Statistical energy analysis of nonconservatively coupled systems

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The axial vibrations of two rods coupled together by a spring and viscous damper are investigated using a modal analysis approach. Exact expressions describing various energy flows are derived in terms of external forcing spectra. The effects of changes to coupling parameters on the relevant power receptances are studied and attention forcused on analysis of the coupling damper. The conditions when significant power is dissipated within this damper are examined, thus highlighting when the nonconservative nature of the coupling cannot be neglected without major error. Finally relationships between the ensemble avarage energy flow and the average total energies of the subsystems are recovered. The basic features of these various relationships are illustrated throughout by the use of numerical examples.

1. Introduction

Statistical energy analysis (SEA) is a tool that may be used for the analysis of complex systems when the usual deterministic methods, such as finite element analysis, are no longer practical. It is used mainly when dealing with complex systems at high frequencies, where the large number of degrees of freedom required for finite element methods lead to dramatically increased computing times and costs, even with powerful computational facilities. Moreover, at high frequencies, the results obtained by deterministic methods suffer from some shortcomings. These arise because of uncertainties in material properties plus the sensitivity of mode shapes and modal resonant frequencies to any changes in boundary conditions or damping distribution, which lead to significant differences in the results obtained for nearly identical structures at such frequencies. It is these uncertainties that require the treatment of high frequency dynamic response prediction as a probabilistic problem. Usually, an ensemble of similar systems is considered which differ in their parameters, and an ensemble average of the response is then predicted, often that of the total energies of the subsystems, which are then used to represent the responses of these subsystems. Engineering applications of SEA normally involve the analysis of complex systems (e.g., buildings designed against earthquakes, jet engines, aerospace structures, ships, etc.). It is standard SEA practice to divide these systems into sets of subsystems, often described by their gross dynamic properties, which receive, dissipate and exchange energy, see for example, Norton (1989). The concept of using energy flows to describe the interaction between such subsystems was first proposed by Lyon and Maidanik (1962), see also Scharton and Lyon (1968) and Lyon (1975).

In the traditional SEA approach, the system is modelled into subsystems with weak conservative coupling, the aim being to estimate the total time-average distribution of energy among the coupled subsystems. The evaluation of the coefficients that relate the energy flows between subsystems to their energy levels thus lies at the heart of SEA. It is extensively discussed by Lyon (1975), where the energy flow between two oscillators coupled by a spring is analysed, leading to the basic principle of SEA, which states that average energy flow is proportional to the difference between the total time averaged energies of the coupled oscillators.

Since that time, extensive studies have focused on the analysis of two coupled multi-modal systems, as many different types of structures may be idealized in this way. Energy flows in beams have been studied by Crandall and Lotz (1971), Goyder *et al.* (1956), Mace (1993) and Fahy and Mohammed (1992) using wave propagation approaches. Remington and Manning (1975) studied the energy flow between two coupled rods using the same approach and compared the results with exact solutions derived using Green functions. Davies (1972*b*) used the modal approach for the analysis of two coupled beams. Keane and Price (1991) and Keane (1988) used the same approach for the analysis of coupled rods. Their results agree with those derived by Remington and Manning (1975). In all these studies, the energy flow equations were derived assuming a conservative coupling and it was found that, in general, the energy flow between two coupled multi-modal subsystems is proportional to the difference in their modal energies (given a number of assumptions which place limitations on the validity of SEA when dealing with certain kinds of problems; these various assumptions and limitations are extensively discussed in many of the cited works, see for example, Keane (1988) or Fahy (1974)).

When the coupling mechanism between two subsystems is nonconservative, it is clear that the standard proportionality relationship is no longer valid. This problem appears to have been investigated first by Lyon (1975) for coupled oscillators, then by Fahy and Yao (1987), Sun, Lalor and Richards (1987) and Chen and Soong (1991) where it was shown that the energy flow between the two oscillators depends also on the sum of the energy levels of the two oscillators, not just their difference.

In the work presented here, the axial vibrations of two rods coupled together by both a spring and a viscous damper are investigated, using modal analysis. Following steps similar to Davies (1972*a*) and Keane (1988), deterministic expressions for the input powers, dissipated powers and energy flow between the subsystems are derived. By introducing a complex coupling stiffness, Ω, which includes both the spring and damper strengths, it is found that the results obtained agree with those derived by Keane (1988) for conservative coupling, except that a complex coupling strength must be used in place of a simple spring constant. Next, two forcing models are considered which allow the separation of frequency and spatial variables in the expressions of modal spectra. The effects of changes in the coupling parameters on the various receptances are then considered, and special attention paid to the power dissipated within the coupling. This is found to remain at relatively low levels except for a specific range of values of the coupling parameters. Moreover, it is shown that the power dissipated within the coupling is relatively small compared with the power transferred through it when dealing with couplings in this range, which suggests that the nonconservative nature of the coupling can often be ignored without introducing significant errors. Finally, a relationship linking the energy flow between the subsystems and their total energy levels is recovered. As expected, the energy flow between the two rods is dependent not only on the difference between the average modal energies of the two subsystems, but also on the sum of their energy

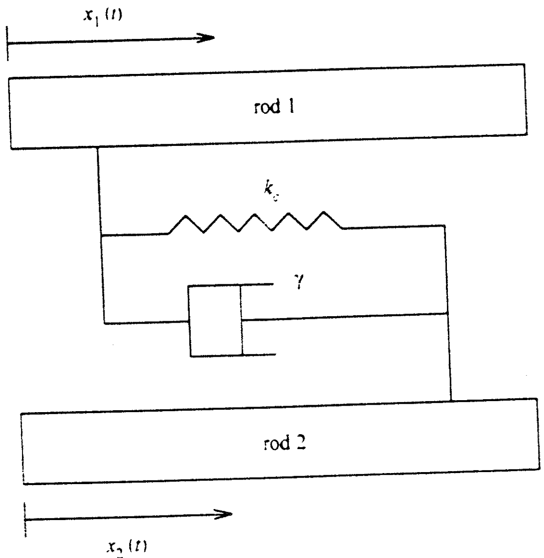

Figure 1. Two point-coupled axially vibrating rods.

levels. This result agrees with that derived by Chen and Soong (1991) for two coupled oscillators, although the constants of proportionality are, of course, different.

2. Derivation of modal summation results

Consider a multi-modal system comprising of two free-free rods coupled together at $x_1 = a_1$ and $x_2 = a_2$ by a spring of stiffness k_c and a damper of strength γ, as is illustrated in figure 1. The mass densities per unit length are $\rho_1(x_1)$ and $\rho_2(x_2)$ and the rigidities per unit length $EA_1(x_1)$ and $EA_2(x_2)$, for rods 1 and 2, respectively. Damping within the two subsystems is assumed proportional to the density per unit length, so that the damping constants for the subsystems, $r_1(x_1)$ and $r_2(x_2)$, are given by,

$$r_1(x_1) = c_1\rho_1(x_1) \quad \text{and} \quad r_2(x_2) = c_2\rho_2(x_2). \tag{2.1,2.2}$$

The differential equations of motion which govern the dynamic behaviour of the rods are then

$$\{\rho_1(x_1)\partial^2/\partial t^2 + r_1(x_1)\partial/\partial t + \Lambda_1\}y_1(x_1, t) = P_1(x_1, t) \tag{2.3}$$

and

$$\{\rho_2(x_2)\partial^2/\partial t^2 + r_2(x_1)\partial/\partial t + \Lambda_2\}y_2(x_2, t) = P_2(x_2, t) \tag{2.4}$$

where Λ_1, Λ_2 are linear spatial differential operators on subsystems 1 and 2, and $P_1(x_1, t)$ and $P_2(x_2, t)$ are the forcing functions on the subsystems. These are given by the expressions

$$P_1(x_1, t) = F_1(x_1, t) + \{k_c[y_2(a_2, t) - y_1(a_1, t)] + \gamma[\dot{y}_2(a_2, t) - \dot{y}_1(a_1, t)]\}\delta(x_1 - a_1) \tag{2.5}$$

and

$$P_2(x_2, t) = F_2(x_2, t) + \{k_c[y_1(a_1, t) - y_2(a_2, t)] + \gamma[\dot{y}_1(a_1, t) - \dot{y}_2(a_2, t)]\}\delta(x_2 - a_2). \tag{2.6}$$

This problem may be solved using modal analysis where the characteristic equations of the subsystems, when uncoupled, are given in terms of their mode shapes Ψ_i and natural frequencies ω_i with

$$\Lambda\Psi_i(x) - \rho(x)\,\omega_i^2\Psi_i(x) = 0. \tag{2.7}$$

This classical eigenvalue problem is readily solved to gain the natural modes and natural frequencies of the two subsystems, see for example Meirovitch (1975). The natural modes are then normalised to satisfy the orthogonality conditions,

$$\int_1 \Psi_i(x_1)\Psi_j(x_1)\rho_1(x_1)\mathrm{d}x_1 = M_1\delta_{ij} \quad \text{and} \quad \int_2 \Psi_r(x_2)\Psi_s(x_2)\rho_2(x_2)\mathrm{d}x_2 = M_2\delta_{rs} \qquad (2.8.2.9)$$

where M_1 and M_2 are the total masses of subsystems 1 and 2, respectively. In the following analysis i, j refer to the first subsystem while r, s refer to the second subsystem. According to the expansion theorem, the displacements of the two coupled subsystems can be written as an expansion in terms of the natural modes with the modal components $W_i(t)$ and $W_r(t)$ so that:

$$y_1(x_1, t) = \sum_{i=1}^{\infty} \Psi_i(x_1)W_i(t) \quad \text{and} \quad y_2(x_2, t) = \sum_{i=1}^{\infty} \Psi_r(x_2)W_i(t). \qquad (2.10.2.11)$$

The modal components of the external forcing are given in terms of the natural modes by the expressions:

$$L_i(t) = \int_1 F_1(x_1, t)\Psi_i(x_1)\mathrm{d}x_1 \quad \text{and} \quad L_r(t) = \int_2 F_2(x_2, t)\Psi_r(x_2)\mathrm{d}x_2. \qquad (2.12.2.13)$$

and the modal components of the coupling forces are given by the expressions

$$V_i(t) = \int_1 \{k_c[y_2(a_2, t) - y_1(a_1, t)] + \gamma[\dot{y}_2(a_2, t) - \dot{y}_1(a_1, t)]\}\delta(x_1 - a_1)\Psi_i(x_1)\mathrm{d}x_1 \tag{2.14}$$

and

$$V_r(t) = \int_2 \{k_c[y_1(a_1, t) - y_2(a_2, t)] + \gamma[\dot{y}_1(a_1, t) - \dot{y}_2(a_2, t)]\}\delta(x_2 - a_2)\Psi_r(x_2)\mathrm{d}x_2 \tag{2.15}$$

so that the external forcing may be written in terms of their modal components as

$$P_1(x_1, t) = \sum_i \Psi_i(x_1)\frac{\rho_1(x_1)}{M_1}[L_i(t) + V_i(t)] \tag{2.16}$$

and

$$P_2(x_2, t) = \sum_r \Psi_r(x_2)\frac{\rho_2(x_2)}{M_2}[L_r(t) + V_r(t)]. \tag{2.17}$$

Considering equations(2.10–2.17) and substituting into the two equations of motion leads to

$$\rho_1(x_1)\sum_i \Psi_i(x_1)[\ddot{W}_i(t) + c_1\dot{W}_i(t) + \omega_i^2W_i(t)] = \sum_i \Psi_i(x_1)\frac{\rho_1(x_1)}{M_1}[L_i(t) + V_i(t)] \tag{2.18}$$

and

$$\rho_2(x_2)\sum_r \Psi_r(x_2)[\ddot{W}_r(t) + c_2\dot{W}_r(t) + \omega_r^2W_r(t)] = \sum_r \Psi_r(x_2)\frac{\rho_2(x_2)}{M_2}[L_r(t) + V_r(t)] \tag{2.19}$$

Multiplying both sides of equation (2.18) by $\Psi_j(x_1)$ and both sides of equation (2.19) by $\Psi_s(x_2)$ and taking the integral over the range $-\infty$ to ∞ yields:

$$M_1[\ddot{W}_j(t) + c_1\dot{W}_j(t) + \omega_j^2 W_j(t)] = [L_j(t) + V_j(t)] \quad (2.20)$$

and

$$M_2[\ddot{W}_s(t) + c_2\dot{W}_s(t) + \omega_s^2 W_s(t)] = [L_s(t) + V_s(t)]. \quad (2.21)$$

