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Characteristic solutions for the statics of repetitive beam-like trusses

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Abstract

This paper concerns two major points: (1) decomposition of functional solutions for the static response of repetitive pin-jointed beam trusses under end loadings into spectrum of elementary function modes; and (2) a mathematical classification of the last. The governing finite difference equation of statics is written as a single matrix form by considering the stiffness matrix of a representative substructure. It is shown that its general solution can be spanned by only $2R$ individual modes, where R is the number of degrees of freedom for a typical nodal pattern inside the truss. These modes are divided into two primary classes: transfer and localised. A unique set of "canonical" transfer solutions is found by a method based on Jordan decomposition of the transfer matrix. Also, a technique of constructing transfer matrices for a wide class of trusses is presented. The canonical modes can be further subclassified as exponential, polynomial and quasi-polynomial. The complete set of $2R$ canonical transfer and localised modes uniquely represents the basic structural response behaviour, and gives a basis for the characteristic (non-harmonic) expansion of static solutions. Several illustrative examples are considered.

Keywords: Repetitive truss; Static solution; Transfer matrix

1. Introduction

The functional approach in the analysis of regular discontinuous structures has a history dating back to the 1957 paper [1] by Ellington and McCallion on static response of rectangular grids under out-of-plane loadings. The methodology first employed in the paper was based on the use of finite

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Nomenclature

$\mathbf{C}$	vector of participation coefficients
C_r	element of $\mathbf{C}$
$\mathbf{d}(n)$	displacement vector function, solution to governing equation
$\mathbf{f}(n)$	external loads
$\mathbf{h}$	eigenvector of the transfer matrix
$\mathbf{H}$	transfer matrix
$\mathbf{I}$	identity matrix
$\mathbf{J}$	Jordan canonical form of the transfer matrix
$\mathbf{J}_k$	Jordan block in $\mathbf{J}$
$\mathbf{j}(n)$	canonical transfer modes
$\mathbf{J}(n)$	fundamental matrix (16)
$\mathbf{K}$	$R \times R$ blocks in the associate cell stiffness matrix
n	discrete spatial parameter
N	maximum value of n
n_k	dimensionality of Jordan block $\mathbf{J}_k$
P	external load
q	dimensionality of the nullspace of matrix $\mathbf{K}_1$
R	dimensionality of vectors $\mathbf{d}(n)$ and $\mathbf{f}(n)$
$\mathbf{T}$	transformation matrix
$\mathbf{u}(n)$	localised modes
$\mathbf{x}(n)$	state vector
$\mathbf{X}(n)$	fundamental matrix
$\mathbf{Y}(n)$	matrix of $2R$ characteristic solutions
$\mathbf{0}$	zero matrix or vector
$\delta_{n,m}$	Kronecker delta
λ	eigenvalue of the transfer matrix

difference calculus, in contrast to the usual differential equations of continuous elastic media. Later it was expanded by various authors to obtain the discrete functional or field solutions for a number of systems represented as a lattice or a pattern of elements [2–4]. The basic idea consisted in taking advantage of geometrical regularity of the pattern and writing equilibrium and compatibility equations for typical joints of the lattice in the form of simultaneous finite difference equations. The position of the joint in the structure was defined by one or several (for multi-dimensional lattices) independent parameters—discrete spatial coordinates—and the corresponding solutions for the governing equations were viewed as discrete vector functions of these parameters.

In most cases, a direct use of this approach yields an involved system of governing equations (also Noor [5]), whose solutions could be normally obtained only in terms of finite Fourier or Taylor's series. Since a series solution provides only formal values of nodal displacements for given external loads and boundary conditions, its use, as an alternative to a direct finite element technique, was mainly justified by a possibility to accomplish a non-linear analysis (for example Refs. [6–8]), or

by achieving an essential saving in computational efforts. As a result, the interest in series solutions has notably faded, due to recent advances in computer performance and the rise of other alternative methods [5].

At the same time, for the case of one-dimensional (beam-like) lattices, one may evaluate solutions in terms of *elementary* functions of the discrete spatial coordinate n (Fig. 2a). Such solutions, if compared with series and finite element solutions, provide considerably more information as to fundamental response behaviour of the lattices. They enable the designer to accomplish a straightforward and reliable assessment of the structural performance before the detailed study of each particular design. Elementary function solutions arose for the first time in the following works: Dean and Tauber [9] wrote solutions for a uniformly loaded planar truss as combinations of power and hyperbolic terms; and Renton [10] obtained polynomial expressions to describe the deflection of various planar latticeworks under point, uniform and linearly varying loads. While Dean and Tauber directly solved the governing finite difference equations, Renton approximated them with differential equations, for which the solutions were readily available. Renton [10] also first pointed at the link between the polynomial solutions and the basic types of deformation of continuum beams: tensile, flexure and shear by comparing deflections of trusses and analogous beams. In a landmark paper [11], the same author showed that these deflections can be presented for various space trusses in terms of only polynomials of order not higher than three, so that all other modes decay exponentially towards the polynomial components, and obtained the equivalent continuum beam properties for the trusses under consideration.

