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Reduced Basis Representation of Large-Scale Random Eigenvalue Problems

Prasanth B. Nair* and Andy J. Keane†
University of Southampton, Highfield, Southampton SO17 1BJ, U.K

Abstract

A reduced basis formulation is presented for efficient solution of large-scale random eigenvalue problems. The present formulation aims to improve the accuracy of the first-order perturbation method, and also allow the efficient computation of higher-order statistical moments of the eigenparameters. In the proposed method, the two terms of the first-order perturbation approximation for the eigenvector are used as basis vectors for Ritz analysis of the random eigenvalue problem. Since only two basis vectors are used to represent each eigenmode of interest, explicit expressions for the random eigenvalues and eigenvectors can be readily derived. A complete statistical description of the eigenvalues and eigenvectors is hence made possible in a computationally expedient fashion. Numerical studies are presented for free and forced vibration analysis of a stochastic structural system. It is demonstrated that the reduced basis method gives significantly better results as compared to the first-order perturbation method, particularly for large stochastic variations in the random system parameters.

Introduction

Linear stochastic differential eigenvalue problems (SDEPs) are frequently encountered in the entire spectrum of computational stochastic mechanics; for example, structural dynamics, stability analysis, fluid dynamics, and aeroelasticity. The application areas of solution methodology for random eigenvalue problems include stochastic structural dynamics[1], robustness analysis of structural and control systems[2], structural model updating and damage identification[3], and parameter-based statistical energy analysis[4].

It is known that spatial discretization techniques can be used in conjunction with random field discretization schemes to represent a linear SDEP in a finite-dimensional setting as an algebraic random eigenvalue problem. These representation schemes have been widely used in the stochastic finite element method (see, for example, Ghanem and Spanos[5]), wherein randomness is treated as an additional dimension of the problem. However, for many problems of practical interest, the size of the discretized equations poses a formidable obstacle to the application of Monte Carlo simulation schemes for accurately estimating the statistics of the eigenvalues and eigenvectors. Hence, in order to compute the eigenparameter statistics in a computationally efficient fashion, the development of approximate solution schemes has been pursued with particular vigor in the computational stochastic mechanics literature.

A recent review of the state of the art[1] reveals that the first-order perturbation method appears to be the most popular technique for approximating the statistics of the eigenparameters. A detailed overview of the perturbation method for random eigenvalue problems can be found in the monographs of Scheidt and Purkert[6], and Kleiber and Hien[7]. The popular use of this method can be primarily attributed to ease of implementation and computational efficiency. However, the perturbation method only gives reasonable quality results for the statistical moments when the coefficient of variation of the random parameters are small. Further, since the higher-order perturbation terms are computationally intensive to compute, it is often difficult to improve the accuracy of first-order approximations in practice.

Lee and Singh[8] proposed an approach based on direct matrix products for approximating the first two statistical moments of the eigenvalues and eigenvectors. It was demonstrated for some simple example problems that improvements over the first-order perturbation method can be achieved. However, the application of this method to large-scale random eigenvalue problems is yet to be demonstrated. Approaches which focus only on approximating the statistics of the eigenvalues have been proposed in the literature; see, for example, Grigoriu[9].

*Ph.D. Student, Computational Engineering and Design Center, School of Engineering Sciences, Member AIAA. P.B.Nair@soton.ac.uk

†Professor of Computational Engineering, Director Computational Engineering and Design Center.

More recently, a hybrid procedure based on the polynomial chaos expansion and the Monte Carlo simulation was proposed by Red-Horse and Ghanem[10]. In this method, the eigenvalues and eigenvectors are represented by a Polynomial chaos expansion scheme. The coefficients in the expansion are then evaluated as generalized Fourier coefficients via a Monte Carlo simulation (MCS) procedure. This representation allows the computation of additional statistics of the eigensolution in an efficient fashion. The main drawback of this approach is the requirement of MCS, which would be computationally expensive for large-scale problems.

Stochastic reduced basis approximation (SRBA) methods for numerical solution of systems governed by stochastic partial differential equations have been proposed by the authors; see, references[11,12]. These techniques are intended for problems where discretization of the governing equations in space together with the random dimension ultimately leads to a linear algebraic system of equations with random coefficients. The formulation presented in this paper is similar in spirit to SRBA methods. However, the choice of basis vectors and the details of the formulation are different. The choice of basis vectors used in the present formulation is motivated by a method proposed earlier in references[13,14] for structural dynamic reanalysis problems. This technique was primarily developed for structural optimization on a limited computational budget[15].

In order to reduce the computational cost of MCS for large-scale problems, the application of model reduction schemes has been investigated; see, for example, Ottarson[16]. However, the computational cost savings using this approach may not be significant for problems where the statistics of a large number of eigenmodes are to be computed. In contrast, the present formulation involves the construction of a sequence of reduced-order problems for each eigenmode of interest. This is expected to lead to better efficiency when the statistics of a large number of eigenmodes are to be computed.

The focus of the present research is to develop a computationally efficient numerical scheme for solution of large-scale random eigenvalue problems. Procedures for discretizing linear SDEPs in space and the random dimension to arrive at an algebraic random eigenvalue problem are outlined. The two terms of the first-order perturbation approximation for the eigenvector are chosen as basis vectors in conjunction with undetermined random functions for representing the random eigenvector of the discretized SDEP. The undetermined random functions in the reduced basis representation are determined using Ritz analysis of the random eigenvalue problem. This leads to a sequence of 2×2 reduced random eigenvalue problems for each eigenmode of interest. Explicit expressions for the random eigenvalues and eigenvectors are derived in terms of the random variables arising from discretization of the underlying random fields. This enables a complete statistical description of the eigenvalues and eigenvectors in a computationally efficient fashion. Numerical studies are presented for free and forced vibration analysis of a stochastic structural system. It is shown that the present method gives significantly better results as compared to the first-order perturbation method, particularly for large stochastic variations in the random structural parameters.

Problem Statement

Consider a linear stochastic differential eigenvalue problem of the form :

$$\mathcal{K}(\alpha(\mathbf{x},\theta))[u(\mathbf{x},\theta)] = \lambda(\theta)\mathcal{M}(\beta(\mathbf{x},\theta))[u(\mathbf{x},\theta)] \quad (1)$$

where $\mathcal{K}(\alpha(\mathbf{x},\theta))$ and $\mathcal{M}(\alpha(\mathbf{x},\theta))$ are stochastic differential operators defined on the domain $\mathcal{D}$; $\mathbf{x} \in \mathcal{D}$ denotes a point on the domain; $\theta \in \Omega$ belongs to the Hilbert space of random variables; $\alpha(\mathbf{x},\theta)$ and $\beta(\mathbf{x},\theta)$ are random fields describing the coefficients of the stochastic differential operators; $\lambda(\theta)$ and $u(\mathbf{x},\theta)$ are the random eigenvalues and eigenfunctions, respectively.

