2 BOUNDARY ELEMENT METHODS IN STRUCTURAL DYNAMIC SYSTEM PROBLEMS

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2.1. INTRODUCTION

The aim of the boundary element method (BEM) is the numerical solution of integral equations derived from various mathematical models of material behaviour. It is desirable that only the variation of unknown quantities over the boundary of the analysed medium appear in those equations so that only boundary meshing and modelling schemes would be required. Hence, no approximations are necessary for the variables within the domain and the dimensionality of the problem is reduced by one. Both the direct as well as the indirect versions of BEM use as weighting functions in the formulation and numerical integration process the fundamental solution of the linear field equations governing the physical problem under consideration. Thus, the method is particularly effective in modelling and analysing unbounded regions since then it permits the discretization of only internal boundaries and, in the case of wave propagation, incorporates the radiation condition at infinity.

Linear structural dynamics is a well established field of study basing its predictions on a wide range of mathematical tools originating from the theory of elasticity. The fundamental solutions to most linear dynamic problems have been obtained and the corresponding integral equations derived long before the emergence of BEM. The most general theory is that for a three-dimensional elastic continuum. Its fundamental solution and integral equation have been extensively studied and applied.[1] BEM formulations of dynamic problems appeared quite early in the development of the method which started almost 60 years ago. The amount of research and published output has increased enormously since computers became widely accessible for the numerical implementation of the generated solution algorithms. There have been already several monographs and collection of articles,[2,3] exclusively dedicated to the theory and applications of BEM in the general area of dynamics.

The emphasis in this contribution will be on recent developments. The basic concepts and all standard formulations are, of course, presented in order to give the reader a broad awareness of the potential and scope of BEM. An outline of the method and a description of the general theory to which it is applicable is followed by the presentation of the special BEM versions developed for the common structural elements, namely rods, beams, membranes, plates and shells. Since fluids interact dynamically with structures, their BEM modelling is discussed as well. Then follows the general BEM solution of the 2D or 3D elastodynamics problem which is relevant to a solid of any shape but in practice it is mostly applied to a semi-infinite soil medium as part of a foundation system. Reference to particular structural applications highlight its usefulness and effectiveness as a predictive tool.

As anticipated, a major section is devoted to the modelling of systems, that is, combinations of interacting solids and/or fluids for which individual BEM solutions need to be coupled through interface conditions. Considerable effort has gone into extending the applicability of the method to problems with material non-linearity and recent developments on this front are reported in some detail. As a powerful analysis tool, BEM has found application to an unexpectedly wide range of problems. A sample of such special applications are presented with descriptions of the particular formulations used. Finally, an important aspect of BEM development has, for quite a while, been its potential for coupling with other numerical methods so that its advantages are exploited to a maximum. The last section is devoted to the description and assessment of such hybrid formulations.

2.2. GENERAL THEORY

The governing field equations result from the application of Newton's Law of motion to the particular dynamic problem. In a multi-variable theory, they have the common form

$$L_{ij}u_j = f_i \tag{1}$$

where L_{ij} are differential operators, u_j the field variables, and f_i the time-dependent external forces in the specified co-ordinate system. Summation is assumed over repeated indices whose range depends on the dimensionality of the problem. Thus, in three-dimensional elastodynamics, u_j are the displacement components relative to a Cartesian frame of reference x_i, and $i, j = 1,2,3$. In compressive fluids, the dynamic problem is often formulated with the scalar pressure p as the field variable. The inertia term containing the second time derivative may be included either in L_{ij} or, as a D'Alembert force, in f_i depending on the adopted approach and the type of analysis.

The Green's functions of a particular problem satisfy

$$L_{ij}\overset{*}{u}_{jk}(\mathbf{x},t|\xi,\tau) = \delta(t-\tau)\delta_{ik}(\mathbf{x}-\xi) \tag{2}$$

they are therefore the responses at the field point $Q(\mathbf{x})$ to unit impulses at times τ acting at source points $P(\xi)$. They take the special name of fundamental solutions if obtained for domains extending to infinity. If displacement is the field variable, the relevant constitutive equations of the respective theory provide the expressions for the various stresses or stress resultants associated with the fundamental solution. The application of the appropriate reciprocity relation leads to an integral equation of the general form

$$\kappa_{ij}u_j(\xi,\tau) = I_b + I_c + \int_{\Omega} \overset{*}{u}_{ij}(\mathbf{x}|\xi) * f_j(\mathbf{x})d\Omega(\mathbf{x}) \tag{3}$$

where Ω is the problem domain, κ_{ij} is $0.5\delta_{ij}$ if $P(\xi)$ is at a smooth portion of the boundary; a star denotes the convolution integral:

$$f * g = \int_0^t f(t-\tau)g(t)dt, \tag{4}$$

I_b is a boundary integral of time convolutions between expressions depending on the fundamental solution $\overset{*}{u}_{ij}$ and the unknown variable u_i. Finally, the initial conditions are accounted for through the domain integral I_c. Once the boundary unknowns are determined by BEM, (3) is applied for any interior source point ξ, that is, with $\kappa_{ij} = \delta_{ij}$, to generate domain values of the unknown variables as well as of their derivatives. This leads to the evaluation of stress time histories.

The time dependence of the problem can be removed by taking the Laplace or the Fourier transform of both sides of (1) and (2). The transformed operator $\hat{L}_{ij}$ can also be obtained by simply assuming that both excitation and response are harmonic at a certain frequency ω in which case $\hat{f}_i$ and $\hat{u}_j$ are, respectively, their amplitudes. In harmonic or transformed lation functions are recommended in problems with significant rotational deformation as in beam, plate or shell-type structures.[5]

To a large extent, the techniques for generating and using interpolation functions in the spatial modelling process are similar to those developed for the finite element method (FEM). The most important distinction between the two methods is made with reference to the integration over elements since the integrands in BEM include kernels with singularities and hence require particular attention. The substitution of expressions such as (14) into the integral equation (3) which is repeated for every nodal point as the source point, leads to a system of algebraic equations for the nodal values of the boundary variables over a specified set of successive time steps. Then a time-marching solution scheme yields all the responses up to the current time step.

In frequency domain analyses, the modelling formula (14) reduces to

$$\hat{u}_j = \sum_k \phi^k(\mathbf{x})\hat{u}_j^k \tag{15}$$

and the system of equations resulting from BEM modelling resemble that obtained from the corresponding static problem. The matrix coefficients depend however on frequency which can be treated either as the transform variable over a specific range leading to the determination of frequency response functions, or as an unknown quantity leading to the evaluation of the natural frequencies and the associated modes of free vibrations.

2.5. STRUCTURAL MODELS

2.5.1. Rods

Plane wave propagation along the axis of a uniform rod or bar is the simplest one-dimensional elasticity model, in which the only non-vanishing displacement component $u_1 = u$ depends only on the axial co-ordinate $x_1 = x$. In the absence of body force, the time-dependent differential operator reduces to

$$(\lambda + 2\mu)\frac{\partial^2}{\partial x^2} - \rho\frac{\partial^2}{\partial t^2} - g\frac{\partial}{\partial t} \tag{16}$$

in which g is the viscous damping coefficient. Such pure longitudinal waves are physically realisable only if the rod is fully constrained laterally. In slender rods with free lateral surfaces, only quasi-longitudinal waves can be generated, governed by the same operator as (16) but with the Young's modulus E replacing $\lambda + 2\mu$.[4]

A time-stepping scheme has been proposed as an alternative to discretizing and modelling the time axis.[6] The formulation is based on the following finite difference approximations for the velocity and acceleration:

$$\dot{u}(x,t+\Delta t) \approx \frac{1}{\Delta t}[u(x,t+\Delta t)-u(x,t)] \tag{17}$$

$$\ddot{u}(x,t+\Delta t) \approx \frac{1}{(\Delta t)^2}[u(x,t+\Delta t)-2u(x,t)]+u(x,t-\Delta t)] \tag{18}$$

Substitution of (17) and (18) into the governing equation of motion leads to the one-dimensional, modified Helmholtz operator

$$\frac{d^2}{dx^2}-k^2 \tag{19}$$

on the displacement at time $t + \Delta t$ and a fictitious forcing term

$$\hat{f}(x)=\frac{1}{(c_1\Delta t)^2}[u(x,t-\Delta t)-(2+\eta)u(x,t)] \tag{20}$$

where the 'wave number' is defined by

$$k=\frac{\sqrt{1+\eta}}{c_1\Delta t}$$

and

$$\eta=\frac{g(\Delta t)}{\rho}$$

can be considered as a damping ratio. With the time dependence removed, the boundary integral equation is of the form (5). In this one-dimensional problem the boundary consists of the two end points of the rod, hence

$$I_b=\left[\hat{u}^*(x,\xi)\frac{\partial u}{\partial x}(x)-\frac{\partial \hat{u}^*}{\partial x}(x,\xi)u(x)\right]_{x=0}^{x=L}$$

Taking the source point $P(\xi)$ to each of the two ends and accounting for the end conditions results in a system of two equations for the remaining boundary unknowns. Then the space variation of the displacement at $t + \Delta t$ can be determined using (5) with $\kappa = 1$ and the transition to the next time

2.5.3. Membranes

For a membrane under uniform tension T and lateral dynamic pressure $q(x_1, x_2, t)$, the differential operator on the dynamic deflection $w(x_1, x_2, t)$ may be reduced to the form

$$L = \nabla^2 - c^{-2}\frac{\partial^2}{\partial t^2} \tag{29}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{30}$$

is the Laplace operator in two dimensions and

$$c = \sqrt{\frac{T}{\rho}} \tag{31}$$

is the wave velocity. The forcing function is given by

$$f = -\frac{q}{Th} \tag{32}$$

where h is the membrane thickness. Operator (29) is the two-dimensional version of the operator in the scalar wave equation which governs a wide range of physical dynamic problems, even outside structural dynamics.

Although analytical transient solutions to the membrane problem can be found in the literature[9], BEM applications have been mainly restricted to steady-state harmonic responses. Transformation of the problem into the frequency domain leads to the well known Helmholtz operator

$$L = \nabla^2 + k^2 \tag{33}$$

with the wave number

$$k = \frac{\omega}{c} \tag{34}$$

The boundary integral in (5) takes the form

$$I_b = \int_\Gamma \left[\hat{u}^*(\mathbf{x}, \xi, \omega)\frac{\partial \hat{w}}{\partial n}(\mathbf{x}) - \frac{\partial \hat{u}^*}{\partial n}(\mathbf{x}, \xi, \omega)\hat{w}(\mathbf{x}) \right] d\Gamma(\mathbf{x}) \tag{35}$$

where Γ represents the boundary and **n** is its unit normal. In order to apply the boundary conditions, it is important to recognise that the edge shear is given by the operator $Q = -T(\partial/\partial n)$ on the deflection.

