Structural Dynamics Recent Advances, Pro. 7th Int. Conf., Southampton, 2000 . 61-73.

Gross modifications in structural dynamics via interpolated modes.

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Abstract

Repeated analysis of a structure for a range of parameter values is often encountered in engineering design and optimisation. In this paper, the problem of approximately predicting the natural frequencies of a system, when parameters undergo gross changes, is addressed. A method of 'interpolated modes' is developed. It is shown that reasonable estimates of the natural frequencies are obtained without a recourse to exact calculations for each value of the system parameter. Illustrative examples are given.

INTRODUCTION

In many design situations, it is required to calculate the natural frequencies of a mechanical vibratory system or a structure for a large set of parameter values. A similar situation is encountered in structural optimisation studies when the objective function is related to the natural frequencies of the system. The problem becomes computationally more and more demanding as the degrees-of-freedom involved in the system increase and as the dimensionality of the parameter space increases. The parameters that describe the system or the structure could be a geometrical dimension (e.g. thickness), a lumped mass, a spring constant, modulus of elasticity etc.

Typically, complex engineering structures are analysed using a finite element (or another approximate) method and the equations of motion are written in terms of second order ordinary differential equations. The natural frequencies are associated with synchronous motion in free vibration; and imposing this condition on the type of motion to be observed leads to the following generalised eigenvalue problem

$$\mathbf{K}\mathbf{u} = \lambda \mathbf{M}\mathbf{u}. \tag{1}$$

The stiffness matrix $\mathbf{K}$ and the inertia matrix $\mathbf{M}$ are both symmetric. In addition, $\mathbf{M}$ is positive definite whereas $\mathbf{K}$, in general, is positive semi-definite to allow for rigid body modes. Square roots of the eigenvalues are the natural frequencies and the eigenvectors represent mode shapes of the system. In a computational cycle of calculating natural frequencies, it is this step of solving the eigenvalue problem (1) that is usually most expensive; setting-up matrices $\mathbf{K}$ and $\mathbf{M}$ are relatively cheaper part of the calculations. Therefore, whenever the eigenproblem (1) needs to be solved repeatedly for a large set of parameter values, the required computational resources are substantially increased. A commonly used approximation that avoids solving equation (1) repeatedly is based on Rayleigh's work [1] and is discussed now.

A change in the parameter p of the system leads to a change in the inertia and the stiffness matrices. If these changes are denoted by $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$ respectively, then the perturbed eigenproblem is given by

$$(\mathbf{K}+\Delta\mathbf{K})(\mathbf{u}_i+\Delta\mathbf{u}_i) = (\lambda_i+\Delta\lambda_i)(\mathbf{M}+\Delta\mathbf{M})(\mathbf{u}_i+\Delta\mathbf{u}_i). \tag{2}$$

Here, $(\lambda_i+\Delta\lambda_i)$ and $(\mathbf{u}_i+\Delta\mathbf{u}_i)$ are the changed eigensolutions. Expanding the two sides of (2), making use of (1), premultiplying by $(\mathbf{u}_i+\Delta\mathbf{u}_i)^T$ and ignoring higher order terms in the expansion leads to

$$\Delta\lambda_i \approx \frac{\mathbf{u}_i^T(\Delta\mathbf{K}-\lambda_i\Delta\mathbf{M})\mathbf{u}_i}{\mathbf{u}_i^T\mathbf{M}\mathbf{u}_i} \tag{3}$$

for the change in the i-th eigenvalue. Note that this expression uses eigensolutions of the original (unperturbed) problem and the changes in the stiffness and mass matrices due to parameter changes. We shall refer to this approximation as the 'classical perturbation'. Based on this expression the exact rate of change of eigenvalue with respect to a parameter can be obtained. While the chassical perturbation (1) gives reasonably good answers for small perturbations, it provides poor approximations for large (or gross) changes in the parameter. This is expected since the classical perturbation formula is based on ignoring higher order terms.

