms to model natural frequency PDF's when studying the higher order statistical behaviour of SEA [12, 13] (and the even more complicated behaviour of the mode shapes in such problems further compounds these difficulties).

For these three studies the numerically estimated statistical overlap factors turn out to be 0·36, 0·83 and 1·17 for the normalized s.d.'s of 10, 20 and 50%, respectively (n.b., for the normalized s.d. of 50% it is rather difficult to relate the peaks in the ensemble to those in the base configuration and so the overlap factor of 1·17 may, in fact, be rather inaccurate, tending to the low side). The previous work cited [5] shows that a statistical overlap factor of 0·83 implies the model is quite random but that an S.E.A. model may still show frequency dependent oscillations of the 5% and 95% confidence bands. The

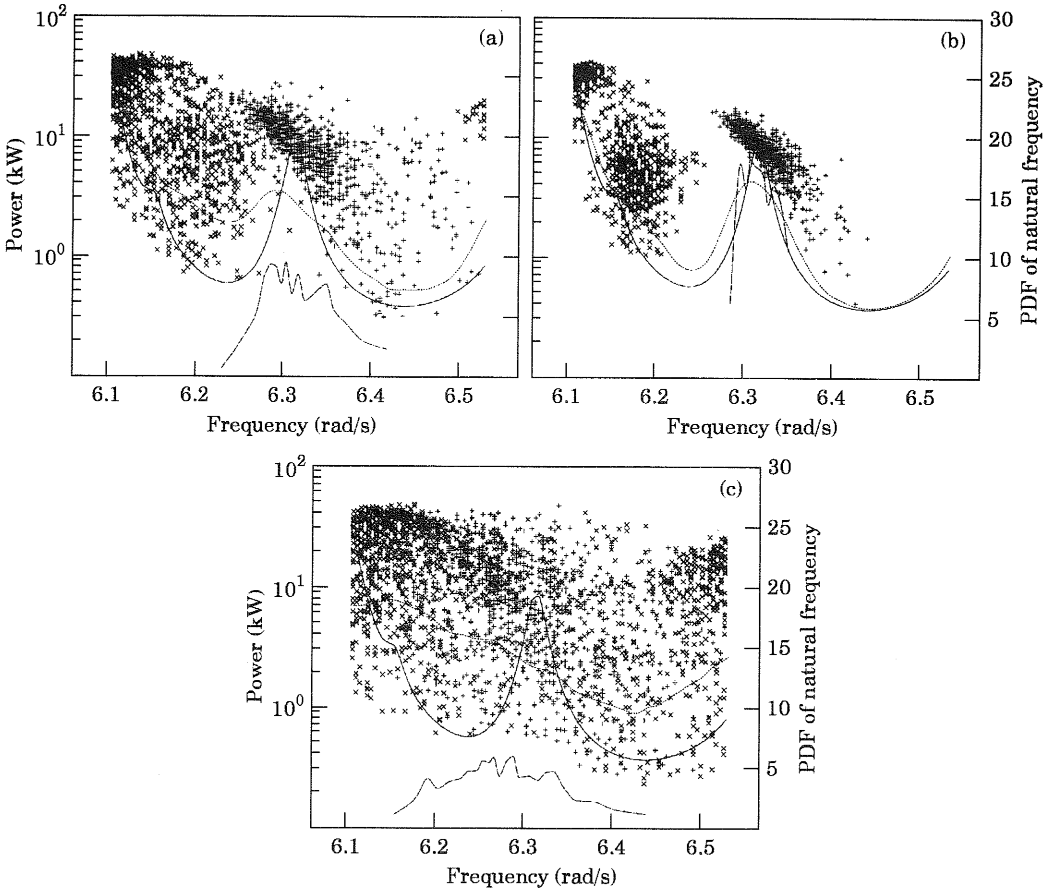

Figure 7. (a) Detail from Figure 6 (solid line) together with peaks from an ensemble of 1000 random systems created by using masses with a normalized standard deviation of 20% (crosses marked '+' are associated with the central peak while those marked '×' are for the others), $E[\Pi_{IN}]$ (dotted line) and probability density function of natural frequency for the central peak (dashed line). (b) As (a) but for masses with a normalized standard deviation of 10%. (c) As (a) but for masses with a normalized standard deviation of 50%.

normalized s.d. of 20% thus represents an intermediate case and is adopted in subsequent sections.