Numerical problems arise when evaluating the fundamental solution at high frequency levels. These may be overcome by adopting a recently proposed approach whereby the Hankel functions making up the fundamental solution (Appendix) are replaced by their asymptotic equivalents at low and high values of the arguments and then transition functions are introduced to approximate the fundamental solutions for intermediate argument values.[10] The transition functions are chosen so that they satisfy the appropriate smoothness conditions at the two ends of their specified ranges. A similar approximation was also proposed for the fundamental solution of the plate harmonic problem.

Using as a weighting function the fundamental solution $\overline{u}^*$ of the Laplace operator (30), the Helmholtz differential equation can be transformed into an alternative integral equation which however includes a domain integral depending on the frequency as well as the unknown deflection amplitude. It is then possible to identify a series of functions $u_1^*, u_2^*, \ldots$ such that $\nabla^2 u_{j+1}^* = u_j^*, j = 0,1,2,\ldots$ with $u_0^* = \overline{u}^*$, through which the domain integral is approximated as a series of boundary integrals. This is known as the Multiple Reciprocity Method (MRM). Boundary element modelling leads to a formulation which can be combined with an element adaptive scheme for the evaluation of eigenvalues to a desired accuracy.[11]

2.5.4. Fluids

Fluid-structure interaction problems have been studied extensively using BEM. In such formulations the fluid is invariably modelled by the BEM while the structure may be modelled by some other approximate method. In most existing BEM solutions, the fluid has been considered irrotational and inviscid. There are however two distinctive models corresponding to compressible and incompressible fluid behaviour.

2.5.5. Compressible Fluid

The dynamic effect on a stationary fluid within closed deformable boundaries is usually termed acoustic waves. The governing equation is directly deduced from the three-dimensional elastodynamics. The state of stress is completely specified in terms of the dynamic pressure p according to

$$\sigma_{ij} = -p\delta_{ij} \tag{36}$$

Thus, the momentum equation and the constitutive relations reduce to

$$p_{,i} = -\rho \ddot{u}_i \tag{37}$$

and

$$p = -K u_{k,k} \tag{38}$$

respectively, where K is the bulk modulus of the fluid substance. It is easily shown that the new dependent variable p is governed by (29) where the Laplace operator may now be given by its more general three-dimensional form and the wave velocity by

$$c = \sqrt{\frac{K}{\rho}}$$

The derivation of the boundary integral equation for the transient problem results in the boundary integral

$$I_b = \int_\Gamma \left[u^*(\mathbf{x}|\xi) * \frac{\partial p}{\partial n}(\mathbf{x}) - p(\mathbf{x}) * \frac{\partial u^*}{\partial n}(\mathbf{x}|\xi) \right] d\Gamma(\mathbf{x}) \tag{39}$$

Introducing the 3D fundamental solution (Appendix) in (39) reduces it to the form

$$I_b = \frac{1}{4\pi} \int_\Gamma \frac{1}{r} \left[\frac{\partial p}{\partial n}(\mathbf{x}, t_r) + \frac{r_{,n}}{r} p(\mathbf{x}, t_r) + \frac{r_{,n}}{c} \dot{p}(\mathbf{x}, t_r) \right] d\Gamma(\mathbf{x}) \tag{40}$$

with $t_r = t - r/c$ the so called retarded time.[12] The simplest possible modelling scheme would involve constant or linear pressure and its normal derivative over boundary elements and a number N of equal time steps. The discrete analogue to (3) is a system of equations in all boundary nodal values at all times. Refined discretization in both space and time may be required to avoid causality violation. The developed numerical algorithm was successfully applied to the assessment of hydrodynamic pressure wave scattering in various engineering environments.

The 3D version of the Helmholtz operator (33) is applicable to the harmonic analysis and the boundary integral I_b has the form (35) with $\hat{p}$ replacing $\hat{w}$ in the integrant. The boundary conditions require that either $\hat{p}$ or the normal component of the velocity be specified. The latter is related to the normal derivative of the pressure by

$$\frac{\partial \hat{p}}{\partial n} = -i\rho\omega\hat{v}_n \tag{41}$$

which is the consequence of (37). The modelling of the boundary leads to the discretized system for the nodal amplitudes:

$$[\mathbf{G}]\{\hat{\mathbf{p}}\} = [\mathbf{H}]\{\hat{\mathbf{v}}\} \tag{42}$$

which is re-arranged into a matrix equation with only boundary unknowns on the left hand side. It was shown[13] that the use of discontinuous elements is numerically advantageous when the field variables are discontinuous between elements or elements of different shapes are combined.

In the case of an infinite fluid medium outside a closed body with surface Γ, the so-called exterior problem, the formulation fails at certain characteristic frequencies which are the eigenfrequencies of the corresponding interior problem. This non-uniqueness is numerically manifested in rank deficiency of the BEM coefficient matrix and various methods have been proposed to circumvent the problem. Singular value decomposition has been lately applied to detect non-uniqueness but also to evaluate the quality of such methods.[14]

The BEM formulation may take a different form when disturbances at low frequency are considered. Then the quantity kL, where L is a representative length for the problem, is small and is adopted as a perturbation parameter. A series expansion of the field variable leads to differential equations involving the Laplace operator. The BEM modelling is essentially applied to the first order perturbation variable and the boundary integrals do not depend on the wave number so that only one matrix assembly is needed for the entire frequency range.[15] A BEM solution for the 3D harmonic pressure problem can be obtained using the fundamental solution of the Laplace operator combined with the Multiple Reciprocity Method[16] in the same manner as for the 2D membrane vibration problem.

2.5.6. Incompressible Fluid

The acoustic problem can be formulated and solved in terms of the velocity potential Φ which is also governed by the 3D version of (29). Incompressibility means infinite stiffness resulting in the field equation

$$\nabla^2\Phi = 0$$

The BEM solution is formulated in the frequency domain with the fundamental solution for the Laplace operator. The boundary integral is similar in form to (35) without however any frequency dependence. The boundary conditions require that either the normal component of the velocity or pressure are specified over the boundary. The former is given by the normal derivative of the potential, the latter is proportional to the potential according to

$$\frac{\partial \hat{p}}{\partial n} = -i\rho\omega\hat{\Phi} \tag{43}$$

2.5.7. SH Waves

Assuming the special displacement and excitation field

$$u_\alpha = b_\alpha = 0,\ u_3 = u_3(x_\alpha, t),\ b_3 = b_3(x_\alpha, t);\ \alpha = 1{,}2$$

in an elastic medium, the three-dimensional equations of elastodynamics reduce to a single scalar equation with the operator (29) on u_3, wave velocity $c = c_2$ and the forcing function given by

$$f(x_\alpha, t) = -c_2^{-2} b_3 \tag{44}$$

For a BEM wave propagation analysis, the boundary integral in (3) is the 2D version of (39) with u_3 replacing p and the traction boundary condition introduced into the formulation by noting that

$$n_{\sigma_3} = \mu \frac{\partial u_3}{\partial n} \tag{45}$$

Given the initial conditions

$$u_3(x_\alpha, 0) = u_0, \quad \dot{u}_3(x_\alpha, 0) = v_0 \tag{46}$$

the domain integral I_c in (3) is written

$$I_c = c_2^{-2} \int_\Omega \left[u^*(\mathbf{x}, t|\xi, 0) v_0(\mathbf{x}) - u_0(\mathbf{x}) \frac{\partial u^*}{\partial t}(\mathbf{x}, t|\xi, 0) \right] d\Omega(\mathbf{x}) \tag{47}$$

The second kernel in (39) has a strong singularity that complicates the evaluation of the integral over the boundary element containing the source point. Various schemes have been proposed to side-step this problem. For instance, the following substitution has been suggested[17]:

$$\int_{\Gamma} \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial \bar{u}^*}{\partial n} d\Gamma + \int_{\Gamma} \left(\frac{\partial u^*}{\partial n} - \frac{\partial \bar{u}^*}{\partial n} \right) d\Gamma \tag{48}$$

where $\bar{u}^*$ is the fundamental solution for the Laplace operator whose normal derivative has the same type of singularity as that of u^*. The second integral on the right of (48) is non-singular while the first is evaluated by an established procedure provided the domain boundary is closed. For this reason, special, 'enclosing' elements have to be used in the case of open boundaries.

On the other hand, it is possible to treat rigorously the spatial and time derivatives of the Heaviside function appearing in the fundamental solution and define the new kernel[18]

$$u_1^* = \frac{c[c(t-\tau)-r]}{2\pi[c^2(t-\tau)^2 - r^2]^{3/2}} H[c(t-\tau)-r] \tag{49}$$

such that

$$-\int_{\Gamma} \left(u_3 * \frac{\partial u^*}{\partial n} \right) d\Gamma = \int_{\Gamma} \frac{\partial r}{\partial n} (u_3 * u_1^* + c_2^{-1} \dot{u}_3 * u^*) \, d\Gamma \tag{50}$$

and

$$-c_2^{-2} \int_{\Omega} \left[u_0 \dot{u}^* \Big|_{\tau=0} \right] d\Omega = c_2^{-1} \int_{\Omega} \left[u_0 \left(\frac{u^* \big|_{\tau=0}}{r} - u_1^* \Big|_{\tau=0} \right) + \frac{\partial u_0}{\partial r} u^* \Big|_{\tau=0} \right] d\Omega \tag{51}$$

resulting in regularised versions of (39) and (47).

