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On transfer matrix eigenanalysis of pin-jointed frameworks

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Abstract

Eigenanalysis of the state vector transfer matrix has previously been employed to obtain Saint-Venant decay rates and continuum beam properties of a repetitive pin-jointed framework. Decay eigenvalues occur as reciprocal pairs, the transfer matrix being symplectic, and three of the unity, transmission, eigenvalues pertain to the trivial rigid body displacements. By setting displacement or force components equal to zero at the remote right-hand end of the structure and, through use of a recurrence relationship, new displacement transfer matrices, **S** or **C**, are derived for the generic cell; these are one-half of the original size, well conditioned, and the redundant information is eliminated. The former, **S**, requires a large value of the recurrence index, *i*, to achieve accurate eigenvalues while the latter, **C**, retains trivial information pertaining to the rigid body displacements. An alternative force transfer matrix, **M**, derived from **S**, retains the maximum amount of relevant information and converges quickly. The method suppresses the redundant right-to-left decay eigenvectors, and calculation of the transmission vectors of tension, bending moment and shearing force is simplified by the need to calculate just one principal vector rather than four for the original eigenproblem. Finally, these transmission vectors are employed to determine the continuum beam properties of the framework.

Keywords: Repetitive; Framework; Continuum beam properties; Saint-Venant decay

1. Introduction

Transfer matrix methods have received considerable recent interest as a means of dynamic analysis of repetitive structures [1–5]. The method is attractive as the behaviour of the complete structures can be determined from a knowledge of the transfer matrix of a single cell. For static analysis, the transfer matrix method has been employed by Stephen and Wang [6] to determine both the Saint-Venant decay rates and the equivalent continuum beam properties of a repetitive pin-jointed framework; the displacement and force components on one side of the cell form a state vector, when left multiplication by the transfer matrix gives the state vector on the other side. Transmission or decay of the state vector, according to whether the force components constitute a cross-sectional force or moment resultant, or are self-equilibrating, is equivalent to a scalar multiplication ($\times\lambda$) of the state vector, which leads immediately to an eigenvalue problem. For the plane framework considered in [6], the transfer matrix is of size (12×12); the decay eigenvalues occur as three reciprocal pairs according to whether decay is from left to right, or vice versa. The transmission eigenvalue ($\lambda = 1$) has a multiplicity of six, three of which pertain to the rigid body displacements, while the other three pertain to the stress resultants of tension, shear and bending moment. (Torsion is, of course, excluded in this plane case.) Eigenvectors for the rigid body displacements are coupled with principal vectors for the stress resultants, and the

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Nomenclature

A	cross-sectional area
$\mathbf{C}$	displacement-transfer matrix, with boundary condition $\mathbf{p}_N = 0$
$\mathbf{d}$	nodal-displacement vector
$\mathbf{D}$	matrix of displacement eigen- and principal vectors
E	Young's modulus of elasticity
$\mathbf{F}$	nodal-force vector (according to the conventions of the finite element method)
G	shear modulus
$\mathbf{G}$	state vector transfer matrix
i	$(-1)^{1/2}$, index
I	second moment of area
$\mathbf{I}$	identity matrix
j, N	index of cell, number of repeated cells in structure
$\mathbf{K}$	stiffness matrix
$\hat{\mathbf{K}}$	stiffness matrix relating force and displacement on left side of restrained structure
L	bar length
M	bending moment
$\mathbf{M}$	force transfer matrix, with boundary condition $\mathbf{d}_N = 0$
O	order (of magnitude)
$\mathbf{p}$	nodal-force vector (according to the conventions of the theory of elasticity)
$\mathbf{P}$	matrix of eigen- and principal force vectors
Q	shearing force
R	beam radius of curvature
$\mathbf{R}$, $\hat{\mathbf{R}}$	rectangular reduction matrices
$\mathbf{s}$	state vector
$\mathbf{S}$	displacement transfer matrix, with boundary condition $\mathbf{d}_N = 0$
T	tensile force
u, v, w	displacement components in x-, y-, and z-directions
x, y, z	cartesian coordinates
$\mathbf{X}$	matrix of eigen- and principal state vectors
γ	shear strain
ε	direct strain
κ	shear coefficient
λ	decay factor, eigenvalue of transfer matrix
Λ	$\lambda + \lambda^{-1}$
ν	Poisson's ratio
ψ	cross-sectional rotation

Superscript

T	transpose of matrix or vector

Subscripts

L	left
R	right

similarity matrix of these vectors transforms the transfer matrix into Jordan canonical form, the transfer matrix being both *defective* and *derogatory.* Consideration of the displacement and force components of the principal vectors allows calculation of exact values for equivalent continuum beam properties of the framework, such as cross-sectional area, second moment of area, shear coefficient, and Poisson's ratio.

The method has been developed by Stephen and Wang [7] as a finite element-transfer matrix procedure for the determination of Saint-Venant decay rates of an elastic prism of arbitrary cross-section subjected to self-equilibrated end loading, and is found to provide excellent agreement with the few exact (according to the spirit of the mathematical theory of elasticity) available solutions, such as the Papkovitch–Fadle solution for the plane strain strip [8], the rod of solid circular section [9], and the rod of hollow circular section [10].

While this transfer matrix approach is very simple, leading as it does to a standard eigenvalue problem, the question arises whether there are preferable methods of implementing the eventual numerical solution. There are two clear disadvantages in operating directly with the transfer matrix: first, difficulties exist in dealing with multiple eigenvalues, a problem, which does not generally arise in dynamic analysis and, as has been pointed out by Golub and Van Loan [11], the Jordan block structure of a defective matrix is very difficult to determine numerically; the second is that the transfer matrix is ill-conditioned. The possibility that preferable numerical procedures may exist is also apparent: first, as the transfer matrix is symplectic [12] and its eigenvalues occur in reciprocal pairs, half of the decay eigenvalues are effectively redundant if one is concerned solely with decay from, say, left to right; second, there is the possibility of reducing the multiplicity of the unity eigenvalue by removing the (trivial) three rigid body displacements if the cell (or structure) can be restrained in some way. These two considerations lead to a potential reduction in size from the initial (12 × 12) eigenproblem to one of (6 × 6); in consequence, the state vector of twelve elements would require trimming to six elements, suggesting that one should redefine the state vector in terms of displacement or force components, but not both.

In this paper, two related approaches are presented, which achieve the above objective; both rely on the imposition of boundary conditions of either zero displacement, or zero force, at some extreme Nth section at the right-hand end of the structure, where N is assumed large, together with the use of a recurrence relationship, which allows one to work back towards the left-hand end of the structure. This results in a (6 × 6) transfer matrix for either displacement, or force components (but not both); the procedure ensures that the new transfer matrix is well conditioned. Although the Jordan block structure cannot be avoided altogether – a shearing force is inevitably coupled to a bending moment, and a rigid body rotation is coupled to a rigid displacement in the y-direction – the reduction in problem size does facilitate calculation of the now single principal vector.

For completeness, a summary of the transfer matrix method is first provided, together with a brief description of the approaches taken by other authors.