These last two equations are in the time domain: to proceed further in the analysis both of them are written in the frequency domain by taking Fourier transforms of each equation. Hence the first equation becomes

$$M_1(-\omega^2 + \omega_j^2 + \mathrm{i}c_1\omega)W_j(\omega) = L_j(\omega) + k_c\Psi_j(a_1)[\textstyle\sum_r \Psi_r(a_2)W_r(\omega) - \sum_i \Psi_i(a_1)W_i(\omega)]$$
$$+\mathrm{i}\gamma\omega\Psi_j(a_1)[\textstyle\sum_r \Psi_r(a_2)W_r(\omega) - \sum_i \Psi_i(a_1)W_i(\omega)]. \quad (2.22)$$

Defining $\Phi_j(\omega)$ and $\Phi_r(\omega)$ as

$$\Phi_j = (-\omega^2 + \omega_j^2 + \mathrm{i}c_1\omega) \quad \text{and} \quad \Phi_r = (-\omega^2 + \omega_r^2 + \mathrm{i}c_2\omega) \quad (2.23.2.24)$$

and the complex coupling strength as

$$\Omega = k_c + \mathrm{i}\gamma\omega, \quad (2.25)$$

the equations become

$$M_1\Phi_j(\omega)W_j(\omega) = L_j(\omega) + \Omega\Psi_j(a_1)[\sum_r \Psi_r(a_2)W_r(\omega) - \sum_i \Psi_i(a_1)W_i(\omega)] \quad (2.26)$$

and

$$M_2\Phi_s(\omega)W_s(\omega) = L_s(\omega) + \Omega\Psi_s(a_2)[\sum_i \Psi_i(a_1)W_i(\omega) - \sum_r \Psi_r(a_2)W_r(\omega)]. \quad (2.27)$$

Multiplying both sides of equation (2.26) by $\frac{\Psi_j(a_1)}{M_1\Phi_j(\omega)}$ and both sides of equation (2.27) by $\frac{\Psi_j(a_1)}{M_1\Phi_j(\omega)}$ and taking the summation over j and s, respectively, the two equations then have the form

$$\sum_j \Psi_j(a_1)W_j(\omega) = \sum_j \frac{L_j(\omega)\Psi_j(a_1)}{M_1\Phi_j(\omega)}$$
$$+\Omega\sum_j \frac{\Psi_j^2(a_1)}{M_1\Phi_j(\omega)}\left[\sum_r \Psi_r(a_2)W_r(\omega) - \sum_i \Psi_i(a_1)W_i(\omega)\right] \quad (2.28)$$

and

$$\sum_s \Psi_s(a_2)W_s(\omega) = \sum_s \frac{L_s(\omega)\Psi_s(a_2)}{M_2\Phi_s(\omega)}$$
$$+\Omega\sum_s \frac{\Psi_s^2(a_2)}{M_2\Phi_s(\omega)}\left[\sum_i \Psi_i(a_1)W_i(\omega) - \sum_r \Psi_r(a_2)W_r(\omega)\right]. \quad (2.29)$$

These two simultaneous equations can be solved for the various summations to give

$$\sum_j \Psi_j(a_1)W_j(\omega) = \sum_j \frac{L_j(\omega)\Psi_j(a_1)}{M_1\Phi_j(\omega)}$$
$$+\frac{\Omega}{\Delta}\sum_j \frac{\Psi_j^2(a_1)}{M_1\Phi_j(\omega)}\left[\sum_r \frac{L_r\Psi_r(a_2)}{M_2\Phi_r(\omega)} - \sum_i \frac{L_i(\omega)\Psi_i(a_1)}{M_1\Phi_i(\omega)}\right] \quad (2.30)$$

and

$$\sum_s \Psi_s(a_2)W_s(\omega) = \sum_s \frac{L_s(\omega)\Psi_s(a_2)}{M_2\Phi_s(\omega)} + \frac{\Omega}{\Delta}\sum_s \frac{\Psi_s^2(a_2)}{M_2\Phi_s(\omega)}\left[\sum_i \frac{L_i\Psi_i(a_1)}{M_1\Phi_i(\omega)} - \sum_r \frac{L_r(\omega)\Psi_r(a_2)}{M_2\Phi_r(\omega)}\right] \quad (2.31)$$

where

$$\Delta = 1 + \Omega\left[\sum_j \frac{\Psi_j^2(a_1)}{M_1\Phi_j(\omega)} + \sum_s \frac{\Psi_s^2(a_2)}{M_2\Phi_s(\omega)}\right]. \quad (2.32)$$

As these equations hold irrespective of the individual natural frequencies and modes shapes, it is then possible to take just the jth term of the summation in equation (2.30) to give

$$W_j(\omega) = \frac{[\kappa_j(\omega) + L_j(\omega)]}{M_1\Phi_j(\omega)} \quad (2.33)$$

where

$$\kappa_j = \frac{\Omega\Psi_j(a_1)}{\Delta}\left[\sum_r \frac{L_r(\omega)\Psi_r(a_2)}{M_2\Phi_r(\omega)} - \sum_i \frac{L_i(\omega)\Psi_i(a_1)}{M_1\Phi_i(\omega)}\right]. \quad (2.34)$$

This result is equivalent to equation 7 of Davies' work (1972*a*) and equation A29 of Keane's work (1988).

3. Long term averages

In the following sections attention is focused on the calculation of energy flows which involve the products of various time-varying functions (forces, displacements and velocities). Before proceeding to the derivation of the energy flows, it is useful to summerize briefly the relations between the energy flow through the coupling and the power dissipated within it.

The time-average energy flow from rod 1 into the coupling is here denoted by $\langle\Pi'_{12}\rangle$. Since the coupling mechanism is not conservative, part of this power will be dissipated within the coupling and this time-average dissipated power is denoted by $\langle\Pi_{dc}\rangle$. The power transferred to rod 2 is then simply calculated by subtracting the power dissipated in the coupling from the total power that leaves rod 1, i.e., the time-average power transferred to rod 2 is given by

$$\langle\Pi_{12}\rangle = \langle\Pi'_{12}\rangle - \langle\Pi_{dc}\rangle. \quad (3.1)$$

Next the time-average power flow from rod 2 into the coupling is denoted by $\langle\Pi'_{21}\rangle$ and the time-average power flow transferred from rod 2 to rod 1 by $\langle\Pi_{21}\rangle$. The relation between these two time-averages is similarly

$$\langle\Pi_{21}\rangle = \langle\Pi'_{21}\rangle - \langle\Pi_{dc}\rangle \quad (3.2)$$

and so it is obvious that

$$\langle\Pi_{12}\rangle = -\langle\Pi'_{21}\rangle \quad \text{and} \quad \langle\Pi_{21}\rangle = -\langle\Pi'_{12}\rangle. \quad (3.3.3.4)$$

The energy balance equation for the coupling is then

$$\langle\Pi_{12}\rangle + \langle\Pi_{21}\rangle + \langle\Pi_{dc}\rangle = 0 \quad (3.5)$$

or

$$\langle \Pi'_{12} \rangle + \langle \Pi'_{21} \rangle = \langle \Pi_{dc} \rangle. \tag{3.6}$$

Also the energy balance equations for subsystems 1 and 2 are

$$\langle \Pi_{1_{in}} \rangle - \langle \Pi_{1_{diss}} \rangle - \langle \Pi'_{12} \rangle = 0 \quad \text{and} \quad \langle \Pi_{2_{in}} \rangle - \langle \Pi'_{2_{diss}} \rangle - \langle \Pi'_{21} \rangle = 0 \tag{3.7,3.8}$$

Since all the processes considered here are assumed ergodic, the time averages taken for any one system will be equal to the ensemble averages taken across an infinite set of similar systems, i.e., all the time averages $\langle \; \rangle$ can be replaced by ensemble averages E[ ].

(a) Energy flow

Consider now the energy flow Π_{21} from rod 1 to rod 2. It is given in the time domain by the expression

$$\begin{aligned}\Pi_{21}(t) &= \int_1 \{k_c[y_2(a_2,t) - y_1(a_1,t)] + \gamma[\dot{y}_2(a_2,t) - \dot{y}_1(a_1,t)]\}\delta(x_1 - a_1)\dot{y}_1(x_1,t)\mathrm{d}x_1 \\ &= \dot{y}_1(a_1,t)\{k_c[y_2(a_2,t) - y_1(a_1,t)] + \gamma[\dot{y}_2(a_2,t) - \dot{y}_1(a_1,t)]\},\end{aligned} \tag{3.9}$$

and the ensemble average of this energy is

$$\begin{aligned}\mathrm{E}[\Pi_{21}(t)] = \mathrm{E}[\Pi_{21}] = k_c\{&\mathrm{E}[\dot{y}_1(a_1,t)y_2(a_2,t)] - \mathrm{E}[\dot{y}_1(a_1,t)y_1(a_1,t)]\} \\ &+\gamma\{\mathrm{E}[\dot{y}_1(a_1,t)\dot{y}_2(a_2,t)] - \mathrm{E}[\dot{y}_1^2(a_1,t)]\}.\end{aligned} \tag{3.10}$$

Since $\mathrm{E}[\dot{y}_1(a_1,t)y_1(a_1,t)] = 0$, see Newland (1975), this last expression becomes

$$\mathrm{E}[\Pi_{21}] = k_c\{\mathrm{E}[\dot{y}_1(a_1,t)y_2(a_2,t)]\} + \gamma\{\mathrm{E}[\dot{y}_1(a_1,t)\dot{y}_2(a_2,t)] - \mathrm{E}[\dot{y}_1^2(a_1,t)]\}. \tag{3.11}$$

Inserting the modal expansions for the displacements leads to

$$\begin{aligned}\mathrm{E}[\Pi_{21}] = k_c\sum_i\sum_r \Psi_i(a_1)\Psi_r(a_2)\mathrm{E}[\dot{W}_i(t)W_r(t)] + \gamma\sum_i\sum_r \Psi_i(a_1)\Psi_r(a_2)\mathrm{E}[\dot{W}_i(t)\dot{W}_r(t)] \\ - \gamma\sum_i\sum_j \Psi_i(a_1)\Psi_j(a_1)\mathrm{E}[\dot{W}_i(t)\dot{W}_j(t)].\end{aligned} \tag{3.12}$$

Writing this last equation in the frequency domain then gives

$$\begin{aligned}\Pi_{21}(\omega) = k_c\sum_i\sum_r \Psi_i(a_1)\Psi_r(a_2)S_{\dot{W}_iW_r}(\omega) + \gamma\sum_i\sum_r \Psi_i(a_1)\Psi_r(a_2)S_{\dot{W}_i\dot{W}_r}(\omega) \\ - \gamma\sum_i\sum_j \Psi_i(a_1)\Psi_j(a_1)S_{\dot{W}_i\dot{W}_r}(\omega)\end{aligned} \tag{3.13}$$

or

$$\begin{aligned}\Pi_{21}(\omega) = \mathrm{i}k_c\sum_i\sum_r \Psi_i(a_1)\Psi_r(a_2)S_{W_iW_r}(\omega) + \gamma\omega^2\sum_i\sum_r \Psi_i(a_1)\Psi_r(a_2)S_{W_iW_r}(\omega) \\ - \gamma\omega^2\sum_i\sum_j \Psi_i(a_1)\Psi_j(a_1)S_{W_iW_r}(\omega).\end{aligned} \tag{3.14}$$

Next, the spectral density of the derived process $S_{W_iW_r}(\omega)$ can be replaced by its equivalent in terms of the modal components of the driving forces acting on the subsystems,

$$S_{W_iW_r}(\omega) = \lim_{T\to\infty}[W_i^*(\omega)W_r(\omega)]\frac{2\pi}{T} = \lim_{T\to\infty}\left[\frac{[L_i(\omega)+\kappa_i(\omega)]^*[L_i(\omega)+\kappa_r(\omega)]}{[M_1\Phi_i(\omega)]^*[M_2\Phi_r(\omega)]}\right]\frac{2\pi}{T} \tag{3.15}$$