Although the basic idea of perturbation of an eigenvalue problem is quite old (e.g. Rayleigh did not use the language of matrices), there have been numerous studies on specific issues. Lancaster [2] and Fox and Kapoor [3] were first to study the change in eigenvalues and eigenvectors of a matrix when the matrix is a function of a parameter. A good review on the subject, with structural dynamics as the main motivation, can be found in Brandon [4]. The issue of sensitivity and derivatives of eigensolutions is a closely related one (see, Haftka et al.,[5, 6] and other citations therein, for example). Stetson et al. [7] and Nagaraj [8] have looked into design and engineering aspects of the eigenvalue perturbation theory.

THE METHOD OF INTERPOLATED MODES

This paper is concerned with an approximation for eigenvalues (and, consequently the natural frequencies) of a system whose parameters undergo gross changes. In an earlier work, Sahu et al. [9] addressed this problem and they proposed an approximation on an interval over which a parameter takes different values. The strategy there is to interpolate between the eigenvalues that are calculated as perturbations from the two ends taking eigenvectors at the left end as the reference in the first instance; and then taking eigenvectors at the right end as the reference in the second instance. In this manner one has two estimates of the eigenvalue for each value of the parameter on the parameter axis. The final estimate is made by calculating the weighted average such that the value calculated from the eigenvector based on the end of the interval that is closer to the parameter value in question receives more weight. The approach of the present work resembles that of [9] in that it also attempts to develop an approximation for eigenvalues based on eigensolutions at the two ends of an interval. The details of the two approximations, however, are different. While [9] directly calculates an average of two different estimates, the present work employs averaging the *eigenvectors* themselves. The details of the procedure follow.

Consider an interval $p_0 \leq p \leq p_f$, where p represents a parameter that describes the system. The eigensolutions at the two ends can be found exactly:

$$\mathbf{K}_0\mathbf{u}_0 = \lambda\mathbf{M}_0\mathbf{u}_0, \quad \text{and} \quad \mathbf{K}_f\mathbf{u}_f = \lambda\mathbf{M}_f\mathbf{u}_f. \tag{4}$$

Defining a non-dimensional parameter t as

$$t = \frac{p - p_0}{p_f - p_0}, \tag{5}$$

we have $0 \leq t \leq 1$. Taking t and $(1-t)$ as the appropriate weighting factors for interpolating mode shapes to be obtained from $\mathbf{u}_0$ and $\mathbf{u}_f$, we have the following expression for the i-th interpolated mode (the subscript i has been dropped):

$$\overline{\mathbf{u}} = (1-t)(\mathbf{u}_0) + (t)(\mathbf{u}_f). \tag{6}$$

This interpolated mode is now taken as an approximation for the true mode-shape. Finally, the approximate eigenvalue $\tilde{\lambda}_i$ is calculated from the Rayleigh quotient

$$\tilde{\lambda}_i = \frac{\overline{\mathbf{u}}_i^T(\mathbf{K} + \Delta\mathbf{K})\overline{\mathbf{u}}_i}{\overline{\mathbf{u}}_i^T(\mathbf{M} + \Delta\mathbf{M})\overline{\mathbf{u}}_i}. \tag{7}$$

The natural frequencies can be calculated in the usual way by taking square root of the eigenvalues, i.e. $\omega_i = \sqrt{\tilde{\lambda}_i}$.

It may be noted that eigenvalues are not interpolated linearly and that the true non-linear dependence of the coefficient matrices $\mathbf{K}$ and $\mathbf{M}$ on the parameter p is incorporated in the Rayleigh quotient. The trial eigenvectors involved in the quotient are, however, linearly interpolated in the interval. The results are expected to be relatively better at the two ends of the interval, whereas the worst approximation errors are expected close to the centre of the interval. In the limiting case of $t = 0$ and $t = 1$, the values of $\tilde{\lambda}_i$ are exact, since the contribution in (6) is zero except due to the exact modes.

The method uses exact eigensolutions at the two end points of the interval. For calculations at the intermediate points on the parameter axis, the computational effort is considerably less: we need to set-up the coefficient matrices (assemble the finite elements) and evaluate two quadratic forms. With respect to the classical perturbation, the extra effort involved is in solving an additional eigenvalue problem. This may well be worth the effort if we require several evaluations within the interval. This would be typical in a design search or optimisation setting.

