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APPROXIMATIONS AND REANALYSIS OVER A PARAMETER INTERVAL FOR DYNAMIC DESIGN

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1. INTRODUCTION

In many design search and optimization situations, the objective of optimization or design improvement is closely related to one or more natural frequencies of a dynamic system. The problem typically involves solving an eigenvalue problem repeatedly for a large set of values of the parameters which describe the system or the structure being designed. When the system is complex, the parameter space tends to be large (i.e., the number of parameters that define the geometry, material properties, etc. is large). The situation is further complicated by the fact that complex geometries usually require a large number of degrees of freedom for a reasonably accurate analysis (e.g., an appropriate finite-element analysis). For this reason, the design search and optimization problem tends to be computationally very demanding. In most situations, it is this step involving the solution of the eigenproblem associated with the free vibration that consumes most computational resources.

The present note is motivated by this engineering need. A method based on interpolation for an approximate estimation of eigenvalues is presented. Instead of the usual approximation around a reference design, the approach here is to find approximations (at possibly several points) over an interval of the parameter of interest. This problem has been attempted via an alternative route by Bhaskar [1] where the approximations are sought for *eigenvectors* by interpolating the mode shapes themselves over the parameter interval of interest; trial vectors obtained in this manner were used in a Rayleigh-quotient approximation. The present scheme differs in that it provides approximations for eigenvalues directly from the estimates that use exact eigensolutions at the terminal points of the design interval.

2. WEIGHTED AVERAGES OF EIGENVALUE ESTIMATES OVER AN INTERVAL

The natural frequencies of a mechanical system or a structure are those frequencies in which the system is capable of executing an in- or out-of-phase free vibratory motion. When

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the non-trivial solutions are sought, the following eigenvalue problem is obtained:

$$\mathbf{Ku} = \lambda \mathbf{Mu}, \tag{1}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the symmetric mass and stiffness matrices that characterize the system. The eigenvector $\mathbf{u}_i$ represents the ith undamped mode shapes whereas the square root of the eigenvalue λ_i is the ith natural frequency. Due to this simple algebraic relationship between the natural frequencies and the corresponding eigenvalues, we shall confine our attention to the eigenvalues of a system. Suppose that a parameter that describes the system is denoted by p and we are presently concerned with solving eigenproblem (1) for a large number of values of p. The parameter p could be a length, thickness, mass, stiffness, density, modulus of elasticity, etc.

A common procedure of estimating natural frequencies of two designs that are fairly close to each other is to use a perturbation technique. The classical perturbation formula (in the spirit of the analysis of Rayleigh [2]; see also Fox *et al.* [3] and Brandon [4]) is based on neglecting higher order terms from the perturbed eigenvalue problem $\mathbf{K}'\mathbf{u}' = \lambda'\mathbf{M}'\mathbf{u}'$; $\mathbf{K}' = (\mathbf{K} + \Delta\mathbf{K})$ and $\mathbf{M}' = (\mathbf{M} + \Delta\mathbf{M})$ where the respective primed quantities refer to the perturbed system. The result for the ith mode is

$$\Delta\lambda_i \approx \frac{\mathbf{u}_i^{\mathrm{T}}(\Delta\mathbf{K} - \lambda_i \Delta\mathbf{M})\mathbf{u}_i}{\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\mathbf{u}_i} = f(\mathbf{u}_i, \lambda_i, \Delta\mathbf{K}, \Delta\mathbf{M}), \tag{2}$$

