Technical Note

On the continuum properties of repetitive beam-like pin-jointed structures

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Abstract: The equivalent continuum beam properties of a one-dimensional repetitive structure have previously been determined through eigenanalysis of the transfer matrix of a single cell. A simpler procedure requires a knowledge of the stiffness matrix of the single cell, together with the ability to deduce the displacement vectors for tension, bending and shear. A *once and for all* application of the principle of minimum potential energy for tension yields the equivalent continuum Poisson's ratio, from which the remaining properties follow.

Keywords: repetitive, periodic structure, equivalent continuum properties

NOTATION

A	equivalent cross-sectional area
d	displacement component
$\mathbf{d}$	displacement vector
E	Young's modulus
F	nodal force component
$\mathbf{F}$	force vector
G	shear modulus
$\mathbf{G}$	transfer matrix
I	equivalent second moment of area
$\mathbf{K}$	stiffness matrix
L	member length
M	bending moment
Q	shearing force
$\mathbf{s}$	state vector of nodal displacement and force components
u, v	displacements in the x and y directions
U	strain energy of the cell
x, y	planar Cartesian coordinate system
γ	shear angle
ε	strain
κ	shear coefficient
ν	Poisson's ratio
ψ	cross-sectional rotation

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1 INTRODUCTION

A structure is said to be repetitive, or periodic, when its construction takes the form of a spatially repeating cell; a honeycomb sandwich panel and rail track supported on equispaced sleepers are examples of two- and one-dimensional repetitive structures respectively. Their manufacture and construction are also repetitive and this leads to cost effective design solutions in a variety of mechanical, civil and aerospace engineering applications. Periodic structures are analysed most efficiently when the periodicity is taken into account; this allows the behaviour of the complete structure to be determined through analysis of a single cell. In turn, the equivalent continuum properties allow the engineer to model the global behaviour, such as vibration and buckling, in a very efficient manner. Eigenanalysis of a state vector transfer matrix $\mathbf{G}$ has previously been employed [1] to determine the Saint Venant decay rates and the continuum beam properties of repetitive one-dimensional (beam-like) structures. The state vectors $\mathbf{s}_L$ and $\mathbf{s}_R$ are comprised of the nodal displacement and force components on the left- and right-hand sides respectively of the single cell of the repetitive structure, while the transfer matrix $\mathbf{G}$ is obtained through manipulation of the stiffness matrix, $\mathbf{K}$, of the single repeating cell. Non-unity eigenvalues of $\mathbf{G}$ pertain to the decay of self-equilibrated end loading, and occur as reciprocals according to whether the decay is from left to right, or vice versa. Multiple unity eigenvalues pertain to the transmission of tension, bending moment and shear (and torsion for non-planar structures), as well as the rigid-body displacements and rotations. From a knowledge of the eigen- and principal vectors associated with the unity eigenvalues (the transfer matrix being both defective and derogatory), the equivalent continuum beam properties of Poisson's ratio, cross-sectional area, second moment of area and shear coefficient were calculated. Since the examples treated in reference [**1**] were pin-jointed, and finite element analysis (FEA) of such structures involves exact elements only, the computational process alone limits accuracy; predictions from the eigenanalysis were thus verified by comparison with 'exact' results from FEA. The equivalent continuum properties were employed to determine the natural frequencies of vibration in reference [**2**] and agreement with FEA predictions was found to be very good so long as the semi-wavelength is greater than the depth of the cell.

In the present note, the continuum properties are found without resorting to eigenanalysis. The cell is first defined by its stiffness matrix, **K**. The approach then relies upon the ability to deduce the displacement vectors for tension, bending moment and shear; this is straightforward for tension and bending. The shear displacement vector is not immediately obvious, but can be deduced employing elementary requirements of force and moment equilibrium of the cell. A *once and for all* application of the principle of minimum potential energy for tension yields the equivalent continuum Poisson's ratio, from which all of the remaining properties follow.

For simplicity, the planar structure treated in reference [**1**] is considered again and then, without derivation, more general expressions for the continuum properties are presented in terms of length and cross-sectional area for this particular cell configuration, allowing more general conclusions to be drawn.

2 EXAMPLE STRUCTURE

Consider the beam-like repetitive pin-jointed framework shown in Fig. 1; the typical repeating cell is shown in bold, together with nodal numbering. Horizontal and vertical members have a cross-sectional area of 1 cm^2, while the diagonal members have an area of 0.5 cm^2. However, since vertical members are regarded as being shared between adjacent cells, *for the single cell* their cross-sectional area is taken to be 0.5 cm^2. Young's modulus for each member is assumed to be 200×10^9 N/m^2 and this, together with the length and depth of the cell of 1 and 2 m respectively, is regarded as applying equally to the equivalent continuum beam.