2. State vector transfer matrix

Consider the jth cell located between the $(j-1)$th and jth sections of the plane framework in Fig. 1. The state vector $\mathbf{s}$ consists of the displacement and force vectors, $\mathbf{d}$ and $\mathbf{p}$, respectively; the state vectors at the left- and right-hand sides are $\mathbf{s}_{j-1} = [\mathbf{d}_{j-1}^{\mathrm{T}}\mathbf{p}_{j-1}^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathbf{s}_j = [\mathbf{d}_j^{\mathrm{T}}\mathbf{p}_j^{\mathrm{T}}]^{\mathrm{T}}$, and are related by the transfer matrix $\mathbf{G}$ through the equation,

$$\mathbf{s}_j = \mathbf{G}\mathbf{s}_{j-1}, \tag{1}$$

or in partitioned form

$$\begin{bmatrix} \mathbf{d}_j \\ \mathbf{p}_j \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{\mathrm{dd}} & \mathbf{G}_{\mathrm{dp}} \\ \mathbf{G}_{\mathrm{pd}} & \mathbf{G}_{\mathrm{pp}} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix}. \tag{2}$$

Setting

$$\mathbf{s}_j = \lambda \mathbf{s}_{j-1}, \tag{3}$$

where λ is the decay factor equivalent to piecewise exponential decay, leads to the eigenvalue problem,

$$(\mathbf{G} - \lambda \mathbf{I})\mathbf{s}_{j-1} = 0, \tag{4}$$

in which $\mathbf{I}$ is the identity matrix.

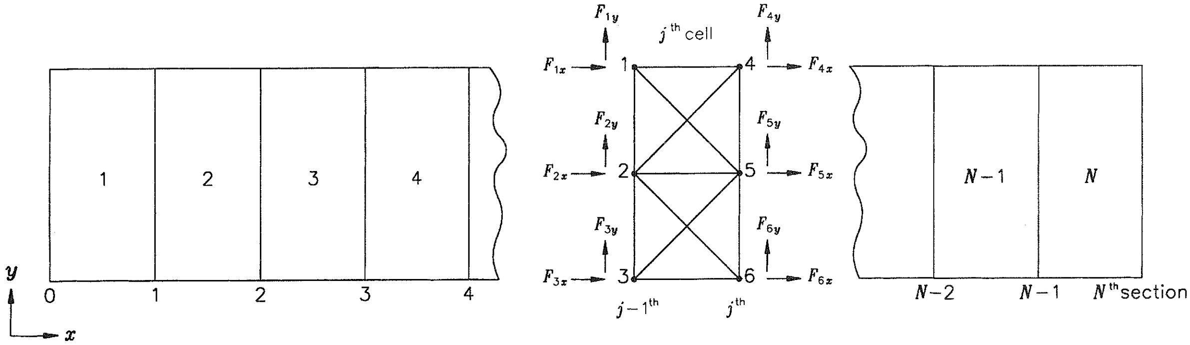

Fig. 1. N cells of repetitive framework.

The transfer matrix for the cell is readily obtained from its stiffness matrix $\mathbf{K}$; referring to Fig. 1, the force and displacement vectors $\mathbf{F}$ and $\mathbf{d}$, are related by the stiffness matrix equation $\mathbf{F} = \mathbf{Kd}$, or in partitioned form

$$\begin{bmatrix} \mathbf{F}_{j-1} \\ \mathbf{F}_j \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{j-1,j-1} & \mathbf{K}_{j-1,j} \\ \mathbf{K}_{j,j-1} & \mathbf{K}_{j,j} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{j-1} \\ \mathbf{d}_j \end{bmatrix}, \tag{5}$$

where the force vector $\mathbf{F}$ is, by convention, defined positive when the components are parallel to the coordinate directions; thus, $\mathbf{F}_{j-1} = -\mathbf{p}_{j-1}$, $\mathbf{F}_j = \mathbf{p}_j$. Since all cells within the structure have identical stiffness matrix $\mathbf{K}$, the subscripts $(j-1)$ and j within $\mathbf{K}$ are replaced by the subscript L and R for left- and right-hand side, respectively; they are, however, retained as an index for the state vector. One then has

$$\begin{bmatrix} -\mathbf{p}_{j-1} \\ \mathbf{p}_j \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{\mathrm{LL}} & \mathbf{K}_{\mathrm{LR}} \\ \mathbf{K}_{\mathrm{RL}} & \mathbf{K}_{\mathrm{RR}} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{j-1} \\ \mathbf{d}_j \end{bmatrix}. \tag{6}$$

Expanding Eq. (6) and re-arranging in accordance with Eq. (2), gives

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{\mathrm{dd}} & \mathbf{G}_{\mathrm{dp}} \\ \mathbf{G}_{\mathrm{pd}} & \mathbf{G}_{\mathrm{pp}} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{\mathrm{LR}}^{-1}\mathbf{K}_{\mathrm{LL}} & -\mathbf{K}_{\mathrm{LR}}^{-1} \\ \mathbf{K}_{\mathrm{RL}} - \mathbf{K}_{\mathrm{RR}}\mathbf{K}_{\mathrm{LR}}^{-1}\mathbf{K}_{\mathrm{LL}} & -\mathbf{K}_{\mathrm{RR}}\mathbf{K}_{\mathrm{LR}}^{-1} \end{bmatrix}. \tag{7}$$

The reason for ill-conditioning of the transfer matrix $\mathbf{G}$ is apparent: the two blocks on the leading diagonal of Eq. (7), $\mathbf{G}_{\mathrm{dd}}$ and $\mathbf{G}_{\mathrm{pp}}$, are independent of Young's modulus E, while the block $\mathbf{G}_{\mathrm{dp}}$ and $\mathbf{G}_{\mathrm{pd}}$ are proportional to E^{-1} and E, respectively,

3. Previous approaches

A quadratic eigenvalue formulation involving the partitioned stiffness matrix can be found as follows: expand Eq. (6) as

$$\begin{aligned} -\mathbf{p}_{j-1} &= \mathbf{K}_{\mathrm{LL}}\mathbf{d}_{j-1} + \mathbf{K}_{\mathrm{LR}}\mathbf{d}_j, \\ \mathbf{p}_j &= \mathbf{K}_{\mathrm{RL}}\mathbf{d}_{j-1} + \mathbf{K}_{\mathrm{RR}}\mathbf{d}_j. \end{aligned} \tag{8}$$

Multiply the first of these by λ, substitute $[\mathbf{d}_j^{\mathrm{T}}\ \mathbf{p}_j^{\mathrm{T}}]^{\mathrm{T}} = \lambda[\mathbf{d}_{j-1}^{\mathrm{T}}\ \mathbf{p}_{j-1}^{\mathrm{T}}]^{\mathrm{T}}$ in both to eliminate the jth state vector and add, to give

$$(\lambda^2\mathbf{K}_{\mathrm{LR}} + \lambda(\mathbf{K}_{\mathrm{LL}} + \mathbf{K}_{\mathrm{RR}}) + \mathbf{K}_{\mathrm{RL}})\mathbf{d}_{j-1} = 0. \tag{9}$$

A similar approach, which has been applied to the dynamic analysis of repetitive two-dimensional (plate-like) structures [13] but which leads to an identical result for the present static case, is as follows: eliminate the right-hand displacement vector, $\mathbf{d}_j$, through the introduction of a rectangular reduction matrix $\mathbf{R}$, according to

$$\begin{bmatrix} \mathbf{d}_{j-1} \\ \mathbf{d}_j \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \lambda\mathbf{I} \end{bmatrix} \mathbf{d}_{j-1} = \mathbf{R}\,\mathbf{d}_{j-1}. \tag{10}$$