The derivation of the algebraic system for the boundary variables is clearly demonstrated in this case of a single field variable with zero initial conditions and zero body forces. The problem is further simplified by taking N equal time increments for the temporal discretization. Then, substitution of (14) into (3) with I_b given by (39) leads to

$$([\mathbf{K}]+[\mathbf{H}]^{(0)}\{\mathbf{u}\}^{(m)} - [\mathbf{G}]^{(0)}\{^n\sigma\}^{(m)} = -\sum_{p=1}^{m-1} \left\{ [\mathbf{H}]^{(m-p)}\{\mathbf{u}\}^{(p)} - [\mathbf{G}]^{(m-p)}\{^n\sigma\}^{(p)} \right\} \tag{52}$$

where $\{\mathbf{u}\}^{(p)}$ and $\{^n\sigma\}^{(p)}$ are, respectively, the one-dimensional arrays of nodal displacements and tractions at time step p, $\mathbf{K}$ is a diagonal matrix filled with the nodal values of κ, while $[\mathbf{H}]^{(p)}$ and $[\mathbf{G}]^{(p)}$ are matrices generated through spatial and temporal integration over elements. A typical matrix coefficient would have the form:

$$H_{ij}^{(p)} = \int_{-\Delta t}^{\Delta t} \int_{-a_e}^{a_e} \frac{\partial u^*}{\partial n}(\mathbf{x},t_p|\xi_i,\tau)\,\phi^{ku}(\mathbf{x})\,\psi^{u}(\tau)\,d\Gamma(\mathbf{x})\,d\tau \qquad (53)$$

$$G_{ij}^{(p)} = \int_{-\Delta t}^{\Delta t} \int_{-a_e}^{a_e} u^*(\mathbf{x},t_p|\xi_i,\tau)\,\phi^{kt}(\mathbf{x})\,\psi^{t}(\tau)\,d\Gamma(\mathbf{x})\,d\tau \qquad (54)$$

where a_e is the half length of an element. The relation between indices j and k depends on the number and order of elements. Superscripts u and t indicate interpolation functions for displacement and traction, respectively. It has been observed that early time disturbances contribute little to later time responses at points where $c_2(t_p - \tau) >> r$, $t_p = p\Delta t$. Time truncation schemes were introduced to improve the efficiency of the numerical algorithm. The truncation can be either complete when the early time contributions are disregarded or partial when simpler expressions for the calculation of the coefficient matrices are derived.[19]

The temporal interpolation function may be chosen such that[20]

$$\psi^n = \begin{cases} \psi(t_n - t) \text{ if } t_n - \Delta t \langle t \langle t_n + \Delta t \\ 0 \qquad\qquad \text{otherwise} \end{cases} \qquad (55)$$

This enhances the efficiency of the time-stepping algorithm but restricts the choice of interpolation function to either constant or linear. The BEM procedure must then be slightly modified if velocity and acceleration continuity is required.[20] Analytical expressions for the coefficients (53) and (54) in the case of zeroth, first and second order spatial interpolation functions have been obtained by Wang and Takemiya.[20] Thus the problem of singular integrals is neatly solved. The accuracy and stability of the general procedure is also improved. Analytical time integrals only, for constant and linear time variation, have been obtained by Israil and Banerjee[21] with emphasis on their behaviour at large time step.

BEM has often been applied to scattering by a surface irregularity in the half space. One such solution[22] considered a free-field propagating in the direction normal to the boundary and consisting of an incident and reflected part thus satisfying the zero traction condition on the free surface. The integral equation for the scattered field used the Green's function for the half space also satisfying the traction-free surface condition. Time histories of displacement amplitudes at surface points were obtained for semi-circular and triangular canyons.

In the frequency domain, the formulation is similar to that for the membranes or compressible fluids in 2D domains. For SH waves propagating in the half-space, the fundamental solution (Appendix) may again be replaced by the Green's function satisfying the traction-free condition on the surface. This is obtained by the method of images[1] and given by

$$\hat{u}^*(\mathbf{x},\xi,\omega)=\frac{i}{4}[H_0^{(2)}(kr)+H_0^{(2)}(kr_1)] \tag{56}$$

where $r_1=\sqrt{(x_1-\xi_1)^2+(x_2+\xi_2)}$ with x_2 in the direction of the inward normal to the free boundary. An indirect BEM is then possible using single-layered potentials with density ϕ so that the amplitude of scattered wave from a finite boundary Γ can be expressed as

$$\hat{u}=\int_\Gamma \hat{u}^*\phi d\Gamma \tag{57}$$

The singularity of the Green's function at the source point can be avoided by defining ϕ over a curve Γ^+ just outside the domain of the BEM solution.

2.5.8. Plates

As for membranes, BEM formulations based on Kirchhoff's thin plate theory involve a single field variable, the dynamic deflection $w(x_1,x_2,t)$, the response to a lateral normal pressure $q(x_1,x_2,t)$. The differential operator is in this problem given by

$$L=D\nabla^4+\rho h\frac{\partial^2}{\partial t^2}$$

where ∇^4 is the bi-harmonic operator in two dimensions, h the plate thickness and D its rigidity defined by

$$D=\frac{Eh^3}{12(1-\nu^2)}$$

There are four boundary variables: the deflection, shear force as well as the rotation and bending moment about the tangent to the boundary. In a well posed plate problem, two of these variables are specified and the remaining two need to be determined at every boundary point. Hence two fundamental solutions u_j^* are needed for a consistent BEM formulation. The first ($j = 1$) of these is the response to a unit force and the second ($j = 2$) to a moment impulse at the source point. The integral equation (3) is thus written twice, first for $u_1 = w$ and then for $u_2 = \theta_n$, the rotation at the source point about the direction tangent to the boundary.

In a time domain formulation, the boundary integral takes the form

$$I_b = \int_\Gamma [u_j^*(\mathbf{x},\xi) * Q(w) - \theta_n(u_j^*) * M_n(w) + M_n(u_j^*) * \theta_n(w) - Q(u_j^*) * w(\mathbf{x})] d\Gamma \tag{58}$$

where the operators

$$\theta_n = \frac{\partial}{\partial n}, \; M_n = -D\left(\frac{\partial^2}{\partial n^2} + \nu\frac{\partial^2}{\partial s^2}\right), \; Q = -D\frac{\partial}{\partial n}\nabla^2 + \frac{\partial M_{ns}}{\partial s}, \; M_{ns} = -D(1-\nu)\frac{\partial^2}{\partial n \partial s} \tag{59}$$

give, respectively, the normal slope, bending moment, shear force and the twisting moment along a boundary with its normal and tangential directions parallel to unit vectors **n** and **s**. The domain integral generated by the initial conditions is written

$$I_c = \rho h \int_\Omega \left[u^*(\mathbf{x}, t|\xi, 0)\frac{\partial w}{\partial t}(\mathbf{x}, 0) - w(\mathbf{x}, 0)\frac{\partial u^*}{\partial t}(\mathbf{x}, t|\xi, 0) \right] d\Omega(\mathbf{x}) \tag{60}$$

For harmonic plate vibrations, the operator is

$$L = D(\nabla^4 - k^4) \tag{61}$$

where

$$k^4 = \frac{\rho h \omega^2}{D} \tag{62}$$

and the boundary integral in (5) is given by

$$I_b = \int_\Gamma [\hat{u}_j^*(\mathbf{x},\xi,\omega) Q(\hat{w}) - \theta_n(\hat{u}_j^*) M_n(\hat{w}) + M_n(\hat{u}_j^*) \theta_n(\hat{w}) - Q(\hat{u}_j^*) \hat{w}(\mathbf{x})] d\Gamma \tag{63}$$

In the case of a non-smooth boundary with N_c corners, the integral equation should include the jump term

$$\sum_{k=1}^{N_c} \left\| M_{ns}(\hat{u}_j^*)\hat{w}(\mathbf{x}_k) - \hat{u}_j^*(\mathbf{x}_k,\xi,\omega)M_{ns}(\hat{w}) \right\| \tag{64}$$

arising from the discontinuity of the twisting moment at the corner points $\mathbf{x}_k$. The formulation has been validated through its application to the vibrations of a circular plate. The result may also be interpreted as the Laplace transform of the response to a transient excitation with the time-dependent response obtained by numerical inversion.[23] The formulation may be extended to orthotropic plates using an approximate fundamental solution obtained either as a double series of Hermitian polynomials or as the superposition[24]

$$\hat{u}_1^* = \overline{u}_1^* + U \tag{65}$$

where $\overline{u}_1^*$ is the fundamental solution of the corresponding static bending problem and U a correction term also expressed as a series of orthogonal functions. In both cases, the coefficients of the assumed solution are obtained by direct substitution to the governing differential equation.

An alternative system of integral equations is derived if

$$\hat{u}_2^*(\mathbf{x},\xi,\omega) = \frac{i}{8}H_0^{(1)}(kr) + \frac{1}{4\pi}K_0(kr) \tag{66}$$

is used as the second fundamental solution instead of $\hat{u}_2^* = \theta_m(\hat{u}_1^*)$ where $\mathbf{m}$ is the unit vector defining the plane of unit couple acting at the source point.[25] The former is the plate vibrational response to a point quadrupole source. Then the boundary integral equations for free vibrations and smooth boundary take the form:

$$\kappa\hat{w} = \int_\Gamma \{\theta_n(\hat{u}_1^*)\nabla^2\hat{w} - \hat{u}_1^*\theta_n(\nabla^2\hat{w}) + k^2[\hat{u}_2^*\theta_n(\hat{w}) - \theta_n(\hat{u}_2^*)\hat{w}]\}d\Gamma \tag{67}$$

$$\kappa^2\hat{w} = \int_\Gamma \{\hat{u}_2^*\theta_n(\nabla^2\hat{w}) - \theta_n(\hat{u}_2^*)\nabla^2\hat{w} + k^2[\theta_n(\hat{u}_1^*)\hat{w} - \hat{u}_1^*\theta_n(\hat{w})]\}d\Gamma \tag{68}$$

The solution procedure is straightforward in the case of clamped plates but it can also be adapted to any other support conditions.

The harmonic problem has also been formulated using the fundamental solution of the static plate bending problem.[26,27] This fundamental solution may be replaced by a Green's function satisfying partially the boundary conditions.[28] As expected, this approach generates an additional domain integral

$$k^4 \int_\Omega \bar{u}_j^*(\mathbf{x},\xi)\hat{w}(\mathbf{x})d\Omega \tag{69}$$

which includes the unknown domain deflection amplitude. It is therefore necessary to model the latter using domain elements and identify the source point with internal nodes as well so that a consistent system of equations is generated. Alternatively, the plate domain may be divided into sectors over which Gauss integration is performed, the deflection values at Gauss points thus becoming the domain unknowns.[26] Apart from this increase in the size of the problem, the solution procedure is in all other respects similar to that based on the frequency-dependent fundamental solution. It has the advantage of being conceptually simple and of providing directly the deflection profiles. It also makes it possible to account for viscous damping as well as the interaction with an elastic foundation by re-defining k as

$$k^4 = \frac{\rho h\omega^2 - ig\omega - \alpha}{D} \tag{70}$$

where $\alpha(x_1, x_2)$ is the coefficient of a non homogeneous subgrade reaction. The same procedure has been applied to the vibration analysis of thin orthotropic plates[29] using the fundamental solution of the corresponding bending problem.

The boundary integral in the integral equation for the deflection includes the hypersingular kernel $Q(\bar{u}_1^*)$ which complicates the numerical integration especially when higher-order elements are employed. It was shown however[27] that

$$\hat{w}(\xi)+\int_\Gamma Q(\bar{u}_1^*)\hat{w}(\mathbf{x})d\Gamma = -\int_\Gamma \left\{ D\frac{\partial(\nabla^2\bar{u}_1^*)}{\partial n}[\hat{w}(\mathbf{x})-\hat{w}(\xi)]+\frac{\partial \hat{w}}{\partial s}M_{ns}(\bar{u}_1^*)\right\} d\Gamma$$
$$+\sum_{k=1}^{N_c} \left\| M_{ns}(\bar{u}_1^*)\hat{w}(\mathbf{x}_k)\right\| \tag{71}$$

Substitution of the above relation removes the singularity from the first integral equation. The second integral equation is regularized by a similar process. Alternatively, it is possible to formulate an indirect non-singular BEM[28] in which an appropriate number of source points are selected in a fictitious region.

