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THE EFFECT OF ATTACHMENTS ON THE NATURAL FREQUENCIES OF A MEMBRANE

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1. INTRODUCTION

The modes of vibration of a tensioned membrane have received much attention in the literature; in fact, many textbooks on acoustics or vibration employ the tensioned membrane as perhaps the simplest example of a two-dimensional vibrating system (see, for example, references [1, 2]). Triangular, rectangular and circular membranes were all considered by Lord Rayleigh in his book *The Theory of Sound* [1], and it might be thought that little could be added to the seminal results contained therein. However, one topic which has not been investigated fully is the effect of attachments (for example, masses and springs) on the natural frequencies and mode shapes of a membrane. The specific case of a point mass attachment was considered by Rayleigh [1], who employed what would now be referred to as a one-term Rayleigh–Ritz procedure to yield approximate formulae for the natural frequencies of mass loaded square and circular membranes [1, see pp. 316–335]. Although these formulae would at first sight appear to be reasonable, there are certain conceptual and mathematical difficulties involved with adding a point mass to a membrane—specifically, it is known that (within the linear theory) the point mobility of a membrane has an infinite imaginary component [1], which suggests that any inertia force generated by a point mass attachment should lead to an infinite displacement. This casts doubt on the validity of Rayleigh's solution, and the resolution of this issue is the subject of the present work.

In section 2 an analytical study of a circular membrane which carries a central attachment is presented. It is shown that Rayleigh's solution is not actually valid, due mainly to the fact that a severe deformation occurs in the vicinity of the attachment—this deformation could be captured only by employing a multi-term Rayleigh–Ritz procedure, and even then severe convergence problems can be envisaged. This type of convergence problem is considered in section 3, where the forced response of a square membrane which carries a point mass attachment is analyzed by using a modal summation technique.

The present work highlights the physical limitations of linear membrane theory with regard to the treatment of point attachments. To obtain a "physical" solution, either the attachment must be considered to be of finite size, a non-linear theory must be adopted, and/or a finite bending stiffness must be considered. This has application to, for example, a study of the effects of attachments on the natural frequencies of a drum.

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2. EXACT ANALYSIS OF A CIRCULAR MEMBRANE

In this section an exact analysis of the free vibration of a circular membrane with a central attachment is presented. The membrane is taken to have radius R, tension T and mass per unit area m; the central attachment is taken to be a rigid disk of radius a to which a device of dynamic stiffness λ is connected. Only axisymmetric vibrations of the membrane are considered, and the out-of-plane displacement at a distance r from the centre of the membrane is written as $w(r)$. With this notation, the membrane boundary conditions take the form

$$2\pi aTw'(a)=\lambda w(a), \qquad w(R)=0, \tag{1, 2}$$

where the prime represents differentiation with respect to r, and a clamped outer boundary has been adopted. The solution of the membrane differential equation for axisymmetric vibrations of frequency ω has the well known form [1]

$$w(r)=A\mathrm{J}_0(kr)+B\mathrm{Y}_0(kr), \qquad k=\omega\sqrt{T/m}, \tag{3, 4}$$

where J_0 and Y_0 are, respectively, the zero order Bessel functions of the first and second kinds, and k is the vibration wavenumber. It is implied by equations (1)–(4) that the natural frequencies of the membrane are governed by the equation

$$[\gamma \mathrm{J}_0'(ka)-\mathrm{J}_0(ka)]\mathrm{Y}_0(kR)=[\gamma \mathrm{Y}_0'(ka)-\mathrm{Y}_0(ka)]\mathrm{J}_0(kR), \qquad \gamma=2\pi Tka/\lambda. \tag{5, 6}$$

Once this equation has been solved to yield the value of k (and hence ω) corresponding to a particular mode, the associated mode shape may be deduced from the relation

$$A/B=-\mathrm{Y}_0(kR)/\mathrm{J}_0(kR). \tag{7}$$

If the central attachment consists of a mass M, then $\lambda=-M\omega^2$ and the parameter γ can be written in the form $\gamma=-2(a/R)(M_0/M)(1/kR)$, where $M_0=m\pi R^2$ is the mass of the membrane. Numerical results for the first four natural frequencies of this configuration are shown in Figure 1 for $M/M_0=0.1$ and in Figure 2 for $M/M_0=0.3$. It can be seen that the natural frequencies are sensitive to the radius a of the attachment: as a is reduced,