Multiplying these various terms and replacing $\lim_{T\to\infty}[L_i^*(\omega)L_r(\omega)]\frac{2\pi}{T}$ by $S_{ir}(\omega)$ and making the necessary mathematical manipulations gives the desired power transmitted to subsystem 1 as

$$\Pi_{21}(\omega) = \frac{[-ik_c\omega+\gamma\omega^2]}{M_1M_2}\sum_i\sum_r\frac{\Psi_i(a_1)\Psi_r(a_2)S_{ir}(\omega)}{\Phi_i^*(\omega)\Phi_r(\omega)}$$

$$+\frac{[-ik_c\omega+\gamma\omega^2]}{\Delta^*}\sum_i\frac{\Omega^*\Psi_i^2(a_1)}{M_1\Phi_i^*(\omega)}\sum_r\frac{\Psi_r^2(a_2)}{M_2\Phi_r^*(\omega)}\left[\sum_r\frac{\Psi_s^2(a_2)S_{sr}(\omega)}{M_2\Phi_s^*(\omega)}-\sum_j\frac{\Psi_j^2(a_1)S_{jr}(\omega)}{M_1\Phi_j^*(\omega)}\right]$$

$$+\frac{[-ik_c\omega+\gamma\omega^2]}{\Delta^*}\sum_r\frac{\Omega\Psi_r^2(a_2)}{M_2\Phi_r^*(\omega)}\sum_i\frac{\Psi_i(a_1)}{M_1\Phi_i^*(\omega)}\left[\sum_j\frac{\Psi_j(a_1)S_{ij}(\omega)}{M_1\Phi_j(\omega)}-\sum_r\frac{\Psi_r(a_2)S_{ir}(\omega)}{M_2\Phi_r(\omega)}\right]$$

$$+\frac{[-ik_c\omega+\gamma\omega^2]}{|\Delta|^2}\sum_i\frac{\Omega^*\Psi_i^2(a_1)}{M_1\Phi_i^*(\omega)}\sum_r\frac{\Omega\Psi_r^2(a_2)}{M_2\Phi_r(\omega)}\left[-\sum_i\sum_j\frac{\Psi_i(a_1)\Psi_j(a_1)S_{ij}(\omega)}{M_1^2\Phi_i^*(\omega)\Phi_j(\omega)}\right.$$

$$\left.-\sum_r\sum_s\frac{\Psi_r(a_2)\Psi_s(a_2)S_{rs}(\omega)}{M_2^2\Phi_r^*(\omega)\Phi_s(\omega)}+\sum_i\sum_r\left[\frac{\Psi_i(a_1)\Psi_r(a_2)S_{ir}(\omega)}{M_1M_2\Phi_i^*(\omega)\Phi_r(\omega)}+\frac{\Psi_i(a_1)\Psi_r(a_2)S_{ri}(\omega)}{M_1M_2\Phi_r^*(\omega)\Phi_i(\omega)}\right]\right]$$

$$-\frac{\gamma\omega^2}{M_1^2}\sum_i\sum_j\frac{\Psi_i(a_1)\Psi_j(a_1)S_{ij}(\omega)}{\Phi_i^*(\omega)\Phi_j(\omega)}$$

$$-\frac{\gamma\omega^2}{\Delta^*}\sum_i\frac{\Omega^*\Psi_i^2(a_1)}{M_1\Phi_i^*(\omega)}\sum_j\frac{\Psi_j(a_1)}{M_1\Phi_j(\omega)}\left[\sum_s\frac{\Psi_s(a_2)S_{sj}(\omega)}{M_2\Phi_s^*(\omega)}-\sum_{j_1}\frac{\Psi_{j_1}(a_1)S_{j_1j}(\omega)}{M_1\Phi_{j_1}^*(\omega)}\right]$$

$$-\frac{\gamma\omega^2}{\Delta}\sum_j\frac{\Omega\Psi_j^2(a_1)}{M_1\Phi_j^*(\omega)}\sum_i\frac{\Psi_i(a_1)}{M_1\Phi_i^*(\omega)}\left[\sum_r\frac{\Psi_r(a_2)S_{ir}(\omega)}{M_2\Phi_r(\omega)}-\sum_{j_1}\frac{\Psi_{j_1}(a_1)S_{ij_1}(\omega)}{M_1\Phi_{j_1}(\omega)}\right]$$

$$-\frac{\gamma\omega^2}{|\Delta|^2}\sum_i\frac{\Omega^*\Psi_i^2(a_1)}{M_1\Phi_i^*(\omega)}\sum_j\frac{\Omega\Psi_j^2(a_1)}{M_1\Phi_j(\omega)}\left[\sum_i\sum_j\frac{\Psi_i(a_1)\Psi_j(a_1)S_{ij}(\omega)}{M_1^2\Phi_i^*(\omega)\Phi_j(\omega)}\right.$$

$$\left.+\sum_r\sum_s\frac{\Psi_r(a_2)\Psi_s(a_2)S_{rs}(\omega)}{M_2^2\Phi_r^*(\omega)\Phi_s(\omega)}-\sum_i\sum_r\left[\frac{\Psi_i(a_1)\Psi_r(a_2)S_{ir}(\omega)}{M_1M_2\Phi_i^*(\omega)\Phi_r(\omega)}+\frac{\Psi_i(a_1)\Psi_r(a_2)S_{ri}(\omega)}{M_1M_2\Phi_r^*(\omega)\Phi_i(\omega)}\right]\right] \tag{3.16}$$

Now this complex expression can be greatly simplified if the driving forces are assumed statistically independent, so that the cross-spectral densities of their modal components

$S_{ir}(\omega) = 0$, and also by recalling the definition of Δ to get

$$\Pi_{21}(\omega) = -ik_c\omega \left[\frac{\left(\sum_j \frac{\Omega\Psi_j^2(a_1)}{M_1\Phi_j(\omega)}\right)^* + \left|\sum_j \frac{\Omega\Psi_j^2(a_1)}{M_1\Phi_j(\omega)}\right|^2}{|\Delta|^2} \right] \sum_r\sum_s \frac{\Psi_r(a_2)\Psi_s(a_2)S_{rs}(\omega)}{M_2^2\Phi_r^*(\omega)\Phi_s(\omega)}$$

$$-ik_c\omega \left[\frac{\sum_r \frac{\Omega\Psi_r^2(a_2)}{M_2\Phi_r(\omega)} + \left|\sum_r \frac{\Omega\Psi_r^2(a_2)}{M_2\Phi_r(\omega)}\right|^2}{|\Delta|^2} \right] \sum_i\sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)S_{ij}(\omega)}{M_1^2\Phi_i^*(\omega)\Phi_j(\omega)}$$

$$+\frac{\gamma\omega^2}{|\Delta|^2}\left(\sum_j \frac{\Omega\Psi_j^2(a_1)}{M_1\Phi_j(\omega)}\right)^* \sum_r\sum_s \frac{\Psi_r(a_2)\Psi_s(a_2)S_{rs}(\omega)}{M_2^2\Phi_r^*(\omega)\Phi_s(\omega)}$$

$$+\frac{\gamma\omega^2}{|\Delta|^2}\left[-1-\left(\sum_r \frac{\Omega\Psi_r^2(a_2)}{M_2\Phi_r(\omega)}\right)^*\right] \sum_i\sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)S_{ij}(\omega)}{M_1^2\Phi_i^*(\omega)\Phi_j(\omega)}. \tag{3.17}$$

Now, when evaluating energy flows. integration is taken over even ranges in the frequency domain so that only the real and even part of the previous expression need be retained. After the necessary mathematical manipulations. and reversing the subscripts, the expression for the power transferred to subsystem 2 can then be recovered as

$$\Pi_{12}(\omega) = \frac{c_2\omega^2 |\Omega|^2}{M_2M_1^2|\Delta|^2} \sum_r \frac{\Psi_r^2(a_2)}{|\Phi_r(\omega)|^2} \sum_i\sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)\mathrm{Re}\{\Phi_i(\omega)\Phi_j^*(\omega)\}S_{ij}(\omega)}{|\Phi_i(\omega)|^2|\Phi_j(\omega)|^2}$$

$$-\frac{c_1\omega^2 |\Omega|^2}{M_1M_2^2|\Delta|^2} \sum_i \frac{\Psi_i^2(a_1)}{|\Phi_i(\omega)|^2} \sum_r\sum_s \frac{\Psi_r(a_2)\Psi_s(a_2)\mathrm{Re}\{\Phi_r(\omega)\Phi_s^*(\omega)\}S_{rs}(\omega)}{|\Phi_r(\omega)|^2|\Phi_s(\omega)|^2}$$

$$-\frac{\gamma\omega^2}{M_2^2|\Delta|^2} \sum_r\sum_s \frac{\Psi_r(a_2)\Psi_s(a_2)\mathrm{Re}\{\Phi_r(\omega)\Phi_s^*(\omega)\}S_{rs}(\omega)}{|\Phi_r(\omega)|^2|\Phi_s(\omega)|^2}. \tag{3.18}$$

Notice that this is exactly as derived by Keane (1988) except that a complex coupling stiffness is used, i.e., Ω in place of k_c, and there is an additional term in $\gamma\omega^2$ (in fact, if the analysis presented by Keane (1988) is carried out starting with a complex coupling stiffness, this expression is exactly recovered).

(b) Input Power

The energy flowing into subsystem 1 from the external forcing is given by the expression

$$\Pi_{1_{in}}(t) = \int_1 \dot{y}_1(x_1,t)F_1(x_1,t)\mathrm{d}x_1. \tag{3.19}$$

The ensemble average of this equation is

$$\mathrm{E}[\Pi_{1_{in}}(t)] = \mathrm{E}[\Pi_{1_{in}}] = \int_1 \mathrm{E}[\dot{y}_1(x_1,t)F_1(x_1,t)]\mathrm{d}x_1. \tag{3.20}$$

By inserting the modal expansions for displacements, the last equation becomes

$$\mathrm{E}[\Pi_{1_{in}}(t)] = \sum_i \sum_j \mathrm{E}[\dot{W}_i(t) L_j(t)] \int_1 \frac{\Psi_i(x_1)\Psi_j(x_1)\rho_1(x_1)}{M_1} \mathrm{d}x_1. \tag{3.21}$$

Next, using the orthogonality property the last equation can be simplified to

$$\mathrm{E}[\Pi_{1_{in}}] = \sum_i \mathrm{E}[\dot{W}_i(t) L_i(t)]. \tag{3.22}$$

Taking spectral densities and Fourier transforms as before gives

$$\Pi_{1_{in}}(\omega) = \sum_i S_{\dot{W}_i i}(\omega) = -\mathrm{i}\omega \sum_i S_{W_i i}(\omega). \tag{3.23}$$

Here $S_{W_i i}$ is the cross-spectral density of the modal displacement and the modal driving force and is given by expression

$$S_{W_i i} = \lim_{T\to\infty} [W_i^*(\omega) L_i^*(\omega)] \frac{2\pi}{T}. \tag{3.24}$$

Substituting $W_i(\omega)$ from equation(2.33) gives

$$\Pi_{1_{in}}(\omega) = -\mathrm{i}\omega \sum_i \left(\lim_{T\to\infty} \left(\frac{\kappa_i^*(\omega) + L_i^*(\omega)}{M_1 \Phi_i^*(\omega)} L_i(\omega) \right) \frac{2\pi}{T} \right). \tag{3.25}$$

Then multiplying the various terms out gives

$$\Pi_{1_{in}}(\omega) = -\mathrm{i}\omega \sum_i \left(\frac{S_{ii}(\omega)}{M_1 \Phi_i^*(\omega)} + \frac{\Omega^* \Psi_i(a_1)}{\Delta^* M_1 \Phi_i^*(\omega)} \left(\sum_r \frac{\Psi_r(a_2) S_{ri}(\omega)}{M_2 \Phi_r^*(\omega)} - \sum_j \frac{\Psi_j(a_1) S_{ji}(\omega)}{M_1 \Phi_j^*(\omega)} \right) \right). \tag{3.26}$$

Assuming that the driving forces acting on the subsystems are uncorrelated as before gives

$$\Pi_{1_{in}}(\omega) = -\mathrm{i}\omega \sum_i \frac{S_{ii}(\omega)}{M_1 \Phi_i^*(\omega)} + \frac{-\mathrm{i}\omega\Omega^*}{M_1^2 \Delta^*} \sum_i \frac{\Psi_i(a_1)}{\Phi_i^*(\omega)} \sum_j \frac{\Psi_i(a_1) S_{ji}(\omega)}{\Phi_j^*(\omega)} \tag{3.27}$$

and taking only the real and even parts of the last expression leads to the desired expression for the input power as

$$\Pi_{1_{in}}(\omega) = \frac{\omega^2 c_1}{M_1} \sum_i \frac{S_{ii}(\omega)}{\mid \Phi_i(\omega) \mid^2} + \frac{\omega}{M_1^2} \mathrm{Im} \left\{ \sum_i \sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)\Omega S_{ji}(\omega)}{\Phi_i(\omega)\Phi_j(\omega)\Delta} \right\}. \tag{3.28}$$