For a plate with internal point, line or area supports, a simple integral representation of the deflection is derived using the numerically determined Green's function for the static problem satisfying homogeneous support conditions along the edges of the plate.[30] A BEM or a Gauss integration scheme is adopted for modelling the continuous internal support reactions and domain accelerations. The eigenvalue problem for free vibrations is formulated in terms of the amplitudes of the Gauss domain points for either spring or rigid internal supports.

2.5.9. Shells

Thin cylindrical shells may be analysed relative to a frame of reference with the x_1, x_2, x_3 axes in the axial, circumferential and radial direction, respectively. For shallow shells, **L** is the Donnell-Mushtari operator. The fundamental solution satisfies a limit absorption principle and is obtained by direct and inverse Fourier transforms.[31] Since it needs to be 2π-periodic relative to x_2, it can be written in the form of a series:

$$u_{ij}^{*} = \sum_{m=-\infty}^{m=\infty} u_{ij}^{m} e^{imx_2} \tag{72}$$

Once the fundamental solution has been either analytically or numerically established, a BEM formulation similar to that for plates can be developed.

2.5.10. Elastic Space

2.5.10.1. Time domain solutions

BEM solutions based on the general theory of elastodynamics have been obtained in both two and three dimensions for transient as well as steady-state problems. The field equations (10) presented in an earlier section should be complemented by the boundary conditions requiring that the displacement satisfies

$$u_i = \tilde{u}_i(x_i, t) \tag{73}$$

on Γ_1 and the stresses satisfy

$$\sigma_{ij} n_j = {}^{n}\tilde{\sigma}_i(x_i, t) \tag{74}$$

on Γ_2 where $\Gamma_1 + \Gamma_2 = \Gamma$, n_i are the direction cosines of the outward normal to the boundary and the stresses are related to displacements by the constitutive equations of isotropic elasticity:

$$\sigma_{ij} = \rho[(c_1^2 - 2c_2^2)u_{k,k}\delta_{ij} + c_2^2(u_{i,j} + u_{j,i})] \tag{75}$$

obtained by substituting (8) into (9) and using the wave velocity definitions (11).

In time domain formulations the unknown variables should also satisfy the initial conditions

$$u_i(x_i,0) = u_{0i}, \quad v_i(x_i,0) = v_{0i} \tag{76}$$

everywhere within the domain Ω as well as the boundary Γ of the solid. The appropriate differential operator is easily deduced from (10) while the boundary integral and the initial conditions domain integral in (3) have the general form

$$I_b = \int_\Gamma [u_{ij}^*(\mathbf{x}|\xi) * {}^n\sigma_{ij}(\mathbf{x}) - u_j(\mathbf{x}) * {}^n\sigma_{ij}^*(\mathbf{x}|\xi)]d\Gamma(\mathbf{x}) \tag{77}$$

$$I_c = \rho\int_\Omega [u_{ij}^*(\mathbf{x},t|\xi,0)v_{0j}(\mathbf{x}) - u_{0j}(\mathbf{x})\underline{u}_{ij}^*(\mathbf{x},t|\xi,0)]d\Omega(\mathbf{x}) \tag{78}$$

In 2D elastodynamics, as in the 2D scalar wave equation, the spatial derivatives in the expression of the traction tensor ${}^n\sigma_{ij}^*$ must be transformed so that they can be computed.[5] A non-explicit kernel is obtained by a cumbersome operation on the Heaviside function. Alternatively, the 2D fundamental solution (Appendix) may be written as[32]:

$$u_{ij}^* = \frac{1}{2\pi\rho}\left(\frac{g_1}{c_1} - \frac{g_2}{c_2}\right) \tag{79}$$

where g_1 and g_2 are functions satisfying the conditions $g_1 = 0$ when $c_1(t - \tau) < r$ and $g_2 = 0$ when $c_2(t - \tau) < r$. An explicit and simple form of the traction kernel is derived from (79) and the subsequent introduction of the Heaviside function to ensure causality of each type of wave. The analytical temporal integration with the linear interpolation model over equal time steps gives explicit formulae for the convoluted kernels.[33]

The singularity of the traction kernel is the same as that of the corresponding elastostatic kernel so that singular boundary integrals can be computed using

$$\int_{\Gamma} {}^{n}\overline{\sigma}_{ij}^{*} d\Gamma + \int_{\Gamma} ({}^{n}\sigma_{ij}^{*} - {}^{n}\overline{\sigma}_{ij}^{*})\, d\Gamma \tag{80}$$

The second integral in (80) is non-singular while the first can be evaluated using the technique of rigid body motion which however requires the body to be bounded. For half-plane problems, a fictitious boundary must therefore be introduced. The interior stresses are given by an integral equation involving spatial derivatives of both kernels. These are derived using (79) and

$$^{n}\sigma_{ij}^{*} = \frac{\mu}{2\pi\rho r}\left(\frac{f_1}{c_1} - \frac{f_2}{c_2}\right) \tag{81}$$

where f_1 and f_2 satisfy the same conditions as g_1 and g_2.[34]

Through mathematical manipulations of the 2D fundamental solution and its derivatives, it is possible to transform the corresponding boundary integral equation so that its kernels have a weak, integrable singularity.[35] An alternative integral representation[36] is possible using a 'fundamental solution' obtained as the response to body forces given by

$$b_{ik} = \frac{H(\tau) - H(\tau - \Delta t)}{\Delta t}\,\delta_{ik}(\mathbf{x} - \xi) \tag{82}$$

where H denotes the Heaviside step function. The resulting BEM formulation is shown to have certain advantages over the more established procedure with respect, in particular, to the computation of singular integrals.

If the whole observation period is divided into equal time intervals Δt with constant time interpolation functions for both displacement and traction, the system of algebraic equations resulting from the BEM modelling has the recurrent form (52) giving, after the application of the appropriate boundary conditions, the nodal displacements $\{\mathbf{u}\}^{(m)}$ and tractions $\{{}^{n}\sigma\}^{(m)}$ at the current time step $m\Delta t$ if the solution at all previous time steps is known. In the absence of singularity, that is, when the field point does not belong to the element containing the source point, the coefficients in the solution matrices are usually computed by numerical quadrature. However, the criterion for enforcing the causality property of the fundamental solution may not always be applied effectively leading to significant computational errors. A scheme has been proposed[37] whereby the source element is divided into a number of sub-elements for a better monitoring of the 'active' portion of that element.

The size of the problem is significantly increased in the case of solids with material discontinuities since then these internal boundaries must be modelled by means of interface nodes. In analyses of non-convex domains, as in vibration isolation by open trenches, the solution was shown to exhibit non-causal behaviour, that is, an excitation at a given point produced responses at other points before the time required by the fastest waves to reach these points.[38]

The general elastic solution has most frequently been applied to the half space representing a semi-infinite soil mass sustaining wave motion or interacting with other solids, usually structures with their foundation. In this case, only the free boundary and any soil-structure interface has to be divided into boundary elements. Wave propagation and scattering is a common problem of practical significance for which BEM was found particularly suitable due to its feature of satisfying the radiation damping condition automatically. The problem was also formulated relative to a cylindrical co-ordinate system[39] by applying co-ordinate transformations on all variables, initially defined relative to a Cartesian frame of reference. Fourier expansions of displacements and tractions results in individual integral equations for each circumferential mode.

A BEM algorithm for the three-dimensional wave scattering problem has been detailed and validated by Hirose[40] who applied it further to the interaction of two cavities in an infinite space subjected to an incident longitudinal wave. This is, in general, a computationally demanding procedure, but it can be simplified by a semi-analytical approach based on the finite strip element whereby the problem is reduced into an one-dimensional one.[41] The boundary domain is discretized into conical frustum or circular strip elements over which the variation of displacements and tractions in the circumferential direction are modelled by a series of trigonometric functions. The usual interpolation models apply in the time domain and the transverse strip direction. The method has been shown to work well with scattering by spherical cavities but its effectiveness with irregular boundaries seems to be in some doubt.

The infinite size of the free boundary itself favours the replacement of the full space fundamental solution by the appropriate Green's function satisfying the traction-free boundary conditions. The general 3D form of the latter has been given[42] as the sum of six parts, the first two representing compressional (P) and shear (S) waves travelling directly from the source to the field point, while the remaining four arise from reflections of a P and an S wave at the free boundary, each resulting in a pair of a P and an S wave. This complex solution is obtained indirectly as the derivative of the total response to a Heaviside force excitation, its use has not therefore been found computationally advantageous. However a closed form 2D version has led to an efficient and effective BEM time domain solution.[43]

The time-dependent fundamental solution of 3D elastodynamics can also be the basis of an indirect BEM formulation.[44] It is used as an influence function which can be substituted into the constitutive equations (75) to generate influence functions for the related stresses.[45] In the indirect BEM, the analysed domain Ω is considered as part of the infinite elastic space. Assuming a layer of unknown fictitious impulses p_j acting on a boundary Γ^+ enclosing the real boundary, the boundary integral equations

$$u_i(\mathbf{x},t) = \int_0^t \int_{\Gamma^+} [u_{ij}^*(\mathbf{x},t|\xi,\tau)p_j(\xi,\tau)]d\Gamma(\xi)d\tau \tag{83}$$

$$\sigma_{ij}(\mathbf{x},t) = \int_0^t \int_{\Gamma^+} [\sigma_{ij}^*(\mathbf{x},t|\xi,\tau)p_j(\xi,\tau)]d\Gamma(\xi)d\tau \tag{84}$$

can be established where $\mathbf{x}$ lies on Γ. The real and the source boundaries may be left separated in which case these integral equations are regular. If they are made to coincide, the generated singularity modifies a little (84). In any case the solution procedure is analogous to that developed in accordance with the direct BEM.

The stability of the algorithm in a time domain indirect BEM solution was rigorously examined and applicable mathematical conditions were identified and tested on simple examples.[46] Considering the predicted response to a displacement pulse emanating from a single element, it was observed that this element could appear to have a greater stress influence on another element than the stress influence it has on itself, the so-called self-effect. The algorithm will be stable if the self-effect dominates all influences over the time history of the model. A scheme was proposed to increase the size of the self-effect as a means of improving the stability of the algorithm.