The force relationship $\mathbf{p}_j = \lambda\mathbf{p}_{j-1}$, (which is a statement of equilibrium of force for the cell when $\lambda = 1$, or a requirement that both load vectors $\mathbf{p}_j$ and $\mathbf{p}_{j-1}$ should be self-equilibrated when $\lambda \neq 1$) may be expressed in the form,

$$[\lambda\mathbf{I}\quad \mathbf{I}] \begin{bmatrix} -\mathbf{p}_{j-1} \\ \mathbf{p}_j \end{bmatrix} = \hat{\mathbf{R}} \begin{bmatrix} -\mathbf{p}_{j-1} \\ \mathbf{p}_j \end{bmatrix} = 0, \tag{11}$$

where $\hat{\mathbf{R}}$ is the complement of $\mathbf{R}$. Substitute Eq. (10) into Eq. (6), and pre-multiplying by $\hat{\mathbf{R}}$, leads to the result,

$$\hat{\mathbf{R}}\mathbf{K}\mathbf{R}\,\mathbf{d}_{j-1} = 0, \tag{12}$$

which, on expansion, leads to Eq. (9); this can be tackled directly as a "lambda" matrix. While the above approaches do result in a halving of the problem size, they have the disadvantage of losing the standard eigenvalue form, and it is preferable to restore the standard form,

$$[\mathbf{A} - \lambda\mathbf{I}] \begin{bmatrix} \mathbf{d}_{j-1} \\ \lambda\mathbf{d}_{j-1} \end{bmatrix} = 0, \tag{13}$$

where the matrix **A** is in the lower companion form,

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{K}_{LR}^{-1}\mathbf{K}_{RL} & -\mathbf{K}_{LR}^{-1}(\mathbf{K}_{LL}+\mathbf{K}_{RR}) \end{bmatrix}. \tag{14}$$

This reduction, together with effective methods for dealing with the non-symmetric eigenvalue problem, is also described by Dong [14].

Zhong and Williams [4] exploited the symplectic properties of the transfer matrix to avoid the numerical ill-conditioning, which occurs on matrix inversion, and re-cast Eq. (6) into the form,

$$\begin{bmatrix} (\mathbf{K}_{LR}-\mathbf{K}_{LR}^{T}) & -(\mathbf{K}_{LL}+\mathbf{K}_{RR}) \\ (\mathbf{K}_{LL}+\mathbf{K}_{RR}) & (\mathbf{K}_{LR}-\mathbf{K}_{LR}^{T}) \end{bmatrix}\begin{bmatrix} \mathbf{d}_{j-1} \\ \mathbf{d}_j \end{bmatrix} = \Lambda\begin{bmatrix} \mathbf{0} & \mathbf{K}_{LR} \\ -\mathbf{K}_{RL} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{d}_{j-1} \\ \mathbf{d}_j \end{bmatrix}, \tag{15}$$

where both matrices are skew symmetric, and $\Lambda = \lambda + \lambda^{-1}$. This form can be solved using the QZ algorithm, but these authors show how the eigenvalue problem can be reduced to normal form, when the QR algorithm can be applied. Finally, note that Lin [15] has exploited the symplectic structure to formulate an alternative to the QR algorithm, resulting in a reduction in the storage requirement and the number of operations.

4. Displacement- and force-transfer matrices

In this new approach, the single cell stiffness matrix is employed to construct a banded global stiffness matrix equation for the complete structure consisting of N identical cells, as shown in Fig. 1; boundary conditions at the remote right-hand end of the structure are then imposed to derive a recurrence relationship from which a displacement or force transfer matrix is obtained.

4.1. The global-stiffness equation of a repetitive structure

In order to see the banded chain nature of the global-stiffness matrix developing, consider the three adjacent cells at the left-hand end of the structure, having $j = 1$–3 in Eq. (6), i.e.,

$$\begin{bmatrix} -\mathbf{p}_0 \\ \mathbf{p}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{LL} & \mathbf{K}_{LR} \\ \mathbf{K}_{RL} & \mathbf{K}_{RR} \end{bmatrix}\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \end{bmatrix}, \tag{16a}$$

$$\begin{bmatrix} -\mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{LL} & \mathbf{K}_{LR} \\ \mathbf{K}_{RL} & \mathbf{K}_{RR} \end{bmatrix}\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix}, \tag{16b}$$

$$\begin{bmatrix} -\mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{LL} & \mathbf{K}_{LR} \\ \mathbf{K}_{RL} & \mathbf{K}_{RR} \end{bmatrix}\begin{bmatrix} \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix}. \tag{16c}$$

Taking the second equation within Eq. (16a) and adding it to the first equation within Eq. (16b), eliminates the vector $\mathbf{p}_1$, to give

$$\mathbf{0} = \mathbf{K}_{RL}\mathbf{d}_0 + (\mathbf{K}_{RR}+\mathbf{K}_{LL})\mathbf{d}_1 + \mathbf{K}_{LR}\mathbf{d}_2.$$

Similarly, the force vector $\mathbf{p}_2$ can be eliminated from Eqs. (16b) and (16c) to give

$$\mathbf{0} = \mathbf{K}_{RL}\mathbf{d}_1 + (\mathbf{K}_{RR}+\mathbf{K}_{LL})\mathbf{d}_2 + \mathbf{K}_{LR}\mathbf{d}_3.$$

Proceeding in this way for the complete structure, i.e. all N cells, leads to the banded matrix equation,

$$\begin{bmatrix} -\mathbf{p}_0 \\ \mathbf{0} \\ \mathbf{0} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{p}_N \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{LL} & \mathbf{K}_{LR} & \mathbf{0} & & & & & \\ \mathbf{K}_{RL} & (\mathbf{K}_{RR}+\mathbf{K}_{LL}) & \mathbf{K}_{LR} & \mathbf{0} & & & & \\ \mathbf{0} & \mathbf{K}_{RL} & (\mathbf{K}_{RR}+\mathbf{K}_{LL}) & \mathbf{K}_{LR} & \mathbf{0} & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \mathbf{0} & \mathbf{K}_{RL} & (\mathbf{K}_{RR}+\mathbf{K}_{LL}) & \mathbf{K}_{LR} \\ & & & & & & \mathbf{0} & \mathbf{K}_{RL} & \mathbf{K}_{RR} \end{bmatrix}\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \\ \mathbf{d}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{d}_{N-1} \\ \mathbf{d}_N \end{bmatrix}, \tag{17}$$

within the banded matrix $\mathbf{0}$, and the blanks out of the band, are the zero matrix having the same size as the sub-matrices $\mathbf{K}_{\mathrm{LL}}$, etc., i.e. of size 6×6 for the plane framework; within the column vectors, $\mathbf{0}$ is a 6×1 zero matrix.