where $\Delta\mathbf{K} = \mathbf{K}' - \mathbf{K}$ and $\Delta\mathbf{M} = \mathbf{M}' - \mathbf{M}$ are the perturbations in the stiffness and mass matrices respectively. The approximate natural frequencies are calculated from $\lambda_i' = \lambda_i + \Delta\lambda_i$ by using the relationship $\omega_i' = \sqrt{\lambda_i'}$. Equation (2) uses the eigensolutions λ_i, $\mathbf{u}_i$ *of the unperturbed problem* and the information related to *the actual mass and stiffness matrices* $\mathbf{M}'$ and $\mathbf{K}'$ (through the *actual* values of $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$) respectively. Since, in most practical applications, the calculation of these two matrices is computationally less demanding (e.g., in a typical finite-element formulation it will involve assembly of elements), overall, the perturbation calculation turns out to be cheap. Also note that the variation of $\lambda_i'(p)$ with the parameter p is not, in general, linear despite having used a "linear" perturbation formula; λ_i being calculated as $\lambda_i' = \lambda_i + \Delta\lambda_i$, and $\Delta\lambda_i$ is given by equation (2). The word linear for the type of perturbation refers to the degree of terms retained in the expansion of the perturbed eigenproblem. This recipe of approximate calculation of frequencies works well when the magnitude of the perturbation is "small". In design search and optimization studies, one is most often encountered with the problem of *large* perturbations $\Delta p = p' - p$. In the present note, the problem is posed as that of approximately calculating frequencies *over an interval* instead of the usual approach of posing it as a problem of approximation *around a reference point*: the words "point", "interval", etc. are referred to the quantities in the parameter space.

Consider calculating the natural frequencies of a vibratory system for several values in the interval $p \in [p_0, p_f]$ (see Figure 1 for a hypothetical situation; the exact variation of $\lambda(p)$ is shown using the dotted line and will be assumed to be not available). The problem can be viewed as one of perturbation from either end of the interval. The estimates obtained by using a perturbation formula from either ends will, in general, be different; and this is shown using the two solid lines in Figure 1. It is assumed that the eigensolutions at the two terminal points are available. The subscript i refers to the relevant quantity for the ith mode throughout. Working "*forward*" from the left end, one has

$$\Delta\lambda_i^o(p) \approx f(\mathbf{u}_i^o, \lambda_i^o, \Delta\mathbf{K}^o(p), \Delta\mathbf{M}^o(p)), \tag{3}$$

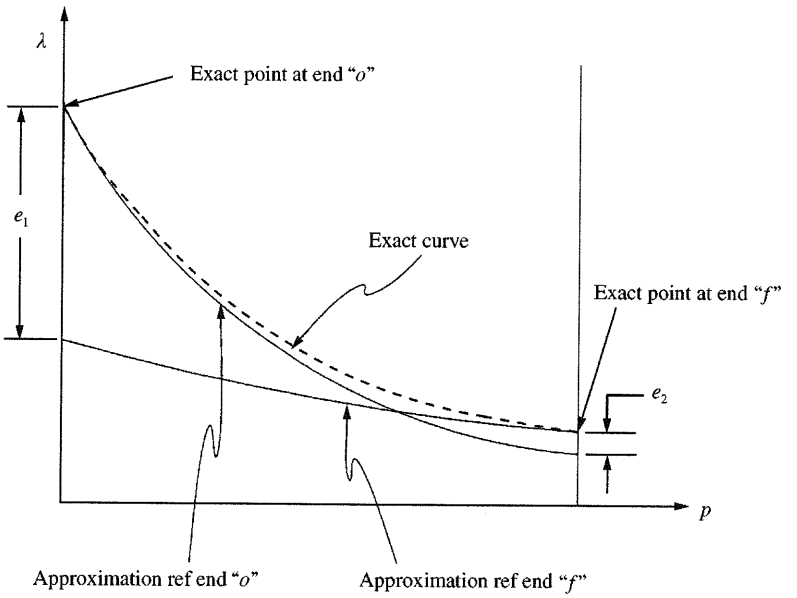

Figure 1. A sketch of the interval over which approximations are sought. The x-axis represents a design parameter, whereas the y-axis represents an eigenvalue (both exact and approximate; the approximations are based on exact eigensolutions at the two terminals of the interval).

where $\Delta\mathbf{K}^o = \mathbf{K}(p) - \mathbf{K}^o$, $\Delta\mathbf{M}^o = \mathbf{M}(p) - \mathbf{M}^o$, with superscript "$o$" referring to the left end throughout. In a similar manner, one can work "*backwards*" from the right end of the interval to obtain

$$\Delta\lambda_i^f(p) \approx f(\mathbf{u}_i^f, \lambda_i^f, \Delta\mathbf{K}^f(p), \Delta\mathbf{M}^f(p)), \tag{4}$$

where the superscript f refers to the quantities at the right end. Therefore, the two (different) estimates of the eigenvalue λ_i are defined as