Figure 8 illustrates a further study of this kind where only eight masses have been used and different positions adopted, again with a normalized s.d. of 20%. As can be seen, this leads to a slight reduction in the overall randomness though not in the character of the plot. This indicates that the model adopted is not sensitive to the exact number and positions of the added masses provided the correct degree of randomness in their masses is adopted. (It is not obvious, but it would seem resonable to presume, that keeping constant masses and varying their positions would yield similar results.)

6. ENERGY FLOW RESULTS

After having investigated modelling of the line coupling and the randomization scheme it is then possible to create an ensemble of random systems based on the L shaped basic model. In this case the same ten added masses are used but positioned to reflect the more complex membrane geometry; see Table 3. Figure 9 illsutrates the input power spectrum

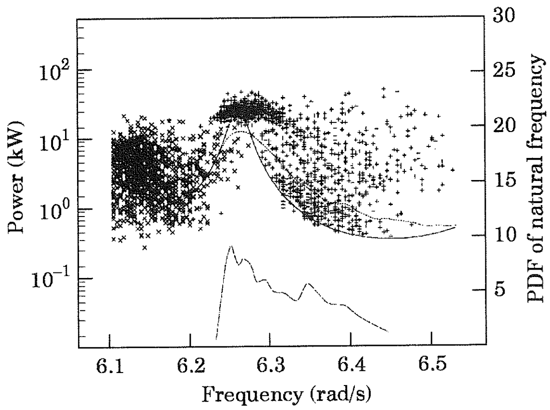

Figure 8. As Figure 7(a) but when using only eight added masses with different mass values and positions.

with previously used point forcing for this case (and also shows the result for the unloaded system given in Figure 5). It is again clear that the added masses have significantly perturbed the response from that of the unloaded system.

The coupling power between the two membranes is more difficult to illustrate since it varies not only with frequency but also position along the joint. Figure 10(a) illsutrates this variation by way of three-dimensional surface where height represents power and the other axes position and frequency. This plot is formed by using the energy flows in the individual springs to represent the flows at the equivalent points in the line junction being simulated, with curve fitting between them. This is not quite the same as power per unit length as the springs are not evenly spaced. Nonetheless, it is clear that the energy flow is a complex function of both position and frequency. Notice in particular, that at any frequency most of the flows are from the driven to the undriven membrane but that some are the reverse. The same is true for variations in frequency at any given position. All that can be said for certain is that the sum of the flows from the driven to the undriven membrane is always positive since this is a requirement for energy balance given the finite losses occurring in the undriven subsystem.

Studies of alternative realizations show that, although the general character of the plot is usually unchanged, the positions and frequencies of negative energy flows

TABLE 3

Final added mass parameters, masses in kg and positions in m from the upper left corner of each membrane (the first five masses are attached to the larger membrane and the second to the smaller one)

Mass no.	Mass	Horz. posn.	Vert. posn.
1	0·009671	0·0995	0·3295
2	0·009260	0·8865	0·8954
3	0·010277	0·7244	0·6639
4	0·010035	0·4711	1·3411
5	0·007733	0·4562	0·8752
6	0·006885	0·8122	0·5841
7	0·008568	0·0348	0·6799
8	0·014238	0·1661	0·6904
9	0·012408	0·0893	0·6310
10	0·011061	0·2774	0·4917

