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REDUCED MODELS FOR STATISTICALLY STATIONARY AND NON-STATIONARY FLOWS

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ABSTRACT

Reduced modelling techniques, based on a Proper Orthogonal Decomposition (POD) method, are applied to an investigation of the incompressible Navier-Stokes equations with inputs. A circular cylinder in uniform flow with and without inputs is studied. Reduced dynamical models are created by POD and by extended POD (EPOD) approaches for the forced flow which is statistically non-stationary. A direct control action is applied to the flow at particular points and this investigation provides insights to the applications of the proposed approaches coupled with a full solver.

INTRODUCTION

The study presented herein describes the construction of explicit low-order models to design controllers for distributed parameter systems which, in this context, are fluid flows. Once a control design has been constructed using such a low order model, it can be tested by comparing its performance against a full high-order simulation.

The difference between modelling for control, and modelling for analysis of dynamical behaviour is that in the latter case the system behaviour is statistically stationary with no external inputs driving the system. However for controlled systems, we are concerned with preserving the relationship between the system behaviour and the system inputs and outputs, or actuators and sensors.

One model reduction method which has been successfully used for dynamical systems' analysis is a Proper Orthogonal Decomposition (POD) method[1]. This method has become popular as a means of extracting dominant energy-containing structures from flow field data and by using these structures as basis functions, generating low order dynamical models for the associated systems. The method has been applied to fluid problems by Sirovich[1] and many other researchers[2,3] to understand the important dynamical features or coherent structures seen in fluid flows.

A full model of the dynamics of such a system is normally represented by a set of high dimensional nonlinear differential equations which can be solved by numerical methods. In this study a cell viscous boundary element method developed by Tan *et al*[4] is used to generate the required data for the reduced model. The POD method describes the system behaviour as an attractor which is a point of evolution for the state space in a subspace of higher dimensions. A reduced solution can be obtained as a linear combination of an optimal set of empirical basis functions using an integral equation method such as the Galerkin projection method.

These bases are created by applying a POD method for statistically stationary data. When a fluid flow is subject to a time-dependent control, the statistical properties of the flow are usually non-stationary. In this case, an extended POD method (EPOD) developed by Glezer *et al*[5] can be adopted. Herein applications of the POD and EPOD methods are investigated in order to derive a reasonable approximation to time-dependent flows associated with vortex shedding.

Several model reduction methods have been proposed and applied for statistically non-stationary systems with control inputs. A control function method has been applied by Ravindran[6] to channel flows. Graham *et al*[7,8] have applied a control function method and penalty method for rotating cylinders in uniform flow. Balanced truncation,[9] optimal Hankel methods and Ott-Grebogi-Yorke (OGY)[10] methods have been proposed to incorporate the control input into the model. Neural networks[11] and reduced basis [12] methods can also be adopted to construct reduced flow models with control inputs. The

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advantages and disadvantages of the applications of some of these approaches in real-time applications in flow control are discussed by Gillies[11].

This paper describes an investigation of POD and EPOD methods to the flow generated by a circular cylinder in uniform flow and reduced dynamical models are established for the non-stationary forced flow.

FULL MODEL

Mathematical Model

A cell viscous boundary element method,[4] developed to solve Navier-Stokes equations, is employed to generate the required data for POD and EPOD analyses by conducting numerical flow simulations for vortex shedding flows behind a circular cylinder. This numerical scheme of study is a hybrid approach combining boundary element and finite element methods. The boundary element method is applied to fluid cells idealising the fluid domain and global equations are obtained by means of finite element procedures. A brief description of the method is included herein whereas, a detailed account is described by Tan *et al*[4].

The governing equations of the flow defined in a body fixed coordinate system translating with a given velocity $\hat{v}_j(t)$, in terms of a non-dimensional velocity field v_j and pressure p relative to a space fixed coordinate system, can be written as

$$\begin{aligned} \dot{v}_j + (v_j v'_k)_{,k} + p_{,j} - [\nu_e(v_{j,k} + v_{k,j})]_{,k} &= 0 \quad (1) \\ v_{j,j} &= 0 \quad (2) \end{aligned}$$

where $\nu_e = 1/\mathrm{Re}$, $v'_k = v_k - \hat{v}_j(t)$ and Re is the Reynolds number.