4.2. Displacement-transfer matrix: suppression of rigid-body displacements

To suppress the rigid-body displacements, one sets $\mathbf{d}_N = \mathbf{0}$. From the penultimate line of Eq. (17), the relationship between the displacement vectors at the Nth, $(N-1)$th, and $(N-2)$th sections is

$$\mathbf{0} = \mathbf{K}_{\mathrm{RL}}\mathbf{d}_{N-2} + (\mathbf{K}_{\mathrm{RR}} + \mathbf{K}_{\mathrm{LL}})\mathbf{d}_{N-1} + \mathbf{K}_{\mathrm{LR}}\mathbf{d}_N,$$

and setting $\mathbf{d}_N = \mathbf{0}$ gives

$$\mathbf{d}_{N-1} = -(\mathbf{K}_{\mathrm{RR}} + \mathbf{K}_{\mathrm{LL}})^{-1}\mathbf{K}_{\mathrm{RL}}\mathbf{d}_{N-2}$$

or

$$\mathbf{d}_{N-1} = (-\mathbf{S}_1^{-1}\mathbf{K}_{\mathrm{RL}})\mathbf{d}_{N-2}, \tag{18}$$

where $\mathbf{S}_1 = (\mathbf{K}_{\mathrm{RR}} + \mathbf{K}_{\mathrm{LL}})$.

Similarly, from the pen-penultimate line of Eq. (17), one has

$$\mathbf{0} = \mathbf{K}_{\mathrm{RL}}\mathbf{d}_{N-3} + (\mathbf{K}_{\mathrm{RR}} + \mathbf{K}_{\mathrm{LL}})\mathbf{d}_{N-2} + \mathbf{K}_{\mathrm{LR}}\mathbf{d}_{N-1},$$

and substituting for $\mathbf{d}_{N-1}$ from Eq. (18) gives

$$\mathbf{d}_{N-2} = (-\mathbf{S}_2^{-1}\mathbf{K}_{\mathrm{RL}})\mathbf{d}_{N-3}, \tag{19}$$

where $\mathbf{S}_2 = (\mathbf{K}_{\mathrm{RR}} + \mathbf{K}_{\mathrm{LL}}) - \mathbf{K}_{\mathrm{LR}}\mathbf{S}_1^{-1}\mathbf{K}_{\mathrm{RL}}$.

Similar manipulations can be performed as one works ones way towards the left-hand end of the structure giving

$$\mathbf{d}_1 = (-\mathbf{S}_{N-1}^{-1}\mathbf{K}_{\mathrm{RL}})\mathbf{d}_0, \tag{20}$$

where $\mathbf{S}_{N-1} = (\mathbf{K}_{\mathrm{RR}} + \mathbf{K}_{\mathrm{LL}}) - \mathbf{K}_{\mathrm{LR}}\mathbf{S}_{N-2}^{-1}\mathbf{K}_{\mathrm{RL}}$.

Now, from the first line of matrix equation (17), one has

$$-\mathbf{p}_0 = \mathbf{K}_{\mathrm{LL}}\mathbf{d}_0 + \mathbf{K}_{\mathrm{LR}}\mathbf{d}_1,$$

substituting for $\mathbf{d}_1$ from Eq. (20), gives

$$\mathbf{d}_0 = -\mathbf{S}_N^{-1}\mathbf{p}_0, \tag{21}$$

where $\mathbf{S}_N = (\mathbf{K}_{\mathrm{LL}}) - \mathbf{K}_{\mathrm{LR}}\mathbf{S}_{N-1}^{-1}\mathbf{K}_{\mathrm{RL}}$. Eq. (21) is a recursive means by which the displacement vector at the left-hand end of the structure, $\mathbf{d}_0$, can be obtained from a knowledge of the load vector, $\mathbf{p}_0$, applied at that end; in effect, one has constructed a 'super element' stiffness matrix $(-\mathbf{S}_N)$ for the complete structure under the assumption that $\mathbf{d}_N = \mathbf{0}$.

In general, if one takes $\mathbf{S}_0 = [\infty]$, such that $\mathbf{S}_0^{-1} = [\mathbf{0}]$, then

$$\mathbf{S}_0 = [\infty], \tag{22a}$$

$$\mathbf{S}_i = (\mathbf{K}_{\mathrm{RR}} + \mathbf{K}_{\mathrm{LL}}) - \mathbf{K}_{\mathrm{LR}}\mathbf{S}_{i-1}^{-1}\mathbf{K}_{\mathrm{RL}}, \tag{22b}$$

$$\mathbf{d}_{N-i} = (-\mathbf{S}_i^{-1}\mathbf{K}_{\mathrm{RL}})\mathbf{d}_{N-i-1} \quad \text{for } i = 1, 2, 3, \ldots (N-1) \tag{22c}$$

$$\mathbf{d}_0 = -\mathbf{S}_N^{-1}\mathbf{p}_0, \tag{22d}$$

$$\mathbf{S}_N = \mathbf{K}_{\mathrm{LL}} - \mathbf{K}_{\mathrm{LR}}\mathbf{S}_{N-1}^{-1}\mathbf{K}_{\mathrm{RL}}. \tag{22e}$$

Note the distinction between the general term $\mathbf{S}_i$, (Eq. (22b)), and the final term $\mathbf{S}_N$ (Eq. (22e)), the latter is applicable only at the (free) extreme left-hand end of the structure.

By writing in Eq. (22c)

$$\mathbf{S} = -\mathbf{S}_i^{-1}\mathbf{K}_{\mathrm{RL}} \tag{23}$$

together with $j = N - i$ and $j - 1 = N - (i + 1)$ when $\mathbf{d}_j = \mathbf{d}_{N-i}$ and $\mathbf{d}_{j-1} = \mathbf{d}_{N-i-1}$, Eq. (22c) becomes

$$\mathbf{d}_j = \mathbf{S}\mathbf{d}_{j-1}, \tag{24}$$

where $\mathbf{S}$ is the displacement-transfer matrix under the assumption $\mathbf{d}_N = \mathbf{0}$.

An eigenvalue problem can now be obtained by setting

$$\mathbf{d}_j = \lambda\mathbf{d}_{j-1}, \tag{25}$$

which leads to the standard form

$$[\mathbf{S} - \lambda\mathbf{I}]\mathbf{d}_{j-1} = 0. \tag{26}$$

It should be noted that transfer matrix $\mathbf{S}$ depends on the recurrence index i, as may be seen from Eq. (23); a brief discussion on the appropriate value for i is provided in Section 5.