The phenomenon of exponential decay of stresses inside statically indeterminate lattices, caused by self-equilibrated components of end loads, represents another aspect of the continuum-like behaviour of lattices—the Saint-Venant's principle. It was first observed by Hoff [12], and later Stephen and Wang [13] showed that the corresponding decay rates can be found as eigenvalues of the transfer matrix for semi-infinite lattices. Stephen and Wang also first demonstrated that the self-equilibrated load might decay fully over the first repeating cell for some lattices. However, analytical discussion of this point did not follow, since the method used by the authors for constructing the transfer matrix was not applicable for this class of structures. We will refer to such structures as *degenerate* henceforth. They may occur as both statically determinate and indeterminate frameworks to feature mathematically insufficient stiffness coupling between two adjacent typical sets of nodes, which appears as singularity of a corresponding block in the structure's stiffness matrix (block $\mathbf{K}_1$ in Eq. (1)). This physically means that degenerate structures are unable to transmit some particular load patterns, and thus display the essentially discontinuous effects of static load filtering or blockage (examples are shown in Figs. 2h and i).

In this paper, we propose a method of finding a complete set of elementary function solutions for a given pin-jointed beam lattice. The governing finite difference equation will be written in a compact matrix form, which immediately follows from the stiffness matrix relationship for a representative substructure. The general solution to this equation will be spanned by $2R$ independent modes of either *localised* or *transfer* spreaded character, to describe, respectively, the effects of boundary loads blockage and the usual states of deformation with non-trivial displacements throughout the lattice.

Further, a full set of transfer solutions will be obtained by the exponentiation of the transfer matrix $\mathbf{H}$, so that the columns of matrix $\mathbf{H}^n$ can serve as such a set. At the same time, a technique of constructing the transfer matrix will be discussed, which is applicable to both non-degenerate and degenerate lattices. Generally, it can be employed for any beam truss, which is stiff, at least when rigidly fixed at one of the ends. For example, the pin-jointed structure depicted in Fig. 2b is stiff, if clamped as shown; therefore, the corresponding transfer matrix can be constructed.

As we will see, among the infinity of linear dependent solutions there exists a *unique* set of transfer modes relating to the Jordan canonical form of transfer matrix, and therefore to be referred to as *canonical* modes. Each of these modes possesses purely exponential, polynomial or quasi-polynomial[1] form. In consistence with the background results, the exponential solutions will be shown to describe deformation of the lattice due to self-equilibrated loads, and the polynomial—due to the basic types of end loads, such as tensile, bending moment, shear and torque. The quasi-polynomial modes, not mentioned in the previous studies, may occur in the analysis of some particular structures with specific discontinuous properties, such as, for example, the oscillating linear growth of displacements in the lattice under the end load depicted in Fig. 2g and discussed in Section 6.2. Finally, we will demonstrate a simple technique for accomplishing the modal expansion of solutions for particular boundary value problems.

Analytical results of the theoretical Sections 2–5 are exact within the pin-jointed model. The approach is applicable for any type of lattices, for which the transfer matrices exist.

2. Governing finite difference equation of statics

For the purposes of this paper, it is useful to clarify the idea of typical nodal pattern and to introduce the concept of *associate cell* of a beam-like lattice. Assume that we can isolate mentally a minimum set of nodes in a repetitive beam, such that it generates the rest of the structural joints if translated along the axial direction of the beam. We call this set the typical (repeating) nodal set or pattern; and define the associate cell as consisting of all the structural bar-elements that interact with the nodes of one typical set, and also of the bars, whose both ends interact with nodes from the left and right neighbour sets. Example frameworks and their associate cells are shown in Figs. 2a–d, where parameter n indicates the spatial locations of the typical nodal sets. The associate cell is a minimum substructure, which represents both the structure's internal and boundary patterns of bar arrangements.

The stiffness matrix of the associate cell can be found by conventional means to give a 3×3 block form to give the static equilibrium for the substructure in the form

$$\begin{pmatrix} \mathbf{K}_2 & \mathbf{K}_1 & \mathbf{0} \\ \mathbf{K}_1^{\mathrm{T}} & \mathbf{K}_0 & \mathbf{K}_1 \\ \mathbf{0} & \mathbf{K}_1^{\mathrm{T}} & \mathbf{K}_3 \end{pmatrix} \begin{pmatrix} \mathbf{d}(n-1) \\ \mathbf{d}(n) \\ \mathbf{d}(n+1) \end{pmatrix} = \begin{pmatrix} \mathbf{f}(n-1) \\ \mathbf{f}(n) \\ \mathbf{f}(n+1) \end{pmatrix}. \quad (1)$$

Here, the blocks are of size $R \times R$, where R is the number of degrees of freedom for a typical nodal set, $\mathbf{d}$ and $\mathbf{f}$ are vectors of the generalised displacements and external loads, related to the degrees of freedom at the corresponding nodal locations. Note that the blocks $\mathbf{K}_1$ and $\mathbf{K}_1^{\mathrm{T}}$ describe the right- and left-hand stiffness coupling between the adjacent nodal locations. Considering the beams under boundary loads only, i.e. when $\mathbf{f}(n)=\mathbf{0}$, $n \neq 0, N$, we can write that any nth non-boundary nodal set

[1] A polynomial pre-multiplied with an exponent.

is in equilibrium, when

$$\mathbf{K}_1^{\mathrm{T}}\mathbf{d}(n-1) + \mathbf{K}_0\mathbf{d}(n) + \mathbf{K}_1\mathbf{d}(n+1) = \mathbf{0}, \quad n = 1,2,\ldots,N-1 \quad (2)$$