An integral equation can be formulated from equations (1) and (2) following the methodology of the boundary element method. That is,

$$\begin{aligned} C(\boldsymbol{\xi})v_s(\boldsymbol{\xi},t) + \int_\Sigma v_j(v'_k n_k v^*_{sj} + R^*_{sj})d\Sigma \\ = \int_\Sigma R_j v^*_{sj} d\Sigma - \int_\Omega f_j v^*_{sj} d\Omega \quad (3) \end{aligned}$$

where $C(\boldsymbol{\xi})$ is a constant, the value of which depends on the location of the field point $\boldsymbol{\xi}$. v^*_{sj} and R^*_{sj} denote the fundamental solution and related function[13] respectively. R_j is the traction force on the cell boundary and f_j is the resultant term derived from the acceleration $\dot{v}_j$ after a finite difference scheme is introduced.

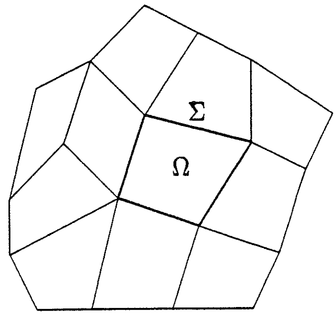


Figure 1: An unstructured mesh idealising the fluid domain.

Discretisation

Integral equation (3) can be applied to each cell in a mesh idealising the fluid domain as shown in fig.1. This generates a set of algebraic equations representing the fluid dynamics on the cell. The application of continuity conditions to fluid velocity and surface traction force at cell interfaces allows a global system of algebraic equations to be obtained. Their solution yields the solutions of the flow velocity in the whole domain.

This method features a high degree of dependence on analytical solutions in the mathematical model, allowing retention of accuracy of solution within the numerical scheme of study. It incorporates a primitive-variable formulation and applies to both structured and unstructured meshes. Furthermore, the use of velocity and surface traction force as the basic unknowns provides a convenient way of expressing boundary conditions.

An extensive validation of this numerical method has been carried out using a number of well documented flow solutions. Good agreement was achieved against data produced from other sources, including theoretical solutions, other numerical predictions, and experimental observations[4,14-17].

Vortex Shedding Problem

Numerical calculations were performed to quantify and describe the vortex shedding behaviour behind a circular cylinder in a uniform current with or without control actions. Forcing of the wake at particular points in the flow is taken as an example of the control actions to be considered. The boundary conditions and geometry description of this cylinder–fluid interaction problem are defined in fig.2. Mixed boundary conditions associated with both traction and velocity are used.

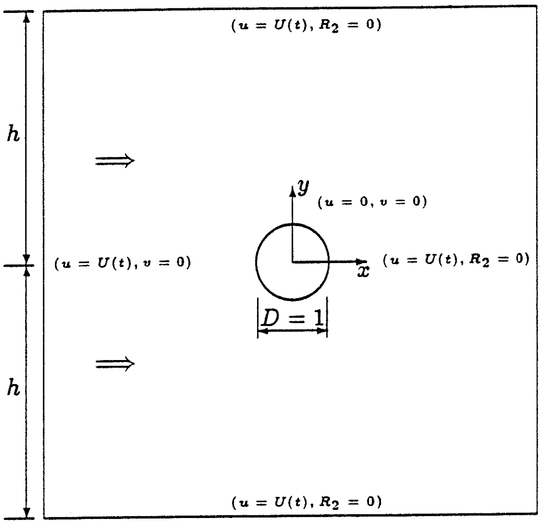


Figure 2: Computational domain definition and boundary conditions for a cylinder in a uniform flow. All the variables and quantities are nondimensional.

MODEL REDUCTION FOR STATIONARY FLOWS

Model reduction methods are applied to the nonlinear stationary flow system which can be expressed formally as,

$$\dot{\boldsymbol{v}}(\boldsymbol{x},t) = NS(\boldsymbol{v}(\boldsymbol{x},t)) \quad (4)$$

where NS denotes the operator of Navier-Stokes equations.

The approach adopted is essentially data based and the type of data we wish to analyze are generated by a nonlinear flow system.

Proper Orthogonal Decomposition Method for Stationary Flows

It is assumed that a velocity field $\tilde{\boldsymbol{v}}(\boldsymbol{x},t)$, described by a set of spatio-temporal data $\tilde{\boldsymbol{v}}(\boldsymbol{x},t_i)_{i=1}^{i=m}$ obtained at discrete time values t_i and at fixed points in space, is expressible in the form

$$\tilde{\boldsymbol{v}}(\boldsymbol{x},t) \;=\; \bar{\boldsymbol{v}}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},t), \quad (5)$$

where $\bar{\boldsymbol{v}}(\boldsymbol{x})$ is the temporal mean of the velocity field.