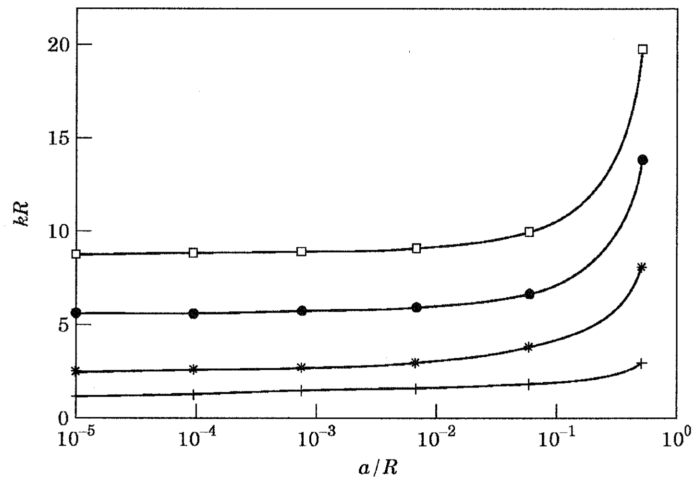

Figure 1. The first four axisymmetric natural frequencies (as measured by kR) of a circular membrane of radius R which carries a central mass of radius a. Mass ratio $M/M_0=0{\cdot}1$. +, Mode 1; $*$, mode 2; ●, mode 3; □, mode 4.

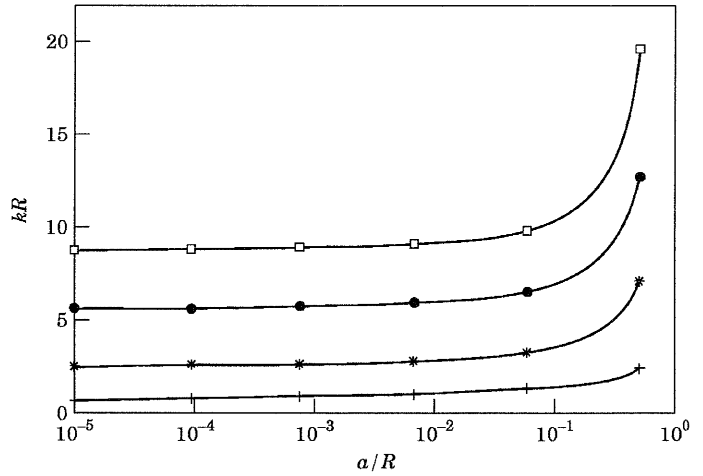

Figure 2. As Figure 1, but $M/M_0=0{\cdot}3$.

the first natural frequency tends (slowly) to zero while the higher natural frequencies asymptotically approach values which are independent of M/M_0. This behaviour can be explained by noting that, for small a, equation (5) takes the form

$$Y_0(kR)\approx[-4T/\lambda+(2/\pi)\ln(ka/2)]J_0(kR), \qquad (8)$$

where use has been made of the asymptotic form of the Bessel functions for small argument [3]. Clearly, for $a\to 0$, equation (8) will yield roots corresponding to $J_0(kR)=0$. These roots ($kR=2{\cdot}405, 5{\cdot}520, 8{\cdot}654, \ldots$) are the asymptotic values which are approached in Figures 1 and 2 for mode 2 onwards, and physically they correspond to the natural frequencies of a circular membrane with no central attachment. With regard to the fundamental natural frequency which is shown in Figures 1 and 2, it can be noted that, for the case of a central mass ($\lambda=-M\omega^2$), equation (8) also admits the following solution as $a\to 0$:

$$kR=\sqrt{-2/(M/M_0)\ln(a/R)}. \qquad (9)$$

This solution (which approximates the fundamental natural frequency for small a) tends very slowly to zero with decreasing a, and this behaviour is exhibited clearly in Figures 1 and 2.