The method outlined here can be readily extended to parameter spaces that are multi-dimensional. For example, if we wish to explore the variations in the natural frequencies simultaneously with respect to two parameters, say, p_x and p_y; then the domain of interpoltion is a rectangle, say, $0 \leq p_x \leq a$, $0 \leq p_y \leq b$. Non-dimensional parameters $0 \leq t_x \leq 1$ and $0 \leq t_y \leq 1$ can be defined as

$$t_x = \frac{p_x}{a}, \quad \text{and} \quad t_y = \frac{p_y}{b}.$$

The interpolated mode (eigenvector) is then calculated as

$$\overline{\mathbf{u}}_i(p_x, p_y) = (1 - t_x)(1 - t_y)\mathbf{u}_{00} + t_x(1 - t_y)\mathbf{u}_{a0} + (1 - t_x)t_y\mathbf{u}_{0b} + t_x t_y \mathbf{u}_{ab} \tag{8}$$

where $\mathbf{u}_{rs} = \mathbf{u}_i(p_x = r, p_y = s)$. The four eigenvectors at the corners of the rectangle $[0, a] \times [0, b]$ need to be calculated exactly. It can be seen from the above equation that everywhere inside the rectangle, the interpolation is bilinear. The final step in the calculation: evaluation of the Rayleigh quotient, remains identical to the previous expression of equation (7). The approximate eigenvalues as calculated from equation (7) coincide with the exact ones at the four corners of the rectangle because the contributions to the interpolated mode from the eigenvectors of the other three corners is exactly zero.

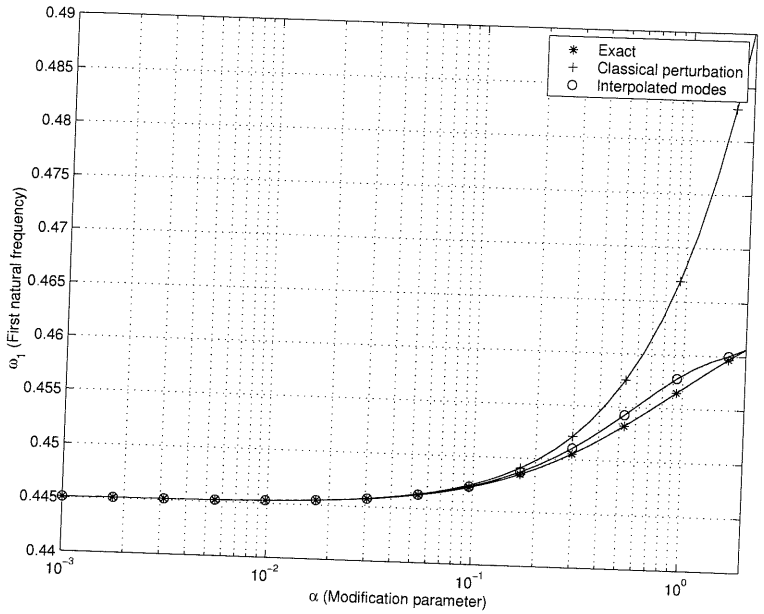

Figure 1: Variation of the first natural frequency with the modification parameter α for the three-degree-of-freedom system (Example 1), $\alpha = 2$ corresponds to 200 % change in the parameter.

EXAMPLES AND DISCUSSIONS

Two examples will be presented now to illustrate the method of interpolated modes developed in the previous section. The first example is of a three-degree-of-freedom discrete mechanical system; and the second example is a finite element model of a cantilever beam with a mass and a stiffness at the tip.