The components of $\tilde{\boldsymbol{v}}(\boldsymbol{x},t_i)$ correspond to scalar values taken at given points in space. Such spatio-temporal data can be usually obtained from either experimental measurements or numerical simulations of the physical process under study.

There are a number of ways to determine quantitatively the underlying spatial structure of a spatio-temporal data set. For example, a simple approach to analyze such data is to perform a Fourier decomposition [18]. This is successful if only a few dominant peaks appear in the power spectrum of the spatial modes, suggesting relatively simple spatial structures. However, this would not be the best approach if there exists a coherent spatial structure composed of many Fourier modes.

A proper orthogonal decomposition method computes these coherent spatial structures directly. The structures computed are optimal for a given data set[1,2,19]. Once the major spatial structures are known, their temporal behaviour can be analyzed using dynamical system theory[1,2,19].

The proper orthogonal decomposition method is a well known analysis technique [1,2,19] with the original concept traced to Pearson[20]. Several different names including principal component analysis, Karhunen-Loève decomposition and total-least-squares estimation have been given to the procedure.

To review briefly the approach, let us consider a data set $\tilde{\boldsymbol{v}}(\boldsymbol{x},t)$ defined over a finite spatial domain Ω and a finite interval $0 \le t \le T$. To investigate the structures of fluctuations in the data, the temporal mean is removed from the velocity field. Thus, the time average of $\boldsymbol{v}(\boldsymbol{x},t)$, written as $\langle \boldsymbol{v}(\boldsymbol{x},t)\rangle$, is then zero as can be seen from equation (5).

A function $\boldsymbol{\phi}_k(\boldsymbol{x})$ can be chosen such that the projection of the data set onto all possible functions of $\boldsymbol{\phi}_k(\boldsymbol{x})$ is maximal with respect to a normalized form of $\boldsymbol{\phi}_k(\boldsymbol{x})$ (*i.e.* $(\boldsymbol{\phi}_k,\boldsymbol{\phi}_k) \equiv \int \boldsymbol{\phi}_k(\boldsymbol{x})\cdot\boldsymbol{\phi}_k(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1$). In some average sense, we are therefore trying to maximize

$$\frac{1}{m}\sum_{i=1}^{m}\frac{|(\boldsymbol{v}(\boldsymbol{x},t_i),\boldsymbol{\phi})|^2}{(\boldsymbol{\phi},\boldsymbol{\phi})} = \frac{(\boldsymbol{K}\boldsymbol{\phi},\boldsymbol{\phi})}{(\boldsymbol{\phi},\boldsymbol{\phi})} = \lambda \quad (6)$$

where m stands for the number of solutions at different time steps.

In this way, an unique orthonormal set of functions $\boldsymbol{\phi}_k(\boldsymbol{x})$ can be found which are the eigenfunctions of the Fredholm type integral equation,

$$\int_{\Omega}\boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')\cdot\boldsymbol{\phi}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' \;=\; \lambda\boldsymbol{\phi}(\boldsymbol{x}) \quad (7)$$

where the kernel $\boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')$ is the time averaged correlation function

$$\boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}') \;=\; \langle \boldsymbol{v}(\boldsymbol{x},t)\,\boldsymbol{v}(\boldsymbol{x}',t)\rangle. \quad (8)$$

These functions ($\boldsymbol{\phi}_k(\boldsymbol{x}),\ k=1,\cdots$) are called the *empirical eigenfunctions* or the *coherent structures*.

It can be shown[1] that any projection of the data onto a finite set of $\phi_k(\boldsymbol{x})$ is given by

$$\boldsymbol{v}_n(\boldsymbol{x},t)=\sum_{k=1}^{n}a_k(t)\phi_k(\boldsymbol{x}). \quad (9)$$

A particular eigenvalue λ_k is used to denote the variance of the data in the direction of the kth eigenfunction. The error is given by $\epsilon_n=\|\boldsymbol{v}-\boldsymbol{v}_n\|^2$ and it is a minimum over all possible sets of orthonormal functions for any given n. Any sample vector using the eigenfunctions can be reconstructed such that

$$\tilde{\boldsymbol{v}}(\boldsymbol{x},t)=\bar{\boldsymbol{v}}(\boldsymbol{x})+\sum_{k=1}^{n}a_k(t)\phi_k(\boldsymbol{x}) \quad (10)$$

where coefficients $a(t)$ are to be determined from the reduced dynamical equations.