(c) Dissipated power

The power dissipated by damping in subsystem 1 is given by the expression

$$\Pi_{1_{diss}}(t) = \int_1 \dot{y}_1(x_1, t) r_1(x_1) \dot{y}_1(x_1, t) \mathrm{d}x_1. \tag{3.29}$$

The ensemble average is

$$\mathrm{E}[\Pi_{1_{diss}}(t)] = \mathrm{E}[\Pi_{1_{diss}}] = \int_1 \mathrm{E}[\dot{y}_1^2(x_1, t)] r_1(x_1) \mathrm{d}x_1 \tag{3.30}$$

so that by inserting modal expansions for the displacements the expression becomes

$$\mathrm{E}[\Pi_{1_{diss}}] = \sum_i \sum_j \mathrm{E}[\dot{W}_i(t)\dot{W}_j(t)] \int_1 \Psi_i(x_1)\Psi_j(x_1)r_1(x_1)\mathrm{d}x_1. \tag{3.31}$$

Now, since $r_1(x_1) = c_1\rho_1(x_1)$ and using the orthogonality property, this gives

$$\mathrm{E}[\Pi_{1_{diss}}] = c_1 M_1 \sum_i \mathrm{E}[\dot{W}_i(t)\dot{W}_i(t)]. \tag{3.32}$$

Taking the spectral densities and Fourier transforms leads to

$$\Pi_{1_{diss}}(\omega) = M_1 c_1 \sum_i S_{\dot{W}_i\dot{W}_i}(\omega) \quad \text{or} \quad \Pi_{1_{diss}}(\omega) = M_1 c_1 \omega^2 \sum_i S_{\dot{W}_i\dot{W}_i}(\omega) \tag{3.33.3.34}$$

where $S_{\dot{W}_i\dot{W}_i}$ is given by the expression

$$S_{\dot{W}_i\dot{W}_i} = \lim_{T\to\infty} [W_i^*(\omega)W_i(\omega)]\frac{2\pi}{T}. \tag{3.35}$$

Now substituting for $W_i(\omega)$ from equation (2.33) gives

$$\Pi_{1_{diss}}(\omega) = M_1 c_1 \omega^2 \sum_i \lim_{T\to\infty} \left\{ \frac{[L_i(\omega) + \kappa_i(\omega)]^* [L_i(\omega) + \kappa_i(\omega)]}{[M_1\Phi_i(\omega)]^* [M_1\Phi_i(\omega)]} \right\} \frac{2\pi}{T}. \tag{3.36}$$

So multiplying out the various terms leads to

$$\Pi_{1_{diss}}(\omega) = \frac{c_1\omega^2}{M_1} \sum_i \frac{S_{ii}(\omega)}{\mid \Phi_i(\omega) \mid^2}$$

$$-\frac{c_1\omega^2}{M_1^2} \sum_i \sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)}{\mid \Phi_i(\omega) \mid^2} \left(\frac{\Omega^* S_{ji}(\omega)}{\Delta^*\Phi_j^*(\omega)} + \frac{\Omega^* S_{ij}(\omega)}{\Delta^*\Phi_j(\omega)} \right)$$

$$+\frac{c_1\omega^2}{M_1} \sum_i \frac{\mid \Omega \mid^2 \Psi_i^2(a_1)}{\mid \Delta \mid^2 \mid \Phi_i(\omega) \mid^2} \left(\sum_r \sum_s \frac{\Psi_r(a_2)\Psi_s(a_2)S_{rs}(\omega)}{M_2^2\Phi_r^*(\omega)\Phi_s(\omega)} + \sum_i \sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)S_{ij}(\omega)}{M_1^2\Phi_i^*(\omega)\Phi_j(\omega)} \right). \tag{3.37}$$

Again taking only the real and even components finally gives

$$\Pi_{1_{diss}}(\omega) = \frac{c_1\omega^2}{M_1} \sum_i \frac{S_{ii}(\omega)}{\mid \Phi_i(\omega) \mid^2}$$

$$-2\frac{c_1\omega^2}{M_1^2} \sum_i \sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)}{\mid \Phi_i(\omega) \mid^2} \mathrm{Re}\left\{ \frac{\Omega S_{ij}(\omega)}{\Delta\Phi_j(\omega)} \right\}$$

$$+\frac{c_1\omega^2}{M_1 \mid \Delta \mid^2} \sum_i \frac{\mid \Omega \mid^2 \Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} \left\{ \sum_r \sum_s \frac{\Psi_r(a_2)\Psi_s(a_2)}{M_2^2} \mathrm{Re}\left\{ \frac{S_{rs}(\omega)}{\Phi_r^*(\omega)\Phi_s(\omega)} \right\} \right.$$

$$\left. + \sum_i \sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)}{M_1^2} \mathrm{Re}\left\{ \frac{S_{ij}(\omega)}{\Phi_i^*(\omega)\Phi_j(\omega)} \right\} \right\}. \tag{3.38}$$

(d) Power dissipated within the coupling

The power dissipated within the coupling is given by the expression

$$\Pi_{dc}(t) = \gamma[\dot{y}_1(a_1,t) - \dot{y}_2(a_2,t)]^2 = -[\Pi_{12}(t) + \Pi_{21}(t)] \tag{3.39}$$

so that in the frequency domain

$$\begin{aligned} \Pi_{dc}(\omega) &= -[\Pi_{12}(\omega) + \Pi_{21}(\omega)] \\ &= \frac{\gamma\omega^2}{M_1^2 \mid \Delta \mid^2} \sum_i \sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)\mathrm{Re}\{\Phi_i(\omega)\Phi_j^*(\omega)\}S_{ij}(\omega)}{\mid \Phi_i(\omega) \mid^2 \mid \Phi_j(\omega) \mid^2} \\ &+ \frac{\gamma\omega^2}{M_2^2 \mid \Delta \mid^2} \sum_r \sum_s \frac{\Psi_r(a_2)\Psi_s(a_2)\mathrm{Re}\{\Phi_r(\omega)\Phi_s^*(\omega)\}S_{rs}(\omega)}{\mid \Phi_r(\omega) \mid^2 \mid \Phi_s(\omega) \mid^2} \end{aligned} \tag{3.40}$$

(e) Energy levels

Here the energy level of a subsystem is taken to be twice its kinetic energy, which for subsystem 1 is given by the expression

$$KE_1(t) = \frac{1}{2}\int_1 \dot{y}_1^2(x_1,t)\rho_1(x_1)\mathrm{d}x_1 \tag{3.41}$$

so that

$$E_1(t) = \int_1 \dot{y}_1^2(x_1,t)\rho_1(x_1)\mathrm{d}x_1 = \Pi_{1_{diss}}(t)/c_1 \tag{3.42}$$

so that, in the frequency domain, the expression for the energy level is given by

$$\begin{aligned} E_1(\omega) &= \frac{\omega^2}{M_1}\sum_i \frac{S_{ii}(\omega)}{\mid \Phi_i(\omega) \mid^2} \\ &-2\frac{\omega^2}{M_1^2}\sum_i\sum_j \frac{\Psi_i(a_i)\Psi_j(a_i)}{\mid \Phi_i(\omega) \mid^2}\mathrm{Re}\left\{\frac{\Omega S_{ij}(\omega)}{\Delta\Phi_j(\omega)}\right\} \\ &+\frac{\omega^2}{M_1 \mid \Delta \mid^2}\sum_i \frac{\mid \Omega \mid^2 \Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2}\left(\sum_r\sum_s \frac{\Psi_r(a_2)\Psi_s(a_2)}{M_2^2}\mathrm{Re}\left\{\frac{S_{rs}(\omega)}{\Phi_r^*(\omega)\Phi_s(\omega)}\right\}\right. \\ &\left.+\sum_i\sum_j \frac{\Psi_i(a_1)\Psi_j(a_1)}{M_1^2}\mathrm{Re}\left\{\frac{S_{ij}(\omega)}{\Phi_i^*(\omega)\Phi_j(\omega)}\right\}\right) \end{aligned} \tag{3.43}$$

4. Two forcing models

The relation between the modal spectrum $S_{ij}(\omega)$ and the subsystem forcing spectrum is simply

$$\int\int S_{F_1F_1}(\omega, x_i, x_j)\Psi_i(x_i)\Psi_j(x_j)\mathrm{d}x_i\mathrm{d}x_j = S_{ij}(\omega). \tag{4.1}$$

It is convenient to consider forcing models that allow the separation of frequency and spatial variables in the expression of modal spectra. The first model considered is point driving.

(a) Point driving

When the driving forces are applied at a single point $x_1 = b_1$ on subsystem 1 and at $x_2 = b_2$ on subsystem 2, the expression for the forcing spectrum of subsystem 1 becomes

$$S_{F_1F_1}(\omega, x_1, x_1) = S_{F_1F_1}(\omega)\delta(x_1 - b_1)\delta(x_1 - b_1). \tag{4.2}$$

Substituting in equation (4.1), the expression for the modal spectra of subsystem 1 becomes

$$S_{ij}(\omega) = S_{F_1F_1}(\omega)\Psi_i(b_1)\Psi_j(b_1). \tag{4.3}$$

The expression for the energy flowing into subsystem 2 for this model of forcing is then

$$\begin{aligned}\Pi_{12}(\omega) = {} & \frac{c_2\omega^2 \mid \Omega \mid^2}{M_2M_1^2 \mid \Delta \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2} \left| \sum_i \frac{\Psi_i(a_1)\Psi_i(b_1)}{\Phi_i(\omega)} \right|^2 S_{F_1F_1}(\omega) \\ & - \frac{c_1\omega^2 \mid \Omega \mid^2}{M_1M_2^2 \mid \Delta \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} \left| \sum_r \frac{\Psi_r(a_2)\Psi_r(b_2)}{\Phi_r(\omega)} \right|^2 S_{F_2F_2}(\omega) \\ & - \frac{\gamma\omega^2}{M_2^2 \mid \Delta \mid^2} \left| \sum_r \frac{\Psi_r(a_2)\Psi_r(b_2)}{\Phi_r(\omega)} \right|^2 S_{F_2F_2}(\omega)\end{aligned} \tag{4.4}$$

that for the power dissipated in the coupling becomes

$$\begin{aligned}\Pi_{dc}(\omega) = {} & \frac{\gamma\omega^2}{M_1^2 \mid \Delta \mid^2} \left| \sum_i \frac{\Psi_i(a_1)\Psi_i(b_1)}{\Phi_i(\omega)} \right|^2 S_{F_1F_1}(\omega) \\ & + \frac{\gamma\omega^2}{M_2^2 \mid \Delta \mid^2} \left| \sum_r \frac{\Psi_r(a_2)\Psi_i(b_2)}{\Phi_r(\omega)} \right|^2 S_{F_2F_2}(\omega)\end{aligned} \tag{4.5}$$

and the expression for the power input into subsystem 1 is

$$\Pi_{1_{in}}(\omega) = \frac{\omega^2 c_1}{M_1} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} S_{F_1F_1}(\omega) - \frac{\omega}{M_1^2} \mathrm{Im} \left\{ \frac{\Omega}{\Delta} \left(\sum_i \frac{\Psi_i(a_1)\Psi_i(b_1)}{\Phi_i(\omega)} \right)^2 \right\} S_{F_1F_1}(\omega). \tag{4.6}$$

(b) Rain-on-the-roof model

In this model of forcing, the modal components of the driving forces are incoherent and this implies that

$$S_{ij}(\omega) = \int\int S_{F_1F_1}(\omega, x_i, x_j)\Psi_i(x_i)\Psi_j(x_j)\mathrm{d}x_i\mathrm{d}x_j = S_{ij}(\omega)\delta_{ij}. \tag{4.7}$$

In this case, the expression for the forcing spectrum for subsystem 1 is

$$S_{F_1F_1}(\omega, x_1, x_1') = S_{F_1F_1}(\omega)\delta(x_1 - x_1')\frac{\rho_1}{M_1}. \tag{4.8}$$

