A boundary element scheme for plate post-buckling analysis

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Abstract

The non-linear post-buckling behaviour of a plate of any shape and support conditions, subjected to loading parallel to its plane, can only be predicted by numerical methods. Recent developments in formulations of this problem, based on the boundary element method (BEM), are reviewed. A new procedure is proposed whereby BEM modelling is combined with domain models for the lateral deflection and the stress function yielding the membrane forces. The non-linear plate response to in-plane loading initiated by imperfections is determined incrementally with the option of increasing accuracy through iterations within each loading step. The curvature and membrane force distribution in the deformed plate is approximated through non-linear interpolation models for the deflection and stress function over domain cells.

1 Introduction

Although thin plates resist very effectively forces parallel to their middle plane, their slenderness may cause instability at loads below their ultimate strength. The possibility of this type of failure must be envisaged in engineering practice. Since the first experimental observation of the plate buckling phenomenon almost 150 years ago, the problem has been extensively investigated both analytically and experimentally.[1] Closed-form solutions for critical loads were obtained for a number of special cases. Rigorous analyses, combined with computational techniques, provided very accurate numerical answers for various plate shapes and a wide range of loading, material as well as support conditions. More recently, approximate methods implemented through computer codes have provided solutions of even wider applicability.

The advantages of boundary element algorithms with regard to computer memory requirements, speed of execution and simplicity of input data structure have been demonstrated in a wide range of applications. However, a genuine BEM formulation for plate stability problems is not possible due to unavailability of closed form fundamental solutions valid for any membrane stress distribution. The approach most commonly adopted by analysts has been the use of the fundamental solutions of the plate bending problem. The resulting integral equations contain irreducible domain integrals arising from in-plane loading and depending on the generated non-uniform membrane stress distribution as well as the unknown curvatures of the deflected plate. Thus the extension of BEM to plate stability problems has not been achieved without penalties since such formulations require domain discretization and modelling which reduce, to a certain extent, the efficiency of the method.[2–6] In addition, a post-buckling analysis imposes the numerical complexities arising from geometric non-linearity as well as coupling between flexural and membrane behaviour.

The boundary element method had originally been applied to the analysis of non-linear plate behaviour induced by lateral loads. A general formulation based on the incremental form of the coupled von Kármán equations has been presented.[7] It was proposed that these equations be solved either by iteration or directly through complementary domain modelling. According to an alternative scheme based solely on the Rayleigh-Green identity for the biharmonic operator,[8] the plate curvatures and the membrane forces are considered as additional unknowns at domain nodes and the resulting non-linear algebraic problem is solved by iteration.

Other incremental and iterative analyses extended the scope of BEM to predict the non-linear plate response to in-plane edge loading. Such numerical simulation can either be initiated by imperfections[9] or generated as a bifurcation path from critical equilibrium.[10,11] An incremental scheme was adopted with the deflection as the only domain unknown requiring modelling and a domain discretization scheme.[9] In the present analysis, the domain modelling is extended to the stress function. Following an earlier BEM plate buckling formulation,[6] domain curvatures and membrane forces are eliminated from the system of integral equations governing the deflection and the stress function. The dependence of nodal curvatures and membrane forces on domain unknowns is established through non-linear interpolation functions over domain cells. Since this scheme reduces the overall number of unknowns, it is expected to increase the efficiency of the method without significant loss of accuracy.

2 Plate post-buckling theory

Plates with initial imperfections loaded in their plane, undergo some deflection before the theoretical critical buckling load is reached. In fact, when these deflections are not negligible compared to the thickness of the plate, the strains in the middle plane of the plate cannot be ignored in the analysis. This coupling between bending and membrane action is modelled by the well known von Kármán equations:[12]

$$N_{\alpha\beta,\beta} = 0 \tag{1}$$

$$D\nabla^4 W = N_{\alpha\beta}(W_{,\alpha\beta} + W^i_{,\alpha\beta}) \tag{2}$$

where $N_{\alpha\beta}$ are the membrane forces, D the rigidity of the plate defined by

$$D = \frac{Eh^3}{12(1-\nu^2)},$$

W is the plate deflection, W^i the initial plate imperfection, E the Young's modulus, ν the Poisson's ratio and h the plate thickness. Greek subscripts indicate mid-plane co-ordinates and a comma denotes differentiation with respect to subscripts, with summation implied over repeated indices.