If the fundamental solution of the elastostatic problem, the so-called Kelvin's solution, is used as a weighting function, the convolutions are removed from (77) but the domain integral

$$\rho \int_{\Omega} \bar{u}_{ij}^*(\mathbf{x}|\xi)\ddot{u}_j(\mathbf{x})d\Omega(\mathbf{x}) \tag{85}$$

is added to the integral equation (3). A method, commonly known as dual reciprocity (DRM), has been proposed[47] for converting this domain integral into an expression containing only boundary integrals. This is achieved by approximating the displacement by the series expansion

$$u_j(\mathbf{x},t) = \sum_{m=1}^{M} \alpha_j^m(t)u^{(m)}(\mathbf{x}) \tag{86}$$

where α_j^m are the unknown time-dependent amplitudes of a family of M known functions $u^{(m)}$. The domain integrals resulting from substituting (86) in (85) are transformed as follows:

$$\ddot{\alpha}_j^m \int_\Omega \bar{u}_{ij}^* u^{(m)} d\Omega = \ddot{\alpha}_k^m \left\{ -\kappa_{ij} u_{kj}^{(m)} + \int_\Gamma \left[\bar{u}_{ij}^* \sigma_{kj}^{(m)} - \bar{u}_{ij}^{(m)} {}^n\bar{\sigma}_{kj}^* \right] \right\} \tag{87}$$

where $u_{kj}^{(m)}$ is the displacement field resulting from the body force $\delta_{kj} u^{(m)}$ and ${}^n\sigma_{kj}^{(m)}$ the corresponding surface tractions. The normal BEM modelling process can next be applied to the integral equation leading to a matrix equation of the form

$$[\mathbf{H}]\{\mathbf{u}\} + [\mathbf{M}]\{\ddot{\mathbf{u}}\} = [\mathbf{G}]\{{}^n\sigma\} \tag{88}$$

where $\{\mathbf{u}\}$ and $\{{}^n\sigma\}$ are the nodal displacement and traction vectors, respectively. The nodal accelerations $\{\ddot{\mathbf{u}}\}$ are related $\ddot{\alpha}_k^m$ to through (86). In the case of plain strain elasticity, functions meeting the requirements of the transformation (87) are

$$u^{(m)}(\mathbf{x}) = r(\xi_m, \mathbf{x}) \tag{89}$$

for which[47]

$$u_{kj}^{(m)} = \left[\left(3 - \frac{10\nu}{3} \right) \delta_{kj} - r_{,k} r_{,j} \right] \frac{r^3}{30(1-\nu)} \tag{90}$$

The matrices in (88) are neither symmetric nor positive definite. This complicates the numerical solution of the associated eigenvalue problem. It was shown however that the introduction of the variational principle of elastodynamics into the formulation process generates a system with equivalent mass and stiffness matrices which are both symmetric and positive definite.[48] Free vibrations as well as transient analyses have been carried out by this procedure.

2.5.11. Transformed domain formulations

Transient problems have been solved by taking the Laplace transform of (10) giving the operator

$$\hat{L}_{ij} = (c_1^2 - c_2^2)\partial_i \partial_j + (c_2^2 \partial_k \partial_k - s^2)\delta_{ij} \tag{91}$$

where s is the transform variable. The appropriate fundamental solution (Appendix) leads to the BEM solution for the transformed variables which may be inverted numerically to yield the time domain solution.[49] For steady-state or Fourier transform analyses, the same operator is obtained by replacing s with $i\omega$ where ω is the frequency. The boundary integral in (5) is written

$$I_b = \int_{\Gamma} [\hat{u}^*_{ij}(\mathbf{x},\xi,\omega)\, {}^n\hat{\sigma}_{ij}(\mathbf{x}) - \hat{u}_j(\mathbf{x})\, {}^n\hat{\sigma}^*_{ij}(\mathbf{x},\xi,\omega)]d\Gamma \tag{92}$$

Simple implementation and validation examples in 2D include the vertical, horizontal and rocking excitation of a soil layer over bedrock due to machine vibrations.[50] When the 3D formulation is applied to the half space, boundary conditions need to be satisfied at infinity. This is accomplished by introducing the concept of infinite boundary element[51] taking advantage of the asymptotic behaviour of the full space fundamental solution. BEM has been shown to yield non-unique solutions in the case of an infinite region with internal boundaries and given surface tractions. The problem arises at the eigenfrequencies of the adjoint interior problem, that is, the dynamic analysis of the elastic solid enclosed by the same finite boundary but with prescribed zero displacement. A new formulation based on higher-order boundary elements was shown to be entirely free of fictitious natural frequencies.[52]

Layered media are more efficiently analysed using, instead of the fundamental solution for the full space, Green's functions satisfying the continuity conditions over the interfaces as well as the boundary conditions on the surface.[38] The Green's function for the whole 3D half space may only be available in integral form but has successfully replaced the full space fundamental solution in BEM frequency domain formulations.[53] A numerical integration scheme was devised for the evaluation of this Green's function taking proper consideration of its singularity. Difficulties arise due to the highly oscillatory nature of integrands within certain intervals but these can be overcome by the use of a variable interval and an adaptive Gaussian integration scheme.

If the geometry and the applied tractions do not vary along one, say the x_3, direction, the problem uncouples into two 2D ones[54]: the SH wave problem already considered and a plane strain problem involving u_1 and u_2. The formulation of the latter is simply a reduced 2D version of the 3D case with the corresponding fundamental solution (Appendix). The Green's function satisfying the traction-free conditions on the surface can be used in the half-space case but it requires numerical integrations for calculating its inverse Fourier transform as in the 3D case. This approach was applied to the steady-state dynamics of a half-space with a canyon.

A boundary element formulation may also be based on the superposition of a set of elementary solutions built by discretization of the full-space fundamental solution expressed in terms of its longitudinal and transverse components.[55] In such an indirect boundary element method, the solid to be analysed is defined by a set of surface points; a point harmonic force is considered acting at each surface point and the displacements and tractions of the infinite space are evaluated at all surface points. These tractions may then be considered as the loads on the finite solid causing the calculated displacements. A consistent system for the determination of the dynamic stiffness matrix of the solid is thus generated. This approach is conceptually simple but displacements and tractions at the point of excitation cannot be evaluated due to the singularity of the solutions there. This problem is overcome by evaluating instead their mean value over the corresponding boundary element.

A formulation based on Kelvin's fundamental solution can be directly derived for a free vibration analysis in a manner analogous to that adopted for the transient problem and led to the introduction of dual reciprocity.[56,57] In the vibration case, the displacement amplitudes are given by (86) where α_j^m are constant coefficients and this approximation is used to eliminate the inertia term from the differential equation before its conversion into a boundary integral equation. This is achieved by seeking particular solutions $u_{kj}^{(m)}$ for the amplitudes satisfying only the inhomogeneous field equations with the assumed inertia terms as body forces. The boundary integral equation is actually formulated for the corresponding elastostatic problem, satisfied by the complementary functions, which are combined with the particular solutions to give the total boundary displacement and traction amplitudes. The equivalent mass matrix of the resulting standard eigenvalue problem is obtained directly from the approximation (86) without any additional integration. The method was compared to FEM through its application to the vibration analysis of twisted plates and an automotive crankshaft. The results from the two methods showed consistency and were obtained in comparable computing times. The application of this approach to axisymmetric solids led to a set of eigenvalue problems each associated with an individual circumferential wave number.

A symmetric and positive definite formulation of the free vibration problem was recently derived[58], on the basis of a modified variational principle. According to this approach, boundary and domain variables are considered as independent from each other and the appropriate form of the potential energy functional is

$$\Pi = \rho \int_{\Omega} \left(\frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + f_i u_i \right) d\Omega - \int_{\Gamma} {}^{n}\Sigma_i (u_i - U_i) d\Gamma - \int_{\Gamma_2} {}^{n}\tilde{\sigma}_i U_i d\Gamma \quad (93)$$

where upper case symbols are used to indicate boundary variables and

$$f_i = g\dot{u}_i + \rho(\ddot{u}_i - b_i) \tag{94}$$

On the constrained boundary Γ_1, the condition

$$U_i = \tilde{u}_i \tag{95}$$

of course applies. Spatial modelling for both U_i and ${}^n\Sigma_i$ is adopted in a manner similar to (15)

$$U_i = \sum_k \Phi_u^k(\mathbf{x}) u_i^k(t) \tag{96}$$

$${}^n\Sigma_i = \sum_k \Phi_\sigma^k(\mathbf{x}) \sigma_i^k(t) \tag{97}$$

In the domain, the displacement model

$$u_i = \sum_k \psi_k(t) \bar{u}_{ik}^*, \quad k = 1,2,\ldots n \tag{98}$$

is adopted in which the necessary number of fundamental solutions to the elastostatic problem are generated by placing the source point outside the domain close to each nodal point. The stationarity of Π provides the condition relating ψ_k tou_i^k as well as the system of ordinary differential equations governing the unknown nodal displacement amplitudes which constitute a subset of u_i^k. The effectiveness of the model was demonstrated through its application to the two-dimensional free vibration analysis of a shear wall.

2.5.12. Poroelasticity

Biot's theory of dynamic poroelasticity consists of a coupled system of equations governing the behaviour of the interacting porous elastic solid and permeating fluid. It shown[59] that there is a direct analogy between this theory and dynamic thermo-elasticity so that a common BEM solution can be formulated for both problems. By taking the Laplace transforms of the governing equations, it is possible to eliminate most dependent field variables and reduce the problem into a system of two coupled, linear second-order differential equations in $\hat{\mathbf{u}}$ and $\hat{p}$, that is, the transforms of displacement and excessive pore pressure. The differential operator acts therefore on four variables and the external action is represented by $\hat{\mathbf{b}}$ and $\hat{d}$, that is, the transforms of body force and volumetric body source rate. The fundamental solution for this expanded operator has been obtained in both two and three dimensions, so that the derived boundary integral equation has the standard form (5). The numerical implementation of BEM modelling included a numerical inverse transform calculation scheme for determining transient responses. The effectiveness of the method was demonstrated with both time-dependent and harmonic problems involving the dynamic responses of poroelastic soil.

2.6. SYSTEM DYNAMICS

2.6.1. Composite Materials

A BEM formulation exists for an incident plane wave in an elastic matrix interacting with a long fibre of uniform cross-section.[60] All wave fields have the same exponential dependence on x_3, the co-ordinate in the fibre direction, the problem is therefore quasi 2D and it is solved in the frequency domain. The boundary integral equation is written for both the matrix assumed as an infinite space and the fibre in the form of a cylindrical inclusion. The bonding between the two materials is represented by linear springs through which tractions are related to displacements over the interface. The possibility of local debonding may be considered by superposing a 3D scattered wave field to the quasi 2D solution. The standard 3D boundary integral equations govern the scattered waves in both matrix and inclusion but the interface conditions are applied over a finite part of the interface which includes the debonding area. Traction-free conditions are assumed over the latter, while the spring contact conditions are again enforced over the remainder of the interface.

2.6.2. Frictional Contact

A problem of practical interest arises when two dissimilar elastic bodies interact dynamically over a rough interface which may be in a state of separation, slip or stick. In two-dimensions, the dynamic excitation may be in the form of transient in-plane or SH waves[61] but also vertical or horizontal impulses.[62] The established BEM modelling process is applied to each body and the formulation is complemented with the various contact conditions in matrix form. These conditions may account for intermittent changes in geometry, as well as impact or release. In the case of a heavy structure in contact with the horizontal boundary of an elastic half-space, modelling, for instance, the dynamic dam-soil interaction, the initial distribution of the normal interface traction balances the self weight of the structure. Separation due to dynamic excitation is detected when the normal traction becomes tensile over certain elements. Then, these elements are removed from the contact area during the previous and the current time step. The displacement over such debonded elements should remain compatible, otherwise rebonding occurs. It was shown that more reliable results are obtained if self-weight is accounted for through the dynamic body force domain integral.