Example 1 Consider a three-degree-of-freedom system with the non-dimensional mass and the stiffness matrices given by

$$\mathbf{M}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{K}_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Suppose one of the stiffnesses is changed from 1 to $(1 + \alpha)$ such that the stiffness matrix changes to

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & (2+\alpha) & -(1+\alpha) \\ 0 & -(1+\alpha) & (1+\alpha) \end{bmatrix}.$$

The mass matrix remains unchanged. Therefore α can be treated as a perturbation parameter. Clearly, $\alpha = 1$ corresponds to 100% change in the stiffness

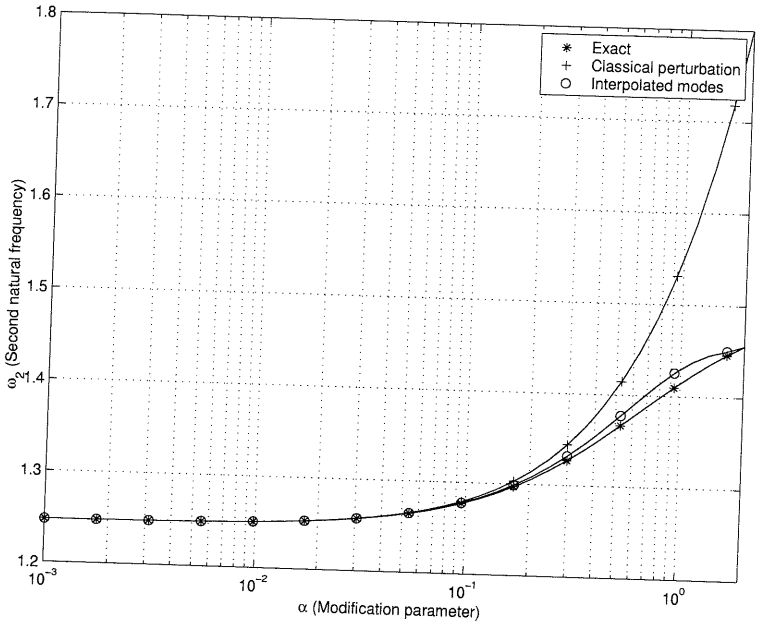

Figure 2: Variation of the second natural frequency with the modification parameter α for the three-degree-of-freedom system (Example 1), $\alpha = 2$ corresponds to 200 % change in the parameter.

parameter, $\alpha = 2$ corresponds to 200% change in the stiffness parameter, and so on.

Eigenvalues and eigenvectors were calculated exactly for $\alpha = 0$ and $\alpha = 2$. Taking $\alpha = 0$ as the 'reference design', frequencies were calculated using the classical perturbation formula (3). Approximate frequencies were also calculated using the interpolated modes as discussed in the previous section. Finally, the exact frequencies were calculated by solving the eigenvalue problem (1) at each intermediate frequency in the range $0 \leq \alpha \leq 2$. Results for the first natural frequency ω_1 are presented in figure (1). Note that the x-axis of the figure is chosen to be logarithmic, since the range of the modifications in the parameter α is reasonably large. It can be seen that the classical perturbation provides excellent approximation to the first natural frequency upto approximately 10% modification in the stiffness parameter (i.e. $\alpha = 0.1$). Beyond this, the higher order terms in the expansion of equation (2) take over and the classical perturbation formula rapidly deviates away from the curve for the variation of the exact natural frequency with respect to α.

The first natural frequency as calculated from the method of interpolated modes also starts deviating from the exact first natural frequency as α