($\mathbf{0}$ is the R-component zero vector); and the boundary loads $\mathbf{f}(0)$ and $\mathbf{f}(N)$ are in equilibrium with structural reactions, if simultaneously

$$\begin{aligned} &\mathbf{K}_2\mathbf{d}(0) + \mathbf{K}_1\mathbf{d}(1) = \mathbf{f}(0), \\ &\mathbf{K}_1^{\mathrm{T}}\mathbf{d}(N-1) + \mathbf{K}_3\mathbf{d}(N) = \mathbf{f}(N). \end{aligned} \quad (3)$$

The matrix finite difference equation (2) gives a system of R scalar second-order equations on the R components of discrete vector function $\mathbf{d}(n)$. According to the theory of difference equations [14,15], its general solution will depend on $2R$ arbitrary constants that can be found from boundary conditions (Section 6). Thus, the finite difference scheme (2) represents the governing statics equation for an arbitrary repetitive beam-like truss under the end loads (3). The displacement solution $\mathbf{d}(n)$ is a function describing fully some equilibrium state of static deformation occurring in the beam in response to these loads.

3. Modal expansion of the general solution: transfer matrix

Assuming that matrix $\mathbf{K}_1$ of Eq. (1) is invertible, we can rearrange Eq. (2) to give the system of $2R$ scalar first-order equations:

$$\mathbf{x}(n+1) = \mathbf{H}\mathbf{x}(n), \quad \mathbf{x}(n) = \begin{pmatrix} \mathbf{d}(n) \\ \mathbf{d}(n+1) \end{pmatrix}, \quad n = 0,1,\ldots,N-1, \quad (4)$$

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_1^{-1}\mathbf{K}_1^{\mathrm{T}} & -\mathbf{K}_1^{-1}\mathbf{K}_0 \end{pmatrix}, \quad (5)$$

where vector $\mathbf{x}(n)$ is referred to as the state vector, $\mathbf{H}$ is the displacement transfer matrix, $\mathbf{0}$ and $\mathbf{I}$ are the zero and identity matrices of dimension R, respectively. Then if solution to Eq. (4) is found, the sought function $\mathbf{d}(n)$ can be taken simply as the upper part of $\mathbf{x}(n)$. The theory of systems of type (4) with non-singular matrices $\mathbf{H}$ is well developed [14,15]. The fundamental matrix for them is built as the matrix function

$$\mathbf{X}(n) = \mathbf{H}^n \equiv (\mathbf{x}_1(n)\ \mathbf{x}_2(n) \ldots \mathbf{x}_r(n) \ldots \mathbf{x}_{2R}(n)), \quad (6)$$

which satisfies the matrix form

$$\mathbf{X}(n+1) = \mathbf{H}\mathbf{X}(n). \quad (7)$$

The $2R$ columns of $\mathbf{X}(n)$ comprise a set of linear independent (for all n) vector functions $\mathbf{x}_1(n)$, $\mathbf{x}_2(n),\ldots,\mathbf{x}_{2R}(n)$ that serve as the fundamental modes of solution for Eq. (4). These functions represent a basis for the linear expansion of $\mathbf{x}(n)$, where the corresponding participation coefficients can be formally taken as the components of vector $\mathbf{x}(0)$:

$$\mathbf{x}(n) = \sum_{r=1}^{2R} C_r\mathbf{x}_r(n) = \mathbf{X}(n)\mathbf{C}, \quad \mathbf{C} = \mathbf{x}(0) = (C_1\ C_2 \ldots C_r \ldots C_{2R})^{\mathrm{T}}. \quad (8)$$

Respectively, a set of $2R$ fundamental solutions for the original Eq. (2) can be obtained by taking upper halves of the corresponding modes $\mathbf{x}_r(n)$ from Eq. (6).

In fact, the above assumption of invertibility of $\mathbf{K}_1$ holds only for a relatively small class of frameworks that we refer to as *non-degenerate*, i.e. able of transmitting any kind of end loads. Their transfer matrices are always non-singular; they can be built according to Eq. (5), or a similar procedure, and were already studied, for example, by Stephen and Wang [13].

For a degenerate lattice, $\det \mathbf{K}_1 = 0$, the transfer matrix cannot be built simply as Eq. (5), and there exist some patterns of deformation at the boundaries, which have no transitive effect upon the internal locations. Indeed, if q is the dimensionality of nullspace of matrix $\mathbf{K}_1$, then we can write $2q$ independent solutions to Eq. (2), which are *localised* at the truss ends, as the following:

$$\mathbf{u}_r(n) = \delta_{n,0}\mathbf{v}_r, \quad r = 1,2,\ldots,q,$$
$$\mathbf{u}_r(n) = \delta_{n,N}\mathbf{v}_r, \quad r = q+1,\ldots,2q. \tag{9}$$

Here, vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ comprise an orthogonal basis-in the nullspace of matrix $\mathbf{K}_1^{\mathrm{T}}$, and $\mathbf{v}_{q+1}, \mathbf{v}_{q+2}, \ldots, \mathbf{v}_{2q}$ in the nullspace of $\mathbf{K}_1$; $\delta_{n,m}$ is the Kronecker delta. In other words, the degenerate truss will be unable to transmit the corresponding patterns of end loads (3); that will be "blocked" at its boundaries, i.e. cause the non-trivial deflections $\mathbf{v}_r$ only at the locations $n = 0, N$ (examples are shown in Figs. 2h and i).