$$\tilde{\lambda}_i^o(p) = \lambda_i^o + \Delta\lambda_i^o(p) \quad \text{and} \quad \tilde{\lambda}_i^f(p) = \lambda_i^f + \Delta\lambda_i^f(p) \tag{5}$$

respectively. The estimates are equal to the exact values at the two terminal points and progressively worsen as we proceed towards the other end. Motivated by this observation, the estimate of the ith eigenvalue is defined as

$$\tilde{\lambda}_i(p) = \frac{(p_f - p)}{(p_f - p_o)}\tilde{\lambda}_i^o(p) + \frac{(p - p_o)}{(p_f - p_o)}\tilde{\lambda}_i^f(p), \tag{6}$$

which is a linear combination of the two estimates based on the first order perturbation formula (of equations (3) and (4)) from the two ends treated as reference. Note that $\tilde{\lambda}_i(p)$ is *not a linear interpolation between the exact values* at the two ends. Rather, it is a weighted average of two different estimates within the interval. It is these *weights that vary linearly* over the interval.

The combination in equation (6) has been formed in such a way that the exactness of the "estimates" at the terminal points "o" and "f" is preserved:

$$\lim_{p \to p_o} \tilde{\lambda}_i(p) = \lambda_i(p_o) \quad \text{and} \quad \lim_{p \to p_f} \tilde{\lambda}_i(p) = \lambda_i(p_f), \tag{7}$$

which results from the limiting properties $\lim_{p\to p_o} \tilde{\lambda}_i^o(p) = \lambda_i(p_o)$ and also $\lim_{p\to p_f} \tilde{\lambda}_i^f(p) = \lambda_i(p_f)$ and that the weights in equation (6) become 1 and 0 as $p \to p_o$ and 0 and 1 as $p \to p_f$. Therefore, one could associate a "trust region" in the vicinity of the two ends of the interval; errors are expected to be relatively large as we move away from these two points.

The procedure of equation (6) is based on estimates that are obtained from *two* first order perturbation analyses. This procedure is on the lines of the method presented by Sahu [5]. This idea can also be used with estimates obtained directly from the Rayleigh quotient since it requires a reasonable guess for the eigenvectors. Assuming that the eigenvectors do not change as the system parameter undergoes a change, Rayleigh quotient estimates for the eigenvalue are given by

$$R_i^o(p) = \frac{\mathbf{u}_i^{oT}\mathbf{K}(p)\mathbf{u}_i^o}{\mathbf{u}_i^{oT}\mathbf{M}(p)\mathbf{u}_i^o} \quad \text{and} \quad R_i^f(p) = \frac{\mathbf{u}_i^{fT}\mathbf{K}(p)\mathbf{u}_i^f}{\mathbf{u}_i^{fT}\mathbf{M}(p)\mathbf{u}_i^f}. \tag{8}$$

The first of these quotients is based on a trial vector equal to the eigenvector corresponding to the left end of the interval; the second is based on a trial vector corresponding to the eigenvector at the right end. Using these estimates for the ith eigenvalue for any parameter value p, a weighted average can be constructed as

$$\tilde{R}_i(p) = \frac{(p_f - p)}{(p_f - p_o)} R_i^o(p) + \frac{(p - p_o)}{(p_f - p_o)} R_i^f(p). \tag{9}$$

Again, as before, the exactness of estimates at the terminal points is maintained since

$$\lim_{p\to p_o} \tilde{R}_i(p) = \lambda_i(p_o) \quad \text{and} \quad \lim_{p\to p_f} \tilde{R}_i(p) = \lambda_i(p_f). \tag{10}$$

For either of the two methods presented via equation (6) or (9), the errors at the two terminal points of the interval are *exactly* known. This is because *two* approximations are available (such as those from equations (3) and (4) in case of the first order perturbation, and similar equations (8) for the Rayleigh-quotient approximation) on the basis of the *exact* eigensolutions at the ends of the interval. Note the two error terms e_1 and e_2 extracted from the two "branches" of approximations in Figure 1. The knowledge of the relative magnitudes of e_1 and e_2 can be used to associate a level of "trust" to the two different approximations (one based on working with the eigensolutions at the left end and one based on working with the eigensolutions at the right end) for either of the two methods. Referring to Figure 1, the error at the left end $e_1 \gg e_2$ the error at the right end. Since e_1 is associated with the performance of the approximation on the basis of eigensolutions at the right end, it is reasonable to say that the approximation based on working with eigensolutions at the left end should receive greater trust.