Since the building blocks of low dimensional attractors need to be identified from spatio-temporally complex data, a high resolution in space is usually required and, to do so, the size of the spatial data $D\gg m$. In this case, the practical approach to calculate the correlation function is not to determine the $D\times D$ correlation matrix but to use the dual approach on the m snapshots[1]. This method is also known as *sample space setting*[19]. Here we consider the snapshot vectors $\boldsymbol{v}(\boldsymbol{x},t_i)$, $i=1,\ldots,m$ and determine the empirical eigenfunctions $\phi_k(\boldsymbol{x})$ as a linear combination of the snapshots given by

$$\phi_k(\boldsymbol{x})=\sum_{i=1}^{m}\alpha_i^{(k)}\boldsymbol{v}(\boldsymbol{x},t_i) \quad (11)$$

such that equation (7) holds. The corresponding eigenvalue problem is to find the eigenvalues and eigenfunctions of a symmetric $m\times m$ matrix defined by,

$$\boldsymbol{A}\,\boldsymbol{\alpha}^{(k)}=\lambda_k\boldsymbol{\alpha}^{(k)} \quad (12)$$

where

$$A_{ij}=\frac{1}{m}\int_\Omega \boldsymbol{v}(\boldsymbol{x},t_i)\cdot\boldsymbol{v}(\boldsymbol{x},t_j)\,\mathrm{d}\boldsymbol{x} \quad (13)$$

and $\boldsymbol{\alpha}^{(k)}$ is a single column array with $\alpha_i^{(k)}$ introduced in equation (11) as elements.

Since the trace of the matrix $\boldsymbol{A}$ represents the averaged energy retained in the snapshots, the energy corresponding to the velocity data is the sum of the eigenvalues of the correlation function, in the sense that

$$E=\sum_{i=1}\lambda_i. \quad (14)$$

An energy percentage can be assigned to each eigenfunction based on the eigenvalue associated with the eigenfunction, such that

$$E_k=\frac{\lambda_k}{E} \quad (15)$$

Under the assumption that the eigenvalues are arranged in descending order from the largest to the smallest, we then have an ordering of the eigenfunctions from most energetic to least energetic.

When the eigenvector $\boldsymbol{\alpha}^{(k)}$ of equation (12) is scaled such that $\|\boldsymbol{\alpha}^{(k)}\|^2=(m\lambda_k)^{-1}$, the eigenfunctions $\phi_k(\boldsymbol{x}), k=1,\cdots,m$ form a set of orthonormal functions. Namely,

$$\int_\Omega \phi_k(\boldsymbol{x})\cdot\phi_l(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\delta_{kl}. \quad (16)$$

The coefficients $a(t)$ at given discrete times t_i can be computed from a projection of the sample vector onto an eigenfunction through the expression

$$\begin{aligned}a_k(t_i)&=\int_\Omega \boldsymbol{v}(\boldsymbol{x},t_i)\cdot\phi_k(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\&=m\lambda_k\alpha_i^{(k)}.\end{aligned} \quad (17)$$

This relationship can be used to determine the initial conditions and projection data required in the time integration of $a(t)$.

Reduced Modelling Based on POD Method

To construct lower order mathematical models a Galerkin projection[21] was used. The objective of this approach is to replace the given dynamics by the dynamics of the subspace in the form

$$\dot{a}(t)=f(a(t)) \quad (18)$$

where $a(t)$ is the time-dependent amplitude of the basis functions.

For reduced modelling of the flow field, it is assumed that

$$\tilde{\boldsymbol{v}}(\boldsymbol{x},t)=\bar{\boldsymbol{v}}(\boldsymbol{x})+\sum_{k=1}^{n}a_k(t)\phi_k(\boldsymbol{x}) \quad (19)$$

with $a_k(t)$ determined from the Navier-Stokes equations given in equation (1).

If the operator $\int_\Omega \phi_{ji}\,()\,\mathrm{d}\boldsymbol{x}$, defined in a Hilbert space, is applied to the Navier-Stokes equation, the following reduced model can be obtained. That is,

$$\frac{da_i}{dt}+b_i+\sum_{j=1}^{N}c_{ij}\;a_j+\sum_{j=1}^{N}\sum_{k=1}^{N}d_{ijk}\;a_j\;a_k=0 \quad (20)$$

$$b_i = \int_\Omega \nu_e \phi_{ji,k} (\bar{v}_{j,k} + \bar{v}_{k,j}) \,\mathrm{d}\boldsymbol{x} + \int_\Omega \phi_{ji}\, \bar{v}_k\, \bar{v}_{j,k} \,\mathrm{d}\boldsymbol{x} + \int_\Sigma \phi_{ji} R_j \,\mathrm{d}\boldsymbol{x}, \tag{21}$$