The transfer matrix of a degenerate beam lattice can be constructed according to the following procedure. Construct the rectangular $R \times 3R$ matrix $\mathbf{Q}$ as

$$\mathbf{Q} = (\mathbf{K}_1^{\mathrm{T}} \ \mathbf{K}_0 \ \mathbf{K}_1), \tag{10}$$

and by accomplishing the elementary row operations, reduce it to the form

$$\mathbf{Q} \to \mathbf{Q}' = \begin{pmatrix} \mathbf{U}_{-1} & \mathbf{U}_0 & \mathbf{0} \\ \mathbf{M}_{-1} & \mathbf{M}_0 & \mathbf{M}_1 \\ \mathbf{0} & \mathbf{L}_0 & \mathbf{L}_1 \end{pmatrix}, \tag{11}$$

where zero blocks $\mathbf{0}$, matrices $\mathbf{U}$ and $\mathbf{L}$ are of size $q \times R$, and matrices $\mathbf{M}$ are of size $(R-2q) \times R$. After substituting the correspondent blocks of $\mathbf{Q}'$ back to Eq. (2), we can write that

$$\mathbf{Q}_{-1}\mathbf{d}(n-1) + \mathbf{Q}_0\mathbf{d}(n) + \mathbf{Q}_1\mathbf{d}(n+1) = \mathbf{0}, \quad n = 1,2,\ldots,N-2, \tag{12}$$

where matrices $\mathbf{Q}_{0,\pm 1}$ read

$$\mathbf{Q}_{-1} = \begin{pmatrix} \mathbf{M}_{-1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{Q}_0 = \begin{pmatrix} \mathbf{M}_0 \\ \mathbf{L}_0 \\ \mathbf{U}_{-1} \end{pmatrix}, \quad \mathbf{Q}_1 = \begin{pmatrix} \mathbf{M}_1 \\ \mathbf{L}_1 \\ \mathbf{U}_0 \end{pmatrix}. \tag{13}$$

Then a singular transfer matrix can be built to relate the consecutive values of state vector $\mathbf{x}(n)$ in Eq. (4), for this class of structures, as

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{Q}_1^{-1}\mathbf{Q}_{-1} & \mathbf{Q}_1^{-1}\mathbf{Q}_0 \end{pmatrix}. \tag{14}$$

According to Eqs. (12) and (13), there exist $2q$ simultaneous restrictions for the components of state vector (4) at non-boundary locations of a degenerate lattice:

$$\begin{pmatrix} \mathbf{L}_0 & \mathbf{L}_{-1} \\ \mathbf{U}_1 & \mathbf{U}_0 \end{pmatrix} \mathbf{x}(n) = \mathbf{0}, \quad n = 1,2,\ldots,N-2; \tag{15}$$

here, $\mathbf{0}$ is the column of $2q$ zeros. Therefore, in the case of degenerate lattices, non-boundary values of the state vector (8) are only spanned by the $2(R-q)$ linear independent columns of matrix $\mathbf{H}^n$, constructed with $\mathbf{H}$ of the form (14).

The singular transfer matrix (14) is not unique, because matrix $\mathbf{Q}'$ (11) is not unique either. Indeed, we can, for example, take q upper/lower rows of $\mathbf{Q}'$ and add them arbitrarily to its $(R-2q)$ medium rows; that will not change the required form (11). However, any matrix (14) will satisfy Eq. (4), i.e. will act as a linear operator in the $2(R-q)$-dimensional subspace of $2R$-component column vectors, whose components obey conditions (15). Therefore, *any* corresponding matrix $\mathbf{H}^n$ will contain $2(R-q)$ independent columns—a basis in this subspace—to expand the vector $\mathbf{x}(n)$ at $n = 1,2,\ldots,N-2$.

Thus, we have found that the static deformation of a beam lattice under end loads can be expanded to a linear combination of $2R$ fundamental modes. For the case of non-singular matrix $\mathbf{K}_1$, the unique transfer matrix (5) can be built for the beam, and the columns of matrix $\mathbf{H}^n$ will give a full set of such modes. When matrix $\mathbf{K}_1$ is singular with rank $R-q$, a family of singular transfer matrices can be found according to formula (14); and the $2(R-q)$ independent columns of $\mathbf{H}^n$, calculated with any of these matrices, will provide $2(R-q)$ transfer modes of the solution to Eq. (2). The remaining $2q$ modes localise and have non-trivial value at the beam's boundaries only. These solutions can be built by using formulas (9) to describe the states of blockage of the boundary loads for the degenerate lattices.

In the next section, we show that for a given lattice, there exist $2(R-q)$ unique transfer modes, relating to the Jordan canonical form of transfer matrix (5) or (14), to feature only purely exponential, polynomial or quasi-polynomial dependence on the value of n.

4. Canonical modes of static deformation

Thus, we have found that the number of transfer fundamental solutions to Eqs. (2) and (4) can be less than $2R$, so that the rank of the fundamental matrix $\mathbf{X}(n)$, $n = 1,2,\ldots,N-2$, for Eq. (7) is generally $2(R-q)$. It is also obvious not to be unique, since any matrix function $\mathbf{J}(n)$, such that

$$\mathbf{J}(n) = \mathbf{T}\mathbf{J}^n, \quad \mathbf{J}^n = \mathbf{T}^{-1}\mathbf{H}^n\mathbf{T} \tag{16}$$

($\mathbf{T}$ is a non-singular transformation matrix), will satisfy Eq. (7) and give the modal decomposition of $\mathbf{x}(n)$, as the following:

$$\mathbf{x}(n) = \mathbf{J}(n)\mathbf{C}, \quad \mathbf{C} = \mathbf{T}^{-1}\mathbf{x}(0). \tag{17}$$

Note that the $2q$ dependent columns of $\mathbf{J}(n)$ will conveniently turn to zeros, and the remaining ones should feature some simpler dependences on the value n, if matrix $\mathbf{J}$ in Eq. (16) is chosen as the Jordan canonical form of $\mathbf{H}$. These points are discussed in more detail below.