To summarize the foregoing discussion, it has been shown that as $a\to 0$ the non-zero natural frequencies of a membrane with a central attachment approach those of a simple membrane with no central attachment, *regardless of the value of the dynamic stiffness* λ. It is interesting to note, however, that the mode *shapes* of the membrane with the central attachment are subtly different from those of the simple membrane. It follows from equation (1) that $\lambda w(a)\to 4BT$ as $a\to 0$; now for those modes which have $J_0(kR)\approx 0$ it follows from equation (7) that $B/A\to 0$, and thus equation (1) ultimately yields $w(a)\to 0$, so that the central attachment is stationary. This behaviour is illustrated in Figures 3 and 4 for the case of a central mass with $M/M_0=0{\cdot}1$. It is clear that the central mass tends to become stationary for modes 2–4 as $a\to 0$, and this is achieved by the appearance of a very localized central depression. A curious limiting process is at work here—one would imagine that a force is required to produce the depression, and yet (in apparent contradiction) no force is generated by the attachment if $w(0)=0$. However, it is known

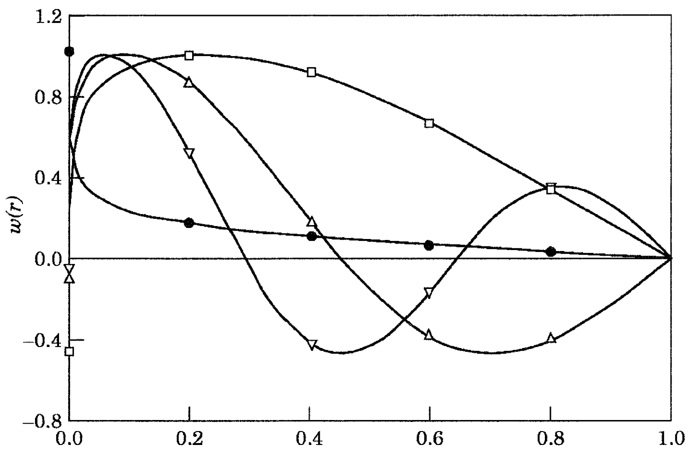

Figure 3. The first four axisymmetric modes $w(r)$ for a circular membrane with a central mass attachment: $a/R=10^{-5}$, $M/M_0=0{\cdot}1$. ●, Mode 1; □, mode 2; △, mode 3; ▽, mode 4.

that the action of any finite force on a membrane produces an infinite deflection [1], and thus (simplistically) a "zero" force is actually required to produce the finite depression.

It can be seen in Figures 3 and 4 that mode 1 involves motion of the mass with very little motion of the membrane other than in the immediate vicinity of the attachment; as $a \to 0$ the mode resembles the zero frequency mode of a "free" mass. Clearly, the mode shapes which are shown in Figures 3 and 4 are likely to cause convergence problems for approximate methods of analysis, and this is investigated in the following section for the case of a square membrane.

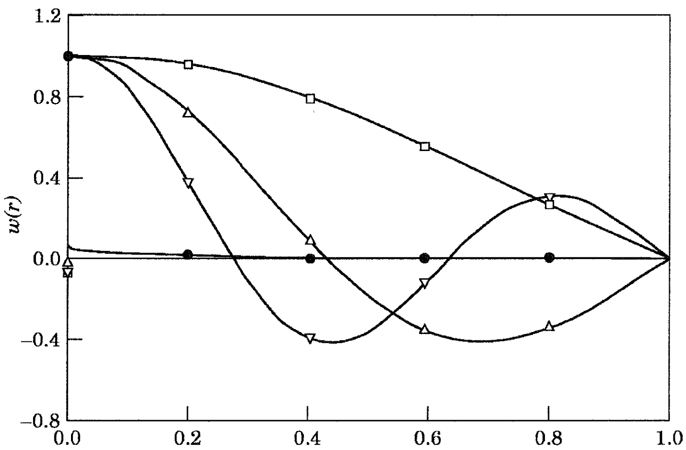

Figure 4. As Figure 3, but $a/R=10^{-70}$. Note the very sharp dip in the vicinity of $r=0$ for modes 2–4, and the very sharp peak in this vicinity for mode 1.