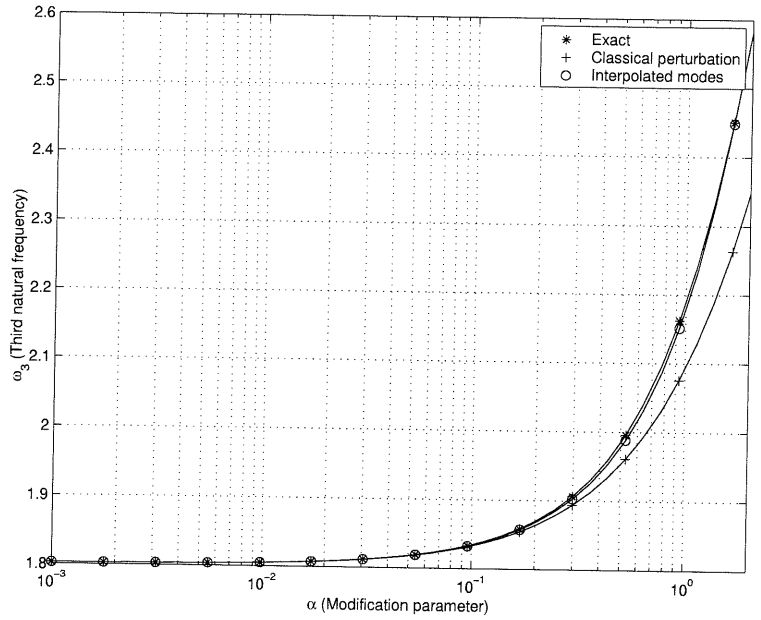

Figure 3: Variation of the third natural frequency with the modification parameter α for the three-degree-of-freedom system (Example 1), $\alpha = 2$ corresponds to 200 % change in the parameter.

increases beyond 15-20%. However, it is contained eventually as we come close to the right side end of the interval corresponding to $\alpha = 200\%$ The errors at the intermediate values can be seen to be substantially less that those obtained from the classical perturbation formula.

The second natural frequency as calculated from equation (7) shows a similar improvement over that calculated from the classical perturbation formula (3). The three curves are plotted in figure (2). Again it can be seen that the values are exact at the two ends of the interval. The third natural frequency as calculated from the perturbation formula (3) underestimates this frequency and is presented as a function of α in figure (3). The discrepancy increases with increase in the parameter α. The performance of interpolated modes, on the other hand, is exceptionally good and the curve for exact eigenvalue is almost indistinguishable from that of the frequencies calculated from the interpolated modes. This indicates that the true mode-shapes, in fact, vary linearly with α in the range shown.

Going back to figure (1), we note that the maximum absolute error in estimating ω_1 is only $\approx 0.49 - 0.461 = 0.029$. However, for a 200% change in α the total change in ω_1 is only $\approx 0.461 - 0.445 = 0.016$. Therefore, to

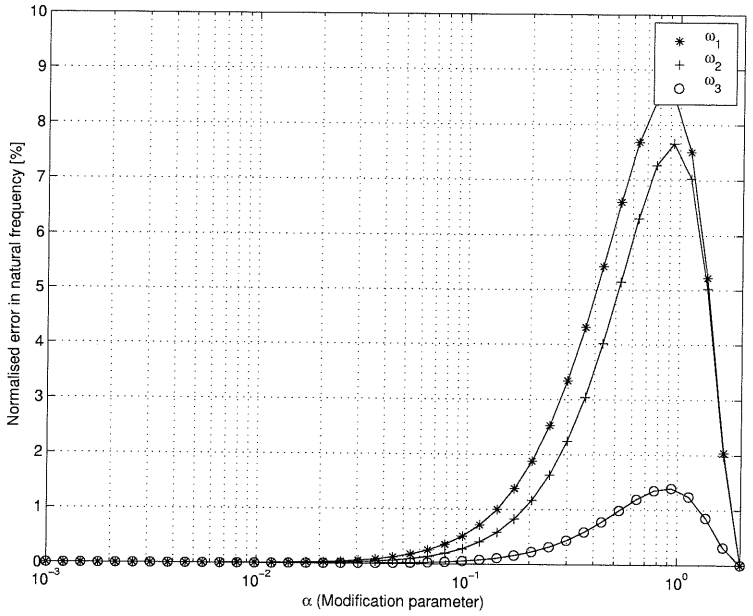

Figure 4: Normalised percentage error in estimating the three natural frequencies as a function of the modification parameter.

represent error better, a normalised error is defined as

$$\epsilon = \frac{\text{Absolute error}}{|\omega_i(p = p_f) - \omega_i(p = p_0)|}.$$

This normalised error (in percentage) is plotted for the three modes in figure (4). The general features of the three error profiles are fairly similar: errors are negligible at the two ends, and are maximum close to the centre of the interval around $\alpha = 80\% - 90\%$. The maximum error for ω_1 is less than 9%, for ω_2 it is less than 8%, and for ω_3 it is less than 2%.