Assume that the Jordan decomposition of a transfer matrix $\mathbf{H}$ is known

$$\mathbf{H}=\mathbf{T}\mathbf{J}\mathbf{T}^{-1}, \quad \mathbf{J}=\mathrm{diag}[\ldots,\mathbf{J}_{k-1},\mathbf{J}_k,\mathbf{J}_{k+1},\ldots], \quad k=1,2,\ldots, \tag{18}$$

where some Jordan block $\mathbf{J}_k$,

$$\mathbf{J}_k=\begin{pmatrix} \lambda & 1 & 0 & \ldots \\ 0 & \lambda & 1 & \ldots \\ 0 & 0 & \lambda & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \tag{19}$$

has dimension n_k, relates to an eigenvalue λ and to a Jordan chain of the eigenvectors:

$$(\mathbf{H}-\lambda\mathbf{I})\mathbf{h}_1=\mathbf{0}, (\mathbf{H}-\lambda\mathbf{I})\mathbf{h}_2=\mathbf{h}_1,\ldots,(\mathbf{H}-\lambda\mathbf{I})\mathbf{h}_{n_k}=\mathbf{h}_{n_k-1}. \tag{20}$$

Here, the generalised eigenvectors $\mathbf{h}_2,\mathbf{h}_3,\ldots,\mathbf{h}_{n_k}$ are unique for a given ordinary eigenvector $\mathbf{h}_1$, if they also satisfy the orthogonality demand

$$\mathbf{h}_1^{\mathrm{T}}\mathbf{h}_2=\mathbf{h}_1^{\mathrm{T}}\mathbf{h}_3=\cdots=\mathbf{h}_1^{\mathrm{T}}\mathbf{h}_{n_k}=0, \tag{21}$$

which will be further discussed at the end of this section.

As soon as

$$\mathbf{H}^n=(\mathbf{T}\mathbf{J}\mathbf{T}^{-1})^n=\mathbf{T}\mathbf{J}^n\mathbf{T}^{-1}, \tag{22}$$

the exponentiation of $\mathbf{H}$ reduces to the separate exponentiations of its Jordan blocks:

$$\mathbf{J}^n=\mathrm{diag}(\ldots,\mathbf{J}_{k-1}^n,\mathbf{J}_k^n,\mathbf{J}_{k+1}^n,\ldots). \tag{23}$$

At the same time, any block $\mathbf{J}_k^n$ can be written explicitly in terms of binomial coefficients, as

$$\mathbf{J}_k^n=\begin{pmatrix} \lambda^n\binom{n}{0} & \lambda^{n-1}\binom{n}{1} & \lambda^{n-2}\binom{n}{2} & \ldots \\ 0 & \lambda^n\binom{n}{0} & \lambda^{n-1}\binom{n}{1} & \ldots \\ 0 & 0 & \lambda^n\binom{n}{0} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$\binom{n}{0}=1, \quad \binom{n}{1}=n, \quad \binom{n}{2}=\frac{(n^2-n)}{2},\ldots \tag{24}$$

(see, for example, Ref. [14]). Therefore, the individual columns of the matrix function $\mathbf{J}(n)$ read

$$\mathbf{J}(n)=\mathbf{T}\mathbf{J}^n=(\cdots\ \mathbf{j}_1(n)\ \mathbf{j}_2(n)\ \cdots\ \mathbf{j}_{n_k}(n)\ \cdots), \tag{25}$$

$$\mathbf{j}_m(n)=\sum_{s=0}^{m-1}\lambda^{n-s}\binom{n}{s}\mathbf{h}_{m-s}, \quad m=1,2,\ldots,n_k. \tag{26}$$

Thus, for each Jordan block (19) of dimension n_k, there exist n_k modes of solution to Eq. (4), given by Eq. (26). For example, if the size of the block is 1×1, we obtain the single mode

$$\mathbf{j}_1(n)=\lambda^n\mathbf{h}_1 \tag{27}$$

and if 2×2—two independent modes

$$\mathbf{j}_1(n)=\lambda^n\mathbf{h}_1, \quad \mathbf{j}_2(n)=\lambda^n\mathbf{h}_2+n\lambda^{n-1}\mathbf{h}_1. \tag{28}$$

If the rank of $\mathbf{H}$ is $2(R-q)$, matrix $\mathbf{J}(n)=\mathbf{T}\mathbf{J}^n$ will contain $2q$ zero columns, due to the corresponding zero blocks in its Jordan form (see the second example in Appendix). Therefore, by considering all the Jordan blocks of $\mathbf{J}$ in a similar way, we can generally obtain $2(R-q)$ independent transfer solutions of type (26).