The random fields describing the coefficients of the differential operators can be discretized using techniques available in the literature, such as Karhunen-Loeve (KL) expansion, Polynomial chaos expansion, and optimal linear estimation; see, for example, reference[17]. Random field discretization involves representation in terms of a finite number of random variables, which are amenable to a numerical treatment. Consider the case wherein the random field $\alpha(\mathbf{x},\theta)$ is discretized using the mean-square convergent KL expansion scheme as shown below :

$$\alpha(\mathbf{x},\theta) = \langle\alpha(\mathbf{x},\mathbf{\Theta})\rangle + \sum_{i=0}^{\infty} \eta_i^{\alpha}\sqrt{\mu_i^{\alpha}}\alpha_i(\mathbf{x}) \quad (2)$$

where μ_i^{α} and $\alpha_i(\mathbf{x})$ are the characteristic functions (eigenvalues and eigenvectors, respectively) of the following deterministic integral eigenvalue problem,

$$\mu_i^{\alpha}\alpha_i(\mathbf{x}) = \int_{\Omega} \mathbf{R}_{\alpha\alpha}(\mathbf{x},\mathbf{x}_2)\alpha_i(\mathbf{x}_1)d\mathbf{x}_1. \quad (3)$$

The vector of zero-mean random variables $\{\eta_i^\alpha\}$ are orthogonal, i.e., $\langle \eta_i^\alpha \eta_j^\alpha \rangle = \mu_i^\alpha \delta_{ij}$; where δ_{ij} denotes the Kronecker delta function. Analytical solutions for the characteristic functions of equation (3) can be readily found for a class of correlation functions defined on simple domains. Further details including approximate schemes for solution of equation (3) for complex domains can be found in the literature; see, for example, references[1,18]. Similarly, the KL expansion of the random field $\beta(\mathbf{x}, \theta)$ can also be carried out.

In practice, depending on the correlation length of the random fields, a small number of terms from the KL expansion can be used to represent the underlying random field without significant loss of accuracy. Using the KL expansions of the random fields, the stochastic differential operators in equation (1) can be written as the sum of a deterministic and stochastic operator as

$$\left(\mathcal{K}^o + \mathcal{K}_\theta\right)\left[u(\mathbf{x},\theta)\right] = \lambda(\theta)\left(\mathcal{M}^o + \mathcal{M}_\theta\right)\left[u(\mathbf{x},\theta)\right] \tag{4}$$

where $\mathcal{K}^o$ and $\mathcal{M}^o$ are deterministic differential operators, and $\mathcal{K}_\theta$ and $\mathcal{M}_\theta$ are stochastic differential operators.

Consider the case wherein the random coefficients of the stochastic differential operators appear as multiplicative terms. Further, let the random variables arising from discretization of the random fields be denoted by the vector $\mathbf{\Theta} = \{\theta_i\}$, $i = 1, 2, \ldots, p$; where p denotes the total number of random variables arising from the discretization procedure. Note that these assumptions are made only for the sake of notational convenience. Hence, without any loss of generality, equation (4) can be rewritten as

$$\left(\mathcal{K}^o + \sum_{i=1}^{p} \theta_i \mathcal{K}_i\right)\left[u(\mathbf{x},\theta)\right] = \lambda(\theta) \times \left(\mathcal{M}^o + \sum_{i=1}^{p} \theta_i \mathcal{M}_i\right)\left[u(\mathbf{x},\theta)\right] \tag{5}$$

where $\mathcal{K}_i$ and $\mathcal{M}_i$ are deterministic differential operators.

A spatial discretization technique such as the finite element method (FEM) can be used to represent equation (5) as an algebraic random eigenvalue problem of the form :

$$\left[\mathbf{A} + \sum_{i=1}^{p} \theta_i \mathcal{A}^i\right]\mathbf{x}(\mathbf{\Theta}) = \lambda(\mathbf{\Theta})\left[\mathbf{B} + \sum_{i=1}^{p} \theta_i \mathcal{B}^i\right]\mathbf{x}(\mathbf{\Theta}) \tag{6}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathcal{A}^i$, $\mathcal{B}^i \in \Re^{n\times n}$ are deterministic matrices while $\lambda(\mathbf{\Theta})$ and $\mathbf{x}(\mathbf{\Theta}) \in \Re^{n\times 1}$ denote the random eigenvalue and eigenvector, respectively.

For problems wherein the coefficients α and β appear nonlinearly in the differential operators, a Taylor series expansion scheme can be used to arrive at a form similar to equation (6). A form similar to equation (6) can be also be readily arrived at for cases where the stochastic system properties are considered as random variables. Here, the matrices $\mathcal{A}^i$ and $\mathcal{B}^i$ denote the sensitivities of the system matrices with respect to the random system parameters.

First-Order Perturbation Method

Let λ^o and $\mathbf{x}^o$ denote the eigenvalue and eigenvector, respectively, of the following deterministic eigenvalue problem

$$\mathbf{A}\mathbf{x}^o = \lambda^o \mathbf{B}\mathbf{x}^o \tag{7}$$

Consider the case where the differential operators in equation (1) are self-adjoint, and the matrices in equation (6) are symmetric positive definite. Further, let the eigenvector of equation (7) be normalized with respect to the matrix $\mathbf{B}$, i.e., $\mathbf{x}^{oT}\mathbf{B}\mathbf{x}^o = 1$. Note that for simplicity of presentation, the eigenmode numbers are not explicitly shown in the equations that follow. First-order approximations for the random eigenvalue and eigenvector based on the deterministic eigenparameters of equation (7) can be written as

$$\lambda(\mathbf{\Theta}) = \lambda^o + \sum_{j=1}^{p} \theta_j \frac{\partial \lambda}{\partial \theta_j} \tag{8}$$

$$\mathbf{x}(\mathbf{\Theta}) = \mathbf{x}^o + \sum_{j=1}^{p} \theta_j \frac{\partial \mathbf{x}}{\partial \theta_j} \tag{9}$$

where $\frac{\partial \lambda}{\partial \theta_j}$ and $\frac{\partial \mathbf{x}}{\partial \theta_j}$ are the sensitivities of the eigenvalues and eigenvectors with respect to the random variables, respectively. The eigenvalue and eigenvector derivatives can be calculated from

$$\frac{\partial \lambda}{\partial \theta_j} = \mathbf{x}^{oT}\left(\mathcal{A}^j - \lambda^o \mathcal{B}^j\right)\mathbf{x}^o \tag{10}$$

and

$$\left(\mathbf{A} - \lambda^o \mathbf{B}\right)\frac{\partial \mathbf{x}}{\partial \theta_j} = \left(\lambda^o \mathcal{B}^j + \frac{\partial \lambda}{\partial \theta_j}\mathbf{B} - \mathcal{A}^j\right)\mathbf{x}^o. \tag{11}$$

There exists a wealth of methods in the literature for solving equation (11); see, for example,

references[19-21]. In the present study, Akgun's first-order method[21] is employed to approximately solve equation (11) and compute the eigenvector derivatives. Note that this formulation considers the eigenvalues of equation (7) to be distinct.

