SAINT-VENANT'S PRINCIPLE AND THE ANTI-PLANE ELASTIC WEDGE

N. G. STEPHEN *Mechanical Engineering Department, The University of Southampton, UK**

P. J. WANG *Mechanical Engineering Department, The University of Southampton, UK***

The stress field due to self-equilibrating loading on the inner or outer radius of a wedge sector, consistent with anti-plane deformation, will be affected by two agencies: a geometric effect of increasing or decreasing area and decay as anticipated by Saint-Venant's principle. When the load is applied to the inner radius the two effects are acting in concert; however, when the load is applied to the outer radius the two effects act in opposition. For a wedge angle in excess of the half-space the geometric effect is dominant over Saint-Venant decay and the stress increases the greater the distance from the loaded outer radius, indicating a breakdown in Saint-Venant's principle. For the wedge angle $2\alpha = 360°$, the unique inverse square root stress singularity at the crack tip, which is at the heart of linear elastic fracture mechanics, can be attributed to this breakdown of Saint-Venant's principle for just one eigenmode.

Key words: Saint-Venant's principle, elastic wedge, stress field, anti-plane deformation

1 INTRODUCTION

Saint-Venant's principle (SVP) underpins much of solid mechanics by allowing the replacement of an actual load system on a structural member by a statically equivalent load distributed in a particular way demanded by the elastostatic solution. The difference between the two load distributions is termed 'self-equilibrating', and since it has no stress resultant or couple which requires reaction at some other location on the structure, there seems no reason why the associated stress field should penetrate any great distance into the structure; according to SVP this depth of penetration should be small.

SVP has been expressed and applied in a variety of ways by many authors; for example, in beam problems, according to Sokolnikoff (**1**),† 'it is commonly assumed that the local eccentricities are not felt at distances that are about five times the greatest linear dimension of the area over which the forces are distributed'. The first mathematical proof of SVP was provided by Toupin (**2**) who considered a prismatic elastic cylinder of arbitrary length and cross-section subjected to a self-equilibrated load system on one end only. Toupin demonstrated the exponential decay of elastic strain energy, and hence stress, but SVP requires also that the rate of decay should be spatially 'rapid'; examples where decay is not rapid include thin-walled structures, and beams and plates of composite and anisotropic construction.

Non-prismatic structural members have received comparatively little attention in the literature: thus, despite being amenable to exact analysis within the spirit of the mathematical theory of elasticity, there has not been a systematic study of the applicability of SVP to the wedge geometry, although Horvay (**3**) employed an approximate variational approach for the plane strain problem and came to the surprising conclusion that stress diffusion into a divergent area did not enhance the rate of decay when compared with the plate of constant thickness.

In contrast, the plane strain elastic wedge subjected to a bending moment at its apex has attracted considerable research attention due to the pathological behaviour of the well-known Carothers' solution (**4**) at the wedge angle $2\alpha \approx 257°$. Only recently (**5**) has this 'wedge paradox' been attributed to a breakdown of SVP, in the sense that an asymmetric self-equilibrating load decays at the same rate as the distance from the loaded apex increases, as does the bending moment, the latter due to stress diffusion into an increasing cross-sectional area. In fact, the breakdown of SVP occurs for the half-space, $2\alpha = 180°$, when a symmetric self-equilibrating load decays at the same rate as the (asymmetric) bending moment.

A systematic study of the plane strain case is undertaken by the present authors in a separate communication (**6**), where it is shown that the breakdown of SVP also manifests itself in the sense that a self-equilibrating load on the outer arc $r = b$ does not decay at all, the increase in stress due to the convergent geometry exactly matching the decrease in stress as anticipated by SVP. In the present note the authors treat the simpler problem of anti-plane deformation, which again appears not to have been considered previously within the context of SVP, although Ting (**7**) has considered the case of anti-plane shearing by loading on the flanks of the wedge. Comparison is made with the exponential decay characteristic of anti-plane shear for the plate of constant thickness (**8**) and also the effect of divergent/convergent geometry on the decay characteristic is investigated; as with the plane strain problem, there is a breakdown of SVP for wedge angles in excess of the half-space. It is also shown that the stress singularity which occurs at a crack tip (mode III) and the unique distribution which is at the heart of linear