The stiffness matrix **K** for the single cell can be found by a variety of means (see, for example, reference [**3**]) and relates the nodal force and displacement components according to

$$\boldsymbol{F} = \mathbf{K}\boldsymbol{d} \tag{1}$$

which is presented explicitly in the Appendix.

It is presumed that the cell is loaded in tension, as shown in Fig. 2, and restrained in such a way as to prevent rigid-body displacement but to allow Poisson's ratio contraction. This immediately implies that displacement components $d_{1x} = d_{2x} = d_{3x} = d_{2y} = d_{5y} = 0$. Set $d_{4x} = d_{5x} = d_{6x} = u$, when the strain in the x direction is $\varepsilon_x = u/L = u$, since $L = 1$. The strain in the y direction is $\varepsilon_y = (d_{4y} - d_{6y})/(2L) = (d_{1y} - d_{3y})/(2L)$, and by virtue of the symmetry of the cell, $d_{1y} = -d_{3y}$, $d_{4y} = -d_{6y}$, so $\varepsilon_y = d_{4y}/L = d_{1y}/L = d_{4y} = d_{1y}$. However, $\varepsilon_y = -\nu\varepsilon_x$, so $d_{1y} = d_{4y} = -\nu u$, $d_{3y} = d_{6y} = \nu u$ and the cell displacement vector for tension is

$$\boldsymbol{d} = [0 \;\; -\nu u \;\; 0 \;\; 0 \;\; 0 \;\; \nu u \;\; u \;\; -\nu u \;\; u \;\; 0 \;\; u \;\; \nu u]^{\mathrm{T}} \tag{2}$$

The strain energy of the cell U is calculated as

$$U = \frac{1}{2}\boldsymbol{d}^{\mathrm{T}}\mathbf{K}\boldsymbol{d} = \frac{Eu^2}{10^4}\left[\nu^2\left(1 + \frac{1}{2\sqrt{2}}\right) - \frac{\nu}{\sqrt{2}} + \frac{3}{2} + \frac{1}{2\sqrt{2}}\right] \tag{3}$$

The cell will deform in such a way as to minimize the above, i.e.

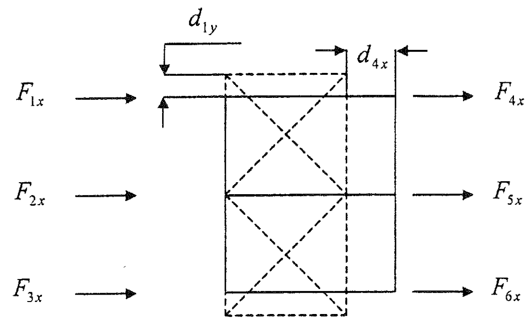

Fig. 2 Single cell loaded in tension. Dotted lines show the initial cell configuration

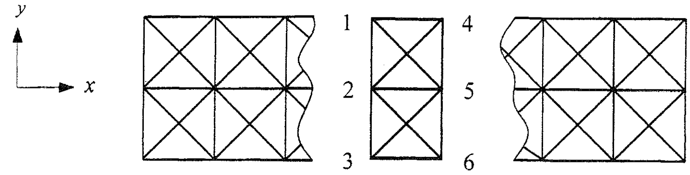

Fig. 1 Planar pin-jointed framework. The typical repeating cell is shown in bold

$$\frac{\partial U}{\partial \nu} = 0 \tag{4}$$

which gives $\nu = 1/(1 + 2\sqrt{2}) = 0.261\,204$. An equivalent shear modulus G can then be defined using $G = E/[2(1 + \nu)]$, with Young's modulus being regarded as invariant.

The tensile force T applied to the cell is

$$T = F_{4x} + F_{5x} + F_{6x} \tag{5}$$

These force components are calculated from equation (1), employing the displacement vector (2) as

$$F_{4x} = F_{6x} = \frac{Eu}{10^4}\left(1 + \frac{1-\nu}{4\sqrt{2}}\right)$$
$$F_{5x} = \frac{Eu}{10^4}\left(1 + \frac{1-\nu}{2\sqrt{2}}\right) \tag{6}$$

and

$$T = \frac{Eu}{10^4}\left(3 + \frac{1-\nu}{\sqrt{2}}\right) \tag{7}$$

For a continuum beam, $T = (EA/L)u$, and with $L = 1$, the equivalent cross-sectional area is $A = [3 + (1 - \nu)/\sqrt{2}] \times 10^{-4} = 3.5224 \times 10^{-4}$ m^2.