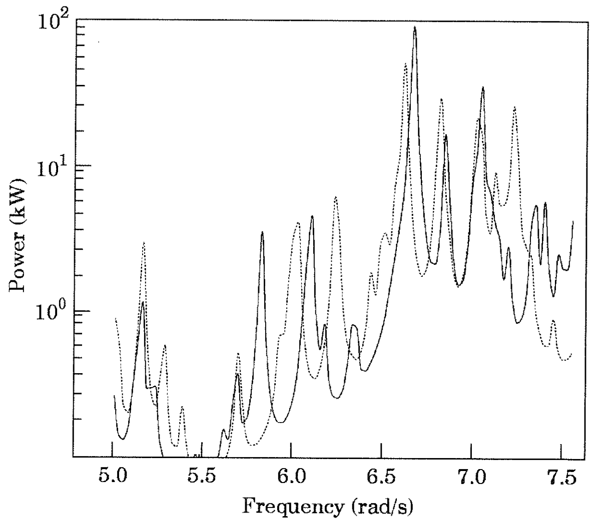

Figure 9. Input power, Π_{IN}, for the L shaped membrane model of Figure 1, when using 30 coupling springs of strength 0·01 N/m, a modal summation bandwidth of 100 rad/s and 10 added masses; also showing response for L shaped membrane with no added masses taken from Figure 5 (dotted line).

vary significantly from realization to realization. Moreover, in some cases, very large circulations, of power occur where the net energy flow is little changed but the flows at particular positions along the coupling are very large and in opposite directions; see Figure 10(b). An important consequence of such features is that the ensemble statistical behaviour, and in particular the higher order statistics, cannot be predicted from frequency averages carried out on any one individual realization. Note also, that although circulating power has been discussed before (see, for instance reference [14]) it is normally considered when there are three or more subsystems whose topology gives rise to loop (as opposed to tree) structures. Here, circulation arises because the junction between the two subsystems is not restricted to a single point and the circulation occurs within a single coupling. It would of course, not be possible to detect such flows experimentally by the normal S.E.A. power injection methods —instead the actual joint would need to be instrumented along its length.

Study of an ensemble of such responses leads to Figures 11–13, which show the mean, 5% and 95% confidence levels for the coupling power in the same form as Figures 10(a) and (b). These have been generated using the same mass realizations used to create

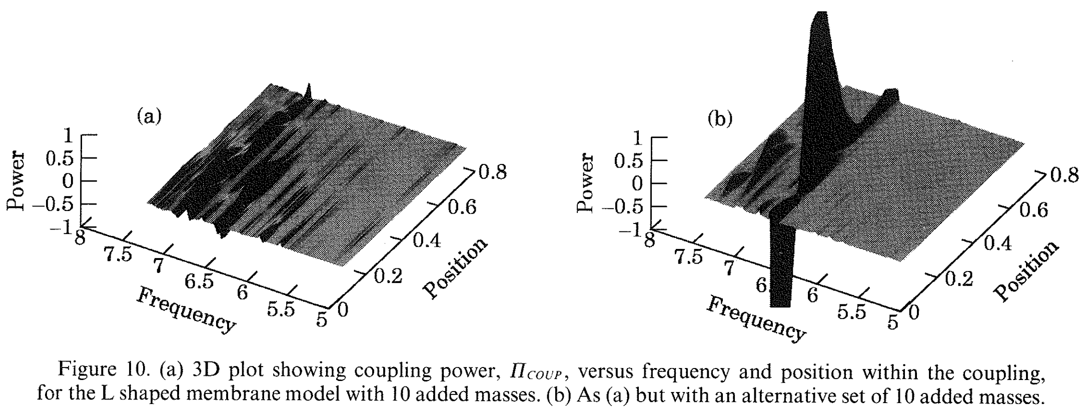

Figure 10. (a) 3D plot showing coupling power, Π_{COUP}, versus frequency and position within the coupling, for the L shaped membrane model with 10 added masses. (b) As (a) but with an alternative set of 10 added masses.