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* *Corresponding author: Mechanical Engineering Department, The University of Southampton, Highfield, Southampton SO17 1BJ, UK*

** *Visiting Professor from Department of Mechanical Engineering, Shanghai Maritime Institute, Shanghai 200135, People's Republic of China*

† *References are given in the Appendix*

elastic fracture mechanics (LEFM) are attributable to a breakdown of SVP for just one eigenmode.

2 THEORY

The uniform isotropic elastic wedge (Fig. 1) is defined by

$$a \leqslant r \leqslant b, \qquad -\alpha \leqslant \theta \leqslant +\alpha, \qquad -L < z < L \tag{1}$$

where r, θ and z are cylindrical coordinates. Body forces are assumed absent and the flanks of the wedge, $\theta = \pm\alpha$, are traction free, the loads being applied on the inner and outer arcs, $r = a$ and $r = b$ respectively, with possible complementary shear stresses on the ends $z = \pm L$. The inner radius may tend to zero, as appropriate for a re-entrant corner and a crack tip when $\alpha = \pi$. The wedge length $2L$ is assumed to be large compared with the other dimensions, so the concern here is stress decay from the loaded arcs.

For anti-plane deformation the displacement components are

$$u_r = u_\theta = 0, \qquad u_z = u_z(r, \theta) \tag{2}$$

and the Navier (displacement) equilibrium equations reduce to the requirement that u_z satisfies Laplace's equation, that is $\nabla^2 u_z = 0$, or

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\,\partial\theta^2}\right)u_z = 0 \tag{3}$$

Taking $u_z \propto (r/r_0)^{-\lambda} \exp(ik\theta)$, where $i = (-1)^{1/2}$ and r_0 is an arbitrary constant having dimension length, leads to the characteristic equation

$$k = \pm\lambda \tag{4}$$

and hence the general solution

$$u_z = \left(\frac{r}{r_0}\right)^{-\lambda}(C_1 \cos \lambda\theta + C_2 \sin \lambda\theta) \tag{5}$$

where $C_{1,\,2}$ are constants having the dimension length.

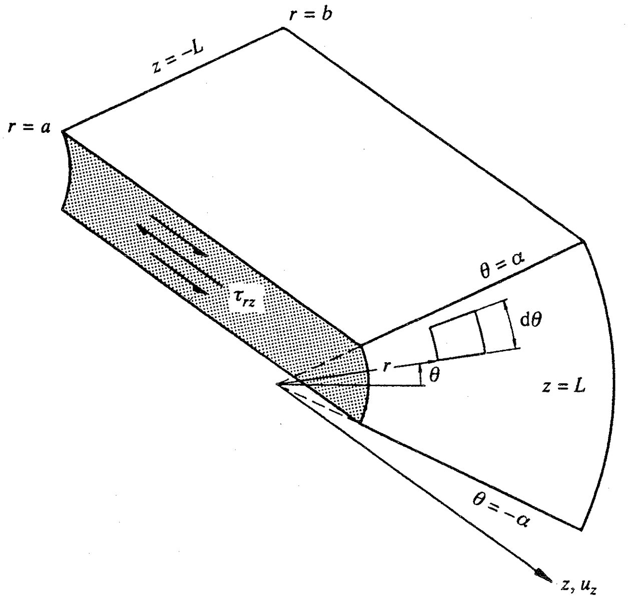

Fig. 1. Wedge sector with self-equilibrated load on inner arc $r = a$

The stress components are

$$\tau_{rz} = \mu\frac{\partial u_z}{\partial r} = -\frac{\mu\lambda}{r_0}\left(\frac{r}{r_0}\right)^{-\lambda-1}(C_1 \cos \lambda\theta + C_2 \sin \lambda\theta)$$