The use of the constitutive equations of plane stress elasticity and the expressions for large strain transforms equations (1) and (2) into a system of non-linear differential equations for the middle-plane displacement V and the deflection.[10,11] Alternatively, the problem can be formulated in terms of a stress function F such that

$$N_{\alpha\beta} = h[(\nabla^2 F)\delta_{\alpha\beta} - F_{,\alpha\beta}] \tag{3}$$

Then equilibrium equations (1) are identically satisfied and are replaced by the compatibility condition for plane strain. The new system of equations for F and W has the form

$$\nabla^4 F = -\frac{E}{2}[S_{\alpha\beta}(\hat{W})\hat{W}_{,\alpha\beta} - S_{\alpha\beta}(W^i)W^i_{,\alpha\beta}] \tag{4}$$

$$D\nabla^4 W = N_{\alpha\beta}\hat{W}_{,\alpha\beta} \tag{5}$$

where $\hat{W}$ is defined as the total deflection

$$\hat{W} = W + W^i.$$

and tensor $S_{\alpha\beta}$ is derivable from the deflection according to

$$S_{\alpha\beta}(W) = (\nabla^2 W)\delta_{\alpha\beta} - W_{,\alpha\beta} \tag{6}$$

it may therefore be considered as a pseudo-stress satisfying identically equilibrium equations (1).

In the absence of imperfection, the obvious solution of (1) and (2) is the pre-buckling, plane-stress state

$$N^0_{\alpha\beta},\ W = 0$$

which bifurcates to the post-buckling solution $N_{\alpha\beta}$, $W \neq 0$ at critical equilibrium. Numerical difficulties associated with this sudden transition are side-stepped if imperfections are included in the formulation. Modelling imperfect plates also makes BEM solutions directly comparable to experimental measurements.

The various proposed methods of solution of the non-linear problem adopt an incremental-iterative approach whereby the non-linear equations in V and W or F and W are replaced by linear equations in their respective increments v and w or f and w. In the case of the more consistent F-W formulation, the differential equations governing the incremental variables are derived from (4) and (5) as

$$\nabla^4 f = -E[S_{\alpha\beta}(\hat{W}) + \frac{1}{2}S_{\alpha\beta}(w)]w_{,\alpha\beta} \tag{7}$$

$$D\nabla^4 w = N_{\alpha\beta} w_{,\alpha\beta} + n_{\alpha\beta}(\hat{W}_{,\alpha\beta} + w_{,\alpha\beta}) \quad (8)$$

where $n_{\alpha\beta}$ are the increments of $N_{\alpha\beta}$. It is noted that the quadratic terms in the incremental quantities have been retained in equations (7) and (8).

3 Integral equations

A BEM solution of (7) and (8) can be based on the Rayleigh-Green identities for the biharmonic operator and the flexural plate theory:

$$\int_\Omega [u(\nabla^4 f) - f(\nabla^4 u)]\mathrm{d}\Omega = I_\mathrm{b}^\mathrm{f}(u,f) \quad (9)$$

$$D\int_\Omega [u(\nabla^4 w) - w(\nabla^4 u)]\mathrm{d}\Omega + I_\mathrm{b}^\mathrm{w}(u,w) = J(u,w) \quad (10)$$

where Ω is the plate domain, Γ its boundary, u any weighting function and

$$I_\mathrm{b}^\mathrm{f}(u,f) = \int_\Gamma \left[u\frac{\partial \nabla^2 f}{\partial n} - \frac{\partial u}{\partial n}\nabla^2 f + \nabla^2 u \frac{\partial f}{\partial n} - \frac{\partial \nabla^2 u}{\partial n} f \right]\mathrm{d}\Gamma \quad (11)$$

$$I_\mathrm{b}^\mathrm{w}(u,w) = \int_\Gamma \left[uV(w) - \frac{\partial u}{\partial n}M_n(w) + M_n(u)\frac{\partial w}{\partial n} - V(u)w \right]\mathrm{d}\Gamma \quad (12)$$

$$J(u,w) = \sum_{j=1}^{N_k} \left(u[M_{ns}(w)] - w[M_{ns}(u)] \right)_{s=s_j} \quad (13)$$

M_n, M_{ns}, V are operators for the bending moment, twisting moment, effective shear, respectively, along a boundary with unit normal **n** and unit tangent vector **s**. A bold-faced square bracket indicates the difference in the enclosed quantity in the positive and negative directions of s at the corner points of Γ. Thus, the term J accounts for the discontinuity jumps of the twisting moment at the N_k corners.