$$c_{ij} = \int_\Omega \phi_{li}\, \bar{v}_{l,k}\, \phi_{kj} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \phi_{li}\, \bar{v}_k\, \phi_{lj,k} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \nu_e \phi_{li,k} (\phi_{lj,k} + \phi_{kj,l}) \,\mathrm{d}\boldsymbol{x}, \tag{22}$$

$$d_{ijk} = \int_\Omega \phi_{li}\, \phi_{mk} \phi_{lj,m} \,\mathrm{d}\boldsymbol{x}, \tag{23}$$

where a tensor index notation with summation convention is adopted in equations (21–23) to simplify the expressions. Here the first subscript of $\boldsymbol{\phi}$ refers to the component of $\boldsymbol{\phi}$ as a vector whereas the second subscript denotes the order of $\boldsymbol{\phi}$ as an eigenfunction. In the problems under examination the boundary integral term in equation (21) is zero since $\boldsymbol{\phi}_i \cdot \boldsymbol{R} = 0$ on the boundary Σ. Note also that equation (2) is satisfied automatically since the basis functions created by the POD method are divergence free because of their definition[1].

MODEL REDUCTION FOR NON-STATIONARY FLOWS

For problems associated with non-stationary flows with forcing, the system under examination is of the form

$$\begin{aligned} \dot{\boldsymbol{v}}(\boldsymbol{x},t) &= NS(\boldsymbol{v}(\boldsymbol{x},t), c(\boldsymbol{x},t)) \\ z(t) &= h(\boldsymbol{v}(\boldsymbol{x},t)) \end{aligned} \tag{24}$$

where c denotes the control input and z represents an output signal.

Proper Orthogonal Decomposition Method for Non-Stationary Flows

If the fluid flow is time-periodically forced, phase relationships may not be captured by statistically stationary two-point correlations and therefore simultaneous measurements are required[5].

The classical POD method is based on two-point correlations of time series and is therefore not the best approach to non-stationary flows. For flows subject to time-dependent excitations, phase-independent correlations may not exist. As a result of the lack of stationarity, correlations therefore depend on the initial and final points in the time series.

Because of the statistical non-stationarity of the data, the computed eigensets derived from snapshots of the velocity field depend on the initial and final times of the portions of the time series under examination. For this purpose the POD method requires extension as described by Glezer *et al*[5] in developing the EPOD method. This is an ensemble average of two-point correlations of velocity. For statistically stationary data, the EPOD is equivalent to the classical POD. However, in a suboptimal control approach as discussed in this paper, the basis of the reduced model can be reset using the velocity field generated from the full flow solver determined from previous iterations. This analysis is similar to the one adopted in an EPOD method.

For a flow simulation subject to an external forcing term $\boldsymbol{F}(\boldsymbol{x},t)$, the reduced model to determine the time-dependent amplitude of the basis functions $a_k(t)$ has a form similar to the one given in equation (20) with the right-hand-side zero term replaced by the generalized forcing term of $f_i(t)$ given by

$$f_i(t) = \int_\Omega \boldsymbol{F}(\boldsymbol{x},t) \cdot \boldsymbol{\phi}_i \,\mathrm{d}\boldsymbol{x}. \tag{25}$$

If the continuous forcing is specified as combinations of point forces acting at given locations in the fluid such that $\boldsymbol{F}(\boldsymbol{x},t) = \sum_l \tilde{\boldsymbol{F}}^{(l)}(t)\Delta(\boldsymbol{x} - \boldsymbol{\xi}^{(l)})$ then $f_i(t)$ can be expressed as

$$f_i(t) = \sum_l \tilde{\boldsymbol{F}}^{(l)}(t) \cdot \boldsymbol{\phi}_i(\boldsymbol{\xi}^{(l)}). \tag{26}$$

This forcing description may be typified by a simplified model of the effect of oscillating wires in a flow field.