2.6.3. Assembled Plate Structures

The vibration of 3D structures consisting of assembled flat plate elements can be performed by combining the BEM formulation for plate vibration with that for 2D plane stress time-harmonic elastodynamics.[63] In both cases, the fundamental solution to the corresponding static problem can be used resulting in domain discretization and an additional set of nodal displacement unknowns. The transformation of the kinematic variables to a global co-ordinate system allows the formulation of the compatibility and equilibrium conditions along the common boundaries and thus to the coupling of the equations for each individual plate to form a single global system for the eigenvalues and eigenvectors.

2.6.4. Soil-Foundation-Structure

The BEM is usually applied only to the soil considered as an elastic half space. Various 2D or 3D time domain solutions have been obtained assuming interaction with rigid foundations. The inertia of the foundation can be accounted for[64] while continuity of displacements and tractions are easily imposed over flat interfaces. Dynamic cross-interaction between active and passive foundations has also been investigated in 2D.[43]

Cross-interaction has also been studied in the frequency domain. A group of rigid railway ties interacting with soil as well as the overlaying flexible rails is one of the special foundation systems considered.[65] The stiffness matrix of the elastic rails, obtained by standard structural analysis procedures is added to the impedance of the track system resulting from the BEM modelling of the soil. In a more wide-ranging approach, foundations of arbitrary three-dimensional shape were assumed and accuracy was improved through higher-order boundary elements.[53] The Green's function for the half space served as a weighting function. A useful outcome from this BEM application was that the cross-interaction between closely spaced footings may not be considered negligible over certain frequency ranges.

The full space fundamental solution was found suitable in modelling a complete system of soil and an embedded structure.[66] The compliance of the interface between soil and an embedded rigid foundation of arbitrary shape can be found indirectly by the subtraction deletion method.[67,68] Two BEM analyses need to be performed: the first for the finite soil volume which is required to re-create the original half-space with a free plane boundary (interior problem); the second for the half-space itself (exterior problem). The full-space fundamental solution is used in the formulation of the interior problem while the exterior problem relies on the Green's function for the half space which can be interpolated from tabulated values for numerical integration over elements not containing the source point. The compliance matrices generated by these solutions are combined through the compatibility and equilibrium conditions into one for the embedded foundation surface. Finally, the impedance matrix is obtained as the inverse of the compliance matrix.

2.6.5. Structure-Fluid

2.6.5.1. Shell-fluid

A BEM frequency domain solution has been formulated for a cylindrical shell with bounded or unbounded domain Ω_S surrounded by a fluid domain Ω_f extending to infinity.[31] In the case of a finite Ω_S, the shell lies between two semi-infinite rigid cylinders, its boundary Γ_S consists of two circular edges and $\Omega_S \subset \Gamma_f$ where Γ_f is the boundary of the fluid domain. In the case of infinite Ω_S, $\Omega_S = \Gamma_f$. The traction on the shell surface takes the form

$$\hat{\mathbf{q}} - \{0, 0, \hat{p}_f(Q)\}^t, \quad Q \in \Omega_S \tag{99}$$

where $\hat{\mathbf{q}}$ is the amplitude of the harmonic shell excitation. The second interface condition is

$$\frac{\partial \hat{p}_f}{\partial x_3} = \rho\omega^2 \hat{u}_{3s}(Q), \quad Q \in \Omega_S \tag{100}$$

with the x_3 axis in the radial direction. In the case of bounded shell, the displacement vanishes over the rigid part of the fluid boundary. The integral representation of pressure is simplified by replacing the fundamental solution by the Green's function satisfying zero pressure gradient condition on the shell surface.

2.6.5.2. Elastic space-compressible fluid

A coupled BEM formulation of the three-dimensional harmonic problem was developed by Poterasu[69] who considered a finite fluid domain Ω_f

surrounded by an infinite elastic domain Ω_e. The interaction is modelled through the continuity conditions for displacement and stress over the common boundary Γ_{fe}. The mathematical form of these conditions are deduced from (37) and (74):

$$\frac{\partial \hat{p}_f}{\partial n} = \rho\omega^2 \mathbf{n}.\hat{\mathbf{u}}_e \tag{101}$$

$$\hat{p}_f \mathbf{n} = \hat{\sigma}_e.\mathbf{n} \tag{102}$$

The dynamic excitation originates from a pulsating fluid boundary Γ_f with given harmonic displacement.

The two-dimensional harmonic problem was formulated in the context of a dam-reservoir analysis.[70] The dam is modelled as a finite elastic space with internal damping, the water as compressible fluid with a free boundary. The Green's function for the fluid pressure that satisfies the zero pressure condition at the water surface is given in terms of the full space fundamental solution (Appendix) by

$$\hat{u}^*(r) - \hat{u}^*(r_1) \tag{103}$$

where r_1 is the distance to the field point from the mirror image of the source point relative to the water surface. Thus, the use of this weighting function obviates the need for descretizing part of the fluid boundary. The BEM modelling can be extended to include a layer of compliant solid interacting with the fluid.

2.6.5.3. *Elastic space-incompressible fluid*

The interface conditions must now be written in terms of the fluid velocity potential. Thus (101) and (102) become

$$\frac{\partial \hat{\Phi}_f}{\partial n} = i\omega \mathbf{n}.\hat{\mathbf{u}}_e \tag{104}$$

$$i\omega\rho_f \Phi_f \mathbf{n} = \hat{\sigma}_e.\mathbf{n} \tag{105}$$

It has been shown[71] that the deformation behaviour of a fluid-saturated porous elastic medium in earthquake motion can be modelled by BEM in exactly the same way as that of an elastic medium. The interaction problem can thus be formulated for a combination of fluid, structure and a porous sea bed.

2.7. NON-LINEAR PROBLEMS

2.7.1. Plasticity

BEM relies on the existence of the fundamental solution which is only available for linear differential operators. It is therefore ideally suitable for solving linear problems. BEM formulations of problems involving material or geometric nonlinearities have however been attempted adopting incremental procedures and assuming linear behaviour over a solution step. Since time-dependence can also be modelled within each step, it was found advantageous in most cases to employ the fundamental solution of the corresponding linear static problem. This approach is clearly demonstrated through a simple incremental formulation of the plate problem.[72]

A more complex dynamic plasticity model for plates[73] involves an elaborate time domain solution which takes into account the interaction with elastic foundation. The integral equation includes the domain integral

$$\int_{\Omega} M_{\alpha\beta}(u^*) * \gamma_{\alpha\beta}^{p} d\Omega \tag{106}$$

generated by the plastic curvatures $\gamma_{\alpha\beta}^{p}$. The adopted fundamental solution has the form

$$u^* = \bar{u}^* \delta(t) + \sum_{n=1}^{\infty} Y_n(\xi, t) w^{(n)}(\mathbf{x}) \tag{107}$$

where $w^{(n)}$ are normalised modes for a finite plate with shape and boundary conditions as close as possible to those of the actual plate so that discretization of the boundary is minimised. Due to the presence of two domain integrals with deflection and curvature as unknown quantities, the domain needs also to be discretized into area elements.

For the dynamic analysis of a 3D elasto-plastic medium, the differential equations (10) are written in incremental form, the incremental strain consists of an elastic and a plastic part:

$$\delta\varepsilon_{ij} = \delta\varepsilon_{ij}^{e} + \delta\varepsilon_{ij}^{p} \tag{108}$$

with only the former related to incremental stress through Hooke's law (9). Using Stoke's fundamental solution of elastodynamics (Appendix), the integral equation governing the incremental displacement[74,75] is very similar to that for the corresponding elastic problem defined by (10), (73), (74) and (76) apart from the presence of the initial stresses

$$\delta\sigma_{ij}^{0} = C_{ijk\ell}\delta\varepsilon_{k\ell}^{p} \tag{109}$$

in the domain integral as part of the body forces. The stress increments at interior points are obtained through differentiation of the boundary integral equation and substitution into the adopted elasto-plastic constitutive equations. In the 2D case, explicit forms of the kernels in the domain integrals for both displacements and stresses have been presented[74] and their singularity discussed in some detail. It was pointed out that the domain integral in the expression for the stresses is strongly singular. A scheme was proposed to evaluate incremental stresses at points on or near the boundary utilising the boundary conditions.

A transient dynamic plasticity analysis in either two or three dimensions can also be based on the Kelvin fundamental solution of static elasticity.[76,77] Then the resulting integral equation is similar to that of the corresponding static problem apart from a domain integral containing the inertia term which is treated as a body force. As in the static case, the stresses related to plastic deformation are treated as initial stresses and this generates a second domain integral extended over only the plastic region. Following spatial modelling over boundary elements and domain cells, the solution algorithm can be formulated through an implicit time integration scheme. Computational economy is achieved by introducing the dual reciprocity scheme described earlier in the context of linear elastic analysis whereby the inertial domain integral is transformed into a boundary integral.[47]

2.7.2. Viscoelasticity

The potential of BEM as an analytical tool for viscoelastic problems has been demonstrated in the context of certain constitutive models.[78,79] Defining the deviatoric stress and strain tensors by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}, \quad e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \tag{110}$$

the elasticity equations (9) can be written

$$s_{ij} = 2\mu e_{ij}, \quad \sigma_{kk} = 3K\varepsilon_{kk} \tag{111}$$

where the bulk modulus of elasticity is given by

$$K = \lambda + \frac{2\mu}{3} \tag{112}$$

The generalised viscoelastic equations of differential operator type, corresponding to (111), are

$$\sum_{k=0}^{N} p'_k \frac{d^{\alpha_k}}{dt^{\alpha_k}} s_{ij} = \sum_{k=0}^{M} q'_k \frac{d^{\alpha_k}}{dt^{\alpha_k}} e_{ij}, \quad \sum_{k=0}^{N} p''_k \frac{d^{\alpha_k}}{dt^{\alpha_k}} \sigma_{kk} = \sum_{k=0}^{M} q''_k \frac{d^{\alpha_k}}{dt^{\alpha_k}} \varepsilon_{kk} \qquad (113)$$

where the use of fractional order time derivatives ($0 \leq \alpha < 1$) has been suggested as providing greater flexibility in fitting measured data. Alternatively, a viscoelastic model of hereditary integral type

$$s_{ij} = 2\int_{-\infty}^{t} \mu(t-\tau)\frac{de_{ij}(\tau)}{d\tau} d\tau, \quad \sigma_{kk} = 3\int_{-\infty}^{t} K(t-\tau)\frac{d\varepsilon_{kk}(\tau)}{d\tau} d\tau \qquad (114)$$

may be adopted. Taking the Laplace or the Fourier transforms of both sides of (113) or (114) leads to relations of the form

$$\hat{s}_{ij} = 2\mu'(i\omega)\hat{e}_{ij}, \quad \hat{\sigma}_{kk} = 3K'(i\omega)\hat{\varepsilon}_{kk} \qquad (115)$$

where the complex, frequency-dependent moduli μ' and K' relate the Fourier transforms of stresses and displacements in exact correspondence with the elasticity relations (111). The viscoelastic moduli are given as ratios of complex polynomials in the case of the differential operator model or as the Fourier transforms of the corresponding time-dependent properties in the case of the hereditary model.