It is important to note that the case when a multiple eigenvalue relates to only one ordinary independent eigenvector (i.e., when $n_k>1$) is the most common, when $\lambda=1$. Indeed, substitute solution (27) into Eq. (2), omitting index "1" at $\mathbf{h}_1$, and pre-multiply Eq. (2) with $\lambda^{-n}\mathbf{h}^{\mathrm{T}}$ to get

$$\lambda^{-1}\mathbf{h}^{\mathrm{T}}\mathbf{K}_1^{\mathrm{T}}\mathbf{h}+\mathbf{h}^{\mathrm{T}}\mathbf{K}_0\mathbf{h}+\lambda\mathbf{h}^{\mathrm{T}}\mathbf{K}_1\mathbf{h}=0, \tag{29}$$

here and further, when constructing a solution to Eq. (2) in terms of the transfer matrix eigenvectors (20), we imply taking their upper halves only. Since $\mathbf{h}^{\mathrm{T}}\mathbf{K}_1\mathbf{h}=\mathbf{h}^{\mathrm{T}}\mathbf{K}_1^{\mathrm{T}}\mathbf{h}$, eigenvalues λ must satisfy the characteristic equation

$$1+b\lambda+\lambda^2=0, \quad b=(\mathbf{h}^{\mathrm{T}}\mathbf{K}_0\mathbf{h})/(\mathbf{h}^{\mathrm{T}}\mathbf{K}_1\mathbf{h}), \tag{30}$$

whose roots λ_1 and λ_2 are always reciprocal, i.e. have the property

$$\lambda_1=1/\lambda_2. \tag{31}$$

This means that, for each mode $\lambda^n\mathbf{h}$, there exists a "twin" to read $(1/\lambda)^n\tilde{\mathbf{h}}$, and the above procedure with the last will yield the same characteristic equation (30). In case $\lambda=\pm 1$, both twins can simultaneously satisfy Eqs. (2) and (30) even for linear dependant vectors $\mathbf{h}$ and $\tilde{\mathbf{h}}$, so that two unity eigenvalues can be related to the same vector $\mathbf{h}$. At the same time, the eigenvalue $\lambda=1$ must be the most common, since it relates, for example, to the structure's rigid-body displacements, when $\mathbf{d}(n)$ reads a constant vector for all n.

Thus, Jordan blocks of size $n_k>1$ with $\lambda=1$ are expected to be common, due to the frequent occurrence of the last, and due to reciprocity (31) that holds for any regular lattice. The occurrence of blocks of sizes $n_k>1$ for the non-unity eigenvalues is rare and to be dictated not by the general property (31), but other specific conditions to hold for particular lattices only (excluding the case of $\lambda=-1$, which should be relatively common).

The above indicates that the transfer modes can be classified as the following:

(a) *Exponential modes* (for $\lambda\neq 1$)—as shown in Eq. (27), where the eigenvalue λ acquires the nature of strain energy decay parameter. Indeed, assume $\mathbf{K}$ is the stiffness matrix for some repeating substructures at locations n of a truss, and their deformation is described by the displacement vector (27). Then the strain energy of the deformation is given by the sequence $E(n)=\lambda^{2n}E_0$, where $E_0=\mathbf{h}_1^{\mathrm{T}}\mathbf{K}\mathbf{h}_1/2$.

(b) *Polynomial modes* (for multiple $\lambda=1$):

$$\mathbf{j}_m(n)=\sum_{s=0}^{m-1}\binom{n}{s}\mathbf{h}_{m-s}, \quad m=1,2,\ldots,n_k. \tag{32}$$

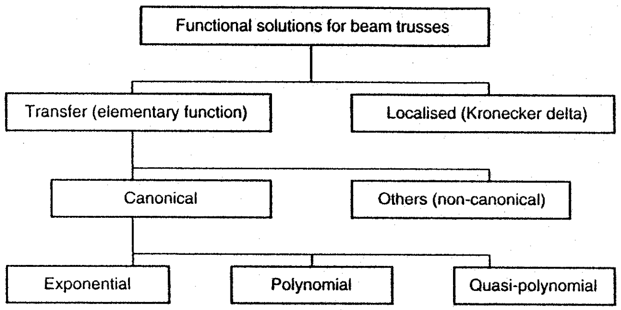

Fig. 1. Classification of non-series functional solutions for the statics of finite repetitive beam trusses under end loads (pin-jointed model).

Here, the components of vector $\mathbf{j}_m(n)$ read some simple power functions of n, since the binomial coefficients are given by the s-order polynomials shown in Eq. (24).

(c) *Quasi-polynomial modes* (for multiple $\lambda \neq 1$)—the modes (26) for $m > 1$.

Note that since the mode (27) with $\lambda = 1$ relates to a rigid-body motion of the structure, the orthogonality demand (21) implies elimination of this motion from the initial values, $\mathbf{j}_{m>1}(0)$, of other polynomial solutions (32) to Eq. (4). More generally, if there occur several Jordan blocks, relating to the same λ, then the generalised eigenvectors in Eq. (26) or (32) should be taken as orthogonal to all the ordinary eigenvectors, existing for this eigenvalue (see the examples in Appendix).

Thus, unlike other transfer solutions (for example, the columns of matrix $\mathbf{H}^n$ in Eq. (6)), the canonical modes can be further subclassified, as discussed above. Then the chart shown in Fig. 1 presents the general classification of non-series solutions for finite beam trusses under end loads.