Stochastic Reduced Basis Approximations

The key idea of the present formulation is to use the two terms of the first-order perturbation approximation (see equation (9)) as basis vectors for representing the eigenvector of the random eigenvalue problem. The assumption made here is that the random eigenvector $\mathbf{x}(\boldsymbol{\Theta})$ can be well approximated in the subspace spanned by $\mathbf{x}^o$ and $\sum_{i=1}^{p} \theta_i \frac{\partial \mathbf{x}}{\partial \theta_i}$, i.e., an approximation for $\mathbf{x}(\boldsymbol{\Theta})$ can be written as

$$\hat{\mathbf{x}}(\boldsymbol{\Theta}) = \zeta_1(\boldsymbol{\Theta})\mathbf{x}^o + \zeta_2(\boldsymbol{\Theta}) \sum_{i=1}^{p} \theta_i \frac{\partial \mathbf{x}}{\partial \theta_j} \tag{12}$$

where $\zeta_1(\boldsymbol{\Theta})$ and $\zeta_2(\boldsymbol{\Theta})$ are undetermined random functions. To compute these undetermined functions, equation (12) is used for Ritz analysis of equation (6), which leads to a 2×2 random eigenvalue problem of each eigenmode of interest. The reduced random eigenvalue problem can be written as

$$\mathbf{A}_R(\boldsymbol{\Theta})\mathrm{Z}(\boldsymbol{\Theta}) = \lambda(\boldsymbol{\Theta})\mathbf{B}_R(\boldsymbol{\Theta})\mathrm{Z}(\boldsymbol{\Theta}) \tag{13}$$

where

$$\mathbf{A}_R(\boldsymbol{\Theta}) = \Psi^T(\boldsymbol{\Theta}) \left[\mathbf{A} + \sum_{i=1}^{p} \theta_i \mathcal{A}^i \right] \Psi(\boldsymbol{\Theta}) \in \Re^{2\times 2}, \tag{14}$$

$$\mathbf{B}_R(\boldsymbol{\Theta}) = \Psi^T(\boldsymbol{\Theta}) \left[\mathbf{B} + \sum_{i=1}^{p} \theta_i \mathcal{B}^i \right] \Psi(\boldsymbol{\Theta}) \in \Re^{2\times 2}, \tag{15}$$

$$\Psi(\boldsymbol{\Theta}) = \left[\mathbf{x}^o, \sum_{i=1}^{p} \theta_i \frac{\partial \mathbf{x}}{\partial \theta_i} \right] \in \Re^{n\times 2}, \tag{16}$$

and $\mathrm{Z} = \{\zeta_1(\boldsymbol{\Theta}), \zeta_2(\boldsymbol{\Theta})\}^T \in \Re^{2\times 1}$.

After some algebra, the elements of the reduced random matrices $\mathbf{A}_R$ and $\mathbf{B}_R$ can be written using tensor notation as

$$\mathbf{A}_R(\boldsymbol{\Theta}) = \begin{bmatrix} \lambda^o + \theta_i \mathbf{a}_i & \theta_i \mathbf{c}_i + \theta_i\theta_j \mathbf{E}_{ij} \\ \text{sym} & \theta_i\theta_j \mathbf{G}_{ij} + \theta_i\theta_j\theta_k \mathcal{Q}_{ijk} \end{bmatrix} \tag{17}$$

$$\mathbf{B}_R(\boldsymbol{\Theta}) = \begin{bmatrix} 1 + \theta_i \mathbf{b}_i & \theta_i \mathbf{d}_i + \theta_i\theta_j \mathbf{F}_{ij} \\ \text{sym} & \theta_i\theta_j \mathbf{H}_{ij} + \theta_i\theta_j\theta_k \mathcal{R}_{ijk} \end{bmatrix} \tag{18}$$

where typical elements of the deterministic tensors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, $\mathcal{Q}$, and $\mathcal{R}$ are given in Table 1. In the notation used here, repeated indices imply summation with respect to the index over the range of 1 to p. It can readily seen from Table 1 that when the system matrices are symmetric, the second and third order tensors in Table 1 will also be symmetric.

Table 1 Elements of Problem Specific Tensors

$\mathbf{a}_i = \mathbf{x}^{oT}\mathcal{A}^i\mathbf{x}^o$	$\mathbf{b}_i = \mathbf{x}^{oT}\mathcal{B}^i\mathbf{x}^o$
$\mathbf{c}_i = \mathbf{x}^{oT}\mathbf{A}\frac{\partial \mathbf{x}}{\partial \theta_i}$	$\mathbf{d}_i = \mathbf{x}^{oT}\mathbf{B}\frac{\partial \mathbf{x}}{\partial \theta_i}$
$\mathbf{E}_{ij} = \mathbf{x}^{oT}\mathcal{A}^i\frac{\partial \mathbf{x}}{\partial \theta_j}$	$\mathbf{F}_{ij} = \mathbf{x}^{oT}\mathcal{B}^i\frac{\partial \mathbf{x}}{\partial \theta_j}$
$\mathbf{G}_{ij} = \frac{\partial \mathbf{x}^T}{\partial \theta_i}\mathbf{A}\frac{\partial \mathbf{x}}{\partial \theta_j}$	$\mathbf{H}_{ij} = \frac{\partial \mathbf{x}^T}{\partial \theta_i}\mathbf{B}\frac{\partial \mathbf{x}}{\partial \theta_j}$
$\mathcal{Q}_{ijk} = \frac{\partial \mathbf{x}^T}{\partial \theta_j}\mathcal{A}^i\frac{\partial \mathbf{x}}{\partial \theta_k}$	$\mathcal{R}_{ijk} = \frac{\partial \mathbf{x}^T}{\partial \theta_j}\mathcal{B}^i\frac{\partial \mathbf{x}}{\partial \theta_k}$