The BEM formulation in the frequency domain can now proceed with the fundamental solution of the elastic problem in which the elastic wave speeds have been substituted by the viscoelastic ones. The inverse Laplace transform of this fundamental solution is required for a time domain formulation. Such an operation has been carried out in the special case of the Maxwell viscoelastic model[79] yielding a closed form solution of considerable complexity. In view of the limitations of the Maxwell model, it would be preferable to retain the versatility of the general constitutive equations, including those with fractional differential operators. The inversion however of the fundamental solution would then require numerical integration.

2.8. OTHER APPLICATIONS

2.8.1. Active Vibration Control

A BEM formulation for the steady-state vibration of beams[8] was combined with active control techniques for an iterative solution of a non-linear optimisation problem. The objective is to minimise a norm of the deflection over a prescribed region through a control harmonic force $\hat{P}(x,t)\delta(x-x_p)$ with $x_1 < x_p < x_2$. This force contributes the terms

$$\hat{P}\hat{u}^*(x_p,\xi),\ \hat{P}\hat{u}^*_\theta(x_p,\xi) \tag{116}$$

which are linearized with respect to $\hat{P}$and x_p for the purpose of the adopted optimisation scheme using Taylor series expansions about initial values of these parameters. Then the system of algebraic equations formulated by BEM as described in the section under beams is solved iteratively until the set of parameter values that minimise deflection is found.

2.8.1.1. Inverse problems

A BEM steady-state solution combined with additional information on boundary displacement can lead to the identification of unknown shape and the location of internal defects.[80] This is formulated as an optimisation problem with objective function the sum of the squares of the displacement residuals between BEM predictions and measured data. The procedure uses established correction techniques and circumvents ill-posedness through the use of data from multiple harmonic forces. It was successfully applied to square plates under plane stress conditions with circular or elliptical defects.

2.8.2. Assessment of Support Conditions

The boundary conditions of a plate can be determined by a combination of BEM modelling and dynamic measurements.[81] The integral equation for the plate vibration problem is used to relate the domain displacement amplitude to boundary unknowns. Amplitude measurements at a sufficient number of points generate a consistent system of equations for these unknowns which can then be interrelated to give the rotational and/or translational compliance or stiffness of the support. It is possible to locate the source point at some boundary nodes and thus reduce the number of required measurements. Boundary damping may also be estimated by changing the real stiffness coefficients into complex ones.

In this application, an approximate fundamental solution has been proposed as a numerically simpler alternative to the exact one given in the Appendix.[81] It has the form (65) with

$$U = \sum_m \sum_n C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{117}$$

where (ab) is the area of the plate. The coefficients C_{mn} are determined from the condition that the mean squared residual function obtained by substituting (65) into the differential equation governing $\hat{u}^*$ is a minimum.

2.8.3. Non Destructive Evaluation

Known BEM formulations and numerical algorithms were combined and optimised for use in ultrasonic non destructive evaluation (NDE).[82] The dynamic problem to be solved is the 3D elastic wave scattering by some arbitrarily shaped defect. Particular reference was made to the suitability and advantages of BEM compared to other numerical methods. The problem was initially solved in the frequency domain with the time domain responses obtained by Fourier transform. This indirect approach can apparently be more efficient than the time-stepping solution which requires more refined boundary meshing to ensure convergence. The defect was represented by an ellipsoidal inhomogeneity. The boundary conditions over its surface are of one type, that is, either the displacements or the tractions vanish corresponding to the cases of a rigid inclusion or a void, respectively. BEM modelling is required for an elastic inclusion which is coupled with that for the surrounding medium.

The scattered field is obtained using (5) with $\kappa_{ij} = \delta_{ij}$. This leads to the assessment of the anticipated response as measured by transducers in the farfield of the defect. The scattering effect is evaluated through the scattering amplitude defined heuristically as the magnitude and relative phase of the spherically spreading wave due to a monochromatic incident plane wave. Open cracks were modelled as ellipsoidal voids with one semi-axis mush smaller than the other two. Frequency, crack orientation and scattering direction have a strong influence on the minimum crack thickness for which the BEM solution is meaningful.

2.8.4. Fracture Mechanics

Dynamic loading has an inertial effect on crack initiation, propagation and branching.[83] The problem has been investigated using the general time domain BEM, the frequency domain BEM combined with Fourier transform calculations[84] but also the special formulation based on the fundamental solution of elastostatics and the dual reciprocity method. In the latter approach, a second boundary integral equation (BIE) for the traction is derived and applied at the nodes along one of the crack surfaces. The BIE for the displacement is applied on the other crack surface as well as on the remaining solid boundary. When (86) is used to relate the nodal accelerations to the second derivatives of the generalised co-ordinates, proper account should be taken of the coincident crack nodes. Only then, a consistent system of dynamic matrix equations can be obtained. Its solution leads, through the calculation of the crack opening displacements, to the modes I and II dynamic stress intensity factors (SIF). It is however possible to compute directly the SIF through a special 3-node element with singular shape functions for the traction.[84]

All three transient dynamic BEM approaches were applied to the same simple crack problems with existing published solutions in order to assess their accuracy and efficiency. While good agreement was achieved by all codes, the time domain approach proved the most accurate over short time periods, the frequency domain approach most effective in incorporating material damping and the dual reciprocity approach most efficient although it tended to smoothen the response.[84]

2.8.5. Wave Barriers

Wave screening by barriers is a means of isolating a vibratory energy source from adjoining structures and sensitive equipment. The frequency domain BEM in half space was applied to the assessment of the effectiveness of such a scheme, measured by the amplitude reduction factor, that is, the ratio of the vertical displacement amplitude in the presence of a barrier to the same amplitude without a barrier.[85] Various arrangements of rigid footings surrounded by open trenches and excited by vertical harmonic forces were analysed and comparisons with field data confirmed the reliability of BEM predictions. Further assessments of the effectiveness of trenches as vibration isolators can be found in the BEM literature.[86]

2.8.6. Arch Dams

An integrated BEM analysis has been applied to arch dams and their foundation with particular reference to earthquake excitation.[87,88] Established BEM models for 3D elastic media with internal damping were adopted for both the dam and the foundation rock. The use of the full-space fundamental solution meant that the discretization of the infinite foundation surface had to be truncated. Parametric studies showed that a BEM mesh extending 2.5 times the dam height gave sufficiently stable results. The input to the analysis was an incident field of time-harmonic waves and its output the scattered field. Moduli of the frequency response function were plotted for a dam on rigid foundation and on compliant rock. The investigation was extended to the effect of a water reservoir, also modelled by BEM using the Green's function for a fluid with a free surface.

2.9. HYBRID FORMULATIONS

In a significant number of applications where dynamic interaction between diverse systems is the predominant feature, the combination of BEM with the finite element (FEM) or another approximate method has been considered by many investigators as the most efficient numerical approach. This is particularly true when one of the interacting bodies has part or the whole of its boundary at infinity so that BEM modelling can be applied to it taking maximum advantage of its features while the second is a structure of finite size and complex shape and material behaviour for which FEM is the most convenient tool. The key step in such formulations is the set up of the coupling conditions which depend on the material and interface properties as well as the analytical models adopted for each interacting system.

Fluid-structure interaction is an area where hybrid solutions have found widespread use due to the convenient modelling of the fluid medium by BEM. An indirect BEM formulation has been combined with the dynamic stiffness method for the prediction of airborne noise transmission into an aircraft fuselage structure.[89]

Hybrid approaches have more often been applied to soil-structure interaction problems where the soil, assumed as an elastic semi-infinite space, is modelled by BEM while the structure is discretized according to an FEM or some other similar procedure. The coupling of the two fields is established through interface compatibility and equilibrium and the radiation condition at infinity is automatically accounted for. The general approach has been tested through the formulation and solution of many practical problems. Several versions of the BEM have been combined with FEM models for a variety of structural forms.

A few examples of implemented transient analyses are first given. In the dynamic excitation of the soil, modelled as 2D elastic half space, coupling may be applied between parts of the same medium with only the soil directly beneath the foundation modelled by FEM, the remainder by BEM.[90] A similar scheme can be applied to the interaction of underground traffic systems using finite elements for only the tunnel and a finite portion of the surrounding soil.[91] The FEM modelling is complemented by a time integration scheme with the time increments identical in the two methods. The performance of this approach was compared to that of conventional time domain BEM as well as frequency domain BEM using Green's functions for layered media and Fourier transform inversion.[38] Considering representative soil-foundation interaction problems, it was shown that FEM-BEM coupling is the most advantageous when inhomogeneities or material non-linearities occur.

The case of a buried pipeline under seismic excitation was analysed[92] using the general BEM formulation for three-dimensional elastodynamics applied to the soil bounded by a horizontal plane and the cylindrical cavity; elastic cylindrical shell elements were used for modelling a finite portion of the pipe. For the vertical and torsional response of single piles embedded in a layered half space, the BEM formulation in cylindrical co-ordinates was proved effective.[93,94] The pile shaft can be modelled by a number of one-dimensional finite elements with linear distributions of the displacements and tractions over the shaft-soil interface in the vertical direction. The dynamic response of railway tracks modelled as flexible beams under moving load has been studied by applying the finite difference method (FDM) to beam analysis with the subgrade reaction determined by the BEM applied to soil.[42] A hybrid methodology can also be applied to the frictional contact problem. A published example[95] examines the uplifting and sliding response of a flexible structure, modelled by FEM, to seismic excitation.

Fluid-structure interaction in the time domain is another problem solved by BEM-FEM coupling with fluid and structure modelled by boundary and finite elements, respectively.[96] Then the boundary integral (39) enters the BIE governing time-dependent fluid pressure. Its second singular term, with kernel the normal derivative of the fundamental solution, can be transformed into a more regular form. The interface conditions in the time domain corresponding to (101) and (102) take the form

$$\frac{\partial p_f}{\partial n} = \rho_f \mathbf{n}.\ddot{\mathbf{u}}_e \tag{118}$$

$$p_f \mathbf{n} = -\sigma_e.\mathbf{n} \tag{119}$$

which lead to the coupling of the two discrete formulations. The solution was applied to dam-water interaction, a typical problem for demonstrating the advantages of both methods, namely the suitability of FEM to finite inhomogeneous domains and that of BEM to homogeneous domains with boundaries at infinity.