The different levels of trust in the two branches of the approximations can be quantitatively incorporated into the process of assigning the weights as follows. The approximation of equation (6) is modified as

$$\hat{\lambda}_i(p) = \frac{[e_1(1-t)\tilde{\lambda}_i^o(p) + e_2 t\tilde{\lambda}_i^f(p)]}{[e_1(1-t) + e_2 t]}, \tag{11}$$

where $t(p) = (p - p_o)/(p_f - p_o)$. Similar modifications to the method based on the Rayleigh quotient leads to the alteration of equation (9) as

$$\hat{R}_i(p) = \frac{[e_1(1-t)R_i^o(p) + e_2 t R_i^f(p)]}{[e_1(1-t) + e_2 t]}. \tag{12}$$

One must be careful while using this approximation. Unless the values of e_1 and e_2 are very different, it is not recommended to use equation (11) or (12), since the degree of trust derived solely on the basis of errors at the ends tends to bias the weights unreasonably in favour of one of the two branches.

Again, the weights are designed in such a manner that the exact values at the terminal points are reproduced from the approximations

$$\lim_{p \to p_o} \hat{\lambda}_i(p) = \lim_{p \to p_o} \hat{R}_i(p) = \lambda_i(p_o) \quad \text{and} \quad \lim_{p \to p_f} \hat{\lambda}_i(p) = \lim_{p \to p_f} \hat{R}_i(p) = \lambda_i(p_f). \tag{13}$$

The working of the approximations presented in equations (6), (9), (11) and (12) will be illustrated in the next section.

3. AN EXAMPLE

Consider a stepped fixed–free cantilever beam as shown in Figure 2. The change in the depth of the beam takes place at the centre of the span. The "parent" design has length L and depth h in the vertical direction. The width in the direction perpendicular to the paper is taken as unity, since the natural frequencies of a beam are independent of this geometrical parameter (stiffness as well as inertia depend linearly on the width b so that the natural frequencies are independent of it). The depth of the beam in the first half (near the clamped end, Figure 2(a)) is modified in the range $h \leqslant h_1 \leqslant 2{\cdot}5h$. In Figure 2(b), the same range of

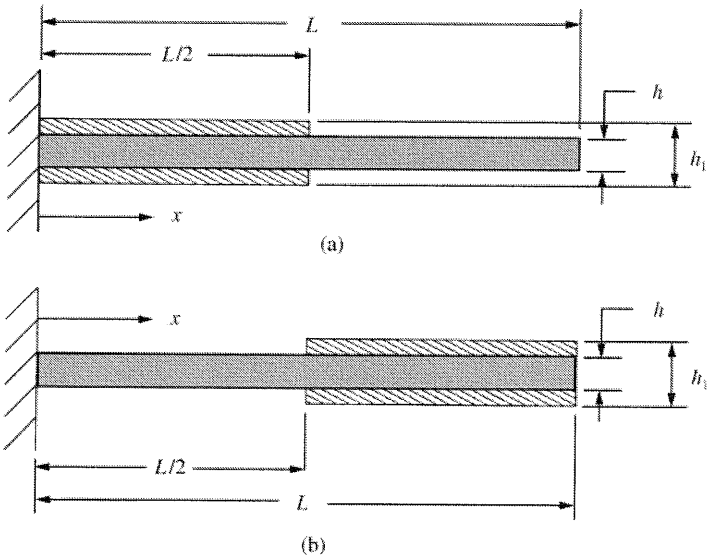

Figure 2. A cantilever beam with a step at the centre. The step is treated as a modification on a uniform beam. (a) When the modification is applied to the "clamped half" and (b) When the modification is applied to the "free half".