Using this formulation, the eigenvalues of the reduced random eigenvalue problem can be computed by solving for the roots of the quadratic

$$(a_{11}a_{22} - b_{12}^2)\lambda^2 + (2a_{12}b_{12} - a_{11}b_{22} - a_{22}b_{11})\lambda + a_{11}a_{22} - a_{12}^2 \tag{19}$$

where a_{ij} and b_{ij} denote the elements of the reduced random matrices $\mathbf{A}_R(\boldsymbol{\Theta})$ and $\mathbf{B}_R(\boldsymbol{\Theta})$, respectively. For the sake of notational convenience, the dependence of these elements on the random variables is not explicitly shown. Note that the quadratic in equation (19) will give two possible values for the approximate eigenvalue. Clearly, for the fundamental eigenmode, the root with the minimum value gives the best approximation. For the higher modes the best approximation is chosen by selecting the root which is closest to the higher-order eigenvalue approximation proposed in reference[22], which can be written using tensor notation as

$$\hat{\lambda} = \lambda^o + \frac{\theta_i \mathbf{a}_i + \theta_i\theta_j \mathbf{E}_{ij} - \lambda^o(\theta_i \mathbf{b}_i + \theta_i\theta_j \mathbf{F}_{ij})}{1 + \theta_i \mathbf{d}_i + \theta_i \mathbf{b}_i + \theta_i\theta_j \mathbf{F}_{ij}} \tag{20}$$

The selection of the appropriate root based on this criteria can be carried out by transforming equation (19) using the substitution $\lambda = \alpha + \hat{\lambda}$, which gives the modified quadratic equation

$$(a_{11}a_{22} - b_{12}^2)\left(\alpha + \hat{\lambda}\right)^2 + (2a_{12}b_{12} - a_{11}b_{22} - a_{22}b_{11})$$

$$\times \left(\alpha + \hat{\lambda}\right) = a_{11}a_{22} - a_{12}^2 \tag{21}$$

The best approximation is hence that root of equation (21) which has smallest absolute value, i.e., $\min \alpha$. The random eigenvalue can hence be evaluated as $\lambda(\boldsymbol{\Theta}) = \min \alpha + \hat{\lambda}$. Using this approximation for the eigenvalue, the random eigenvector is approximated such that it satisfies the normalization condition with respect to $\left[\mathbf{B} + \sum_{i=1}^{p} \theta_i \mathcal{B}^i\right]$ with probability one. After some further algebra, an approximation for the normalized random eigenvector can be then written as

$$\hat{\mathbf{x}}(\boldsymbol{\Theta}) = \frac{1}{\sqrt{\beta}} \left[\mathbf{x}^o + \left(\frac{a_{11} - \lambda(\boldsymbol{\Theta}) b_{11}}{a_{12} - \lambda(\boldsymbol{\Theta}) b_{12}} \right) \sum_{i=1}^{p} \theta_i \frac{\partial \mathbf{x}}{\partial \theta_i} \right] \tag{22}$$

where

$$\beta = b_{11} + \left(\frac{a_{11} - \lambda(\boldsymbol{\Theta}) b_{11}}{a_{12} - \lambda(\boldsymbol{\Theta}) b_{12}} \right) b_{12} + \left(\frac{a_{11} - \lambda(\boldsymbol{\Theta}) b_{11}}{a_{12} - \lambda(\boldsymbol{\Theta}) b_{12}} \right)^2 b_{22} \tag{23}$$

Conceptually, the statistics of the eigenvalues and eigenvectors can be computed using equations (20-23). However, since the resulting expressions for the eigenvalues and eigenvectors are highly nonlinear functions of the random variables, analytical solutions for the statistical moments are not readily possible. Fortunately, the solution of equations (20-23) requires only a few operations (of the order p^3). Hence, a complete probabilistic description of the eigenvalues and eigenvectors is within reach using simulation techniques. The formulation presented in the section is henceforth referred to as RBA.

Simplification of the Formulation

This section introduces some approximations to simplify the reduced basis method and to improve the computational efficiency. As mentioned earlier, p^3 operations are required to compute the eigenvalue and eigenvector of each mode for each realization of the random variables. Reduction of the operations count would require simplifying the third order tensors which appear in the expression for $\mathbf{A}_R$ and $\mathbf{B}_R$. Consider for example a typical term involving third-order terms such as

$$a_{22} = \theta_i \theta_j \mathbf{G}_{ij} + \theta_i \theta_j \theta_k \mathcal{Q}_{ijk}. \tag{24}$$

This term can be simplified by replacing the third-order term with its ensemble average, i.e.,

$$\bar{a}_{22} = \theta_i \theta_j \mathbf{G}_{ij} + \langle \theta_i \theta_j \theta_k \rangle \mathcal{Q}_{ijk} \tag{25}$$

Similarly, the third-order term appearing in b_{22} can also be replaced by its ensemble average. The expectation operation in equation (25) can be readily computed using the probability density function (pdf) of the vector $\boldsymbol{\Theta}$. This simplification allows for the solution of the reduced random eigenvalue problem using only of the order of p^2 operations for a given realization of the random system parameters.

An approach based on crossing theory developed by Grigoriu[9] could also be applied for analytically computing the statistics of the eigenvalues of equation (13).

A Note on Computational Aspects

The steps involved in the present formulation are summarized below :

Step 1 : The first step involves representing the random eigenvalue in the form of equation (6). This can readily be done either by discretizing the underlying random fields of the governing SDEP, or by computing the sensitivities of the system matrices with respect to the random physical parameters.

Step 2 : The deterministic eigenvalue problem in equation (7) is solved for the eigenmodes of interest and the eigenvector derivatives are computed with respect to the random variables.

Step 3 : The problem specific deterministic tensors given in Table 1 are computed for each eigenmode of interest. The computational complexity of this step is of the order n^2, since only matrix vector multiplications are involved.

Step 4 : The constants computed in Step 3 are then used to expedite the statistical analysis of the eigenvalues and eigenvectors via Monte Carlo simulation using equation (20-23).

It is of interest to note that the statistics of the eigenvalues and eigenvectors of each eigenmode can be computed independently of each other. This enables the possibility of exploiting massively parallel computing systems for solving large-scale random eigenvalue problems.

Demonstration Examples, Results and Discussion

Numerical studies are presented for free and forced vibration analysis of the network of stochastic Euler-Bernoulli beams with random Young's modulus and mass density shown in Figure 1. The structure is modeled using 4 elements for each beam member, which leads to a finite element model with a total of 210 dof. The axial and flexural rigidity of each structural member are modeled as $EoA(1+\theta_i)$ and $EoI(1+\theta_i)$, $i = 1, 2, \ldots, 20$, and the mass

density of each member is modeled as $\rho_o(1+\theta_i)$, $i = 21, 22, \ldots, 40$. θ_i are considered as uncorrelated zero-mean Gaussian random variables with standard deviation σ_θ, while $EoA = 6.987 \times 10^6$ N, $EoI = 1.286 \times 10^3$ Nm^2, and $\rho_o = 2.74$ kg/m. This leads to a total of 40 random system parameters for this problem.