The canonical transfer (26) and singular modes (9) comprise a full set of $2R$ independent solutions to Eq. (2) that display only a particular character of functional dependence on the value n. We call them, therefore, *characteristic* modes of a beam-like structure. For a given lattice, the canonical solutions are unique up to a constant factor, and the deformation patterns assigned by them can be often associated with response to the basic types of external loads (Section 6). There is an analogy with the spectrum of dynamic normal modes, whose linear superposition yields a complicated vibration of the structure: the canonical modes span an arbitrary state of free static deformation,[2] and serve also as a set of unique internal characteristics of the structure. While a dynamic mode has an intrinsic parameter—the normal frequency—each of the canonical modes has a static parameter λ.

[2] That is, deformation due to boundary loads only, but on the areas excluding the boundaries.

5. Boundary value problem

Expressions (8) and (17) provide solutions to the *initial* value problem, i.e. when the vectors $\mathbf{d}(2), \mathbf{d}(3), \ldots, \mathbf{d}(N-1)$ are found from a knowledge of $\mathbf{d}(0)$ and $\mathbf{d}(1)$. Form (8) can give the decomposition of these vectors into the $2R$ columns of matrix $\mathbf{H}^n$, of which $2(R-q)$ are generally independent, and Eq. (17) into the $2(R-q)$ canonical transfer modes (26). However, in practice, one normally faces the boundary value problem, when vectors $\mathbf{d}(1), \mathbf{d}(2), \ldots, \mathbf{d}(N-1)$ are to be found by employing the boundary displacements $\mathbf{d}(0), \mathbf{d}(N)$ and/or loads $\mathbf{f}(0)$, $\mathbf{f}(N)$. We can find solutions to this kind of problems in the form of linear superpositions of the characteristic modes, as the following.

Construct the $R \times 2R$ matrix of all $2R$ independent characteristic solutions to Eq. (2), using the $2q$ singular modes (9) and $2(R-q)$ transfer modes, obtained as Eq. (26) for all Jordan blocks of $\mathbf{H}$:

$$\mathbf{Y}(n) = (\mathbf{u}_1(n) \ldots \mathbf{u}_{2q}(n)\,\mathbf{j}_1(n) \ldots \mathbf{j}_{2(R-q)}(n)). \tag{33}$$

Here, we have used a unified numbering for all the canonical transfer modes. The general solution to Eq. (2) can then be written compactly as

$$\mathbf{d}(n) = \mathbf{Y}(n)\mathbf{C}, \quad n = 0, 1, \ldots, N, \tag{34}$$

where the vector of participation coefficients $\mathbf{C}$ read

$$\mathbf{C} = \begin{pmatrix} \mathbf{Y}(0) \\ \mathbf{Y}(N) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{d}(0) \\ \mathbf{d}(N) \end{pmatrix} \tag{35}$$

if the boundary displacements are known, or

$$\mathbf{C} = \begin{pmatrix} \mathbf{K}_2\mathbf{Y}(0) + \mathbf{K}_1\mathbf{Y}(1) \\ \mathbf{Y}(N) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}(0) \\ \mathbf{d}(N) \end{pmatrix} = \begin{pmatrix} \mathbf{Y}(0) \\ \mathbf{K}_1^{\mathrm{T}}\mathbf{Y}(N-1) + \mathbf{K}_3\mathbf{Y}(N) \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{d}(0) \\ \mathbf{f}(N) \end{pmatrix} \tag{36}$$

if there is given a pair of vectors $\mathbf{f}(0), \mathbf{d}(N)$ or $\mathbf{d}(0), \mathbf{f}(N)$. Expressions (36) were obtained by employing Eq. (34) and also Eq. (3), where the matrices $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ read the corresponding $R \times R$ blocks of the associate cell stiffness matrix in Eq. (1).

6. Illustrative examples

To illustrate the analytical results of Sections 3–5, consider the characteristic solutions for the pin-jointed X-braced-1, X-braced-2, Warren and two-layered plane trusses shown in Fig. 2. Assume that Young's modulus and cross-section area of all members of these trusses are 2×10^{11} N/m^2 and 1 cm^2, respectively. All the Warren truss members and non-diagonal members of the X-braced lattices have length 1 m, and the length of all diagonal members is $\sqrt{2}$ m. Throughout this section, the displacement solutions will be given in mm.

The stiffness matrices for the corresponding associate substructures (Fig. 2) can be found by conventional means, where the block $\mathbf{K}_1$ is non-singular for the X-braced lattices, and singular

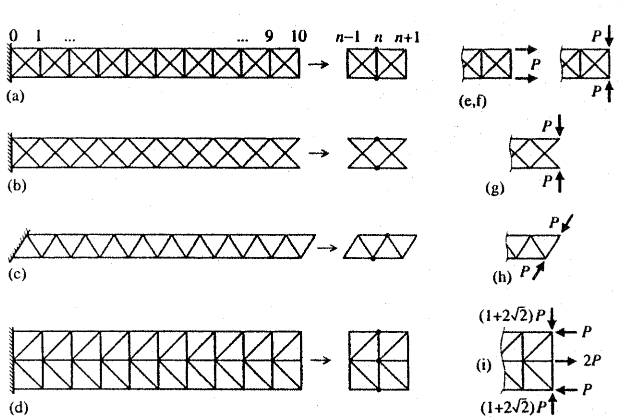

Fig. 2. Repetitive planar latticeworks and their associate cells: (a) X-braced-1; (b) X-braced-2; (c) Warren; and (d) two-layered. Example boundary value problems (e)–(g) and blockage of end loads (h), (i); the deformation is described with polynomial (e), exponential (f), quasi-polynomial (g) and localised (h), (i) modes.

for the Warren and two-layered structures. The non-singular and degenerate transfer matrices are to be found according to formulas (5) and (14) respectively. The Jordan decompositions of **H** (i.e. matrices **J** and **T** of Eq. (18)) for the X-braced-1 and Warren trusses are given in the appendix.