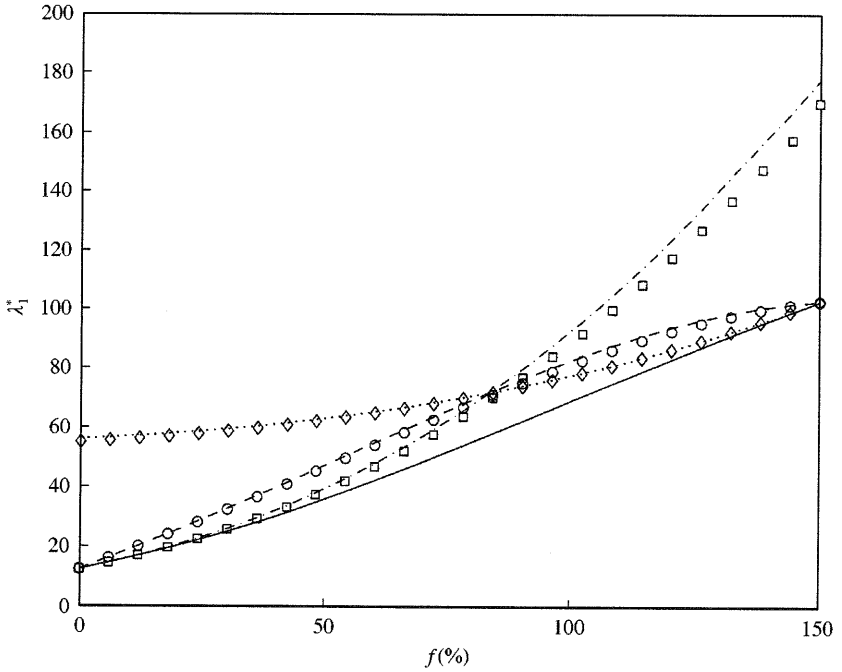

Figure 3. Approximations for the example in Figure 1(a). The first order perturbation as well as the Rayleigh-quotient approximate are presented on the basis of eigensolutions at the two terminals. The two estimates are combined as a weighted average over the interval [(— — —), equation (6) and (O O O), equation (9)]. ——, Exact; ·–·–·–·–·, first order perturbation ref "o"; · · · · , first order perturbation ref "f"; □, Rayleigh-quotient approximation ref "o"; ◇, Rayleigh-quotient approximation ref "f".

modification is applied to the dimension h_1 for the half of the beam at the free end. All other parameters that describe the structure (e.g., length, width, density, modulus of elasticity, etc.) are kept unchanged.

A finite-element model was created using 80 beam elements, each having two degrees of freedom at each node. The modified design parameter h_1 is expressed as the original dimension h plus a fractional change over this; so that $h_1 = (1 + f)h$, $0 \leqslant f \leqslant 150\%$. The interval is sub-divided into 25 smaller segments, each representing a modification of 6% over the original dimension. The eigenvalues are presented in a non-dimensional form by using

$$\lambda_i^* = \frac{\lambda_i}{(EI_o/m_o L^4)}, \tag{14}$$

where EI_o is the bending stiffness of the original beam, m_o its linear density, and L the length as shown in Figure 2.

The exact values are calculated for the first eigenvalue λ_1 of eigenproblem (1), and the results are shown in Figure 3 for the beam in Figure 2(a). The first order perturbation based on the known exact eigensolutions at the left end (corresponding to $f = 0$) are shown using line with dots and dashes. The "exact value" of $\lambda_i^* = 12{\cdot}3624$ for $f = 0$ (as calculated from the finite-element model) matches very closely with the analytical solution for a fixed–free uniform cantilever beam; this shows a good convergence for the level of discretization chosen. The same approximation based on the eigensolutions at the right end of the interval