To study the performance of the proposed method of interpolated modes for extremely large modifications, α was varied in the range 0-500%. Again exact calculations were carried out only at the two ends of interval. The approximate calculations and the exact calculations are presented in figure (5) for ω_1. The maximum error is observed around $\alpha = 200\%$ and its value equals approximately 40%. Elsewhere in the interval, it is substantially less. This demonstrates the usefulness of the method developed here, when one is interested only in rough estimates of natural frequencies. The possibility of integrating the method with a design search tool, therefore, exists: one could start with very rough estimates and scan a very large parameter space very cheaply. This could be followed by increasingly more accurate calculations

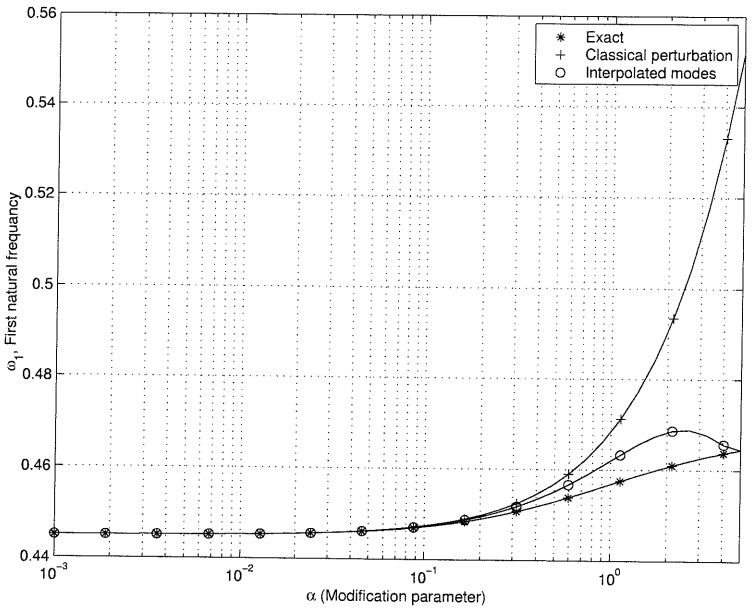

Figure 5: Variation of the first natural frequency with the modification parameter α. Note that the maximum perturbation in α is 500%.

(by reducing the interval successively) as we begin to be more and more committed to a particular design.

Yet another point to note in figure (5) is the non-monotonic nature of the curve corresponding to the interpolated modes. From one of the well-known theorems of Rayleigh [1], it follows that it is impossible to observe this trend. This is because increasing the parameter α means increasing the stiffness of the system, and therefore, natural frequencies can only increase or remain constant. Hence, it is possible to improve the performance of the method of interpolated modes further by incorporating this information that the true curve cannot cross the horizontal line passing through $\omega_1|_{p=p_f}$. Details of imposing this ceiling will not be discussed further and will be presented elsewhere. This scheme of further improvement will work only when a parameter affects *either* the mass properties *or* the stiffness properties.

Recall that the error for $\alpha = 200\%$ was zero in figure (1) because it corresponded to the right end of the interval. As the range increases it is expected that the maximum error will also increase. This variation is presented in figure (6). On the x-axis, maximum value of α (which corresponds to the right end of the interval) is plotted. On the y-axis, the maximum value of the normalised error (maximum being taken over the relevant interval) is plotted for the three natural frequencies. It is seen that the method performs better for ω_2 for relatively smaller values of α_{max} than for ω_1. This trend,