4.3. Displacement-transfer matrix: suppression of force and moment transmission modes

An alternative end condition at the Nth section is to take the force vector $\mathbf{p}_N = \mathbf{0}$, which has the effect of suppressing the force and moment transmission modes, i.e., tension, shear and bending moment. Again a recursive relationship is obtained, which may be summarised as

$$\mathbf{C}_0 = \mathbf{K}_{\mathrm{RR}}, \tag{27a}$$

$$\mathbf{C}_i = (\mathbf{K}_{\mathrm{RR}} + \mathbf{K}_{\mathrm{LL}}) - \mathbf{K}_{\mathrm{LR}}\mathbf{C}_{i-1}^{-1}\mathbf{K}_{\mathrm{RL}}, \tag{27b}$$

$$\mathbf{d}_{N-i} = (-\mathbf{C}_i^{-1}\mathbf{K}_{\mathrm{RL}})\mathbf{d}_{N-i-1} \quad \text{for } i = 1, 2, 3, \ldots (N-1) \tag{27c}$$

$$\mathbf{d}_0 = -\mathbf{C}_N^{-1}\mathbf{p}_0, \tag{27d}$$

$$\mathbf{C}_N = \mathbf{K}_{\mathrm{LL}} - \mathbf{K}_{\mathrm{LR}}\mathbf{C}_{N-1}^{-1}\mathbf{K}_{\mathrm{RL}}. \tag{27e}$$

As with the previous analysis, an eigenvalue problem can now be obtained by writing

$$\mathbf{C} = -\mathbf{C}_i^{-1}\mathbf{K}_{\mathrm{RL}}, \quad \mathbf{d}_j = \lambda\mathbf{d}_{j-1} \tag{28}$$

to give the eigenequation

$$[\mathbf{C} - \lambda\mathbf{I}]\mathbf{d}_{j-1} = 0, \tag{29}$$

where $\mathbf{C}$ is the displacement-transfer matrix under the assumption that $\mathbf{p}_N = \mathbf{0}$. Again, it should be noted that transfer matrix $\mathbf{C}$ depends on the recurrence index i, as may be seen from Eq. (28).

Once the eigenvalues and (displacement) eigenvectors have been found from either of Eq. (26) or Eq. (29), the force vectors are determined as follows: form the column vector comprising the displacement eigenvectors $\mathbf{d} = [\mathbf{d}_{j-1}^{\mathrm{T}}\ \mathbf{d}_j^{\mathrm{T}}]^{\mathrm{T}}$ when the force eigenvector $\mathbf{p} = [\mathbf{p}_{j-1}^{\mathrm{T}}\ \mathbf{p}_j^{\mathrm{T}}]^{\mathrm{T}}$ can be obtained from Eq. (6) as

$$\mathbf{p} = \begin{bmatrix} -\mathbf{K}_{\mathrm{LL}} & -\mathbf{K}_{\mathrm{LR}} \\ \mathbf{K}_{\mathrm{RL}} & \mathbf{K}_{\mathrm{RR}} \end{bmatrix}\mathbf{d}. \tag{30}$$

The complete state vectors may then be formed as

$$\mathbf{s}_{j-1} = \begin{bmatrix} \mathbf{d}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ -(\mathbf{K}_{\mathrm{LL}} + \mathbf{K}_{\mathrm{LR}}\mathbf{C}) \end{bmatrix}\mathbf{d}_{j-1},$$

$$\mathbf{s}_j = \begin{bmatrix} \mathbf{d}_j \\ \mathbf{p}_j \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{K}_{\mathrm{RL}} + \mathbf{K}_{\mathrm{RR}}\mathbf{C} \end{bmatrix}\mathbf{d}_{j-1} \tag{31}$$

under the assumption that $\mathbf{p}_N = \mathbf{0}$. The alternative end condition, $\mathbf{d}_N = \mathbf{0}$, leads to similar expressions, but with matrix $\mathbf{C}$ replaced by matrix $\mathbf{S}$.

4.4. Force-transfer matrix

The approaches described in the above sections result in a reduced *displacement*-transfer matrix **S** or **C** according to the two different end conditions at the Nth section of the structure. In this section, an alternative *force*-transfer matrix is derived in which case the appropriate end condition is $\mathbf{d}_N = \mathbf{0}$. Expanding Eq. (30) and employing Eq. (24), gives

$$\begin{aligned} \mathbf{p}_{j-1} &= -\mathbf{K}_{LL}\mathbf{d}_{j-1} - \mathbf{K}_{LR}\mathbf{d}_j = -(\mathbf{K}_{LL} + \mathbf{K}_{LR}\mathbf{S})\mathbf{d}_{j-1}, \\ \mathbf{p}_j &= \mathbf{K}_{RL}\mathbf{d}_{j-1} + \mathbf{K}_{RR}\mathbf{d}_j = (\mathbf{K}_{RL} + \mathbf{K}_{RR}\mathbf{S})\mathbf{d}_{j-1}. \end{aligned} \tag{32}$$

Recalling $j = N - i$ and $j - 1 = N - (i + 1)$, and noting $\mathbf{d}_{N-(i+1)} = \mathbf{S}^{-1}\,\mathbf{d}_{N-i}$ gives

$$\begin{aligned} \mathbf{p}_{N-(i+1)} &= -(\mathbf{K}_{LL} + \mathbf{K}_{LR}\mathbf{S})\mathbf{d}_{N-(i+1)} = -(\mathbf{K}_{LL} - \mathbf{K}_{LR}\mathbf{S}_i^{-1}\mathbf{K}_{RL})\mathbf{d}_{N-(i+1)}, \\ \mathbf{p}_{N-i} &= (\mathbf{K}_{RL}\mathbf{S}^{-1} + \mathbf{K}_{RR})\mathbf{d}_{N-i} = -(\mathbf{K}_{LL} - \mathbf{K}_{LR}\mathbf{S}_{i-1}^{-1}\mathbf{K}_{RL})\mathbf{d}_{N-i}; \end{aligned} \tag{33}$$

hence, in general, one can write

$$\mathbf{p}_{j-1} = \hat{\mathbf{K}}_{j-1}\mathbf{d}_{j-1}, \tag{34}$$

where

$$\hat{\mathbf{K}}_{j-1} = \hat{\mathbf{K}}_{N-(i+1)} = -(\mathbf{K}_{LL} - \mathbf{K}_{LR}\mathbf{S}_i^{-1}\mathbf{K}_{RL}) \tag{35}$$

is the stiffness matrix relating the force and displacement vectors on the left-hand side of the cell. Let $i = N - 1$ in Eq. (35), and comparing with Eq. (22e) gives $\hat{\mathbf{K}}_0 = -\mathbf{S}_N$ and $\mathbf{p}_0 = \hat{\mathbf{K}}_0\mathbf{d}_0$, which is consistent with Eq. (22d).

Eliminating the displacement vector $\mathbf{d}_{j-1}$ in Eq. (32) gives the required transfer equation,

$$\mathbf{p}_j = \mathbf{M}\mathbf{p}_{j-1}, \tag{36}$$

where the force-transfer matrix is

$$\mathbf{M} = -(\mathbf{K}_{RL} + \mathbf{K}_{RR}\mathbf{S})(\mathbf{K}_{LL} + \mathbf{K}_{LR}\mathbf{S})^{-1}. \tag{37}$$

Alternatively, it can be derived from Eqs. (34) and (35) as

$$\mathbf{p}_j = \hat{\mathbf{K}}_j\mathbf{d}_j = \hat{\mathbf{K}}_j(\mathbf{S}\mathbf{d}_{j-1}) = \hat{\mathbf{K}}_j\mathbf{S}\left(\hat{\mathbf{K}}_{j-1}^{-1}\mathbf{p}_{j-1}\right) = \left(\hat{\mathbf{K}}_{N-i}\mathbf{S}\hat{\mathbf{K}}_{N-(i+1)}^{-1}\right)\mathbf{p}_{j-1}, \tag{38}$$

where the force-transfer matrix is accordingly defined as

$$\mathbf{M} = \hat{\mathbf{K}}_{N-i}\mathbf{S}\hat{\mathbf{K}}_{N-(i+1)}^{-1}. \tag{39}$$

The assumption of piecewise decay of the force vector, i.e.,

$$\mathbf{p}_j = \lambda\mathbf{p}_{j-1} \tag{40}$$

leads to the eigenvalue problem for the force vector, as

$$[\mathbf{M} - \lambda\mathbf{I}]\mathbf{p}_{j-1} = 0. \tag{41}$$

Again, note that the force-transfer matrix **M** is dependent on i.