The expression for the energy flowing into subsystem 2 for this model of forcing is

$$\Pi_{12}(\omega) = \frac{c_2\omega^2 \mid \Omega \mid^2}{M_2 M_1^2 \mid \Delta \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} S_{F_1F_1}(\omega)$$
$$- \frac{c_1\omega^2 \mid \Omega \mid^2}{M_2 M_2^2 \mid \Delta \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2} S_{F_2F_2}(\omega)$$
$$- \frac{\gamma\omega^2}{M_2^2 \mid \Delta \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2} S_{F_2F_2}(\omega) \qquad (4.9)$$

and that for the power dissipated in the coupling becomes

$$\Pi_{dc}(\omega) = \frac{\gamma\omega^2}{M_1^2 \mid \Delta \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} S_{F_1F_1}(\omega) + \frac{[\gamma\omega^2]}{M_2^2 \mid \Delta \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2} S_{F_2F_2}(\omega). \quad (4.10)$$

Finally, the expression for the power input into subsystem 1 becomes

$$\Pi_{1_{in}}(\omega) = \frac{\omega^2 c_1}{M_1} \sum_i \frac{1}{\mid \Phi_i(\omega) \mid^2} S_{F_1F_1}(\omega) + \frac{\omega}{M_1^2} \mathrm{Im} \left\{ \frac{\Omega}{\Delta} \left(\sum_i \frac{\Psi_i(a_1)}{\Phi_i(\omega)} \right)^2 \right\} S_{F_1F_1}(\omega). \qquad (4.11)$$

(c) Power Receptances

The various expressions for the input power, coupling power and power dissipated within the coupling can be written in terms of power receptances for the two cases of forcing as follows:

$$\Pi_{1_{in}}(\omega) = H_{11}(\omega) S_{F_1F_1}(\omega). \qquad (4.12)$$

$$\Pi_{2_{in}}(\omega) = H_{22}(\omega) S_{F_2F_2}(\omega). \qquad (4.13)$$

$$\Pi_{dc}(\omega) = H_{dc_1}(\omega) S_{F_1F_1}(\omega) + H_{dc_2}(\omega) S_{F_2F_2}(\omega), \qquad (4.14)$$

$$\Pi_{12}(\omega) = H_{12}(\omega) S_{F_1F_1}(\omega) - H_{21}(\omega) S_{F_2F_2}(\omega) - H_{dc_2}(\omega) S_{F_2F_2}(\omega), \qquad (4.15)$$

$$\Pi_{21}(\omega) = H_{21}(\omega) S_{F_2F_2}(\omega) - H_{12}(\omega) S_{F_1F_1}(\omega) - H_{dc_1}(\omega) S_{F_1F_1}(\omega), \qquad (4.16)$$

where the expressions for the various receptances are

(i) point driving

$$H_{11}(\omega) = \frac{\omega^2 c_1}{M_1} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} + \frac{\omega}{M_1^2} \mathrm{Im} \left\{ \frac{\Omega}{\Delta} \left(\sum_i \frac{\Psi_i(a_1)\Psi_i(b_1)}{\Phi_i(\omega)} \right)^2 \right\},$$

$$H_{22}(\omega) = \frac{\omega^2 c_2}{M_2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2} + \frac{\omega}{M_2^2} \mathrm{Im} \left\{ \frac{\Omega}{\Delta} \left(\sum_r \frac{\Psi_r(a_2)\Psi_r(b_2)}{\Phi_r(\omega)} \right)^2 \right\},$$

$$H_{12}(\omega) = \frac{c_2\omega^2 \mid \Omega \mid^2}{M_2 M_1^2 \mid \Delta \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2} \left| \sum_i \frac{\Psi_i(a_1)\Psi_i(b_1)}{\Phi_i(\omega)} \right|^2,$$

$$H_{21}(\omega) = \frac{c_1\omega^2 \mid \Omega \mid^2}{M_1 M_2^2 \mid \Delta \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} \left| \sum_r \frac{\Psi_r(a_2)\Psi_r(b_2)}{\Phi_r(\omega)} \right|^2,$$

Table 1. *Parameters used in the examples*

Parameter	Rod 1	Rod 2	Units
Mass density(ρ)	4.156	4.156	kg/m
Length (l)	5.182	4.328	m
Rigidity (EA)	17.85	17.85	MN
Coupling Position (x/l)	0.1176	0.7042	-
Damping strength (c)	88.95	106.49	s^{-1}

$$H_{dc_1}(\omega) = \frac{\gamma\omega^2}{M_1^2 \mid \Delta \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2}.$$

$$H_{dc_2}(\omega) = \frac{\gamma\omega^2}{M_2^2 \mid \Delta \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2}:$$

(ii) rain-on-the-roof driving

$$H_{11}(\omega) = \frac{\omega^2 c_1}{M_1} \sum_i \frac{1}{\mid \Phi_i(\omega) \mid^2} + \frac{\omega}{M_1^2} \mathrm{Im}\left\{ \frac{\Omega}{\Delta} \left(\sum_i \frac{\Psi_i(a_1)}{\Phi_i(\omega)} \right)^2 \right\},$$

$$H_{22}(\omega) = \frac{\omega^2 c_2}{M_2} \sum_r \frac{1}{\mid \Phi_r(\omega) \mid^2} + \frac{\omega}{M_2^2} \mathrm{Im}\left\{ \frac{\Omega}{\Delta} \left(\sum_r \frac{\Psi_r(a_2)}{\Phi_r(\omega)} \right)^2 \right\},$$

$$H_{12}(\omega) = \frac{c_2\omega^2 \mid \Omega \mid^2}{M_2 M_1^2 \mid \Delta \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2},$$

$$H_{21}(\omega) = \frac{c_1\omega^2 \mid \Omega \mid^2}{M_1 M_2^2 \mid \Delta \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2},$$

$$H_{dc_1}(\omega) = \frac{\gamma\omega^2}{M_1^2 \mid \Delta \mid^2} \sum_i \frac{\Psi_i^2(a_1)}{\mid \Phi_i(\omega) \mid^2}.$$

$$H_{dc_2}(\omega) = \frac{\gamma\omega^2}{M_2^2 \mid \Delta \mid^2} \sum_r \frac{\Psi_r^2(a_2)}{\mid \Phi_r(\omega) \mid^2}.$$

5. Variations in the coupling parameters

In the following examples rod 1 is loaded by unit forcing, so that the energy flow into subsystem 2 is equal to $H_{12}(\omega)$ and the power dissipated within the coupling is equal to $H_{dc_1}(\omega)$. The energy flow from rod 1 into the coupling is simply $H_{12}(\omega) + H_{dc_1}(\omega)$, whereas that input into rod 1 is $H_{11}(\omega)$. The properties of these various receptances are illustrated for the case of 'rain-on-the-roof' forcing and light damping. To aid comparison of the various results the parameters values adopted here are the same as those used in several previous studies (Remington and Manning, 1975; Keane and Price, 1987, 1991).

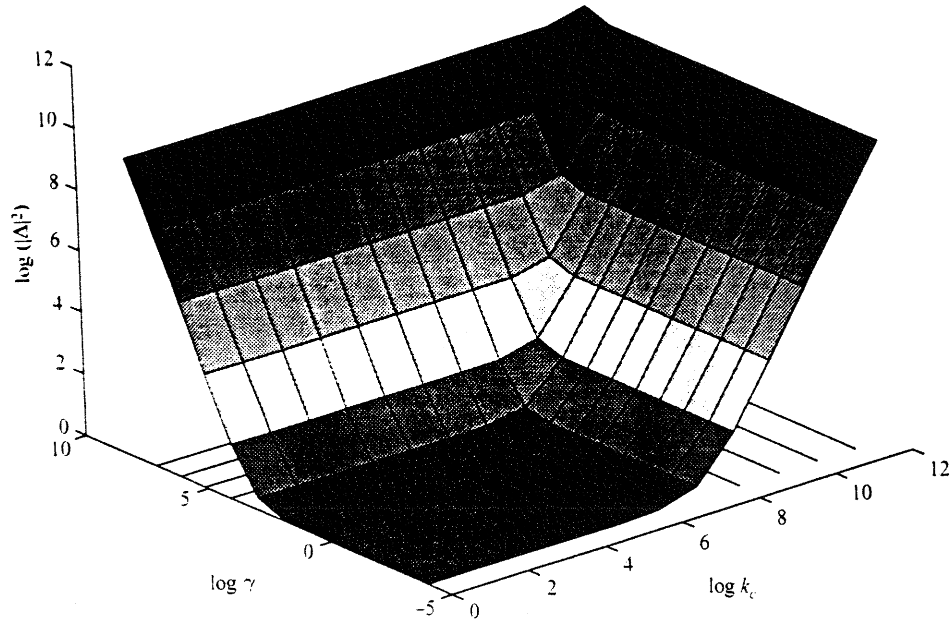

Figure 2. Variation in log($|\Delta|^2$) with γ and k_c at $\omega = 10\,000$ rad/s.

(a) Coupling Strength

In the previous equations Δ may be written as

$$\Delta = 1 + \Omega Z \tag{5.1}$$

where Z is a complex quantity which is independent of the coupling parameters γ and k_c. It is given by the expression

$$Z = \sum_i \frac{\Psi_i^2(a_1)}{M_1 \Phi_i(\omega)} - \sum_r \frac{\Psi_r^2(a_2)}{M_2 \Phi_r(\omega)}. \tag{5.2}$$

Weak coupling may be defined by the criterion

$$|\Delta|^2 \approx 1$$

which requires that $|\Omega|$ is small. Then, as the coupling strength $|\Omega|$ increases, $|\Delta|^2$ also increases until the coupling becomes infinitely strong as $|\Delta|^2$ approaches infinity. The variation of $|\Delta|^2$ with the coupling parameters γ and k_c is illustrated in figure 2.

(b) Energy flow into rod 2

The cross-receptance $H_{12}(\omega)$ for the two cases of forcing discussed above can be written as

$$H_{12}(\omega) = P_{12}\frac{|\Omega|^2}{|\Delta|^2} \tag{5.3}$$

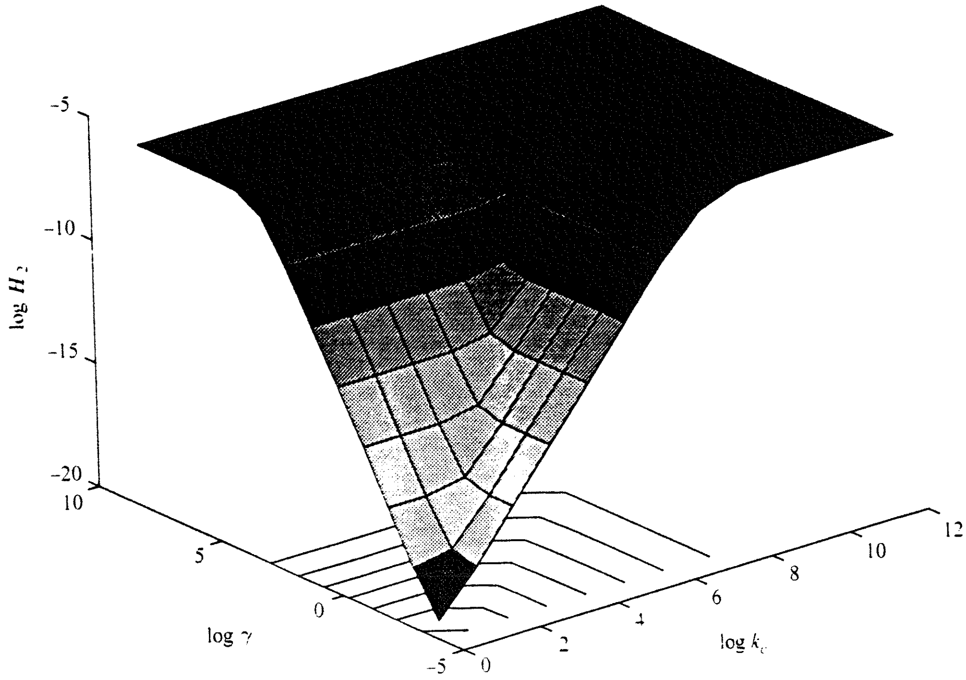

Figure 3. Variation in $\log(H_{12})$ with γ and k_c at $\omega = 10\,000$ rad/s.

where P_{12} contains all the terms that are independent of the coupling parameters. For the case of weak coupling it is clear that this reduces to

$$H_{12}(\omega) = P_{12}|\Omega|^2 \tag{5.4}$$

which implies that for weak coupling and a given magnitude of $|\Omega|^2$, the energy flow into rod 2 is independent of the ratio $\frac{\gamma\omega}{k_c}$. Then, as the coupling strength increases, the energy flow into rod 2 increases until $k_c \to \infty$ or $\gamma \to \infty$, when the energy flow reaches a constant level given by

$$\lim_{\gamma\to\infty} H_{12}(\omega) = \lim_{k_c\to\infty} H_{12}(\omega) = \frac{P_{12}}{|Z|^2}. \tag{5.5}$$