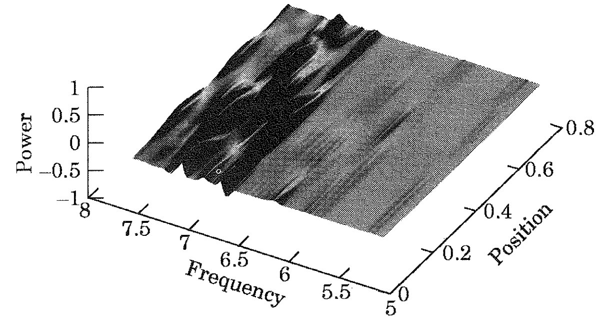

Figure 11. 3D plot showing $E[\Pi_{COUP}]$, versus frequency and position within the coupling, for an ensemble of 1000 random systems based on the L shaped membrane model of Figure 1 and created by using masses with a normalized standard deviation of 20%.

Figure 7(a) and show that the mean is usually positive but sometimes negative, the 5% confidence level is mostly negative and the 95% level mostly positive (n.b., the 5% confidence level is that level below which 5% of all the ensemble members lie at any given frequency and position, the 95% level being similarly defined; moreover, the ensemble members forming the subsets that lie below these levels change from frequency to frequency and position to position, depending on the responses of individual members). Additionally, there are certain frequencies, such as those around 6·8 rad/s where the confidence levels are extremely widely spaced: i.e., where it is very difficult to predict the behaviour at any point in the coupling and, in consequence, where the mean level may be of little use. This wide spacing would seem to be a direct consequence of the circulating power seen in some ensemble members such as that used to produce Figure 10(b).

Given this ensemble of data it is then possible to calculate the coupling loss factor used in S.E.A on the traditional assumption that, for a two subsystem problem, the factor is simply the energy flow between the subsystems divided by $\omega(E[E_1] - N_1 E[E_2]/N_2)$, where N_1 and N_2 are the mode counts of the two membranes. Now, because of the composite subsystems used here, this means that

$$\eta_{12} = \frac{\sum_{i=1}^{30} E[\Pi_{COUPi}]}{\omega\left\{E[E_{MEM1}] + \sum_{i=1}^{5} E[E_{MASSi}] - \frac{N_1}{N_2}\left(E[E_{MEM2}] + \sum_{i=6}^{10} E[E_{MASSi}]\right)\right\}}, \qquad (1)$$

since the energies of the two compound subsystems may be found by summing over their constituents, see Figure 14. It may be seen from the figure that η_{12} is mostly in the range

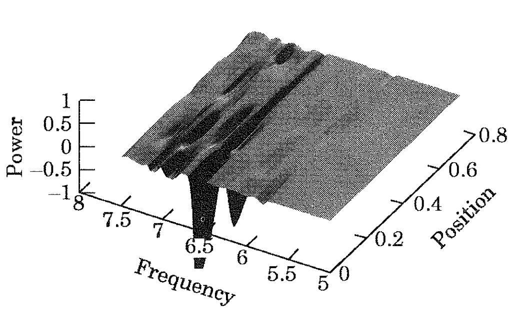

Figure 12. 3D plot showing the 5% confidence level of Π_{COUP}, versus frequency and position within the coupling; other details as for Figure 11.

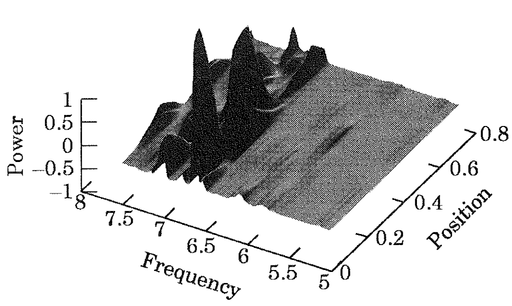

Figure 13. 3D plot showing the 95% confidence of Π_{COUP}, versus frequency and position within the coupling; other details as for Figure 11.