The associated state vectors can be obtained as

$$\begin{aligned} \mathbf{s}_{j-1} &= \begin{bmatrix} \mathbf{d}_{j-1} \\ \mathbf{p}_{j-1} \end{bmatrix} = \begin{bmatrix} -(\mathbf{K}_{LL} + \mathbf{K}_{LR}\mathbf{S})^{-1} \\ \mathbf{I} \end{bmatrix}\mathbf{p}_{j-1}, \\ \mathbf{s}_j &= \begin{bmatrix} \mathbf{d}_j \\ \mathbf{p}_j \end{bmatrix} = \begin{bmatrix} -\mathbf{S}(\mathbf{K}_{LL} + \mathbf{K}_{LR}\mathbf{S})^{-1} \\ \mathbf{M} \end{bmatrix}\mathbf{p}_{j-1} \end{aligned} \tag{42}$$

under the assumption that the Nth end condition is $\mathbf{d}_N = 0$.

5. Example

Consider the plane framework shown in Fig. 1, which has the properties: Young's modulus $E = 200 \times 10^9$ N m^{-2}, vertical and horizontal bars have length $L = 1$ m while diagonal bars have length $\sqrt{2}$ m; vertical and horizontal bars have cross-sectional area $A = 1$ cm^2 while diagonal bars have $A = 0.5$ cm^2. The objectives are threefold: first, to elucidate the redundant information and the ill conditioning, which arise from the original eigenproblem formulation, Eq. (4). Second, to apply the reduction procedures derived in Section 4, and to gauge the sensitivity of the eigenvalues to the recurrence index i, and third, to demonstrate the advantage in calculation of the now single principal vector.

Employing the (12×12) state transfer matrix $\mathbf{G}$, Eq. (7), the eigenvalue problem (4) gives the decay eigenvalues:

$$\begin{bmatrix} 16.779756 \\ 0.0595956 \end{bmatrix}, \quad \begin{bmatrix} 3.5345841 \\ 0.2829187 \end{bmatrix}, \quad \begin{bmatrix} -14.243501 \\ -0.0702075 \end{bmatrix}$$

occurring as reciprocal pairs; thus three of these are effectively redundant. The rigid body and transmission modes should have an eigenvalue of exactly unity, but the MATLAB QR algorithm returns the values:

$$1.000145,\ 0.999855,\ 1.000000 \pm O(10^{-4})i,\ 1.000000 \pm O(10^{-9})i;$$

again, three of these eigenvalues are redundant, as they pertain to the rigid body displacements. The MATLAB command rcond(**G**) gives the reciprocal of the condition of **G** in 1-norm, and is close to unity (zero) if **G** is well (badly) conditioned; for this example, rcond(**G**) is $O(10^{-17})$, and the ill conditioning is clear.

For the reduced eigenproblems, Eqs. (26), (29) and (41), various values of the recurrence index i within matrices **S**, **C** and **M** are considered; the eigenvalues are shown in Table 1, where the first three in each cell are the eigenvalues of the decay modes and the remaining three are the unity (or close to unity) eigenvalues associated with the rigid body or the force transmission modes. First, it is noted that for all the above transfer matrices, the MATLAB command *rcond* returns a value $O(10^{-2})$, indicating a much improved numerical condition. Second, it is seen from Table 1, that the right to left decay modes ($\lambda > 1$) have been eliminated, leaving just the left to right decay eigenvalues; these can be determined accurately with a recurrence index i as small as 10. On the other hand, in order to obtain accurately the unity eigenvalues, a large recurrence index i is required if one employs the displacement-transfer matrix **S**, otherwise there is the danger of misinterpreting what should be a transmission mode as a decay mode. However, a small recurrence index i suffices if one employs the displacement-transfer matrix **C**, or the force-transfer matrix **M**; indeed, there is a loss of accuracy if i is taken to be too large, as one might expect from the increased number of matrix multiplications. The disadvantage of employing **C** is that the unity eigenvalues pertain to the trivial rigid body modes. Overall, the maximum (non-trivial) information is retained if one employs the transfer matrices **S** or **M**, and the latter is preferable as it gives the greater accuracy with less computational effort.

Now, consider the eigenvectors obtained from the force-transfer matrix **M** within which one has taken $\mathbf{S} = -\mathbf{S}_{10}^{-1}\mathbf{K}_{RL}$; the matrix of force eigenvectors is

Table 1
Eigenvalues for the displacement and force-transfer matrices **S**, **C** and **M**

Value of index i	$\mathrm{eig}(\mathbf{S}) = \mathrm{eig}(-\mathbf{S}_i^{-1}\,\mathbf{K}_{RL})$	$\mathrm{eig}(\mathbf{C}) = \mathrm{eig}(-\mathbf{C}_i^{-1}\,\mathbf{K}_{RL})$	$\mathrm{eig}(\mathbf{M}) = \mathrm{eig}(-(\mathbf{K}_{RL} + \mathbf{K}_{RR}\mathbf{S})\times(\mathbf{K}_{LL} + \mathbf{K}_{LR}\mathbf{S})^{-1})$
$i=5$	0.282916, 0.059596, −0.070207, 0.831935, 0.751161 ± 0.150334i	0.282921, 0.059596, −0.070207, 1.000000, 1.000000, 1.000000	0.282921, 0.059596, −0.070207, 1.000000, 1.000000, 1.000000
$i=10$	0.282919, 0.059596, −0.070207, 0.908677, 0.839081 ± 0.104987i	0.282919, 0.059596, −0.070207, 1.000000, 1.000000, 1.000000	0.282919, 0.059596, −0.070207, 1.000000, 1.000000, 1.000000
$i=100$	0.282919, 0.059596, −0.070207, 0.990094, 0.980230 ± 0.013785i	0.282919, 0.059596, −0.070207, 1.000000, 1.000001, 0.999999	0.282919, 0.059596, −0.070207, 1.000000, 1.000001, 0.999999
$i=1000$	0.282919, 0.059596, −0.070207, 0.999001, 0.998002 ± 0.001411i	0.282919, 0.059596, −0.070207, 1.000000, 1.000009, 0.999991	0.282919, 0.059596, −0.070207, 1.000000, 1.000010, 0.999990
$i=10{,}000$	0.282919, 0.059596, −0.070207, 0.999900, 0.999801 ± 0.000139i	0.282919, 0.059596, −0.070207, 1.000000, 1.000139, 0.999928	0.282920, 0.059594, −0.070207, 1.000000, 1.000115, 0.999922

$$\mathbf{P}_{j-1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0.89645 \\ -0.77990 & -2.4351 & 1 & O(10^{-8}) & O(10^{-7}) & O(10^{-8}) \\ -2 & -2 & 0 & O(10^{-9}) & O(10^{-9}) & 1 \\ 0 & 0 & -2 & O(10^{-8}) & O(10^{-7}) & O(10^{-15}) \\ 1 & 1 & 0 & -1 & -1 & 0.89645 \\ 0.77990 & 2.4351 & 1 & O(10^{-8}) & O(10^{-7}) & O(10^{-8}) \end{bmatrix}. \tag{43}$$