NUMERICAL RESULTS

Reduced Modelling for Stationary Flows

Using a cell viscous boundary element method, numerical data were derived describing the characteristics of vortex shedding in the wake of a circular cylinder in uniform flow at Re=200. Forty snapshots per cycle were collected. Fig.3 shows the magnitudes of the eigenvalues produced from this set of data. It confirms the rapid decrease of the magnitudes of the eigenvalues of higher modes and since they indicate the energy level in each mode, the results show that the first few modes contain most of the energy associated with this steady vortex shedding flow. Thus it is possible to capture the main

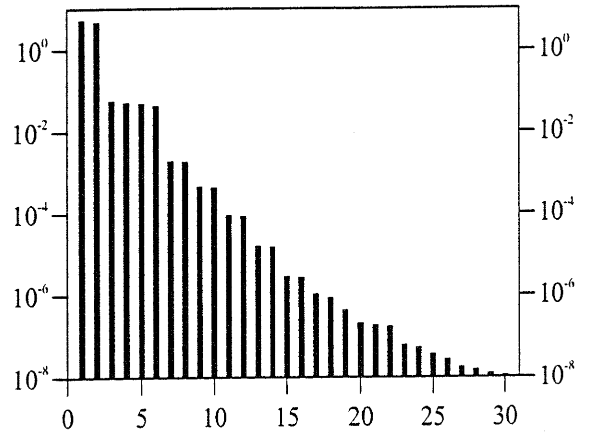


Figure 3: Magnitudes of the eigenvalues associated with different modes characterising the wake flow behind a cylinder in uniform flow at Re=200.

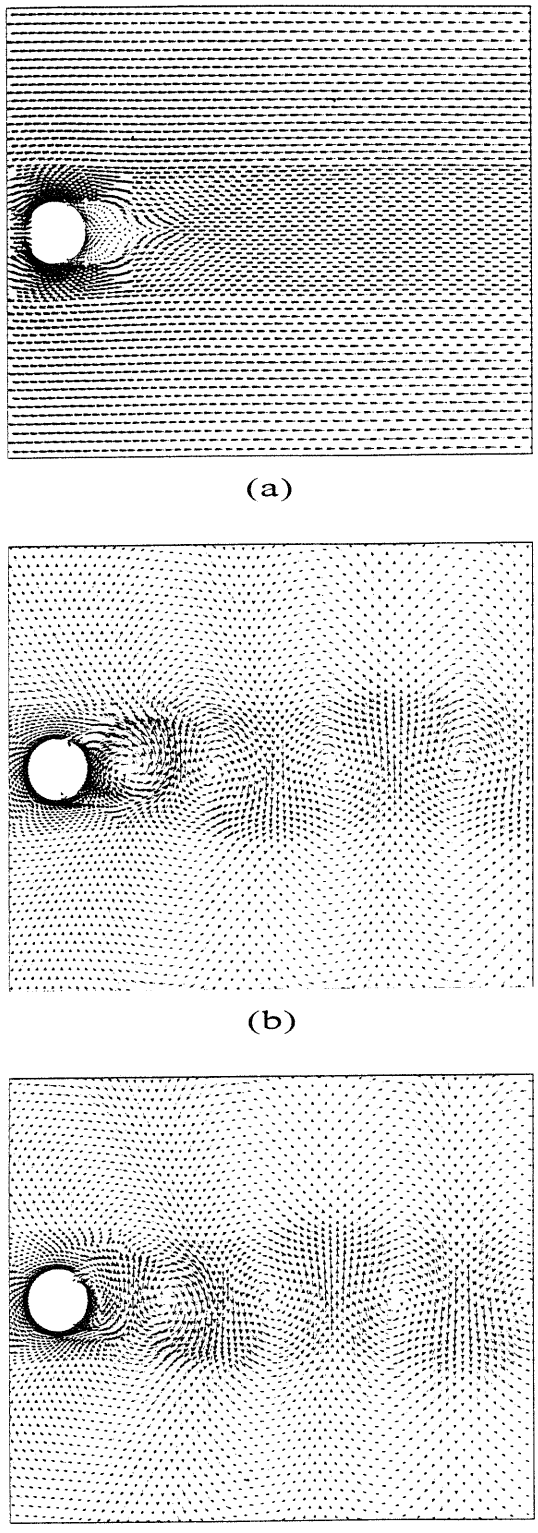
features of the flow with a much reduced model as long as the dominant modes capturing this energy are included in the model.

Fig.4(a) shows the vector plots of a flow field represented by the time averaged flow in one cycle and fig.4(b,c) show the first two most energy dominant dynamic modes. The main features of the vortex shedding process can be constructed from these modes involving a phase shift in time.

When these eigenfunctions are used as basis functions in the POD method, their amplitudes can be found by integrating equation (20). Here a 4th order Runge-Kutta method is used for the time integration and the results are plotted in fig.5 for the first 6 functions. The good agreement between the predicted and projected amplitudes of these basis functions confirms the validity and benefit of the POD method to model the type of flows under examination using low order models.