The soil-pavement interaction problem has been formulated[75] using the time-domain elastoplastic BEM model for the soil and FEM for the plate. The interaction is simplified by assuming frictionless contact and neglecting the horizontal motion of the soil surface. A discrete system of equations is obtained by dividing (i) the contact area into rectangular boundary elements with constant traction and displacement; (ii) the anticipated plastic zone into volume cells with constant stresses and (iii) the time interval into equal time steps with constant or linear variation of traction and displacement during each step. The dynamic FEM equations for the pavement are combined with the Newmark-β time domain discretization scheme. The coupled system is solved by an iterative algorithm. The procedure was modified to deal efficiently with a moving applied load. The displacement and stress fields are considered moving with the load so that the results obtained at the previous time step can be used to estimate the initial values for the current step. This reduces the number of load increments and iterations.

Hybrid formulations have also been developed in the frequency domain. The two methods can be combined within the soil medium itself subjected to seismic waves. Considering SH wave excitation, an integrated symmetric BEM-FEM formulation has been derived using an energy approach based on Hamilton's variational principle. The boundary integral representation (57) of the displacement in the infinite region is incorporated in the variational functional and the problem is discretized using models for the displacement in the finite region representing a valley as well as the boundary traction and layer potential.[97]

An irregular finite region is conveniently modelled by FEM while the remaining unbounded region may be considered homogeneous and thus modelled by BEM. A two-dimensional version of this approach was applied to the analysis of strong earthquake motion in Mexico City. The numerical results were in good agreement with observed records.[98] The soil-structure transfer functions have been obtained for elastic beams and shear walls excited by horizontally propagating waves.[55] A soil stiffness matrix is generated by an indirect boundary element approach allowing the soil to enter the finite element code as a new type of element. Soil interaction with coupled shear walls on rigid foundation has been assessed using a hierarchical finite element model for the structure based on the assumption of an equivalent continuum for the lintel beams.[99]

An alternative to FEM in solving scattering problems in two dimensions is the wavefunction expansion technique.[100] The elastic half-space is divided into an interior domain which includes a long cylindrical embedded structure with, possibly, a cavity and the exterior unbounded domain. The former is modelled by BEM, in the latter the free field as well as scattered displacements are represented by wavefunction expansions while their interface is a circular cylindrical surface extending to infinity in the third direction. Finally, the continuity and boundary conditions for displacements and tractions at nodes over interfaces and boundaries yield the coefficients of the exterior expansion as well as the displacements and tractions over the boundary of the embedded structure. The solution is separately formulated for the plane strain and the SH wave problem and has been applied to scattering of SH waves due to an underground circular cylindrical shell and a tunnel.

The coupling of BEM with FEM has been attempted for solving elasto-plastic problems. A recently developed two-dimensional time domain formulation[101] applies the BEM only in the domain of the structure which is expected to remain elastic throughout the excitation time history while the elasto-plastically deformed portion is modelled by the FEM. The scheme is particularly efficient whenever the plastic region is small compared to the whole problem domain.

2.10. CONCLUSIONS

Results from the application of the method have not been presented here but numerous can be found in the literature. The majority of these results were obtained from various well known, simple or complex engineering problems basically for validation purposes, that is, for comparison of BEM predictions to those of exact analytical solutions or other numerical methods. Good agreement between these results have led to confidence as to the reliability and effectiveness of the method. The degree of accuracy achieved obviously depends on the complexity of the problem but there are no clearly defined rules relating these two important factors.

It should be remembered that, apart from the presence of non-linearities and inhomogeneities in the solution domain, another important factor influencing the choice of BEM as an analysis tool is the size of the problem domain. As stated earlier, BEM can be used to a maximum advantage on domains of infinite extent. For this reason, the development of hybrid schemes for dynamic systems consisting of finite and infinite, solid and/or fluid volumes has been promoted.

The prospective user should be aware of evidence of numerical instabilities in published codes implementing both direct and indirect time domain BEM formulations.[46] This can be observed even in applications to simple problems. In a simple case, the rate at which instability develops is low but it increases rapidly with the complexity of the solution domain and the boundary element mesh. Numerical instabilities cannot always be removed by simply reducing the time step. Further research is needed in developing time-stepping algorithms that are the least likely to go unstable and thus render BEM more competitive compared to FEM and FDM.

The range of structural dynamics problems that have been or can be solved by some version of BEM is certainly very impressive. Such applications of the method continue to attract the interest of many engineering analysts so that new solutions or improvements on existing solutions are still being developed. The next challenge facing developers and computer programmers is the dissemination of all this expertise among practising engineers so that BEM is established as an effective and easily accessible design tool. Although commercially available BEM computer codes exist, it seems that the dynamic applications of BEM have not as yet emulated the success of corresponding FEM formulations in being part of large, user-friendly, wide-ranging packages. This may be attributed to the mathematical complexity of the method which increases disproportionally as its scope expands beyond linear elastodynamics. Thus, the method relies on the expertise of knowledgeable, well trained users.

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(ii) frequency domain

$$\hat{u}^*(\mathbf{x},\xi,\omega)=\frac{e^{-ikr}}{4\pi r} \tag{A.6}$$

Plates — frequency domain

$$\hat{u}_1^*(\mathbf{x},\xi,\omega)=\frac{i}{8k^2}H_0^{(1)}(kr)-\frac{1}{4\pi k^2}K_0(kr) \tag{A.7}$$

where $H_0^{(1)}(kr)$ is the zero-order Hankel function of the first kind while K_0 is the zero-order modified Bessel function.

Elastodynamics[1]
(i) time domain in 2D

$$\hat{u}_{ij}^*(\mathbf{x},t;\xi,\tau)=\frac{1}{2\pi\rho}\left\{\frac{H[c_2(t-\tau)-r]}{c_2R_2}\delta_{ij})\right.$$
$$\left.-\sum_{\alpha=1}^{2}(-1)^{\alpha}\frac{H[c_\alpha(t-\tau)-r]}{c_\alpha R_\alpha r^2}\left[(2c_\alpha^2(t-\tau)^2-r^2)r,_i r,_j-R_\alpha^2\delta_{ij}\right]\right\} \tag{A.8}$$

where

$$R_\alpha=\sqrt{c_\alpha^2(t-\tau)^2-r^2},\quad \alpha=1,2 \tag{A.8}$$

(ii) time domain in 3D

$$u_{ij}^*(\mathbf{x},t;\xi,\tau)=\frac{1}{4\pi\rho r}\left\{\frac{t-\tau}{r^2}(3r,_i r,_j-\delta_{ij})\left[H\left(t-\tau-\frac{r}{c_1}\right)-H\left(t-\tau-\frac{r}{c_2}\right)\right]\right.$$
$$\left.+r,_i r,_j\left[c_1^{-2}\delta\left(t-\tau-\frac{r}{c_1}\right)-c_2^{-2}\delta\left(t-\tau-\frac{r}{c_2}\right)\right]+\delta_{ij}c_2^{-2}\delta\left(t-\tau-\frac{r}{c_2}\right)\right\}$$

(A.9)

Elastodynamics — frequency domain

General form

$$\hat{u}^*(x,\xi)=\frac{1}{\alpha\pi\rho c_2^2}[U_1(r,\omega)\delta_{ij}-U_2(r,\omega)r,_i r,_j] \tag{A.10}$$

Traction

$$^{n}\hat{\sigma}^{*}_{ij} = \frac{1}{\alpha\pi r}[(rU_1' - U_2)(r,_n \delta_{ij} - n_i r,_j) + 2U_2 r,_i (n_j + 2r,_n r,_j)$$

$$-2rU_2' r,_n r,_i r,_j + (2-\beta^2)(rU_1' - rU_2' - 0.5\alpha U_2) r,_i n_j] \qquad \text{(A.11)}$$

where a prime indicates differentiation with respect to r.

In 2D, $\alpha = 2$ and

$$U_1 = K_0(ik_2 r) + \frac{1}{k_2 r}[K_1(ik_2 r) - \beta^{-1} K_1(ik_1 r)] \qquad \text{(A.12)}$$

$$U_2 = K_2(ik_2 r) - \beta^{-2} K_2(ik_1 r)] \qquad \text{(A.13)}$$

where K_i ($i = 0, 1, 2$) are the modified Bessel functions of the second kind and β the ratio of the longitudinal to shear wave velocity.

In 3D, $\alpha = 4$ and

$$U_1 = \left(1 - \frac{1}{k_2^2 r^2}\right)\frac{e^{-ik_1 r}}{r} + \left(\frac{1}{k_2 r} + \frac{i}{\beta}\right)\frac{e^{-ik_1 r}}{k_2 r^2} \qquad \text{(A.14)}$$

$$U_2 = \left(1 - \frac{3}{k_2^2 r^2} - \frac{3i}{k_2 r}\right)\frac{e^{-ik_2 r}}{r} + \left(\frac{3}{k_2^2 r^2} + \frac{3i}{k_2 \beta} + \frac{1}{\beta^2}\right)\frac{e^{-ik_1 r}}{r} \qquad \text{(A.15)}$$

NOTATION

A	cross-sectional area (beams)
b	body force per unit mass
C	elastic constant
c	wave velocity
D	plate rigidity
d	volumetric body source rate (porous media)
E	Young's modulus

$\mathbf{e}$	deviatoric strain tensor
$\mathbf{f}$	external force function
g	damping coefficient
h	plate or membrane thickness
I	second moment of area (beam sections)
K	bulk modulus
k	wave number
$\mathbf{L}$	differential operator
p	fluid pressure
Q	shear force (beams, plates, shells)
q	lateral load on beams, plates, shells
r	distance from source to field point
$\mathbf{s}$	deviatoric stress tensor
T	membrane tension
t	time
$\mathbf{u}$	displacement
v	velocity
w	deflection of beams, membranes, plates or shells
$\mathbf{x}$	field point (x_1, x_2, x_3)
α	coefficient of subgrade reaction (plates)
β	c_1/c_2
Γ	Boundary
γ	plate curvature
δ	Kronecker delta, delta function
ε	strain
η	damping ratio
θ	rotation of beam or plate section
λ	Lame constant isotropic elasticity

μ	Lame constant isotropic elasticity
ν	Poisson's ratio
ξ	source point (ξ_1, ξ_2, ξ_3)
ρ	density
σ	stress
Φ	velocity potential
ϕ	space interpolation function
ψ	time interpolation function
Ω	domain
ω	frequency

Superscripts

t	transpose
*	fundamental solution
n	traction/unit normal **n**
$(\hat{\ })$	steady-state amplitude or transformed variable
$(\bar{\ })$	fundamental solution to the static problem
$(\tilde{\ })$	specified boundary values