The first three columns pertain to the decay modes while column six is the transmission mode of tension; columns four and five are (sensibly) identical, and describe the bending moment. The shear force (principal) vector is obtained from the chain rule,

$$\begin{aligned} \mathbf{MP}_{j-1}(:,4) &= \mathbf{P}_{j-1}(:,4), \\ \mathbf{MP}_{j-1}(:,5) &= \mathbf{P}_{j-1}(:,5) + \mathbf{P}_{j-1}(:,4) \end{aligned} \tag{44}$$

giving, $\mathbf{P}_{j-1}(:,5) = [-8.7326 \quad 0.46935 \quad -0.77786 \quad 1.0613 \quad 7.3380 \quad 0.46935]^{\mathrm{T}}$, which is a linear combination of shear, bending moment and tension; subtracting multiples of columns four and six gives the new principal vector for shear force as

$$\mathbf{P}_{j-1}(:,5) = [0 \quad 0.46935 \quad 0 \quad 1.0613 \quad 0 \quad 0.46935]^{\mathrm{T}}$$

The final matrix of force eigen- and principal vectors is found by replacing the repeated column in Eq. (43) by the shear-force vector, to give

$$\mathbf{P}_{j-1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0.89645 \\ -0.77990 & -2.4351 & 1 & 0 & 0.46935 & 0 \\ -2 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1.0613 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0.89645 \\ 0.77990 & 2.4351 & 1 & 0 & 0.46935 & 0 \end{bmatrix}, \tag{45}$$

and this transforms the force-transfer matrix $\mathbf{M}$ into Jordan canonical form, as

$$\mathbf{P}_{j-1}^{-1}\mathbf{MP}_{j-1} = \begin{bmatrix} 0.2829187 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0595956 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0702075 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{46}$$

A major advantage of the present procedure is now clear. The original eigenproblem required the calculation of four principal vectors – tension coupled to rigid body displacement in the x-direction, rotation coupled to rigid body displacement in the y-direction, bending moment coupled to rotation, and shearing force coupled to bending moment. In the present method, only the latter is required.

Next, consider the determination of the equivalent "beam" properties of the framework based upon the force eigen- and principal vectors associated with the unity eigenvalues, that is $\mathbf{P}_{j-1}(:,4:6)$, columns four to six in Eq. (45). The complete state vectors of displacement and force components on both sides of the cell are required; these are obtained from Eq. (42) as

$$\begin{aligned} \mathbf{X}_{j-1}(:,4:6) &= \begin{bmatrix} -(\mathbf{K}_{\mathrm{LL}} + \mathbf{K}_{\mathrm{LR}}\mathbf{S})^{-1} \\ \mathbf{I} \end{bmatrix} \mathbf{P}_{j-1}(:,4:6), \\ \mathbf{X}_{j}(:,4:6) &= \begin{bmatrix} -\mathbf{S}(\mathbf{K}_{\mathrm{LL}} + \mathbf{K}_{\mathrm{LR}}\mathbf{S})^{-1} \\ \mathbf{M} \end{bmatrix} \mathbf{P}_{j-1}(:,4:6), \end{aligned} \tag{47}$$

which results in the left-hand side state vector as

$$\mathbf{X}_{j-1}(:,4:6) = \begin{bmatrix} -5.1585\times10^{-7} & -2.8348\times10^{-6} & -4.3510\times10^{-7} \\ -2.8381\times10^{-6} & -2.2286\times10^{-5} & -1.0355\times10^{-8} \\ 0 & 0 & -4.3510\times10^{-7} \\ -2.8319\times10^{-6} & -2.2286\times10^{-5} & 0 \\ 5.1585\times10^{-7} & 2.8348\times10^{-6} & -4.3510\times10^{-7} \\ -2.8381\times10^{-6} & -2.2286\times10^{-5} & 1.0355\times10^{-8} \\ 1 & 0 & 0.89645 \\ 0 & 0.46935 & 0 \\ 0 & 0 & 1 \\ 0 & 1.0613 & 0 \\ -1 & 0 & 0.89645 \\ 0 & 0.46935 & 0 \end{bmatrix}, \tag{48}$$

and the right-hand side state vector as

$$\mathbf{X}_{j}(:,4:6) = \begin{bmatrix} -4.6892\times10^{-7} & -2.8113\times10^{-6} & -3.9546\times10^{-7} \\ -2.3457\times10^{-6} & -1.9324\times10^{-5} & -1.0355\times10^{-8} \\ 0 & 0 & -3.9546\times10^{-7} \\ -2.3396\times10^{-6} & -1.9318\times10^{-5} & 0 \\ 4.6892\times10^{-7} & 2.8113\times10^{-6} & -3.9546\times10^{-7} \\ -2.3457\times10^{-6} & -1.9324\times10^{-5} & 1.0355\times10^{-8} \\ 1 & 1 & 0.89645 \\ 0 & 0.46935 & 0 \\ 0 & 0 & 1 \\ 0 & 1.0613 & 0 \\ -1 & -1 & 0.89645 \\ 0 & 0.46935 & 0 \end{bmatrix}. \tag{49}$$

To illustrate the procedure, first determine the tensile properties of the framework, which requires $\mathbf{X}_{j-1}(:,6)$ and $\mathbf{X}_j(:,6)$, the third columns in Eqs. (48) and (49), respectively. These displacement and force components are shown in Fig. 2, where it is seen that the force components are equal on either side; the only differences are the displacements: axial elongation and the Poisson's ratio contraction. The elongation of the cell is $4.3510\times10^{-7} - 3.9546\times10^{-7} = 3.9645\times10^{-8}$, which is equal to the axial strain ε_x, as the cell has unit length; the transverse strain $\varepsilon_y = -1.0355\times10^{-8}$, and hence the Poisson's ratio is $v = 0.2612$. The total tensile force is $T = 1 + 2\times0.89645 = 2.7929$ N, and hence the equivalent cross-sectional area is $A = T/(E\varepsilon_x) = 3.522386\times10^{-4}$ m^2. It should be noted that, by virtue of the symmetry of the cell about the mid-plane, there is no coupling between extensional and shear/bending displacements.

The first columns, $\mathbf{X}_{j-1}(:,4)$ and $\mathbf{X}_j(:,4)$, define bending of the cell with a bending moment of magnitude $M = 2$, as shown in Fig. 3. The elongations on the top and at the bottom are

$$\Delta u_{\text{top}} = -4.6892\times10^{-7} - (-5.1585\times10^{-7}) = 4.6935\times10^{-8},$$
$$\Delta u_{\text{bottom}} = 4.6892\times10^{-7} - 5.1585\times10^{-7} = -4.6935\times10^{-8}.$$

Then, the beam curvature is

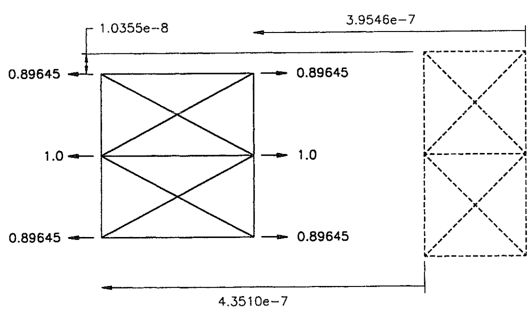

Fig. 2. Nodal displacements and forces for tension.