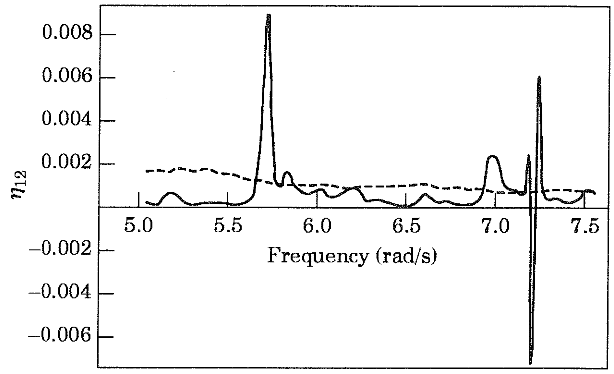

Figure 14. Ensemble average coupling loss factor, η_{12}, versus frequency; other details as for Figure 11 (solid line), together with that found by using a moving frequency average of 2·5132 rad/s on a single ensemble member excited by "rain-on-the-roof" forcing (dotted line).

0·0002 to 0·04 but that at around 7·2 rad/s it goes negative. This arises because, at that frequency, the ensemble average energy level of the undriven membrane goes above that for the driven one, see the solid and chain dotted lines in Figures 15 between 0·7 and 7·5 rad/s (the figure also shows the mean energy levels of the individual masses as dotted lines). Clearly, a negative coupling loss factor is nonsensical and indicates that, for this level of coupling, the preceding equation is no longer strictly valid, a point that has been made before [9, 14]. Also shown on Figure 14 is the coupling loss factor curve that results if the averaging processes of the previous equation are carried out by using a moving frequency average on a single ensemble member excited by "rain-on-the-roof" forcing (with a bandwidth 2·5132 rad/s, i.e., equal to the width of the plot). Note that, this kind of forcing averages the responses over all possible points of forcing and thus prevents the mode shapes at the point of forcing from influencing the calculations. Clearly, although such an approach is commonly used to predict coupling loss factors, yielding representative values of η_{12}, it is unable to indicate the detailed variations seen when using the ensemble average.

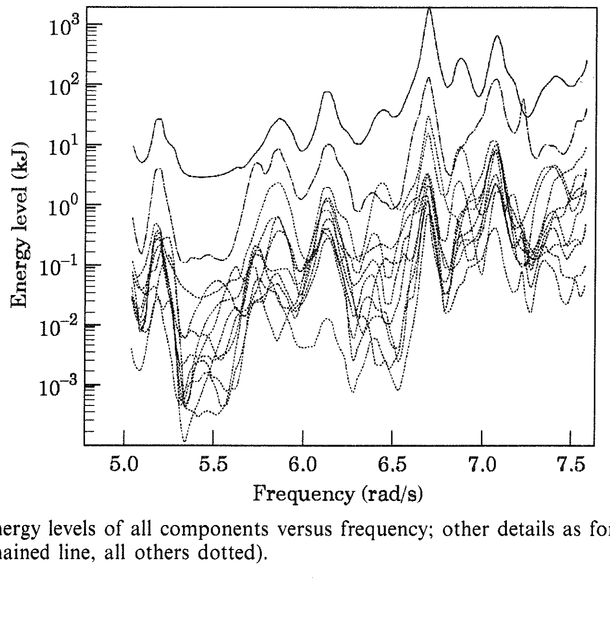

Figure 15. Mean energy levels of all components versus frequency; other details as for Figure 11 (E[E_{MEM1}] solid line, E[E_{MEM2}] chained line, all others dotted).

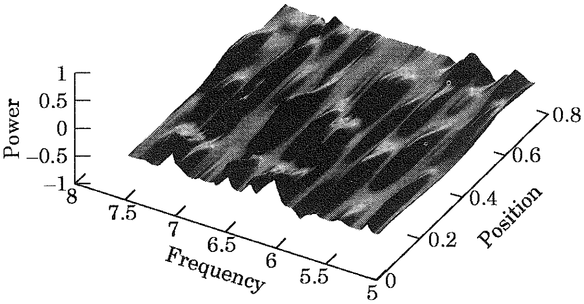

Figure 16. As Figure 11 but for "rain-on-the-roof" forcing.