3. MODAL ANALYSIS OF A SQUARE MEMBRANE

Some of the difficulties inherent in the analysis of mass loaded membranes when dealt with in closed form have been demonstrated. Attention is next turned to a modal expansion solution. Consider a square membrane of side length L, again with fixed boundary conditions. The modes of the unloaded membrane are well known, and the mode shapes ψ_{NM} and natural frequencies ω_{NM} are given by

$$\psi_{NM}(\mathbf{x}) = (2/L\sqrt{m}) \sin(N\pi x_1/L) \sin(M\pi x_2/L), \qquad \omega_{NM} = \pi[(N^2+M^2)T/mL^2]^{1/2}, \tag{10a, b}$$

where N and M are integers indicating the number of half-wavelengths in each direction across the membrane. Notice that, for a square membrane, this leads to degenerate modes for $(N, M) = (M, N)$ when the mode shapes cannot be specified independently of each other.

Given the natural frequencies ω_n (with n being used here in preference to the double index NM appearing in equation (10)) and mode shapes ψ_n of such a system, it is then possible to construct the Green function $G(\mathbf{x}, \mathbf{y}, \omega)$ which relates the response at $\mathbf{x}$ due to unit harmonic forcing at $\mathbf{y}$ of frequency ω:

$$G(\mathbf{x}, \mathbf{y}, \omega) = \sum_{n=1}^{\infty} \psi_n(\mathbf{x})\psi_n(\mathbf{y})/[(\omega_n^2 - \omega^2) - \mathrm{i}C\omega], \tag{11}$$

where C is the viscous damping constant for the system, which is zero for undamped membranes.

The problem of a mass loaded membrane can then be dealt with by considering the unknown force F exerted by the mass on the membrane and setting this equal to the size of the mass times its acceleration. Given that the motion of the mass is given by the response of the attachment point, it is then possible to calculate the behaviour of the

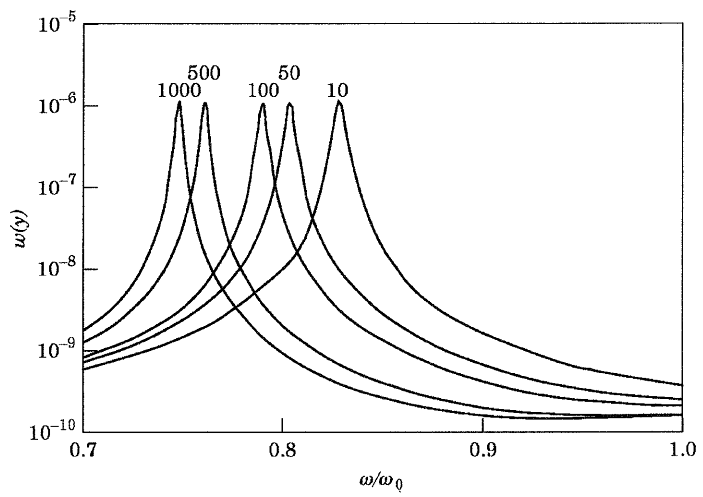

Figure 5. The response of a square membrane with a central point mass ($M/M_0 = 0{\cdot}1$) for unit forcing applied at ($0{\cdot}333\,L$, $0{\cdot}75\,L$). The number written above each curve represents the number of terms included in the modal summation. ω_0 represents the fundamental natural frequency of the membrane in the absence of the mass.