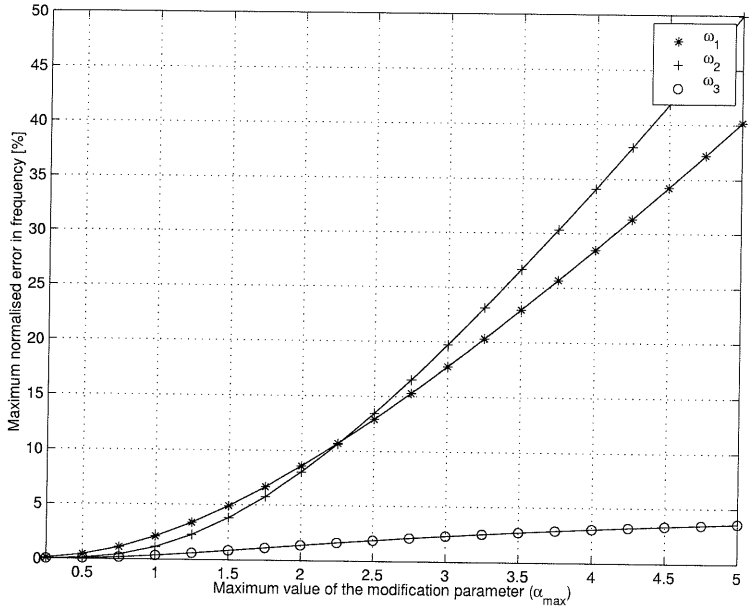

Figure 6: Maximum over interpolation range of the normalised error as a function of $\alpha_{\max}$

however, is reversed for higher values of α_{max}. The third natural frequency is consistently calculated very accurately, the worst error being less than 5% for a modification equal to 500%.

Example 2 Consider a clamped-free cantilever beam. The tip of the beam has a concentrated mass m_0 and a spring k_0. The beam is characterised by three parameters: m, the mass per unit length of the beam; EI its bending rigidity; and L the length. Non-dimensional parameters are defined as

$$m_0^* = m_0/(mL), \quad k_0^* = k_0/(EI/L^3), \quad \lambda_i^* = \lambda_i/(EI/mL^4).$$

We shall treat the two parameters m_0^* and k_0^* as modification over the reference structure which is a cantilever beam without any tip mass or tip spring. The parameter space, therefore is two dimensional.

The beam was discretised using a finite element procedure using 10 elements. Each node of an element possesses two degrees-of-freedom: one in the transverse displacement direction and one for rotation (slope).

The four corners of the domain of interpolation are defined by the points: $(m_0^* = 0, k_0^* = 0)$, $(m_0^* = 2, k_0^* = 0)$, $(m_0^* = 0, k_0^* = 2)$, and $(m_0^* = 2, k_0^* = 2)$. Percentage error is now defined as the ratio of the difference of exact eigenvalue and the eigenvalue calculated via interpolated modes to the exact value at each point in the parameter space. This error is plotted as a function

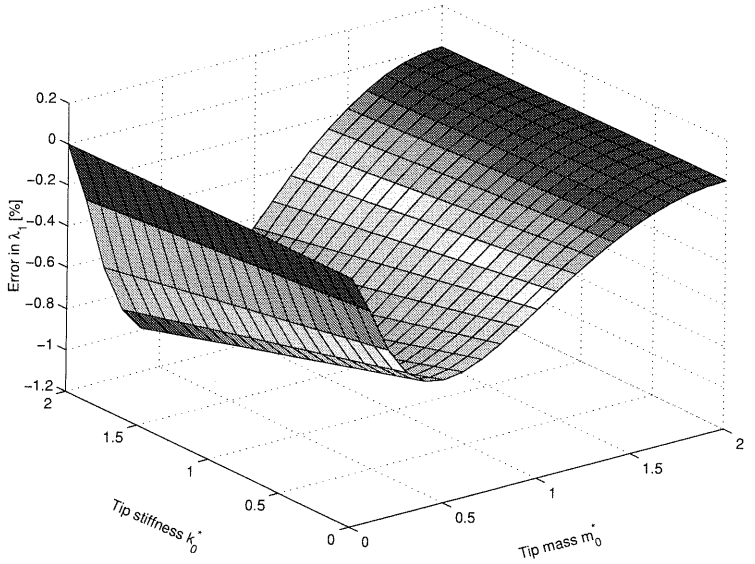

Figure 7: Percentage error in λ_1^* as a function of m_0^* and k_0^*. Maximum modification in both parameters is 200%.

of m_0^* and k_0^* for λ_1^* in figure (7). It is seen that this error is less than 1% throughout the domain when the parameters at the corner points are modified by 200%. Note the values of error at the four corners which is exactly zero, as expected. The same plot for λ_2^* is shown in figure (8). This time the maximum error is of the order of 35%.