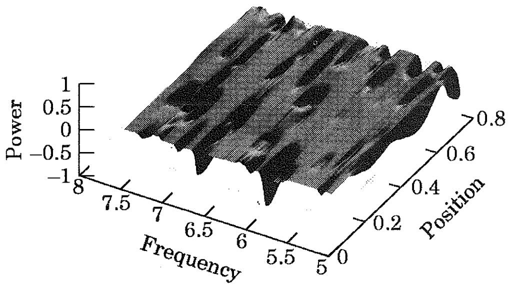

Figure 17. As Figure 12 but for "rain-on-the-roof" forcing.

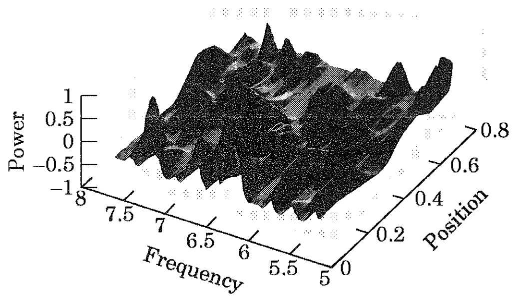

Figure 18. As Figure 13 but for "rain-on-the-roof" forcing.

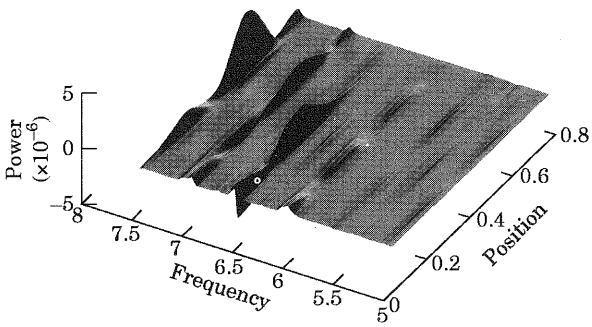

Figure 19. As Figure 11 but with coupling springs 10^3 times weaker.

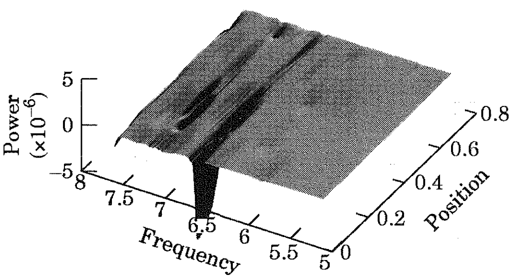

Figure 20. As Figure 12 but with coupling springs 10^3 times weaker.

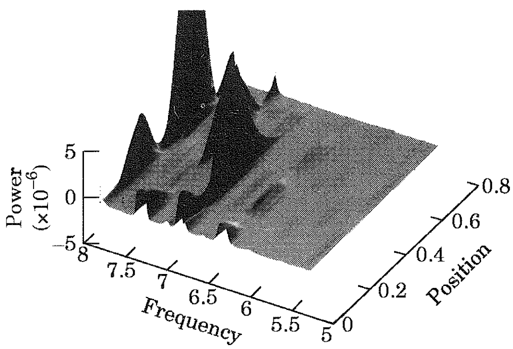

Figure 21. As Figure 13 but with coupling springs 10^3 times weaker.

Figure 16–18 and 19–21 illustrate the results of two further studies into the statistics of this problem. For Figures 16–18 the system and ensemble are as before but now the forcing is of the distributed "rain-on-the-roof" type. By way of contrast, Figures 19–21 are again for point forcing but with the coupling spring joining the two membranes reduced in strength by a factor of 10^3, leading to a weakly coupled problem (see again Figure 3). These six figures show that when details concerning the point of forcing are removed from the model, the confidence bands surrounding the mean show modest fluctuations throughout the frequency range considered, but no extreme variations. Conversely, even a weakly coupled model with point forcing can exhibit the circulations of power that lead to occasionally widely spaced confidence bands, albeit at much lower absolute values.