$$\tau_{\theta z} = \mu\frac{\partial u_z}{r\,\partial\theta} = \frac{\mu\lambda}{r_0}\left(\frac{r}{r_0}\right)^{-\lambda-1}(-C_1 \sin \lambda\theta + C_2 \cos \lambda\theta) \tag{6}$$

where μ is the shear modulus. Applying the traction-free boundary condition $\tau_{\theta z} = 0$ on $\theta = \pm\alpha$ leads to the eigenequation

$$\sin 2\lambda\alpha = 0 \tag{7}$$

Hence

$$\lambda = \frac{n\pi}{2\alpha}, \qquad n = 0, \pm 1, \pm 2, \pm 3, \text{etc.} \tag{8}$$

For even n, the constant C_2 is zero, and for odd n, constant C_1 is zero. It is straightforward to show that the stress resultant $\int_{-\alpha}^{\alpha} \tau_{rz} r \, d\theta$ is zero. Assuming completeness of the solution, an arbitrary self-equilibrating shear on the arcs $r = a$ and $r = b$, consistent with anti-plane deformation, can be expanded as a summation of terms as in the first of equations (6).

The root $\lambda = 0$ for $n = 0$ requires special consideration, as it leads to the double root $k = 0$, and hence the displacement $u_z = C_1 + C_2\theta$. This leads to the stress field $\tau_{rz} = 0$, $\tau_{\theta z} = \mu C_2/r$. However, this field violates the boundary condition $\tau_{\theta z} = 0$ on $\theta = \pm\alpha$ unless $C_2 = 0$, when this case may be discounted, leading as it does to just the rigid body displacement C_1 in the z direction. An alternative displacement field that also satisfies the Navier equation and leads to a stress variation equivalent to $\lambda = n = 0$ is

$$u_z = (C_1 + C_2\theta)\ln\left(\frac{r}{r_0}\right) \tag{9}$$

This leads to the stress components

$$\tau_{rz} = \mu\frac{C_1 + C_2\theta}{r}$$

$$\tau_{\theta z} = C_2\frac{\mu}{r}\ln\left(\frac{r}{r_0}\right) \tag{10}$$

The traction-free boundary condition $\tau_{\theta z} = 0$ on $\theta = \pm\alpha$ requires that constant $C_2 = 0$, and hence $\tau_{rz} = \mu C_1/r$, $\tau_{\theta z} = 0$, which corresponds to an axial shear force resultant which is uniformly distributed over any arc $r =$ constant. The radial dependence is consistent with the expectation that, since stress is force divided by area and the area carrying the load increases linearly with radius, so stress should vary inversely with the radius. It is straightforward to show that the wedge is in force and moment equilibrium under this surface loading, including the required contribution on the ends $z = \pm L$.

3 DISCUSSION

The variation of λ with wedge semi-angle α is shown in Fig. 2 and is the initial focus for discussion. It is first noted that the root loci are symmetric about $\lambda = 0$; since stress varies as $(r/r_0)^{-\lambda-1}$, the stress field is independent of radius for $\lambda = -1$. For $\lambda > -1$, the stress decreases as