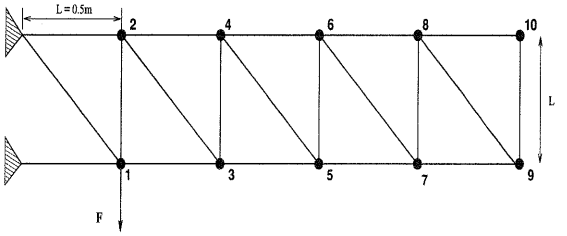

Figure 1 : Network of 20 Euler-Bernoulli Beams

Three cases are considered to compare the accuracy of the methods for increasing values of σ_θ. The value of σ_θ is kept at 0.05, 0.15, and 0.25 for case 1, case2, and case 3, respectively. Numerical studies were conducted to compute the statistics of the first 20 eigenmodes, and the transverse component of the displacement response at node 9 in the region of 0-500 Hz. The structure subjected to transverse harmonic excitation at node 1.

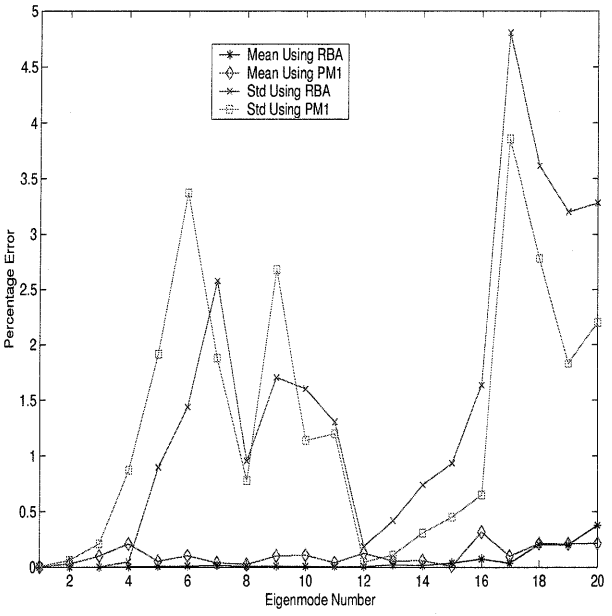

Figure 2 : Comparison of Errors in Mean and Standard Deviation of Eigenvalues for Case 1, $\sigma_\theta = 0.05$

Monte Carlo simulation using exact eigensolution with a sample size of 10000 is used to generate benchmark results against which the reduced basis formulation and first-order perturbation method are compared. The benchmark results are referred to as exact results throughout the discussion. For the reduced basis approximation (RBA) method, the integrals for the response statistics were computed using a sample size of 10000. The same sample size was also used to compute the forced response statistics using the first-order perturbation method (PM1). For PM1, the statistics of the eigenvalues can be computed analytically, since a linear approximation is involved. Note that the pdfs have been normalized with respect to the mean value predicted using exact MCS.

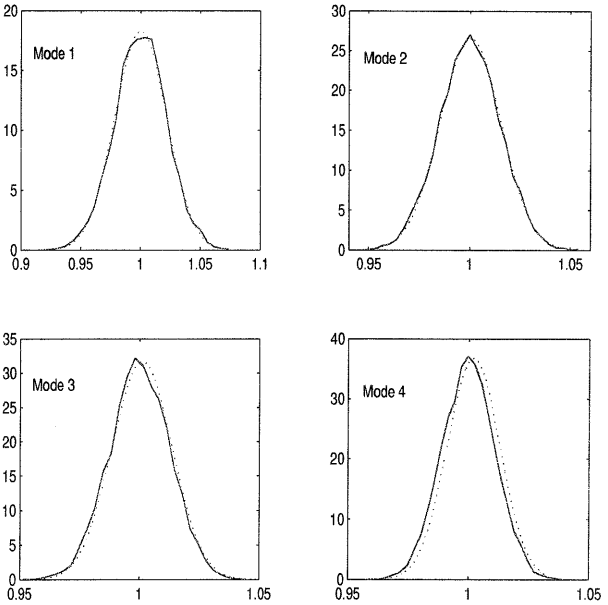

Figure 3 : Comparison of pdf of Eigenvalues for Case 1, $\sigma_\theta = 0.05$, solid line - Exact MCS, dashed lined - RBA, dotted line - PM1

A comparison of the errors in the mean and standard deviation of the eigenvalues for Case 1 using RBA and PM1 are shown in Figure 2. The pdfs of the first four eigenvalues are shown in Figure 3. It can be seen that when the standard deviation of η_i is 0.05, highly accurate results can be obtained for the first two statistical moments of the eigenvalues using the approximate methods. The accuracy of the RBA is seen to be better than PM1 for most of the eigenmodes of interest. The skew and kurtosis of the eigenvalues computed using exact analysis and RBA are shown in Table 1. Since PM1 uses a linear approximation of the eigenvalues, it gives a skew of 0 and kurtosis 3 for all the eigenvalues. The results in Table 1 indicate that for small values of σ_θ, the eigenvalue pdfs can be approximated by a Gaussian distribution.

Table 1: Comparison of Skew and Kurtosis for Case 1, σ_θ=0.05

Skew		Kurtosis	
Exact	RBA	Exact	RBA
0.0457	0.0457	3.0884	3.0884
0.0089	0.0090	2.9429	2.9429
0.0420	0.0418	3.0137	3.0133
0.0027	0.0037	2.9758	2.9696
0.0232	0.0091	2.9549	3.0760
0.0332	0.0116	3.0064	3.2708
0.0313	0.0099	2.9573	3.3629
0.0322	0.0271	2.9868	3.1280
0.0558	0.0469	2.9494	3.1620
0.1581	0.1946	3.0274	3.2662
0.1084	0.0927	2.9779	3.2767
0.0392	0.0397	2.9548	2.9868
0.0282	0.0323	3.0133	3.0674
0.0336	0.0341	2.9345	3.0379
0.0295	0.0524	3.0585	3.2162
0.0451	0.0172	2.9799	3.2175
0.1446	0.1039	3.1210	3.8941
0.0253	0.0244	3.0044	3.5012
0.0086	0.0334	2.9614	3.4200
0.0375	0.1058	2.9634	3.4374