This is illustrated in figure 3, which shows the variation of the cross-receptance $H_{12}(\omega)$ with respect to the two coupling parameters for a constant value of driving frequency ω. The variations of $H_{12}(\omega)$ with respect to the driving frequency ω and the coupling damping γ are illustrated in figure 4 for a fixed value of k_c. For $\gamma = 0$, the results are identical to those illustrated in figure 10 of Keane and Price's work (1991), which is plotted for conservative coupling. Figure 4 clearly shows peaks at the natural frequencies of both rods and marked dips between these frequencies. However, as the coupling damping strength increases, although the curves show peaks and dips at the same frequencies, the magnitude of $H_{12}(\omega)$ increases, as expected. This figure is also consistent with figure 8 of Chen and Soong's work (1991). Similarly, the variations of $H_{12}(\omega)$ with respect to driving frequency ω and spring stiffness k_c, for a fixed value of γ, is illustrated in figure 5. This is very similar to figure 4, which is as expected, since the relation between $H_{12}(\omega)$ and

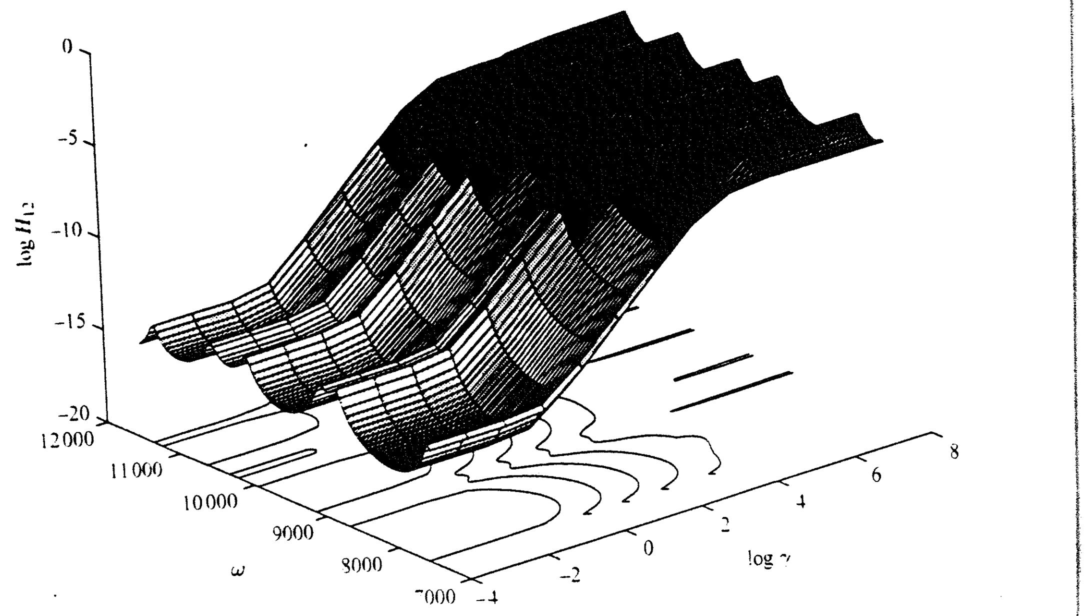

Figure 4. Variation in $\log(H_{12})$ with γ and ω for $k_c = 10\,000$ n/m.

k_c is identical to that between $H_{12}(\omega)$ and $\gamma\omega$. Both figures show that as the coupling becomes infinitely strong energy flow becomes independent of the coupling strength.

(c) Power dissipated within the coupling

For the case of conservative coupling, when $\gamma = 0$, there is no power dissipation within the coupling. However, even for a nonconservative coupling, when either $k_c \to \infty$ or $\gamma \to \infty$ the power dissipated within the coupling is also zero. This means that there must be certain values of γ and k_c for which the dissipated power is maximized. Maximum power is found to be dissipated within the coupling when

$$\gamma = \sqrt{\frac{1 + k_c^2 |Z|^2 + 2k_c \mathrm{Re}\{Z\}}{\omega^2 |Z|^2}}. \tag{5.6}$$

Figure 6 clearly shows this behaviour, with the power dissipated within the coupling staying at relatively low levels except for a specific range of values of γ and k_c. Moreover, this behaviour is sensibly independent of k_c until k_c reaches sufficiently high levels that the damping element becomes rigidly blocked and the dissipated power then falls to low levels. It is straightforward to show that when the coupling is weak the power dissipated within the coupling is independent of k_c and takes the form

$$H_{dc_1}(\omega) = C\gamma, \tag{5.7}$$

where C is independent only on the subsystem parameters, see figure 7. The variation of $H_{dc_1}(\omega)$ with driving frequency ω and damping stiffness γ for a specific value of k_c is further illustrated in figure 8 and that for fixed γ with varying ω and k_c in figure 9. Both of these figures show peaks at the natural frequencies of the directly-driven subsystem and dips between these natural frequencies. Notice that figure 8 shows a maximum for

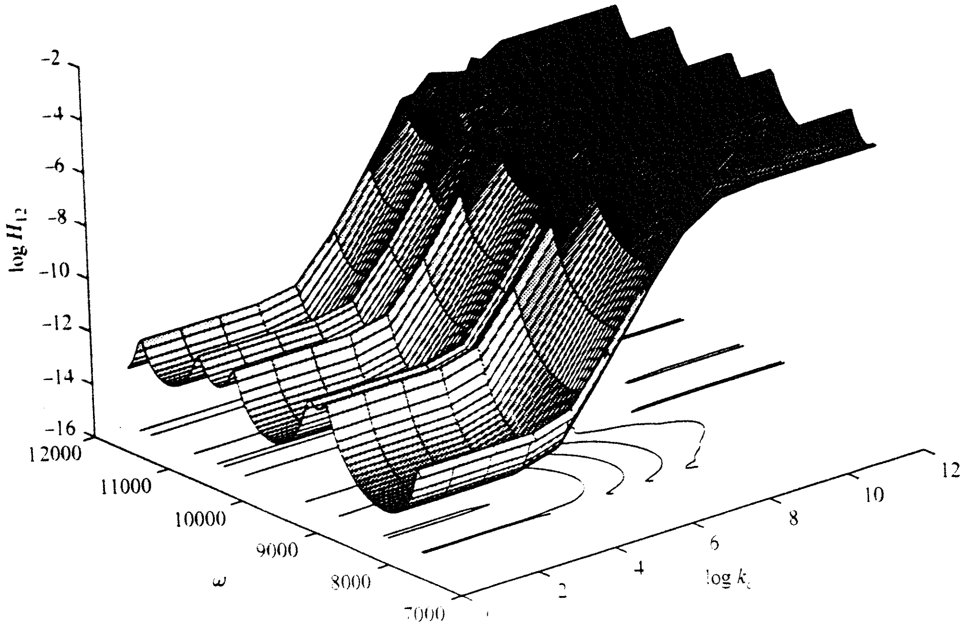

Figure 5. Variation in $\log(H_{12})$ with k_c and ω for $\gamma = 1$ ns/m.

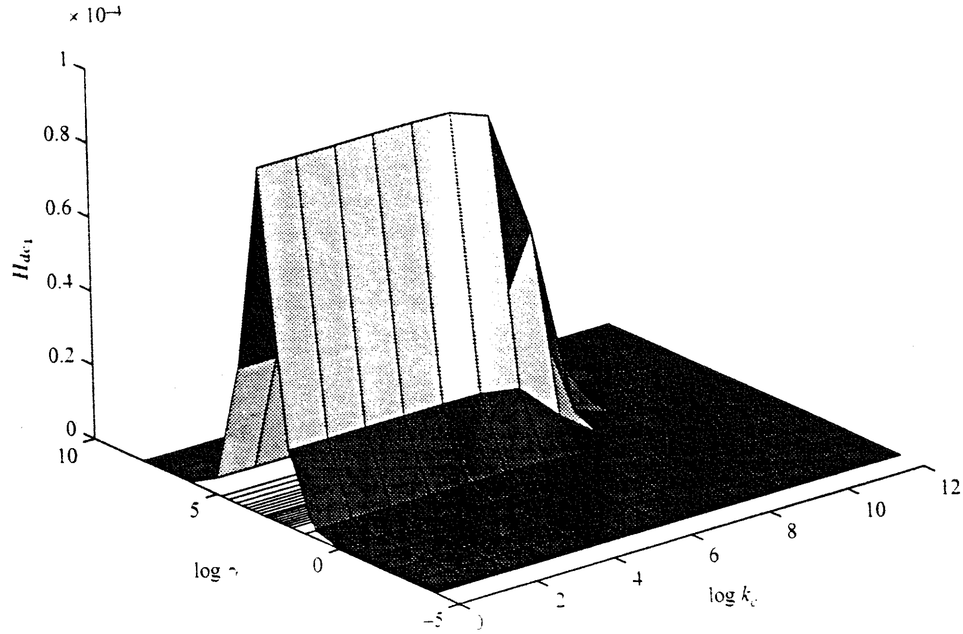

Figure 6. Variation in (H_{dc_1}) with γ and k_c at $\omega = 10\,000$ rad/s.

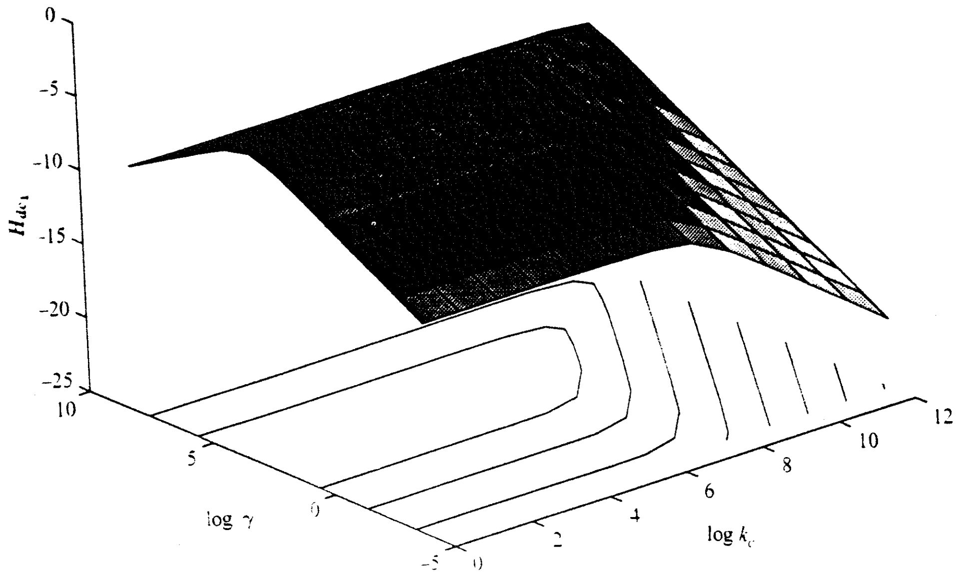

Figure 7. Variation in log(H_{dc_1}) with γ and k_c at $\omega = 10\,000$ rad/s.

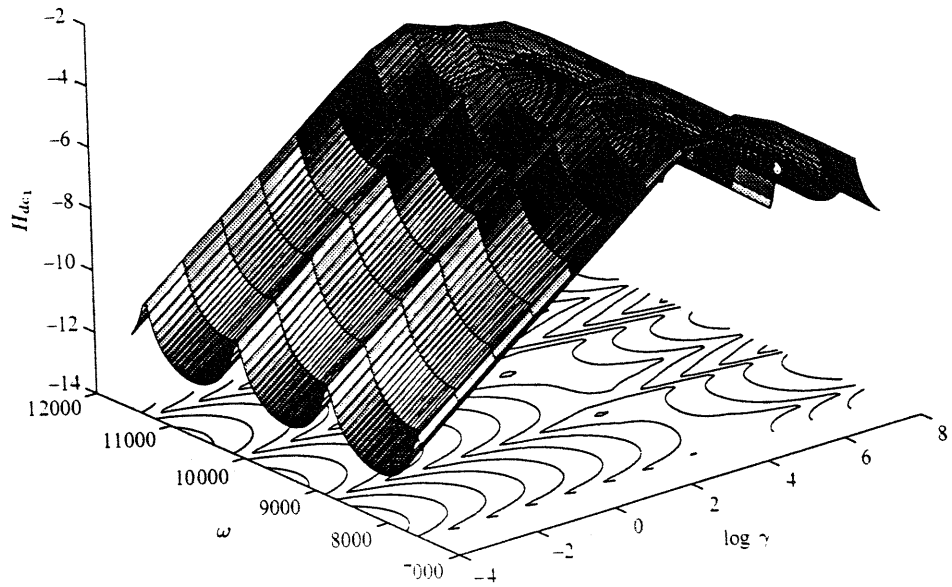

Figure 8. Variation in log(H_{dc_1}) with γ and ω for $k_c = 1000$ n/m.

given ω at the same values of γ, while there are no maxima in figure 9 for variations in k_c, i.e., as per figure 6.