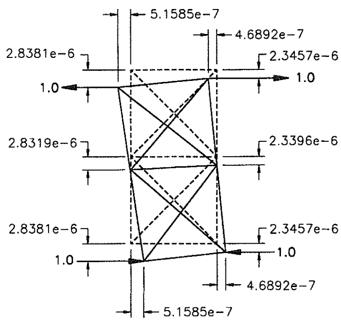

Fig. 3. Nodal displacements and forces for bending.

$$1/R = (\Delta u_{\text{top}} - \Delta u_{\text{bottom}})/2 = 4.6935 \times 10^{-8};$$

thus, the equivalent second moment of area is found to be

$$I = (MR)/E = 2.13061 \times 10^{-4}\ \text{m}^4.$$

Finally, consider $\mathbf{X}_{j-1}(:,5)$ and $\mathbf{X}_j(:,5)$, the second columns in Eqs. (48) and (49), respectively, which define a shearing force coupled with a bending moment and the associated displacement, as shown in Fig. 4, where the shearing forces of magnitude two on both sides are balanced by a bending moment of magnitude two on the right hand side. The shearing force Q and the shear angle γ are related by

$$Q = \kappa A G \gamma, \tag{50}$$

where κ is the shear coefficient, A is the cross-sectional area, G is the shear modulus; and the shear angle has the following relationship:

$$\gamma = \psi - \frac{\mathrm{d}v}{\mathrm{d}x}, \tag{51}$$

where $\mathrm{d}v/\mathrm{d}x$ is the centre-line slope And ψ is the rotation of the cross-section. The cross-sectional rotations on either side of the cell are different, and taking the average gives the rotational angle of the cell, $\psi = 0.5 \times (2.8348 \times 10^{-6} + 2.8113 \times 10^{-6}) = 2.823076 \times 10^{-6}$, and the centre-line slope is $\mathrm{d}v/\mathrm{d}x = -1.9318 \times 10^{-5} + 2.2286 \times 10^{-5} = 2.96756 \times 10^{-6}$ hence the shear angle is $\gamma = \psi - \mathrm{d}v/\mathrm{d}x = -1.44486 \times 10^{-7}$. Employing $Q = 2$, the equivalent shear modulus $G = E/2(1+\nu) = 79.2896 \times 10^9\ \mathrm{N\,m^{-2}}$ and taking the equivalent area as calculated above, the equivalent shear coefficient is calculated as $\kappa = 0.4956$.

The above continuum stiffness properties have been successfully employed in well-known continuum beam and rod theories, suitably modified, in order to predict natural frequencies of vibration of the complete truss [16], wherein it is shown that the present approach provides excellent agreement so long as the semi-wavelength of vibration exceeds the depth of the beam. (Once the semi-wavelength approaches the beam depth, there is the possibility of depthwise modes of vibration, and this defines the extent to which a one-dimensional approximation is useful for the prediction of natural

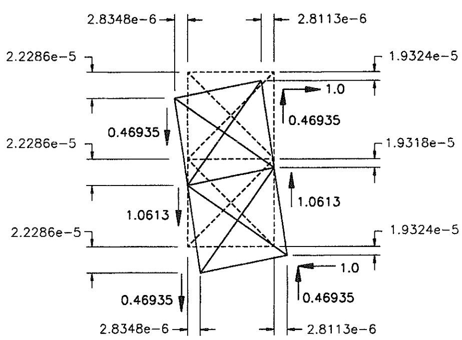

Fig. 4. Nodal displacements and forces for shear coupled with bending.

frequencies.) This accuracy appears to be better than, for example, the continuum models based upon energy equivalence, which according to Lee [17] are expected to give good results when the wavelength of vibration spans many repeating cells; however a full comparison of the merits of the various approaches is beyond the scope of the present paper.

6. Conclusion

By imposing boundary conditions of either zero displacement or zero force at some extreme Nth section at the right-hand end of a repetitive structure together with the use of a recurrence relationship, the nodal displacement or force components on either side of the typical cell can be related by a transfer matrix of half the original size; calculation of the left to right Saint-Venant decay rates is then reduced to a standard eigenvalue problem. A complete characterisation of the framework, including its equivalent beam properties, is achieved most effectively using the force-transfer matrix formulation. Besides the reduction in transfer matrix size, the procedure has the advantage of reducing the number of principal vectors to just one, and results in a well-conditioned eigenproblem.

References

[1] Zhong WX, Lin JH, Qiu CH. Computational structural mechanics and optimal control – the simulation of substructural chain theory to linear quadratic optimal control problems. Int J Num Meth in Engng 1992;33:197–211.
[2] Zhong WX, Williams FW. Wave problems for repetitive structures and symplectic mathematics. Proc IMechE. Part C: J Mech Engng Sci 1992;206:371–9.
[3] Zhong WX, Williams FW. Physical interpretation of the symplectic orthogonality of a Hamiltonian or symplectic matrix. Comput Struct 1993;49:749–50.
[4] Zhong WX, Williams FW. On the direct solution of wave propagation for repetitive structures. J Sound Vib 1995;181:485–501.
[5] Langley RS. A transfer matrix analysis of the energetics of structural wave motion and harmonic vibration. Proc Roy Soc, Ser A 1996;452:1631–48.
[6] Stephen NG, Wang PJ. On Saint-Venant's principle in pin-jointed frameworks. Int J Solids Struct 1996;33:79–97.
[7] Stephen NG, Wang PJ. Saint-Venant decay rates: a procedure for the prism of general cross-section. Comput Struct 1996;58: 1059–66.
[8] Timoshenko SP, Goodier JN. Theory of elasticity. 3rd ed. New York: McGraw-Hill; 1982 [article 26].
[9] Klemm JL, Little RW. The semi-infinite elastic cylinder under self-equilibrated end loading. SIAM J Appl Math 1970;19:712–29.
[10] Stephen NG, Wang MZ. Decay rates for the hollow circular cylinder. Trans ASME J Appl Mech 1992;59:747–53.
[11] Golub GH, Van Loan CF. Matrix computations. Oxford: North Oxford Academic; 1983. p. 197.
[12] Stengel RF. Stochastic optimal control. New York: Wiley; 1987. p. 506.
[13] Langley RS. A note on the force boundary conditions for two-dimensional periodic structures with corner freedoms. J Sound Vib 1993;167:377–81.
[14] Dong SB. A block-Stodola eigensolution technique for large algebraic systems with non-symmetrical matrices. Int J Num Meth Engng 1977;11:247–67.
[15] Lin WW. A new method for computing the closed-loop eigenvalues of a discrete-time algebraic Riccati equation. Linear Algebra Applic 1987;96:157–80.
[16] Stephen NG. On the vibration of one-dimensional periodic structures. J Sound Vib 1999;227:1133–42.
[17] Lee U. Dynamic continuum modeling of beamlike space structures using finite-element matrices. AIAA J 1990;28:725–31.