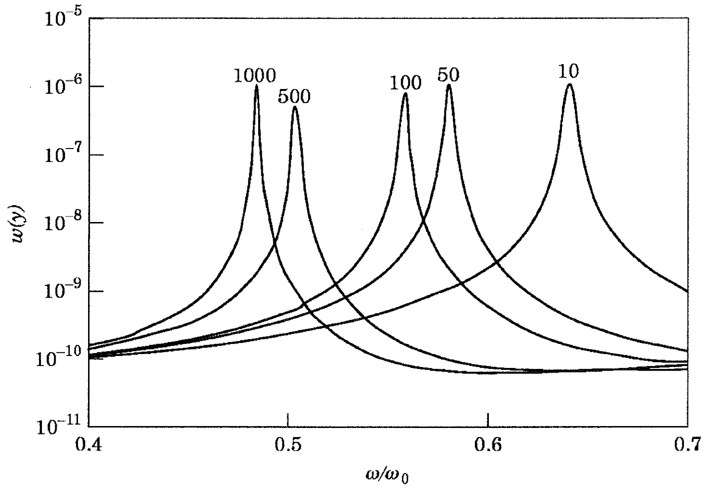

Figure 6. As Figure 5, but $M/M_0 = 0{\cdot}3$.

combined system. Specifically, consider a system with point force P applied at $\mathbf{x}$ and a mass at $\mathbf{y}$:

$$w(\mathbf{x}) = PG(\mathbf{x}, \mathbf{x}, \omega) + FG(\mathbf{x}, \mathbf{y}, \omega), \qquad w(\mathbf{y}) = PG(\mathbf{y}, \mathbf{x}, \omega) + FG(\mathbf{y}, \mathbf{y}, \omega),$$
$$\ddot{w}(\mathbf{y}) = -F/M. \tag{12–14}$$

Eliminating F and noting that for harmonic motion the double differentiation yields $-\omega^2$ leads to

$$w(\mathbf{x}) = P\{G(\mathbf{x}, \mathbf{x}, \omega) + M\omega^2[G(\mathbf{y}, \mathbf{x}, \omega)]^2/[1 - M\omega^2 G(\mathbf{y}, \mathbf{y}, \omega)]\}. \tag{15}$$

This function can then be evaluated for various numbers of terms in the summations. Clearly, $w(\mathbf{x})$ will show peaks at resonance frequencies of the combined mass and membrane, thus enabling the behaviour of the overall system to be studied.

In Figures 5 and 6 is shown the behaviour of the first mode of the combined system as obtained by using such calculations, for the two mass ratios used previously. The vertical scale shown in these figures is rather arbitrary, in the sense that the present work is concerned mainly with the frequency location of the peak response, rather than its magnitude. As can be seen from the figures, as the number of terms in the summations is increased, the peaks migrate to the left, indicating that the frequencies reduce in an analogous fashion to those for the circular membrane considered in the previous section. Thus the effect of using only a finite number of terms in such modal sums can be considered as equivalent to a variation in the size of the attachment region, with a point mass being correctly modelled only when all the terms in the summation are used. The slow convergence seen in the previous section (for the fundamental mode) is seen again here, although it now manifests itself in the slow convergence of the summations.

4. CONCLUSIONS

It has been shown that some care is needed when considering the dynamic behaviour of a membrane which carries an attachment. If the attachment is considered to act at a point then this point will be held stationary for the non-zero frequency modes of vibration, regardless of the value of the (non-zero) dynamic stiffness of the attachment. This is achieved by the occurrence of a very local deformation of the membrane in the vicinity of the attachment. Strictly this result has been demonstrated here only for a central attachment on a circular membrane, but a "physical" argument based on the nature of the point mobility of a membrane (as outlined in section 2) demonstrates the more general validity of the result: were the attachment to move, then a force would be generated, which would in turn produce an infinite displacement and thus an infinite force—thus either the attachment does not move, or the natural frequency of the motion must tend to either infinity or zero, depending on the dynamic stiffness of the attachment (infinity would be achieved for positive dynamic stiffness and zero for negative dynamic stiffness).

The present results are in disagreement with earlier results due to Rayleigh [1], who calculated the effect of a point mass attachment on the natural frequency of a membrane by employing what would now be referred to as a one-term Rayleigh–Ritz solution technique. The disagreement arises from the fact that many terms are in fact required to model correctly the local deformation of the membrane in the vicinity of the mass; this has been demonstrated here by the modal solution presented in section 3. As mentioned in section 2, the difficulties encountered by the linear membrane theory can be overcome by considering the attachment to be of finite dimension; for very small attachments, the inclusion of bending and non-linear effects may also be required.

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