Table 1 illustrates the energy retained in the reduced model against the number of modes admitted in the analysis as a percentage of the total energy of all the dynamic modes. For the three cases examined, i.e. Reynolds number Re=100, 150 and 200, more than 99.9% of energy is retained if only the first 6 modes are adopted in the reduced models.

Although Reynolds number (i.e. $\nu_e = 1/\mathrm{Re}$) is one of the parameters in the reduced model described by equations (20–23), equation (20) cannot be used to model flows at different Reynolds number without modifying the modes involved as well. In some cases, however, the snapshots of flows at different Reynolds numbers may be combined to approximate

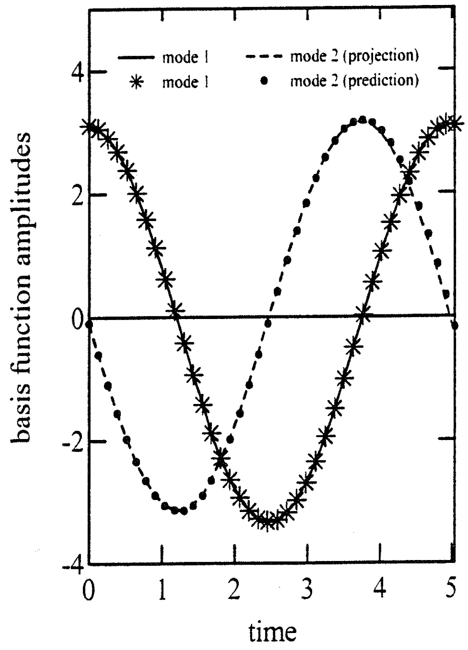


(a)

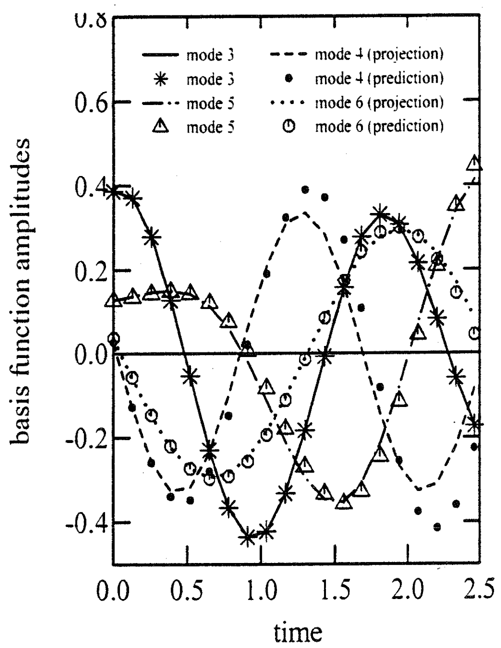
(b)

(c)

Figure 4: The vector plots of velocity of (a) mean flow, (b) mode 1, (c) mode 2.



(a)



(b)

Figure 5: Comparisons of predicted and projected amplitudes of (a): first two modes; (b): modes 3-6.

No. of	Energy retained (%)		
modes	Re=100	Re=150	Re=200
2	97.195	97.754	97.996
4	99.161	98.955	99.058
6	99.970	99.955	99.952
8	99.997	99.992	99.989

Table 1: Percentage of energy retained in the reduced models versus number of modes included.

Reynolds	Strouhal number		Relative
number	Predicted	Williamson[22]	error (%)
100	0.1650	0.1643	0.4
110	0.1680	0.1690	0.6
120	0.1720	0.1731	0.6
130	0.1760	0.1768	0.5
140	0.1805	0.1802	0.2
150	0.1845	0.1834	0.6
160	0.1875	0.1864	0.6
170	0.1905	0.1892	0.7
180	0.1935	0.1919	0.8
190	0.1955	0.1945	0.5
200	0.1980	0.1970	0.5

Table 2: Predicted shedding frequencies versus Reynolds number.

the snapshots at another Reynolds number which then can be used to produce basis functions in the POD method. As an example, 40 snapshots were collected from each flow simulation at Re=100 and Re=200. New snapshots at other Reynolds numbers (i.e. 100 <Re< 200) were generated by interpolation of these snapshots and this information was incorporated in the reduced models to predict shedding frequencies. The results of the calculated Strouhal number are presented in table 2 for a series of Reynolds numbers. Also included are data generated by the *universal* empirical relationship given by Williamson.[22] The experimentally determined Strouhal values of Williamson have an accuracy claimed *to the 1% level* and the *universal* empirical relationship was obtained through interpolation of the observed data. As can be seen from table 2, the agreement between the predicted Strouhal numbers and Williamson's data is very good.