The integral equations are derived from eqns (9) and (10) using the following two fundamental solutions of the biharmonic operator:

$$u_1(Q) = \frac{r^2}{8\pi}\ln r \tag{14}$$

$$u_2(Q) = -\frac{r}{8\pi}(2\ln r+1)\cos\alpha \tag{15}$$

The physical interpretation of these solutions in the context of both plane stress and plate bending analyses has been given in an earlier account of a BEM solution of the general plate buckling problem.[6] By accounting for eqns (7) and (8), governing f and w, and by entering the fundamental solutions u_i, (i=1,2) as weighting functions in eqns (9) and (10), the following system of integral equations is obtained:

$$\kappa f_i + I_b^f(u_i,f) + EI_d(u_i,\hat{W},w) + \frac{E}{2}I_d(u_i,w,w) = 0 \tag{16}$$

$$-\kappa D w_i + I_b^w(u_i,w) + I_d(u_i,N,w) + I_d(u_i,n,\hat{W}+w) = J(u_i,w) \tag{17}$$

where

$$\kappa = \begin{cases} 0.5 & \text{if P on } \Gamma \\ 1 & \text{if P in } \Omega \end{cases}$$

$$f_1=f(P), f_2=\frac{\partial f}{\partial n}(P), w_1=w(P), w_2=\frac{\partial w}{\partial n}(P),$$

and

$$I_d(u,W,w) = \int_\Omega u S_{\alpha\beta}(W) w_{,\alpha\beta} d\Omega$$

$$I_d(u,N,w) = \int_\Omega u N_{\alpha\beta} w_{,\alpha\beta} d\Omega$$

Using the identities

$$I_d(u,W,w) - I_d(w,W,u) = I_t(u,W,w),$$

$$I_d(u,N,w) - I_d(w,N,u) = I_t(u,T,w),$$

where T(s) is the edge traction and

$$I_t(u,W,w) = \int_\Gamma \left[\frac{\partial^2 W}{\partial s^2}\left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right) - \frac{\partial^2 W}{\partial n \partial s}\left(u\frac{\partial w}{\partial s} - w\frac{\partial u}{\partial s}\right)\right] d\Gamma,$$

$$I_t(u,\mathrm{T},w) = \int_\Gamma \left[T_n\left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right) + T_s\left(u\frac{\partial w}{\partial s} - w\frac{\partial u}{\partial s}\right)\right] d\Gamma,$$

integral equations (16) and (17) are transformed to

$$\kappa f_i + I_b^f(u_i,f) + EI_t(u_i,\hat{W},w) + EI_d(w,\hat{W},\mu_i) + \frac{E}{2} I_d(u_i,w,w) = 0 \quad (18)$$

$$-\kappa D w_i + I_b^w(u_i,w) + I_t(u_i,\mathrm{T},w) + I_t(u_i,\hat{W},f)$$
$$+ I_d(w,\mathrm{N},\mu_i) + I_d(f,\hat{W},\mu_i) + I_d(u_i,\mathrm{n},w) = J(u_i,w) \quad (19)$$

Boundary element modelling leads to the solution of integral equations (18) and (19) which govern the incremental plate post-buckling behaviour. The quadratic terms in these equations are initially omitted from the formulation. It is also assumed that the total forces, moments, displacements and curvatures are known at the start of a load step. Accounting for the specified support and loading conditions, there are six unknown variable quantities in the boundary integrals $I_b^f(u_i,f)$, $I_b^w(u_i,w)$, $I_t(u_i,\hat{W},w)$, $I_t(u_i,\mathrm{T},w)$, $I_t(u_i,\hat{W},f)$ and the jump term $J(u_i,w)$. More specifically, the boundary values of f and its normal derivative are functions of the edge traction increment while deflection-related boundary variables depend on the support conditions.[6] The deflection, appearing in the domain integrals $I_d(w,\hat{W},\mu_i)$ and $I_d(w,\mathrm{N},\mu_i)$, and the stress function f, appearing in the domain integral $I_d(f,\hat{W},\mu_i)$, are additional unknown variables. For consistency, two domain equations, obtained by setting $\kappa=1$ and $i=1$ in eqns (18) and (19), complement the formulation