These comparisons lead to the conclusion that the statistics for energy flows through joints in point forced systems may differ significantly from those where averages over the point of forcing are taken. They thus indicate that if details of the joint powers are being examined, "rain-on-the-roof" forcing should only be adopted where the excitation for the problem being considered genuinely does load the structure evenly in space: i.e., "rain-on-the-roof" forcing may be an acceptable model for turbulent boundary layer or acoustic noise loading but could result in misleading results if applied to machinery vibrations, which tend to act at specific points.

7. APPLICATION OF S.E.A.

After having produced a model that enables the statistics of a line coupling to be studied in some detail and the coupling loss factor to be derived on an ensemble or frequency averaged basis, it is then possible to consider what the application of traditional S.E.A. theory to such a problem might yield. Consider the design problem where η_{12} is taken as given and the behaviour of a pair of subsystems is to be deduced (or course, calculating η_{12} for a system as complicated as that examined here is, itself, no easy task. Here, however, attention is focussed on the variations to be found even when this parameter is correctly estimated). It is assumed that all energy flows are normalized on the basis of input power and, again, just subsystem one driven. The designer wishes to predict the energy flowing from the driven to the undriven subsystem as a function of frequency, without knowing more than η_{12}, the damping factors and the modal densities: i.e., by using

$$\Pi_{12}(\omega) = \omega\eta_{12}\left\{E_1(\omega) - \frac{N_1}{N_2}E_2(\omega)\right\} \tag{2}$$

with

$$E_1 = \frac{\Pi_{1DISS}}{\omega\eta_1} \quad \text{and} \quad \Pi_{1DISS} = \Pi_{1IN} - \Pi_{12}, \text{ etc.},$$

where η_1 is the loss factor for subsystem 1 and so on. Rearranging these gives

$$\frac{\Pi_{12}}{\Pi_{1IN}} = 1\bigg/\left(1 + \frac{\eta_1}{\eta_{12}} + \frac{N_1}{N_2}\frac{\eta_1}{\eta_2}\right). \tag{3}$$

Here the modal density ratio is the ratio of the numbers of modes in the bandwidth used for the summations, i.e. 2176/1990 (= 1·0935), and the ratio of loss factors unity so, given that $\omega\eta_1$ is 0·0272 rad/s, this becomes just

$$\Pi_{12}/\Pi_{1IN} = 1/(2{\cdot}0935 + (0{\cdot}0272/\omega\eta_{12}). \tag{4}$$

This result can then be compared to the actual non-dimensional coupling powers in a variety of ways. First, η_{12} can be taken from the ensemble average data (i.e., the solid curve

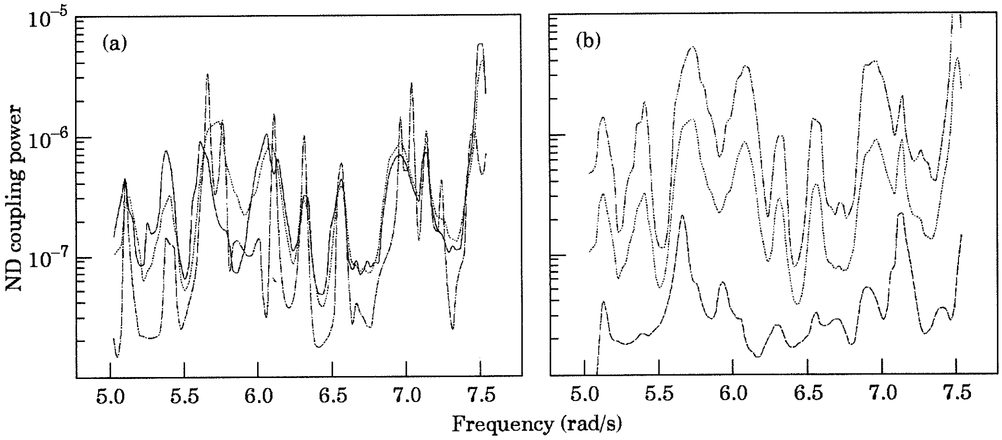

Figure 26. (a) Non-dimensional coupling powers Π_{12}/Π_{1IN} for point forcing and weak coupling; key as Figure 22(a). (b) As (a) but key as Figure 22(b).