Reduced Modelling with Control Actions

Oscillatory body forces applied to the fluid are treated as an example of control actions. In the case considered, four oscillatory point forces are applied at the four points $(0.34, \pm 0.43)$ and $(1.05, \pm 0.68)$ behind the cylinder and their frequency of oscillation is close to that observed in the vortex shedding process.

Three flow simulations were carried out using the reduced model with (a) no forcing included and (b) forcing applied in the x-direction only and (c) forcing applied in the y-direction only. The streakline patterns of these simulations are shown in fig.6 which illustrates clearly the effect of forcing on the vortex shedding process.

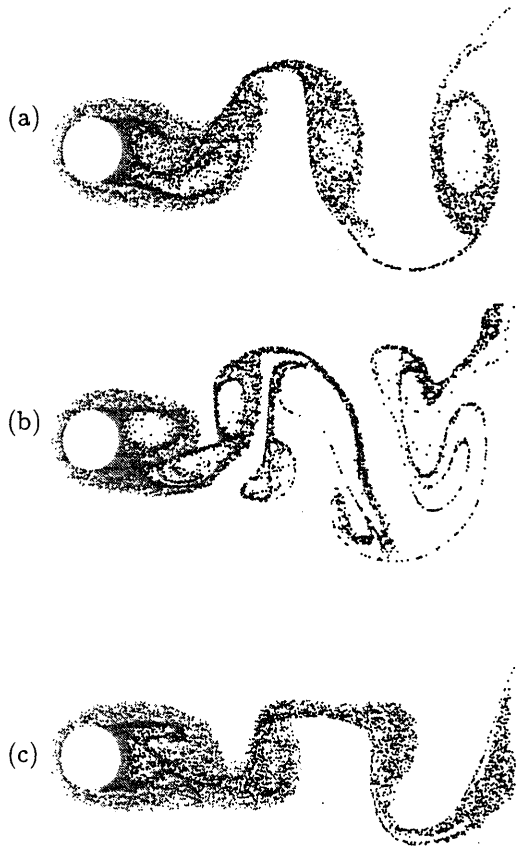


Figure 6: The streakline patterns of flows with (a): no forcing; (b): forcing in x direction; (c): forcing in y direction.

In order to investigate further the effect of the control actions in the reduced model, calculations were performed using forcing functions with different frequencies and amplitudes. The power spectrum of the velocity field generated by the reduced model was studied under different forcing conditions and data were obtained for the entrainment region defined by Gillies[11]. The data are illustrated in fig.7

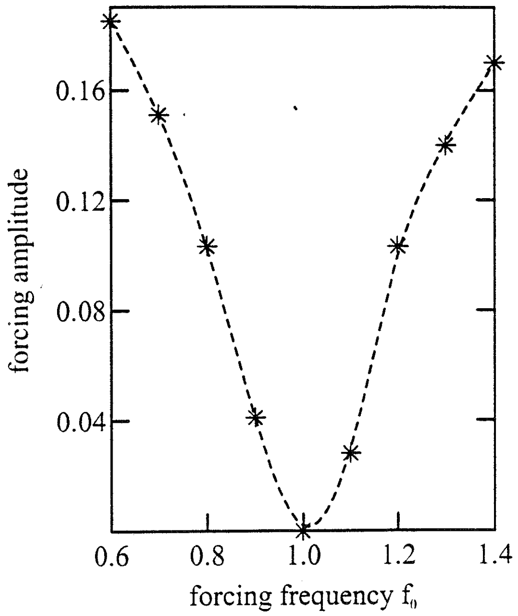


Figure 7: Forcing entrainment region[11] with stars $(*)$ indicating points of calculation and the dash line $(- - -)$ as an interpolation of the data points.

and the best fit curve shows a familiar V shape similar to Gillies' results.

CONCLUDING REMARKS

A Proper Orthogonal Decomposition method is used to investigate reduced flow modelling of the vortex shedding wake exhibited behind a circular cylinder. The method is found very effective in creating reduced models to describe vortex shedding processes. The method can also be applied for different Reynolds number cases with modified snapshots or data sets.

A reduced model has also been constructed where forcing terms are treated as control actions in the fluid domain and flow simulations with different forcing descriptions have been conducted.

The method discussed herein can also be adopted to the situation when the control action is the forced oscillation of the cylinder. In this case the oscillation of the cylinder is treated as an inertial force term in the Navier-Stokes equations, if the problem is formulated in a body fixed reference system.

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