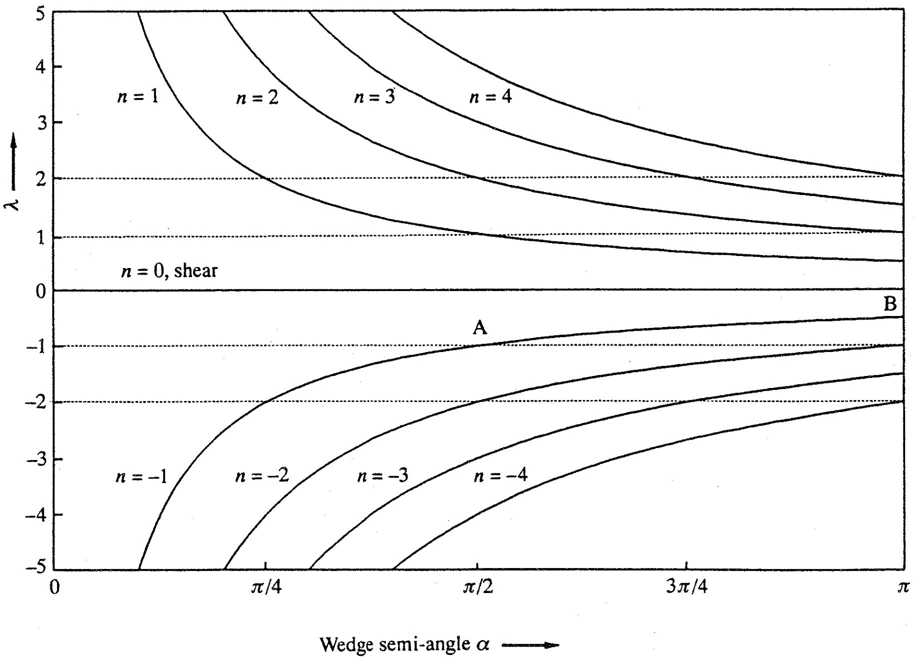

Fig. 2. Variation of λ with wedge semi-angle α

the radius increases, and this would correspond to a decay of self-equilibrated loading on the arc $r = a$. On the other hand, for $\lambda < -1$, the stress decreases as the radius decreases, which corresponds to decay of self-equilibrated loading on the arc $r = b$; in general the two cases correspond to $n > 0$ ($\lambda > 0$) and $\lambda < -1$ for $n < 0$. However, there is one exception, the case of $\lambda > -1$ for $n = -1, \alpha > \pi/2$ (the locus AB on Fig. 2), and this will be discussed in detail in terms of a breakdown of SVP.

Firstly, however, a comparison is made between the decay characteristic of the wedge, loaded on the arc $r = a$, and that of the anti-plane elastic strip (**8**), for which the minimum rate of decay is $\exp(-\pi z/t)$, where z is coordinate (distance from the loaded edge) and t is plate thickness. For the wedge the radial variation $(r/r_0)^{-\lambda-1}$ may be written in the form

$$\left\{1 + 2\alpha\left(\frac{\text{distance from arc}}{\text{arc length}}\right)\right\}^{-\lambda-1}$$

such that comparison can be made according to the spirit of SVP. The result is shown in Fig. 3 for various wedge angles; for all wedge angles the rate of decay is initially more rapid than for the plate—the larger the angle the greater the decay rate. However, as the distance from the loaded arc increases, the decay rate becomes less than the exponential decay of the plate, again for all wedge angles. Thus, if decay to 20 per cent is taken as a criterion for rapidity of decay, then the wedge is quicker; on the other hand, if the criterion is decay to 2 per cent