even if only on an empirical basis, since this would enable practitioners to gain some view on the reliability of their calculations given some knowledge of the variability in their subsystem parameters. Such a linkage lies outside the scope of this paper, but the methodology proposed would clearly allow a number of surveys to be carried out to provide such information for a variety of configurations of interest. Such a survey might even reveal trends that could be used to class structures by topological type, leading to the prospect of designing structures with inherently reduced variability in their structural vibrations.

8. CONCLUSIONS

In this paper the problem of modelling a pair of two-dimensional membrane subsystems coupled along a common boundary has been considered. This has been accomplished by using a series of discrete, point couplings to represent the line junction. It has been shown that this model can be made acceptably accurate for the chosen frequency range provided sufficient coupling points are included. Consideration has also been given to the theoretical difficulties inherent in applying point forces to membranes and note made that

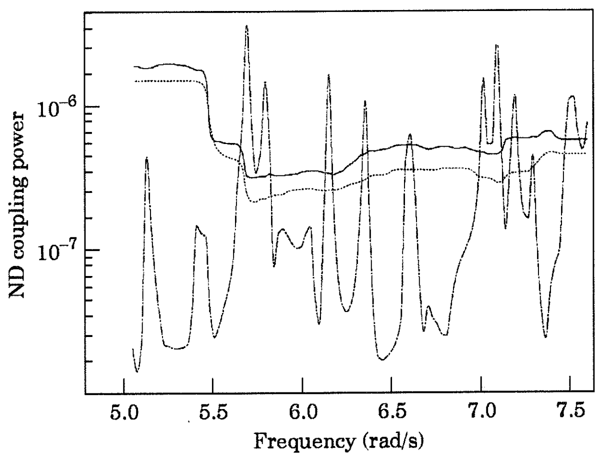

Figure 27. Non-dimensional coupling powers Π_{12}/Π_{1IN} for weak coupling; key as Figure 23.

using finite modal summations implies a slight smearing of the points of action of the applied forces.

Having produced a reasonably accurate model this has been considered in statistical terms by perturbing the two-dimensional subsystems using a number of random point masses. The number and degree of randomness of these masses has been investigated and it is seen that relatively few are needed to significantly randomize the overall behaviour of the system. The statistics of the resulting ensembles, which cannot readily be modelled by simple mathematical formulations, have been considered from the viewpoint of Statistical Energy Analysis (S.E.A.) and comparisons made between "exact" and S.E.A. calculations.

These comparisons reveal that S.E.A. is reasonably good at predicting net energy transfers given a reliable estimate of the coupling loss factor, although it can only give a detailed picture of the frequency variations to be found in such transfers if ensemble average coupling loss factors are used. Moreover, given the approach adopted by S.E.A., it is unable to reveal some of the complexities that occur in energy transfers along line couplings, such as energy flow vortices where power circulates with little net transfer of energy. These effects give rise to considerable variations in the confidence bands applying to localized measures of energy transfer but fortunately do not affect those applying to net transfers between subsystems. In addition, it seems to be the case that the confidence bands applying to net energy transfers are relatively unaffected by the exact details of the forcing or the strength of the coupling itself. This suggests that further studies may be able to link variations in the statistical overlap factors of subsystems, which measure their randomness, and the variabilities in the net energy transfers between them. Finally, although this whole study has been based on models of tensioned membranes, it does not seem unreasonable to suggest that similar findings would be obtained for a system of two coupled plates, since the use of plates would lead only to slight changes in mode shapes and natural frequency spacings.

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