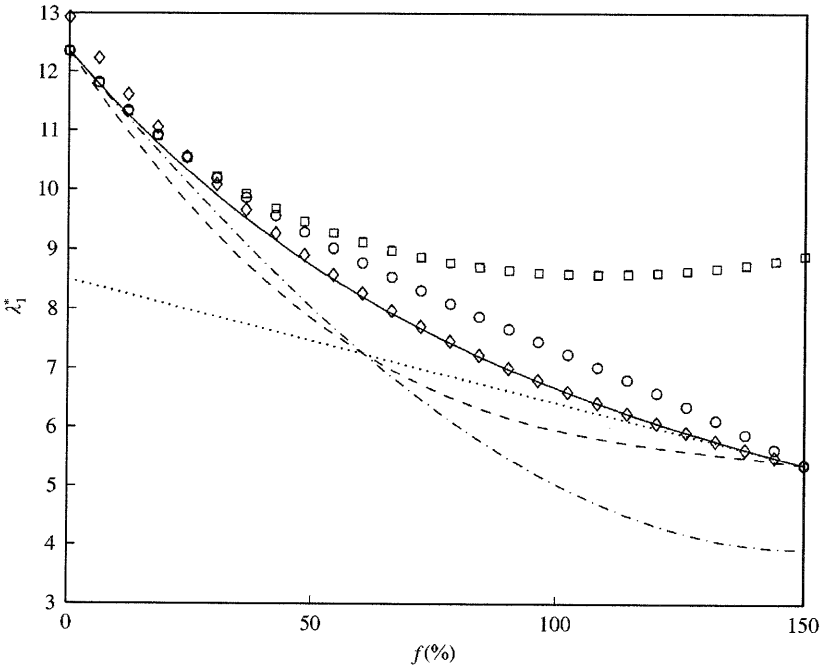

Figure 4. Approximations for the example in Figure 1(b). The first order perturbation as well as the Rayleigh-quotient approximation are presented on the basis of eigensolutions at the two terminals. The two estimates are combined as a weighted average over the interval [(– – – –), equation (6) and (○○○), (equation 9)]. ——, Exact; ·–·–·– ·–, first order perturbation ref "o"; · · · ·, first order perturbation ref "f"; ——, □, Rayleigh-quotient approximation ref "o"; ◇, Rayleigh-quotient approximation ref "f".

is shown using the dotted line. In this manner, there are two "branches" of the first order perturbation originating from either end. A second approximation based on the Rayleigh quotient with eigenvectors at either of the two ends shows a fairly similar behaviour. Both approximations work well close to the vicinity of the point whose eigensolution is taken as a reference for the approximation. However, as the perturbation becomes large, errors start to grow for each branch of the approximation. Equation (6) (for the first order perturbation) and equation (9) (for the Rayleigh-quotient approximation) combine the two branches into one approximation for each of the two methods. The results are presented using dashed line for the approximation of equation (6) and using circles for the approximation of equation (9). They illustrate that the errors are contained within reasonable limits by combining the results of the branches of approximations. Both equations show very similar performance in terms of achieving accuracy for this case.

The approximations for the example in Figure 2(b) are presented in Figure 4. Again, the two branches for each method of approximation increase progressively as one proceeds away from the "reference design" when the approximations around a reference point ("o" or "f") are used. Note that the "exact value" of λ_i^* at the left end is the same as that in Figure 3. When the information regarding approximations around the terminal points is combined into a single expression (as in equations (6) and (9)) the approximations "home in" at the two ends and keep the errors within a reasonable limit over the parameter range of interest.

The first eigenvalue increases as the thickness of the first half of the beam increases as in Figure 2(a), whereas it decreases in case of the modification as in Figure 2(b). Thickening the beam increases inertia as well as stiffness of the system. For the first mode of a cantilever