$$f + I_b^f(u_1,f) + EI_t(u_1,\hat{W},w) + EI_d(w,\hat{W},\mu_1) + \frac{E}{2} I_d(u_1,w,w) = 0 \quad (20)$$

$$-Dw + I_b^w(u_1,w) + I_t(u_1,\mathrm{T},w) + I_t(u_1,\hat{W},f)$$
$$+ I_d(w,\mathrm{N},\mu_1) + I_d(f,\hat{W},\mu_1) + I_d(u_1,\mathrm{n},w) = J(u_1,w) \quad (21)$$

into the one-dimensional array $\{Z_f\}$ while those in $I_b^w(u_i,w)$ and $I_t(u_i,\hat{W},w)$ are grouped into $\{Z_w\}$. Nodal deflection and stress function values are assembled in $\{w\}$ and $\{f\}$, respectively. The terms arising from the known values of f and its normal derivative on the boundary make up arrays $\{A_f\}$ and $\{A_w\}$. Finally, $\{Q_f\}$ and $\{Q_w\}$ result from the integration of the quadratic terms in eqns (18) and (19) and are evaluated only if iteration is envisaged.

An additional system of $2N_d$ algebraic equations is obtained by placing the source point on the N_d domain nodes and applying eqns (20) and (21). The final result is matrix equations of the form

$$\delta\lambda[D_b^f]\{Z_f\} + [D_t^f]\{Z_w\} + \{f\} + [D_d^f]\{w\} = \delta\lambda\{B_f\} + \{R_f\} \quad (27)$$

$$([D_b^w] + \lambda[D_t^w])\{Z_w\} + [D_d^{wf}]\{f\} + [D_d^w]\{w\} = \{B_w\} + \{R_w\} \quad (28)$$

where arrays $\{B\}$ and $\{R\}$ are defined in the same way as $\{A\}$ and $\{Q\}$, respectively.

At the beginning of each load step the solution is carried out without the arrays $\{Q\}$, and $\{R\}$ containing the integrals of the quadratic terms in the unknown increments. This results in a consistent system of equations in the boundary and domain unknowns. It is noted that several coefficient arrays in eqns (25)–(28) are fixed and need to be calculated only once, at the beginning of the incremental solution process. After incremental deflection and stress function values are determined, the nodal curvatures and membrane stresses are calculated using eqns (24). An alternative, more rigorous but also more time consuming approach would be to differentiate the linear parts of eqns (16) and (17) with $\kappa=i=1$, then generate and solve a coupled system of equations in $\{y\}$ and $\{n\}$. The result may be further corrected by including the integrals of the quadratic terms and solving the system iteratively within the current load step. Finally, the total values of all variables are calculated to be used in the next load step.

6 Discussion and Further Work

A new BEM formulation for plate post-buckling has been developed. It is based on a neat set of integral equations, mathematically symmetric relative to the domain variables, namely deflection and stress function. Through thoughtful application of Green's theorem, second derivatives of the unknown variables, representing curvatures and membrane stresses have been eliminated from the algebraic system resulting from BEM and domain modelling.

This is an essentially incremental procedure whereby small changes in the field variables are determined at each load step with the option of performing iterations in order to enhance accuracy in the case of large load steps or near bifurcation points. This contrasts with other formulations which rely solely on iteration and, therefore, may not be as numerically robust as the proposed method which has the additional advantage of yielding directly all the important design quantities as part of the solution process.

The proposed numerical procedure needs to be validated by comparing its predictions with published experimental data or other analytical results. The versatility of the method can also be demonstrated by analysing plates with a variety of geometrical, support or loading characteristics. The successful implementation of a similar BEM solution[9] is a positive indicator for the effectiveness and reliability of the new algorithm.

A disadvantage of the proposed scheme in its present form is that it cannot admit in-plane edge constraints. In this respect, it compares unfavourably with schemes which consider the in-plane displacements rather than the stress function as field variables.[10,11] One aim of further work would be the extension of the analysis to account for such constraints through minor changes in the boundary integral formulation. Other, more long term aims would be the development of stability analysis for elastoplastic and stiffened plates.

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