(d) Energy dissipated within the coupling compared with that transferred to rod 2

The energy that leaves rod 1 may be divided into two parts: some of it is dissipated within the coupling and the remainder is transferred to rod 2. It is of interest to see how

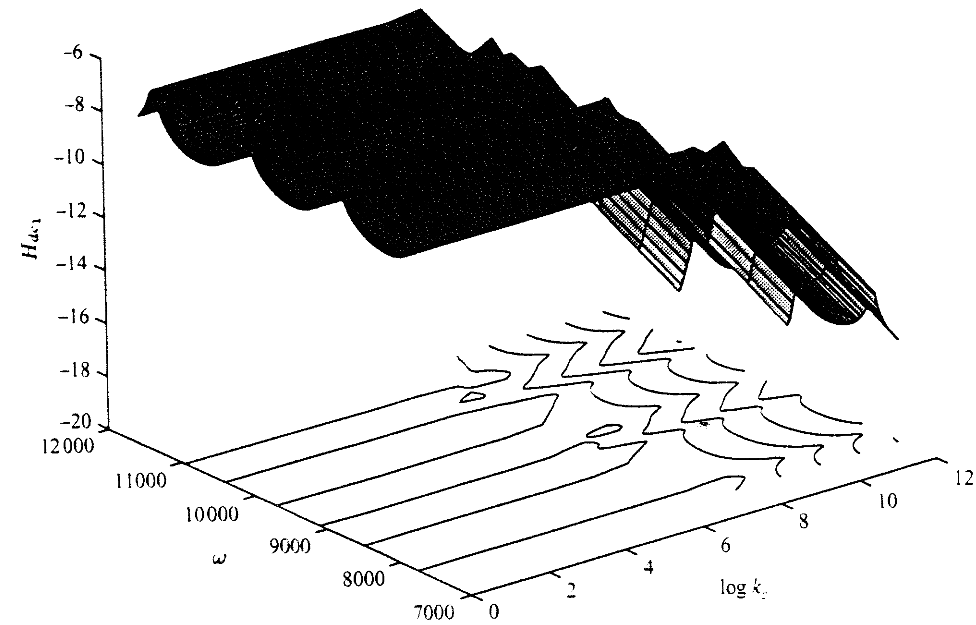

Figure 9. Variation in log(H_{dc_1}) with k_c and ω for $\gamma = 1$ ns/m.

the ratio of these two quantities is affected by changes in the coupling parameters. This ratio is given by the expression

$$\frac{H_{dc_1}(\omega)}{H_{12}(\omega)} = \frac{\gamma B}{|\,\Omega\,|^2} = R \quad \text{where} \quad B = \frac{M_2}{c_2 \sum_r \dfrac{\Psi_r^2(a_2)}{|\,\Phi_r(\omega)\,|^2}}. \tag{5.8,5.9}$$

It is clear that the ratio is dependent only on the second rod and the coupling parameters, i.e., the ratio is not affected by any changes in the first rod's parameters. As the spring strength k_c increases, more power is transferred to rod 2 and less power is dissipated within the coupling, so that the ratio R decreases. However, as the coupling damping increases, R increases until it reaches a maximum value, when $\gamma = k_c/\omega$, after which it begins to fall again, see figure 10. This figure also shows the contour for which $|\,\Delta\,|^2 = 2$, i.e., where the strength of the coupling is transitional between weak and strong coupling, and it is clear that large values of R (i.e.. those for which the losses in the coupling are a significant fraction of the coupling power) arise both when the coupling is weak and as it becomes strong.

A comparison of figure 6, which shows the absolute magnitude of the energy absorbed in the coupling, and figure 10 (which further shows the locus of maximum $H_{dc_1}(\omega)$ taken from figure 6) thus reveals that for moderate values of γ and k_c, the power dissipated in the coupling is usually much larger than that transmitted and that when it is not, this mostly arises because the coupling is essentially 'short-circuited' by either a very stiff spring or a virtually rigid damper. It is also clear from the elliptical nature of the K contours in figure 10, that it is possible to have low energy absorption in the damper if it is made too weak. Thus, there is a strictly bounded region of values of the two coupling parameters where the damper is absorbing both a significant proportion of the

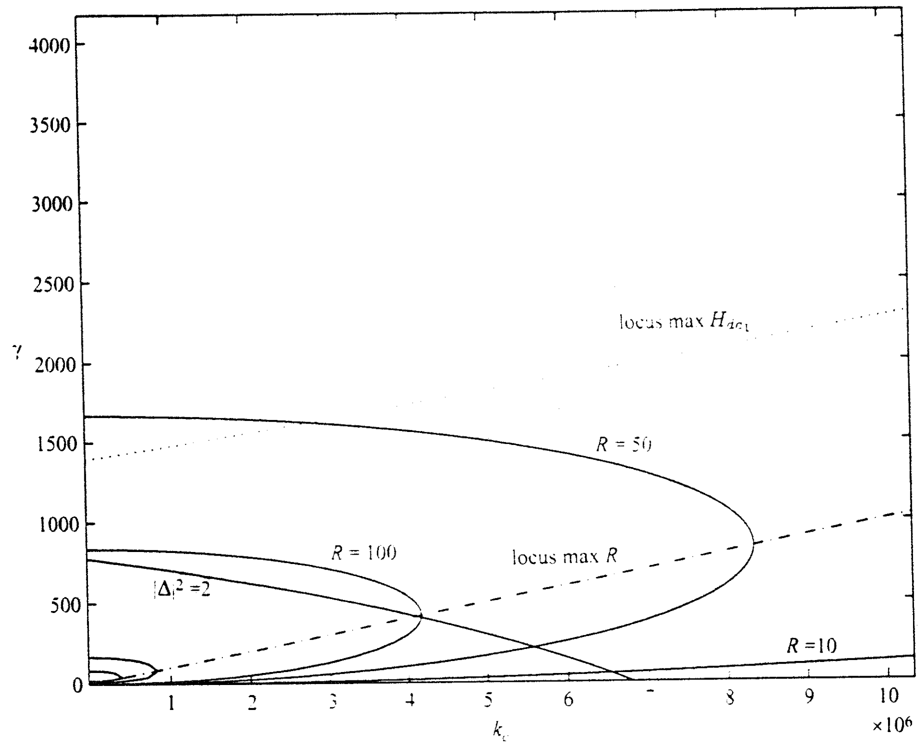

Figure 10. Variation in R with γ and k_c at $\omega = 10\,000$ rad/s, also showing contour of $|\Delta|^2 = 2$ and locus of manimum H_{dc_1}.

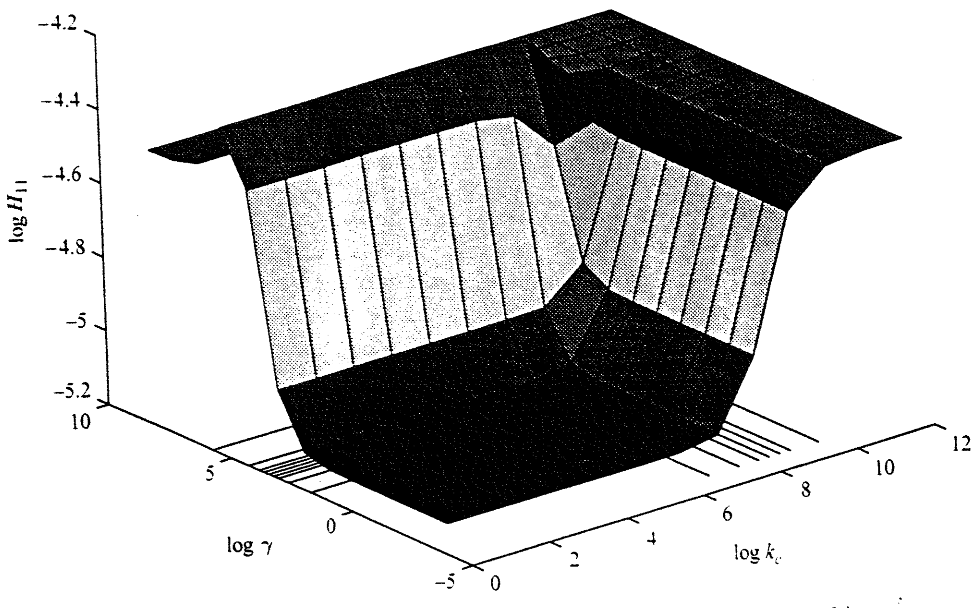

Figure 11. Variation in $\log(H_{11})$ with γ and k_c for $\omega = 8000$ rad/s.

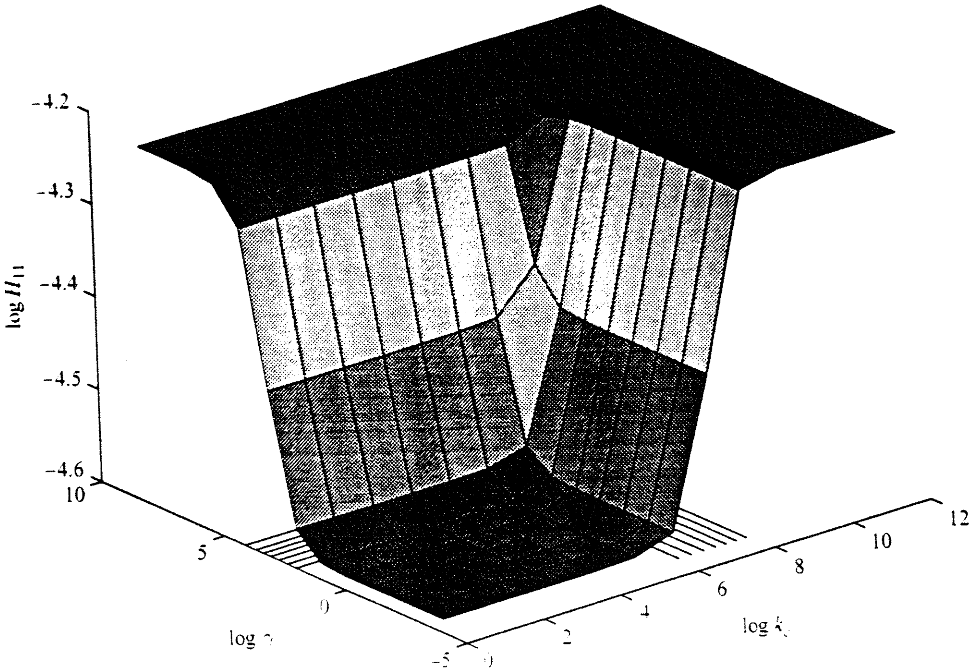

Figure 12. Variation in $\log|H_{11}|$ with γ and k_c for ω = 9000 rad/s.

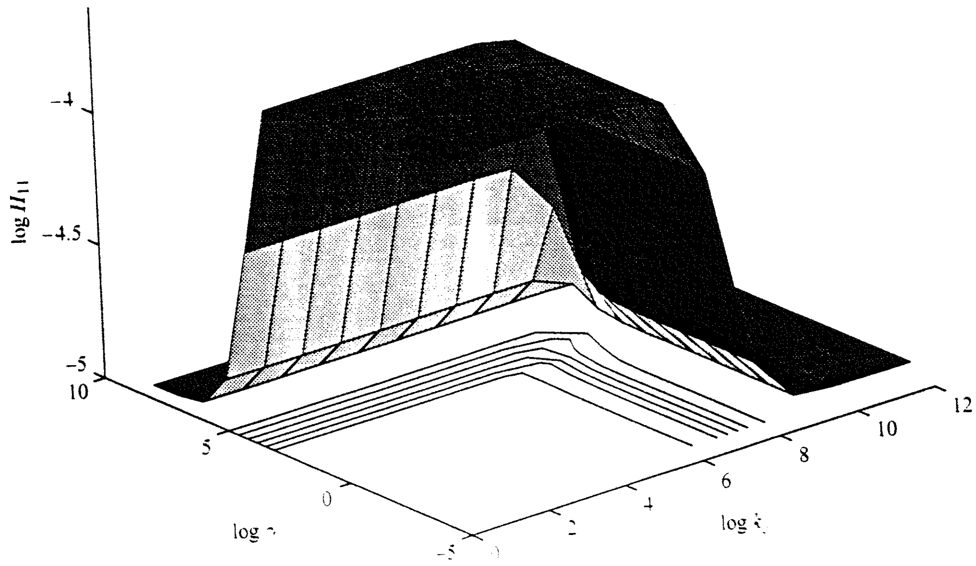

Figure 13. Variation in $\log|H_{11}|$ with γ and k_c for ω = 10 000 rad/s.