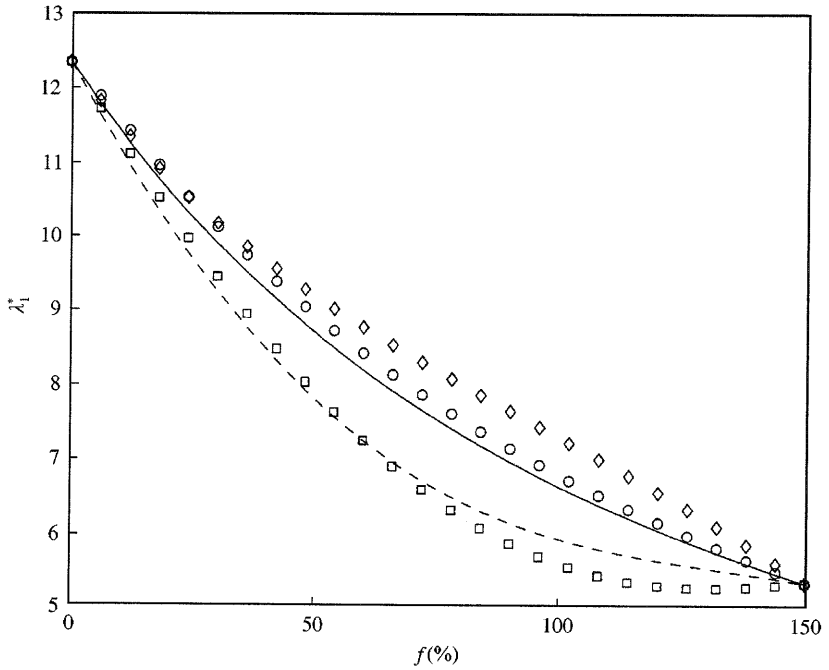

Figure 5. Approximations using weights that incorporate the knowledge of errors at the terminal points. ——, Exact; -- Equation (6); □, Euqation (11); ◇, Equation (9); ○, Equation (12)

beam, it is seen that thickening the clamped half has an overall *stiffening* effect. On the other hand, when the same modification is applied to the half that contains the free end, the overall effect is that of *softening* for the first mode. This could be explained by the fact that the overall gain in the kinetic energy due to extra mass is more than compensated by the gain in the potential energy due to stiffening of the root for the modification in Figure 2(a). On the other hand, for a modification of the type in Figure 2(b), the gain in the kinetic energy dominates the ratio of the potential energy over the kinetic energy; hence a decrease in the eigenvalue.

The mechanics of this is easily understood by considering the expressions for the kinetic energy $T(t)$ and the potential energy $V(t)$,

$$T(t) = \int_0^L (m/2)\dot{w}(x, t)^2 \, dx \quad \text{and} \quad V(t) = \int_0^L (EI/2)w''(x, t)^2 \, dx, \tag{15}$$

where EI is the flexural rigidity and m the mass per unit length of the beam. The shape of the first mode (see, for example, standard texts such as reference [6]) is such that it has small values of the transverse displacement $w(x, t)$ in the range $0 \leqslant x \leqslant L/2$, whereas it has relatively large values of the curvature $w''(x, t)$ in the range. As opposed to this, the transverse displacement $w(x, t)$ is large for the first mode (and hence, for synchronous motion at a frequency ω, $\dot{w}(x, t) = \omega w(x, t)$ is large) in the range $L/2 \leqslant x \leqslant L$ but the curvature is small in this portion of the beam. Indeed, the curvature is zero at the free end. Hence, due to the expressions in equation (15), the contributions to the kinetic energy come primarily from the free end of the beam, whereas those for potential energy come from the clamped end. Taking Rayleigh's approach, since an eigenvalue is the ratio of these two energy terms, the difference in the observed behaviour of the two modifications is readily

explained: in the case of a modification as in Figure 2(a), increase in the potential energy dominates; for a modification as in Figure 2(b), increase in the kinetic energy dominates.

Another point of interest is the fact that Rayleigh-quotient-based approximations always provide an estimate of eigenvalue greater than the actual one due to the well-known theorem of Rayleigh. This one-sided boundedness is not a property of the first order perturbation (compare various approximations with respect to the exact values in Figures 3 and 4).

Here (in Figure 4), the Rayleigh-quotient-based approximation having the right terminal point as the reference must be given a high degree of confidence (see the diamonds; this information is available in the absence of the knowledge of the exact value due to the small value of e_1 at the left end of the interval). Therefore, the situation is ideal for application of the method of equation (12). Expectedly, the results of the approximation are excellent: the circles (in Figure 5) lie very lose to the true values. On the other hand, application of equation (11) does not improve the approximation because e_1 and e_2 are roughly of the same order of magnitude.

4. CONCLUSIONS

A method of combining approximations around the two terminal points of a parameter interval for calculating approximate values of eigenvalues is presented. The approximations include the first order perturbation and the one based on the Rayleigh-quotient approximation. It was shown through a simple example that reasonable estimates of the eigenvalues can be obtained by combining "cheap calculations" over an interval judiciously. The method was further refined by the use of an additional weight reflecting the trust of approximations inferred from the errors at the terminals of the interval.

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