Next consider the displacements during bending, as shown in Fig. 3. The displacements $d_{1x} = d_{6x} = -u$, $d_{3x} = d_{4x} = u$ are assumed, which is consistent with rotations of the side faces of the cell, and also $d_{2x} = d_{5x} = 0$, which is consistent with zero axial strain on the neutral axis. For a continuum beam, a fibre coinciding with the member joining nodes 1 and 4 would have strain $\varepsilon_x = y/R$, where y is distance from the neutral axis and R is the radius of curvature. However, the strain $\varepsilon_x = 2u/L$ and $y = 1$; hence

$$\frac{M}{EI} = \frac{1}{R} = \frac{2u}{L} = 2u \tag{8}$$

Also shown in Fig. 3 is an apparent shift of the neutral axis (the member joining nodes 2 and 5) towards the tension (upper) face of the cell. In fact, the upper face moves towards the neutral axis while the lower, compression, face moves away by an equal amount, both displacements being Poisson's ratio effects. For a continuum beam, Sokolnikoff (reference [**4**], equation 32.9) gives the transverse displacement during pure bending as $v = -[M/(2EI)](x^2 + \nu y^2) + c$; here the x^2 term represents the curvature due to bending and may be ignored (or equivalently absorbed into the constant c) if the origin of coordinates is located mid-way between nodes 2 and 5 and the slope of the cell is horizontal, as depicted. The constant c represents a rigid-body displacement in the y direction and is adjusted such that $v = 0$ on $y = \pm 1$ to give $v = [M\nu/(2EI)](1 - y^2)$. Nodes 2 and 5 have $y = 0$ and employing expression (8) gives $d_{2y} = d_{5y} = \nu u$. The cell displacement vector for bending is then

$$\boldsymbol{d} = [-u \;\; 0 \;\; 0 \;\; \nu u \;\; u \;\; 0 \;\; u \;\; 0 \;\; 0 \;\; \nu u \;\; -u \;\; 0]^{\mathrm{T}} \tag{9}$$

The bending moment is

$$M = (F_{4x} - F_{6x})L \tag{10}$$

where the force components are calculated from equation (1) employing the displacement vector (9) as

$$F_{4x} = -F_{6x} = \frac{Eu}{10^4}\left(2 + \frac{1-\nu}{4\sqrt{2}}\right) \tag{11}$$

Hence

$$M = \frac{2Eu}{10^4}\left(2 + \frac{1-\nu}{4\sqrt{2}}\right) \tag{12}$$

from which the second moment of area is $I = [2 + (1 - \nu)/(4\sqrt{2})] \times 10^{-4} = 2.130\,602 \times 10^{-4}$ m^4.

Figure 4 shows the cell subjected to a shearing force, together with a bending moment; again the nodal displacements are guided by the solution for a cantilevered continuum beam subjected to a shearing force (see reference [**4**], equation 55.2). Rotations on both sides of the

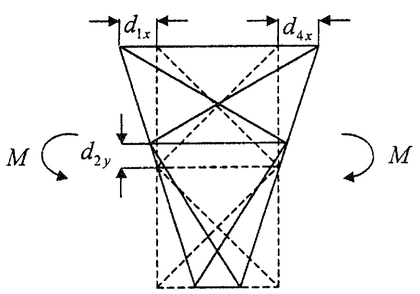

Fig. 3 Single cell loaded in pure bending

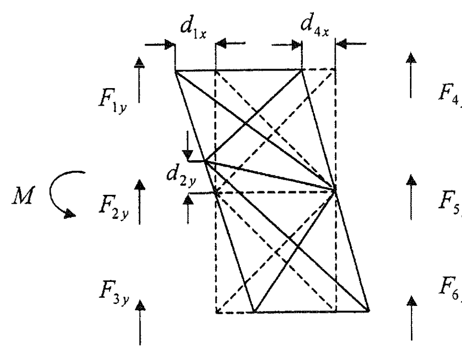

Fig. 4 Single cell subject to shear and bending moment

cell are different, so $d_{1x} = -u_1$, $d_{3x} = u_1$ and $d_{4x} = -u_2$, $d_{6x} = u_2$. As with pure bending, there is also a Poisson's ratio effect of an apparent shift of the neutral axis in the y direction (d_{2y}), but this effect is absent on the right face of the cell where the bending moment is zero, i.e. $d_{5y} = 0$. The shear displacement vector is written initially as

$$\boldsymbol{d} = [-u_1 \; 0 \; 0 \; d_{2y} \; u_1 \; 0 \; -u_2 \; 0 \; 0 \; 0 \; u_2 \; 0]^{\mathrm{T}} \tag{13}$$