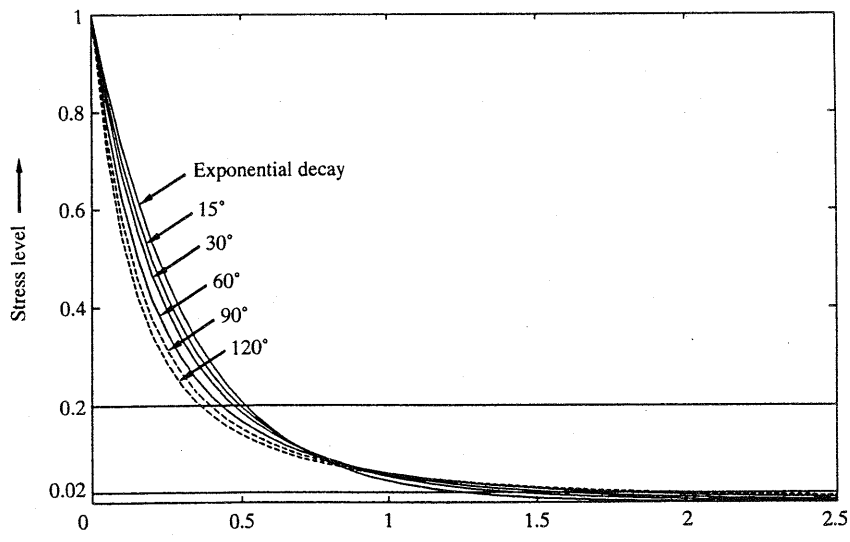

Fig. 3. Decay of stress from loaded arc $r = a$

then the plate is the quicker. (Of course, an attempt is being made to compare the exponential decay of the plate with the power law decay of the wedge. Therefore such differences should not be surprising. What is more surprising is that the decay characteristics should be so similar when expressed in the above manner.) The largest difference between the wedge and plate decay characteristics occurs close to the loaded arc $r = a$, where the rate of decay is very sensitive to the wedge angle. However, after a distance one arc length ($2a\alpha$) from the loaded arc, the difference is small. In the plane strain case, Horvay (**3**) was surprised that attenuation should be more rapid in the case of the plate, expecting that diffusion into a divergent area would enhance the rate of decay. This was attributed to the greater interference between stresses that occurs at the free edge, an effect that would be reduced with the wedge geometry; indeed, the largest wedge angle in Fig. 3, $\alpha = 120°$, has the slowest decay in the far field. The above observations suggest that the geometric effect of divergent area is dominant close to the loaded arc (a near-field effect) while free-edge interference is a far-field effect.

Returning to a comparison of the rates of decay of self-equilibrated loading on the arcs $r = a$ and $r = b$, intuitively the decay is expected to be affected by two agencies, the divergent or convergent area into which the stress field diffuses and SVP. For loading on the inner arc, $r = a$, both of these effects work in concert to give a rate of decay greater than for loading on the outer arc, $r = b$, when the two effects work in opposition. Thus, for a load on the outer arc, stress is expected to decay by virtue of SVP but to increase by virtue of the convergent geometry; moreover, as seen above, the geometric effect is expected to be of greater importance close to the loaded arc, and since it is possible to move only a maximum radial distance $(b - a)$ from the loaded arc, there may not even be a far field, in which case the free-edge interference effect will be absent. Referring to the branch $n = -1$ in Fig. 2, it can be seen that the geometric effect becomes dominant over the Saint-Venant decay for the semi-wedge angle $\alpha > \pi/2$, when $\lambda > -1$ (the locus AB); the self-equilibrating shear defined by this eigenvalue on the arc $r = b$ will then *increase* the greater the distance from the loaded arc.

For this same branch ($n = -1$), $\lambda = -\frac{1}{2}$ for $\alpha = \pi$ (point B), which is the familiar inverse square root stress singularity for the tearing mode III of LEFM. It can also be seen that the branch $n = -2$ for $\alpha = \pi$ has $\lambda = -1$, which pertains to a stress field independent of radius and will therefore also reach the crack tip. The crack tip stress field becomes

$$\begin{aligned} \tau_{rz} &= -\frac{\mu C_2}{2r_0^{1/2}}\frac{1}{r^{1/2}}\sin\frac{\theta}{2} + \frac{\mu C_1}{r_0}\cos\theta \\ \tau_{\theta z} &= -\frac{\mu C_2}{2r_0^{1/2}}\frac{1}{r^{1/2}}\cos\frac{\theta}{2} - \frac{\mu C_1}{r_0}\sin\theta \end{aligned} \tag{11}$$

where the latter terms are ignored by most writers (see, for example, reference (**9**)).

4 CONCLUSIONS

The solution to the eigenvalue problem of an elastic wedge undergoing anti-plane deformation and having traction-free flanks has been presented. Self-equilibrated loading on the inner arc will decay by virtue of the two agencies of SVP and diffusion into a divergent area, working in concert. For a self-equilibrated load on the outer arc the stress will decay by virtue of SVP but increase by virtue of the convergent geometry, and the latter effect becomes dominant for wedge angles in excess of the half-space. The unique crack tip stress singularity, which is at the heart of LEFM, can thus be attributed to the failure of SVP to provide a rate of decay that is sufficient to overcome the stress increase caused by the effect of convergent geometry for the single eigenmode pertaining to $n = -1$.

APPENDIX

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