Although the method discussed in this paper works quite well for the examples presented here, there are some unresolved difficulties. Firstly, it is recognised that eigenvectors showing sharp changes with respect to a parameter will not be properly interpolated. The results for natural frequencies calculated from such interpolated modes are likely to be fairly inaccurate. Secondly, when two eigenvalues cross each other in the the parameter space leading to reversal of the mode order, extra care needs to be taken in automating the interpolation procedure. If this is not properly done, interpolation with two different 'types' of modes will take place resulting in inaccurate estimation of the corresponding natural frequencies. Finally, the procedure needs to be adapted to systems that possess degeneracy, e.g. periodic structures. The present formulation is robust only for distinct eigenvalues.

CONCLUSIONS

A method of approximate calculation of natural frequecies of a mechanical system based on interpolated modes was presented. This method offers substantial computational saving when one is interested in approximate values

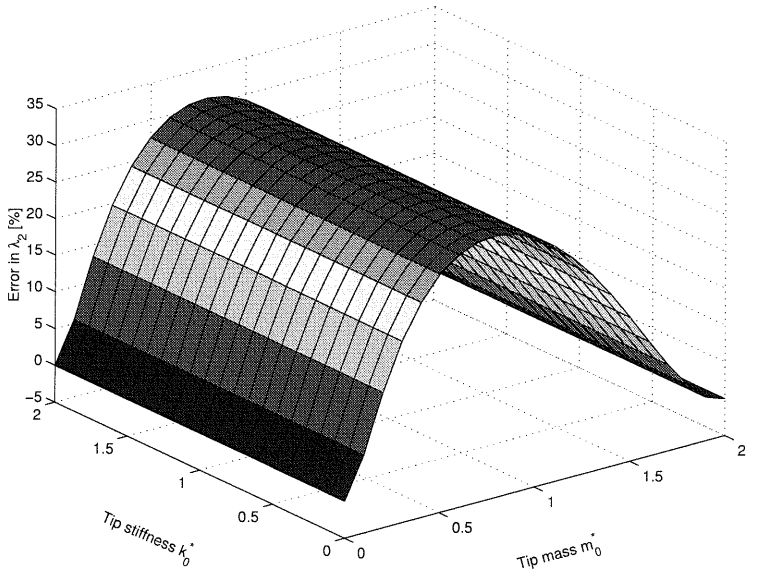

Figure 8: Percentage error in λ_2^* as a function of m_0^* and k_0^*. Maximum modification in both parameters is 200%.

at a large number of points inside an interval. The interval could be multi-dimensional. It was shown, through two simple examples, that reasonably accurate estimates of the natural frequencies are obtained for fairly large values of modification. It is believed that the method will give very useful inputs to optimisation and search porblems associated with structural dynamic design.

Acknowledgement: I thank Professor Andy J. Keane for very useful comments at an early stage of this work.

References

[1] Lord Rayleigh 1945 *Theory of Sound*, vol. 1, Dover, New York.

[2] Lancaster, P., On eigenvalues of matrices dependent on a parameter, *Numerische Math.*, 1964, **6**, 377-387.

[3] Fox, R.L. and Kapoor, M.P., Rate of Change of eigenvalues and eigenvectors, *AIAA Journal*, 1968, **6**, 2426-2429.

[4] Brandon, J.A., *Strategies for Structural Dynamic Modification*, 1990, John Wiley, New York.

[5] Adelman, H.M. and Haftka, R.T., Sensitivity analysis of discrete structural systems, *AIAA Journal*, May 1986, **24**, No. 5, 823-832.

[6] Murthy, D. V. and Haftka, R.T., Derivatives of eigenvalues and eigenvectors of a general complex matrix, *Int. J. for Num. Meth. in Eng.*, 1988, **26**, 293-311.

[7] Stetson, K.A. and Palma, G. E, Inversion of first-order perturbation theory and its application to structural design, *AIAA Journal*, 1976, **14**, No. 4, 454-460.

[8] Nagaraj, V.T., Changes in vibration characteristics due to changes in the structure, *J. Spacecraft*, 1970, **7**, No. 12, 1499-1501.

[9] Sahu, S.S., Bhaskar, A. and Nakra, B.C., Interpolation approach for gross dynamic modifications of structures, 2000, manuscript in preparation.