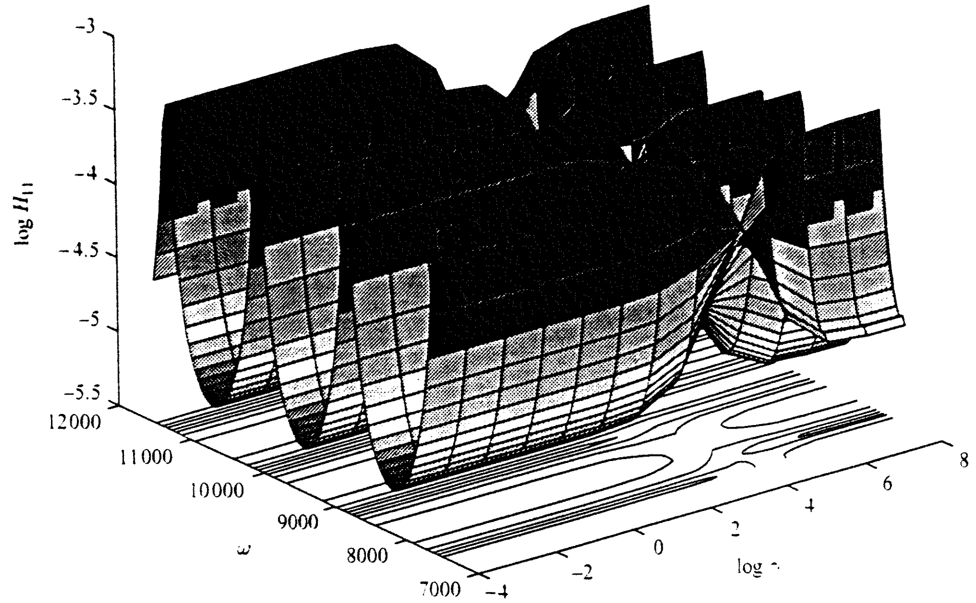

Figure 14. Variation in $\log(H_{11})$ with γ and ω for $k_c = 1000$ n/m.

transmitted energy and an amount which is also a significant fraction of that injected into the overall system. This kind of behaviour is well known to those who design shock mounts for sensitive equipment, where the damping in the mount must not be either too strong or too weak. In either case little energy is absorbed, in the first because the damper is hardly deflected and in the second because, although it deflects, it is too weak to have much effect.

(e) Input power

It can be seen from equation (4.11) that, for weak coupling, the first term in the expression dominates so that the input power is constant for small changes in k_c or γ. Since the second term in the expression may have positive or negative values depending on ω, variation of the input power for larger changes in the coupling parameters shows a variety of behaviours depending on the value of ω chosen, see figures 11–13. However, all these figures show two sensibly constant power levels and a transition from one to the other over roughly the same range of values of the coupling parameters, i.e., those that separate weak and strong coupling. The variation of $H_{11}(\omega)$ with driving frequency and γ for a constant value of k_c is illustrated in figure 14, and that with the driving frequency and k_c for constant value of γ in figure 15. Both figures show peaks at the natural frequencies of the directly-driven subsystem at low coupling strengths and shifts in these as the coupling becomes strong, as expected.

6. Average energy flows and subsystem energies

In the previous sections exact expressions for the various energy flows have been derived. Assuming that deterministic knowledge of the subsystems is not available, the subsystems characteristics can only be described probabilistically. All the previous equations are then expressed as ensemble averages, which are taken across a supposedly infinite set

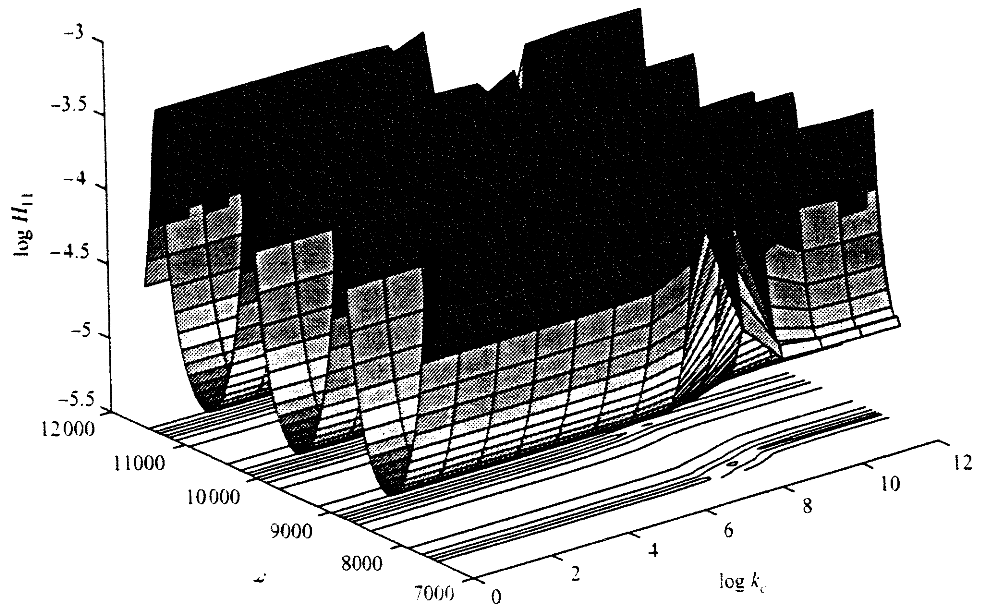

Figure 15. Variation in $\log(H_{11})$ with k_c and ω for $\gamma = 1$ ns/m.

of grossly similar systems, in which the individual members differ in some unpredictable detail. Ensemble averages are denoted by E[ ] and are functions of the driving frequency ω. Thus,

$$\mathrm{E}[\Pi_{1_{in}}(\omega)] = \mathrm{E}[H_{11}(\omega)]S_{F_1}S_{F_1}(\omega), \quad \mathrm{E}[\Pi_{1_{diss}}(\omega)] = c_1\mathrm{E}[E_1(\omega)] \tag{6.1,6.2}$$

and

$$\mathrm{E}[\Pi'_{12}(\omega)] = \mathrm{E}[H_{12}(\omega)]S_{F_1}S_{F_1}(\omega) + \mathrm{E}[H_{dc_1}(\omega)]S_{F_1}S_{F_1}(\omega) - \mathrm{E}[H_{21}(\omega)]S_{F_2}S_{F_2}(\omega). \tag{6.3}$$

Also

$$\mathrm{E}[\Pi_{dc}(\omega)] = \mathrm{E}[H_{dc_1}(\omega)]S_{F_1}S_{F_1}(\omega) + \mathrm{E}[H_{dc_2}(\omega)]S_{F_2}S_{F_2}(\omega). \tag{6.4}$$

The energy balance equations can be written as follows:

$$\mathrm{E}[\Pi_{1_{in}}(\omega)] - \mathrm{E}[\Pi_{1_{diss}}(\omega)] - \mathrm{E}[\Pi'_{12}(\omega)] = 0. \tag{6.5}$$

$$\mathrm{E}[\Pi_{2_{in}}(\omega)] - \mathrm{E}[\Pi_{2_{diss}}(\omega)] - \mathrm{E}[\Pi'_{21}(\omega)] = 0 \quad \text{and} \quad \mathrm{E}[\Pi'_{12}(\omega)] - \mathrm{E}[\Pi'_{21}(\omega)] = \mathrm{E}[\Pi_{dc}(\omega)]. \tag{6.6,6.7}$$

Rearranging these leads to

$$\mathrm{E}[\Pi'_{12}(\omega)] = \alpha_1\mathrm{E}[E_1(\omega)] - \alpha_2\mathrm{E}[E_2(\omega)] + \beta_1\mathrm{E}[E_1(\omega)] \tag{6.8}$$

and

$$\mathrm{E}[\Pi'_{21}(\omega)] = \alpha_2\mathrm{E}[E_2(\omega)] - \alpha_1\mathrm{E}[E_1(\omega)] + \beta_2\mathrm{E}[E_2(\omega)] \tag{6.9}$$

where

$$\alpha_1 = c_1 \frac{\mathrm{E}[H_{12}(\omega)]\mathrm{E}[H_{22}(\omega)]}{D}, \quad \alpha_2 = c_2 \frac{\mathrm{E}[H_{21}(\omega)]\mathrm{E}[H_{11}(\omega)]}{D}. \tag{6.10.6.11}$$

$$\beta_1 = c_1 \frac{(\mathrm{E}[H_{22}(\omega)] - \mathrm{E}[H_{21}(\omega)] - \mathrm{E}[H_{dc_2}(\omega)])\mathrm{E}[H_{dc_1}(\omega)] - \mathrm{E}[H_{12}(\omega)]\mathrm{E}[H_{dc_2}(\omega)]}{D}. \tag{6.12}$$

$$\beta_2 = c_2 \frac{(\mathrm{E}[H_{11}(\omega)] - \mathrm{E}[H_{12}(\omega)] - \mathrm{E}[H_{dc_1}(\omega)])\mathrm{E}[H_{dc_2}(\omega)] - \mathrm{E}[H_{21}(\omega)]\mathrm{E}[H_{dc_1}(\omega)]}{D}, \tag{6.13}$$

and

$$D = (\mathrm{E}[H_{11}(\omega)] - \mathrm{E}[H_{dc_1}(\omega)])(\mathrm{E}[H_{22}(\omega)] - \mathrm{E}[H_{dc_2}(\omega)]) - \mathrm{E}[H_{12}(\omega)](\mathrm{E}[H_{dc_2}(\omega)])$$
$$-\mathrm{E}[H_{21}(\omega)](\mathrm{E}[H_{11}(\omega)] - \mathrm{E}[H_{dc_1}(\omega)]).$$

These equations are similar to those derived by Keane (1988). It is clear that the constants α_1, α_2, β_1 and β_2 are frequency dependent, since these equations deal with energy flows at a particular frequency. They also depend in a complicated way on both k_c and γ, and it is impossible to separate the expressions into distinct terms distinguishing the spring and damper dominant terms. They do, however, show that the energy flows are not simply related to the difference in the energy levels, even after taking ensemble averages.

7. Conclusions

Energy flow relationships for nonconservatively coupled rods have been established using a modal approach. It has been shown that the expressions derived for the various receptances are consistent with those for a conservative coupling when the stiffness is replaced by a complex coupling stiffness which includes the contributions of both spring and damper, but that additional terms proportional to the damper strength also arise (in fact, if the analysis leading to the standard results for conservatively coupled systems is carried out, but using a complex coupling stiffness, the extra terms are readily obtained). The effects of changes in the coupling parameters on the various power receptances have been illustrated through the use of numerical examples in which one rod is excited by 'rain-on-the-roof' forcing.

It has further been shown that the energy transferred to rod 2 through the coupling has a similar qualitative behaviour to that appertaining to the case of conservative coupling. Additionally, it is seen that the power dissipated within the coupling takes relatively low absolute levels except for a specific range of coupling damper rates. Outside of this range the damper is either so weak that it absorbs almost no power or is so strong that it virtually locks the two subsystems rigidly together. Moreover, within this range it is quite easy to arrange for almost no power to be transmitted through the coupling to the undriven subsystem, as might be expected.

Finally, a relationship between the average energy flows and the average total energies has been recovered. The results are consistent with those derived by Chen and Soong (1991) for two coupled oscillators although the constants of proportionality are, of course, different. They are seen to depend in a complicated way on the contributions from both the stiffness and the damping within the coupling and cannot be readily separated into distinct forms containing stiffness and damping dominant terms.

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