The associated force components are then calculated from equation (1) as

$$\begin{aligned}
F_{1x} &= -F_{3x} = \frac{E}{10^4}\left[u_2 - \left(1 + \frac{1}{4\sqrt{2}}\right)u_1\right] \\
F_{4x} &= -F_{6x} = \frac{E}{10^4}\left[u_1 - \frac{1}{4\sqrt{2}}d_{2y} - \left(1 + \frac{1}{4\sqrt{2}}\right)u_2\right] \\
F_{2x} &= F_{5x} = 0 \\
F_{1y} &= F_{3y} = \frac{E}{10^4}\left(\frac{u_1}{4\sqrt{2}} - \frac{d_{2y}}{2}\right) \\
F_{2y} &= \frac{E}{10^4}\left[\left(1 + \frac{1}{2\sqrt{2}}\right)d_{2y} + \frac{u_1 + u_2}{4\sqrt{2}}\right] \\
F_{4y} &= F_{6y} = -\frac{E}{10^4}\left(\frac{u_2 + d_{2y}}{4\sqrt{2}}\right) \\
F_{5y} &= -\frac{E}{10^4}\left(\frac{u_1}{2\sqrt{2}}\right)
\end{aligned} \tag{14}$$

These components satisfy vertical force equilibrium for the complete cell and there is zero horizontal resultant on both sides. Moment equilibrium requires the relationship

$$u_1 = \frac{d_{2y}}{4\sqrt{2}} + \left(1 + \frac{1}{4\sqrt{2}}\right)u_2 \tag{15}$$

while asymmetry of the shear force vector requires

$$F_{1y} = -F_{4y}, \qquad F_{2y} = -F_{5y}, \qquad F_{3y} = -F_{6y} \tag{16}$$

which yield the single relationship

$$u_1 = u_2 + (1 + 2\sqrt{2})d_{2y} \tag{17}$$

Hence

$$u_1 = (16 + 6\sqrt{2})d_{2y}, \qquad u_2 = (15 + 4\sqrt{2})d_{2y} \tag{18}$$

The shearing force is then

$$Q = F_{1y} + F_{2y} + F_{3y} = \frac{E}{10^4}(5 + 8\sqrt{2})d_{2y} \tag{19}$$

In Timoshenko beam theory, the shear angle γ is defined as

$$\gamma = \psi - \frac{\mathrm{d}v}{\mathrm{d}x} \tag{20}$$

where ψ is the cross-sectional rotation and $\mathrm{d}v/\mathrm{d}x$ is the centre-line slope. The rotation is taken as the average of the rotations on either side of the cell when the shear angle is (Fig. 4) equal to

$$\gamma = \frac{u_1 + u_2}{2} + d_{2y} \tag{21}$$

bearing in mind that the cell has unit length. Finally, the above expressions are introduced into the shear equation $Q = GA\kappa\gamma$ to give the shear coefficient as

$$\kappa = \frac{4(5 + 8\sqrt{2})(1 + \nu)}{(33 + 10\sqrt{2})[3 + (1 - \nu)/\sqrt{2}]} = 0.495\,62 \tag{22}$$

The equivalent properties as derived above are in agreement with those determined in reference [**1**].

A cell, as in Fig. 1, is considered last, but having more general lengths and cross-sectional areas. In particular, the longitudinal members have length L and cross-sectional area A_L, while the vertical and diagonal members have lengths H and $D = \sqrt{L^2 + H^2}$ and areas A_H and A_D respectively; the stiffness matrix $\mathbf{K}$ is written in terms of these parameters. The equivalent properties are found using the above procedures, and are expressed first in terms of the absolute parameters of the cell and then more simply in terms of derived equivalent properties, in particular the Poisson's ratio, which is

$$\nu = \frac{A_D H L^2}{A_H D^3 + A_D H^3} \tag{23}$$

For isotropic materials Poisson's ratio can take values within the range $-1 \leqslant \nu \leqslant 0.5$. For this particular cell configuration, ν cannot be negative; it has a minimum value of zero when A_D is zero, when the cell can withstand tension and bending (for shear it is a mechanism). The equivalent cross-sectional area is

$$A = 3A_L + \frac{4A_D A_H L^3}{A_H D^3 + A_D H^3} = 3A_L + 4\nu\frac{L}{H}A_H \qquad (24)$$

which is equal to that of the three longitudinal members, together with a necessarily positive contribution from the vertical and diagonal members.