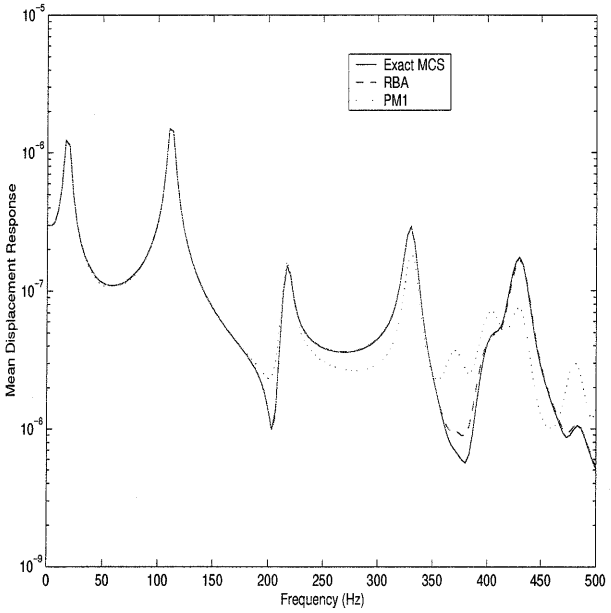

Figure 4 : Comparison of Mean Displacement Response for Case 1, σ_θ=0.05

The mean and standard deviation of the forced response computed using RBA and PM1 are compared with the exact results in Figures 4 and 5. It can be observed that the RBA shows excellent correlation with the exact results for both statistical moments of the forced response. In comparison, the errors in PM1 are rather high at some frequency points. These trends indicate that the eigenvector statistics computed using RBA is more accurate than PM1.

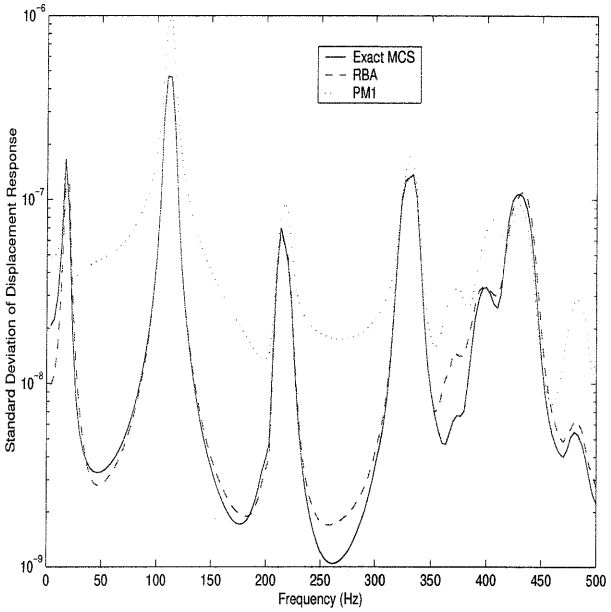

Figure 5 : Comparison of Standard Deviation of Displacement Response for Case 1, σ_θ=0.05

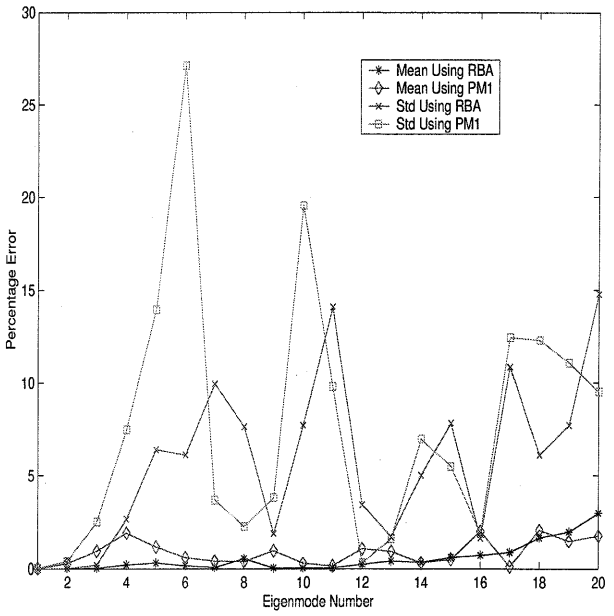

Figure 6 : Comparison of Errors in Mean and Standard Deviation of Eigenvalues for Case 2, σ_θ=0.15

A comparison of the percentage error in the mean and standard deviation of the eigenvalues computed using RBA and PM1 for Case 2 is shown in Figure 6. As observed earlier for Case 1, RBA gives better results, as compared to PM1, for the statistical

moments of most of the eigenvalues. However, for some eigenmodes, the mean and standard deviation predicted by PM1 can be seen to be marginally more accurate as compared to RBA.

Table 2: Comparison of Skew and Kurtosis for Case 2, σ_θ=0.15

Skew		Kurtosis	
Exact	RBA	Exact	RBA
0.1419	0.1413	3.1895	3.1888
0.0306	0.0291	2.9819	2.9803
0.1176	0.1070	3.1082	3.0706
0.1169	0.0209	3.2762	2.7457
0.0320	0.0100	3.0634	3.8477
0.1203	0.1388	2.9578	3.7544
0.1012	0.1784	3.0240	4.4861
0.0773	0.0343	3.0836	4.0062
0.0142	0.0224	3.0693	3.3256
0.0274	0.1027	2.9985	4.0908
0.1732	0.2454	2.8698	5.3373
0.0598	0.0988	2.9803	3.4708
0.0429	0.0267	3.0475	3.2559
0.1683	0.2043	3.0060	3.6467
0.0284	0.0099	3.0322	4.0797
0.0465	0.0407	2.9643	3.1036
0.1141	0.0987	3.0192	4.6442
0.0265	0.0116	3.0009	3.7757
0.0291	0.0428	2.9922	4.0900
0.0564	0.2530	2.9711	5.7394

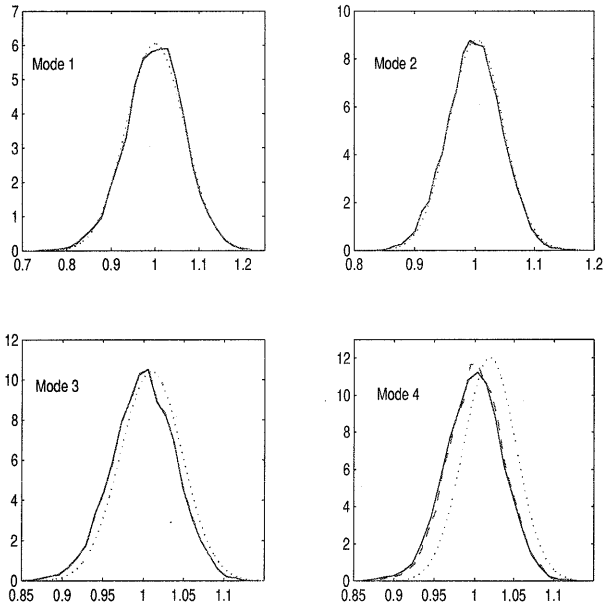

Figure 7 : Comparison of pdf of Eigenvalues for Case 2, σ_θ=0.15

The pdfs of the first four eigenvalues computed using various methods are compared in Figure 7. The skew and kurtosis computed using RBA is compared with the exact results in Table 2. Clearly, the exact results indicate that the pdfs of some of the eigenvalues are non-Gaussian, a trend which cannot be captured using PM1. However, RBA accurately captures this trend for the first few eigenvalues.