The equivalent second moment of area is

$$I = 2A_L H^2 + \frac{A_D A_H H^2 L^3}{A_H D^3 + A_D H^3} = 2A_L H^2 + \nu A_H LH \qquad (25)$$

and consists of a 'parallel-axes theorem' contribution from the top and bottom longitudinal members ($2A_L H^2$) together with a positive contribution from the vertical and diagonal members. Moreover, this additional contribution is consistent with a 'parallel-axes theorem' treatment of the additional area term, $4\nu(L/H)A_H$, in expression (24). In particular, it is reasonable that one-half of this additional area should be placed symmetrically about the neutral axis, at distance $\pm H/2$, when the parallel-axes theorem gives

$$2\left(2\nu\frac{L}{H}A_H\right)\left(\frac{H}{2}\right)^2 = \nu A_H LH \qquad (26)$$

It is interesting to note that I reduces to $2A_L H^2$, and not zero, when the diagonal members are absent, as it might be argued that these members are required to transmit shear from the upper (tensile) to the lower (compressive) members of the cell; in practice, these diagonals are clearly necessary. On the other hand, the assumed displacement vector for bending prescribes that the cell deforms in the required manner, i.e. with the upper horizontal member in tension and the lower in compression.

The shear coefficient becomes

$$\kappa = \frac{8(A_H D^3 + A_D H^3 + A_D HL^2)(2A_L A_H D^3 + 2A_L A_D H^3 + A_D A_H L^3)A_D H^2 L}{(3A_L A_H D^3 + 3A_L A_D H^3 + 4A_D A_H L^3)(2A_L D^3 + A_D L^3)(A_H D^3 + A_D H^3)}$$
$$= \frac{8(1+\nu)(2A_L H + \nu A_H L)A_D H^3 L}{(3A_L H + 4\nu A_H L)(2A_L HD^3 + \nu A_H D^3 L + \nu A_D LH^3)} \qquad (27)$$

This reduces to zero when $A_D = 0$, as the cell cannot withstand shear.

3 CONCLUDING REMARKS

The determination of the equivalent continuum properties has relied upon the ability to deduce the cell displacement vectors for tension, bending moment and shearing force, a process aided by the planar nature and symmetry of the structure. For less symmetric structures, and those involving torsion, deduction of the vectors is slightly more complicated, but still quite straightforward. Extension of the process to two-dimensional, plate-like, structures is also possible.

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APPENDIX (see over)

APPENDIX

$$
\begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \\ F_{5x} \\ F_{5y} \\ F_{6x} \\ F_{6y} \end{bmatrix} = \frac{E}{10^4}
$$

$$
\times \begin{bmatrix}
1+\frac{1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 & 0 & 0 & -1 & 0 & \frac{-1}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} & 0 & 0 \\
\frac{-1}{4\sqrt{2}} & \frac{1}{2}+\frac{1}{4\sqrt{2}} & 0 & \frac{-1}{2} & 0 & 0 & 0 & 0 & \frac{1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 \\
0 & 0 & 1+\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & -1 & 0 & \frac{-1}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} \\
0 & \frac{-1}{2} & 0 & 1+\frac{1}{2\sqrt{2}} & 0 & \frac{-1}{2} & \frac{-1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 & \frac{1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} \\
0 & 0 & 0 & 0 & 1+\frac{1}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} & 0 & 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & -1 & 0 \\
0 & 0 & 0 & \frac{-1}{2} & \frac{1}{4\sqrt{2}} & \frac{1}{2}+\frac{1}{4\sqrt{2}} & 0 & 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 \\
-1 & 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 & 1+\frac{1}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 & \frac{1}{4\sqrt{2}} & \frac{1}{2}+\frac{1}{4\sqrt{2}} & 0 & \frac{-1}{2} & 0 & 0 \\
\frac{-1}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} & -1 & 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 & 1+\frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\
\frac{1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & \frac{-1}{2} & 0 & 1+\frac{1}{2\sqrt{2}} & 0 & \frac{-1}{2} \\
0 & 0 & \frac{-1}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} & -1 & 0 & 0 & 0 & 0 & 0 & 1+\frac{1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} \\
0 & 0 & \frac{1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & \frac{-1}{4\sqrt{2}} & \frac{1}{2}+\frac{1}{4\sqrt{2}}
\end{bmatrix}
\begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \\ d_{5x} \\ d_{5y} \\ d_{6x} \\ d_{6y} \end{bmatrix}
$$