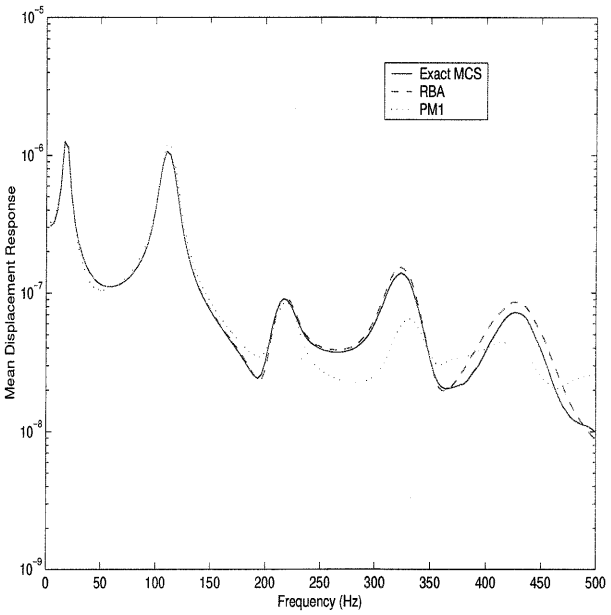

Figure 8 : Comparison of Mean Displacement Response for Case 2, σ_θ=0.15

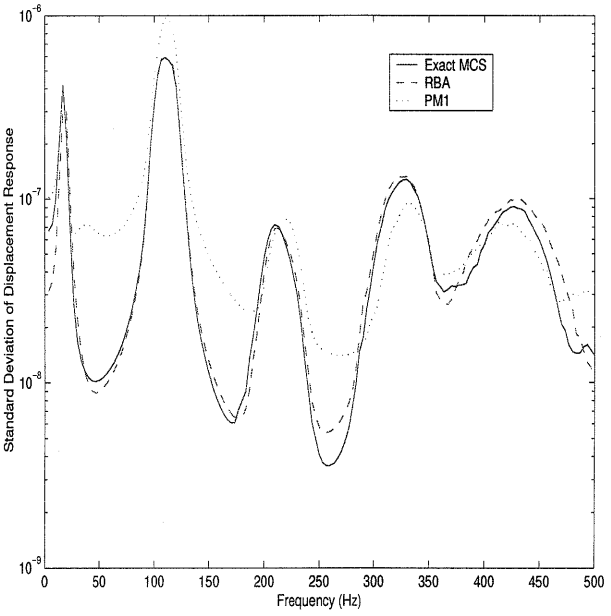

Figure 9 : Comparison of Standard Deviation of Displacement Response for Case 2, σ_θ=0.15

The mean and standard deviation of the forced response computed using RBA and PM1 are compared with the exact results in Figures 8 and 9. It

can be seen that the RBA shows better agreement with the exact results for both the mean and the standard deviation of the response.

For Case 3, the random system parameters are considered as Gaussian random variables with standard deviation of 0.25. This pathological test case is expected to give insights in to when the RBA method will break down and give unacceptable results. For this case, some realizations of the random variables may lead to negative stiffness and mass properties. During the simulation, the small fraction of the realizations of θ_i which lead to negative stiffness and mass are excluded.

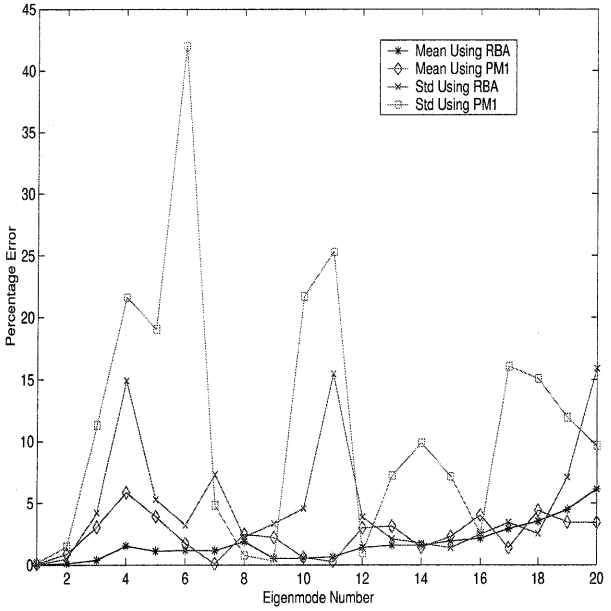

Figure 10 : Comparison of Errors in Mean and Standard Deviation of Eigenvalues for Case 3, σ_θ=0.25

The percentage errors in the mean and standard deviation of the first 20 eigenvalues for Case 3 are shown in Figure 10. It can be seen that RBA gives significantly better approximations for the standard deviation as compared to PM1 for the first few eigenmodes. As expected, PM1 gives acceptable approximations only for the mean of the eigenvalue. The pdfs of the first four eigenvalues computed using various methods are compared in Figure 11.

It was observed that due to the large value of σ_θ, the statistical overlap factor[23] increases drastically as compared to the earlier cases, i.e., the pdfs of all the eigenvalues show a great degree of overlap. The extent of overlap was found to increase drastically for the higher eigenmodes. The skew and kurtosis of the first 20 eigenvalues are shown in Table 3. It can be clearly seen that RBA accurately captures the non-Gaussian distribution of the fundamental eigenvalue. However, the errors for the other modes are seen to be rather high. The trends indicate the RBA can only be used for reliably computing the statistics of the fundamental eigenvalue when $\sigma_\theta = 0.25$.

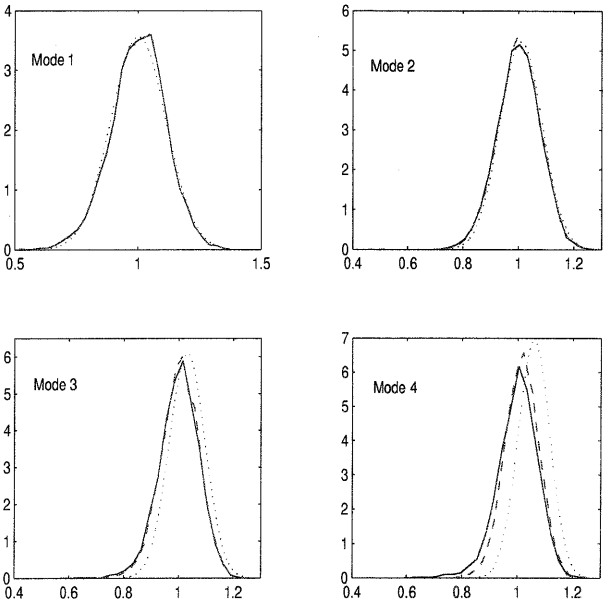

Figure 11 : Comparison of PDF of Eigenvalues for Case 3, σ_θ=0.25

Table 3 : Comparison of Skew and Kurtosis for Case 3, σ_θ=0.25

Skew		Kurtosis	
Exact	RBA	Exact	RBA
0.2531	0.2494	3.3782	3.3719
0.1170	0.0758	3.2573	3.0115
0.6070	0.2764	5.1004	3.0949
0.7058	0.0352	5.2691	1.6734
0.1404	0.0003	3.4669	3.6272
0.0610	0.0206	3.0883	3.2995
0.0562	0.1619	3.1414	4.0981
0.1737	0.1059	3.2428	3.2514
0.0199	0.0100	3.2919	2.6349
0.0871	0.1214	3.1947	3.7026
0.1341	0.2958	2.9642	5.1441
0.0033	0.0865	3.0904	3.4496
0.0167	0.0131	3.1771	2.7979
0.0828	0.0445	3.1819	3.2265
0.0327	-0.0456	3.1273	3.1286
0.0722	0.0554	3.1420	2.6586
0.0271	0.0086	3.2159	3.5129
0.0006	0.0036	3.0754	3.3322
0.0292	0.0056	3.0294	4.1757
0.0458	0.2746	3.0443	6.0725

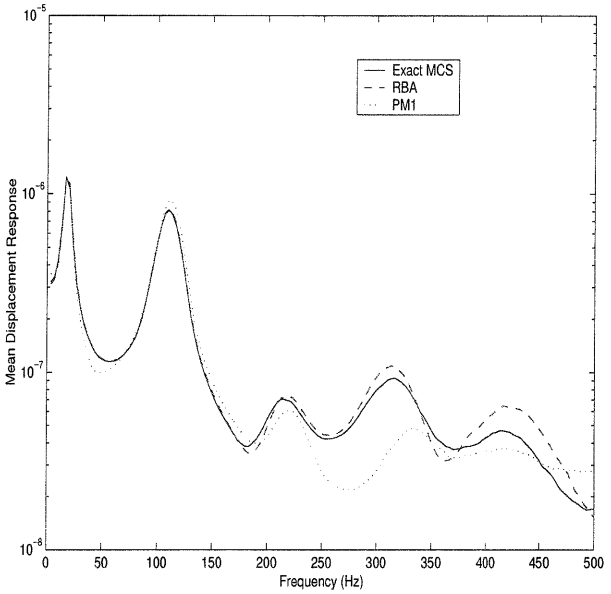

Figure 12 : Comparison of Mean Displacement Response for Case 3, σ_θ=0.25

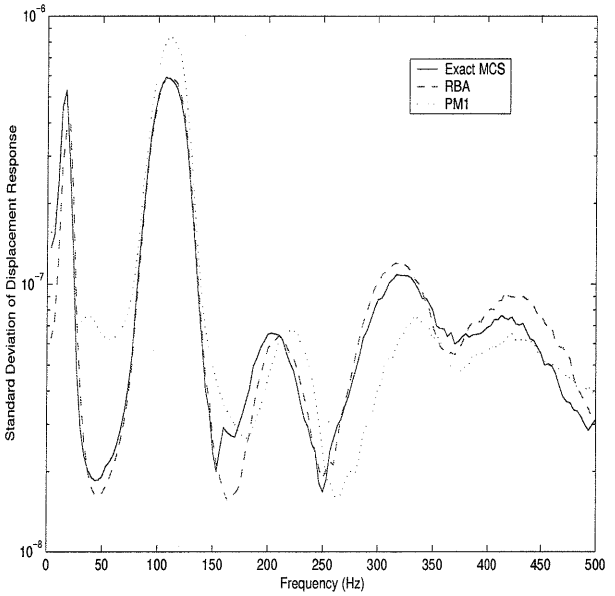

Figure 13 : Comparison of Standard Deviation of Displacement Response for Case 3, σ_θ=0.25

The mean and standard deviation of the forced response computed using RBA and PM1 are compared with exact results in Figures 12 and 13. It can be seen that RBA gives better approximations for both statistical moments of the forced response as compared to PM1. This indicates that the eigenvector statistics computed using RBA are also of reasonable accuracy. It is of interest to note that, in spite of the high magnitude of error in the eigenvalue statistics, the first two statistical moments of the forced response are of reasonable accuracy.

Computation of the first 20 modes of this structure using the Lanczos method requires around 1.7 seconds on a SGI Origin2000 with R10000 processors. Hence, the Monte Carlo simulation procedure using a sample size of 10000 involved nearly 5 hours of computer time on a single processor. In contrast, the reduced basis formulation required only 2.3 minutes, with the first-order perturbation method taking around 1.7 minutes. Note that the routines implementing the approximate methods have not been fully optimized. It is expected that the difference between the computational cost of the methods will become even more significant with increase in the problem size, i.e., both RBA and PM1 will require only a very small fraction of the computation cost required for MCS using exact eigensolution.

Concluding Remarks

An efficient numerical scheme based on reduced basis methods is presented for solution of large-scale algebraic random eigenvalue problems. The proposed method reduces the original eigenvalue problem into a sequence of 2 $\times$ 2 reduced-order random eigenvalue problems for each mode of interest. The terms of the reduced-order eigenvalue problem can be efficiently computed by solving a deterministic eigenvalue problem and computing its sensitivities. Further, the present formulation allows explicit expressions for the random eigenvalues and eigenvectors to be derived in terms of the random system properties. This enables a complete statistical description of the random eigenvalues and eigenvectors in a computationally efficient fashion.

Numerical studies have been presented for free and forced vibration analysis of a stochastic structural system to demonstrate that significant improvements over the first-order perturbation method can be achieved, particularly for large stochastic variations in the system properties. In particular, nearly exact results can be obtained for the statistics of the first few eigenvalues. In contrast to the first-order perturbation method, the reduced basis method accurately predicts the first two statistical moments of the forced response. It is also demonstrated that this improvement is achieved with only a small increment in the computational effort as compared to the first-order perturbation method.

It is of interest to note that the proposed method can be readily extended to linear algebraic random eigenvalue problems with general non-Hermitian matrices, and quadratic random eigenvalue problems. Extension of the proposed reduced basis method to interval eigenvalue problems[24] is also a topic for further research.

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