6.1. By employing expression (26) for each of the Jordan blocks, we obtain eight canonical transfer modes, comprising the complete set of characteristic solutions for the X-braced-1 truss (here, $R=4$ and $q=0$), and present them in the matrix form (33):

$$\mathbf{Y}(n)=\begin{pmatrix} \lambda_1^n 1.4565 & \lambda_2^n 1.4565 & 1 & 2n-1 & 0 & -1 & 2-2n & -9-6\sqrt{2}+18n-6n^2 \\ -\lambda_1^n 6.8802 & \lambda_2^n 6.8802 & 0 & 1-\sqrt{2} & 1 & 2n-1 & 1-4n+2n^2 & -3+20n-18n^2+4n^3 \\ \lambda_1^n 1.4565 & \lambda_2^n 1.4565 & 1 & 2n-1 & 0 & 1 & -2+2n & 9+6\sqrt{2}-18n+6n^2 \\ \lambda_1^n 6.8802 & -\lambda_2^n 6.8802 & 0 & -1+\sqrt{2} & 1 & 2n-1 & 1-4n+2n^2 & -3+20n-18n^2+4n^3 \end{pmatrix},$$

$$\lambda_1=-9.55217,\quad \lambda_2=-0.10469. \tag{37}$$

Here and below in this section, $n=0,1,\ldots,10$. Thus, the first two of the modes are exponential, and the rest are polynomial. The exponential solutions describe deformation of the structure due to self-equilibrated end loads. The third, fifth and sixth modes obviously describe rigid-body motions: translations in x- and y-axis, and rotation of the structure in the xy-plane. The fourth, seventh, and a combination of the eighth and seventh columns can be, respectively, associated with continuum-like extensional, flexural and shear deformation of the framework.

Indeed, consider the boundary value problems with the four basic types of end loads: tensile (Fig. 2e), bending, shear and self-equilibrated (Fig. 2f):

$$\begin{aligned} \mathbf{d}(0)=\mathbf{0},\ \mathbf{f}(10)&=P(1\ \ 0\ \ 1\ \ 0)^{\mathrm{T}},\\ \mathbf{f}(10)&=P(1\ \ 0\ \ -1\ \ 0)^{\mathrm{T}},\\ \mathbf{f}(10)&=P(0\ \ -1\ \ 0\ \ -1)^{\mathrm{T}},\\ \mathbf{f}(10)&=P(0\ \ -1\ \ 0\ \ 1)^{\mathrm{T}}. \end{aligned} \tag{38a–e}$$

Solutions for these problems read the matrix form (34), where the vectors of participation coefficients **C** can found according to Eq. (36) with $N=10$ to read, respectively,

$$\begin{aligned} \mathbf{C}&=10^{-5}P(-8.8256\times10^{-12}\ \ 0.12469\ \ 1.8895\ \ 2.0711\ \ 0\ \ 0\ \ 0\ \ 0)^{\mathrm{T}},\\ \mathbf{C}&=10^{-5}P(0\ \ 0\ \ 0\ \ 0\ \ -5/2\ \ -5\ \ -5/2\ \ 0)^{\mathrm{T}},\\ \mathbf{C}&=10^{-5}P(0\ \ 0\ \ 0\ \ 0\ \ -54.571\ \ -99.571\ \ -42.500\ \ 0.83333)^{\mathrm{T}},\\ \mathbf{C}&=10^{-16}P(5.1440\ \ 5.1440\ \ -14.984\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^{\mathrm{T}}. \end{aligned} \tag{39a–e}$$

It is noteworthy that the fourth mode describes the structure's elongation (39a) accurate to a rigid-body motion and also small deflections (given by the exponential modes) due to restriction of the Poisson effect at the truss left end. For the second problem, the solution reads a combination of the bending and rigid-body motion/rotation modes, and acquires an exclusively simple form:

$$\mathbf{d}(n)=5\times10^{-5}Pn(1\ \ -n\ \ -1\ \ -n)^{\mathrm{T}}. \tag{40}$$

6.2. The X-braced-2 truss (Fig. 2b) is remarkable to have a rare pair of modes, exponential and quasi-polynomial, which relates to a 2×2 Jordan block with $\lambda=-1$,

$$\mathbf{j}_1(n)=(-1)^n\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\quad \mathbf{j}_2(n)=(-1)^n\begin{pmatrix}1\\(1+2\sqrt{2})(1-2n)\\1\\-(1+2\sqrt{2})(1-2n)\end{pmatrix}. \tag{41}$$

Up to a rigid-body displacement in the x-axis, a superposition of these modes gives the solution to the boundary value problem with a self-equilibrated end load shown in Fig. 2g:

$$\begin{aligned} &\mathbf{d}(0)=\mathbf{0},\quad \mathbf{f}(10)=P(0\ \ -1\ \ 0\ \ 1)^{\mathrm{T}},\\ &\mathbf{d}(n)=(5/2)\times10^{-5}P(-(1+2\sqrt{2})\mathbf{j}_1(n)+\mathbf{j}_2(n)). \end{aligned} \tag{42}$$

The remaining six modes of this structure are of polynomial character. They comprise two groups of two and four modes, having similar mathematical forms and physical meanings as the corresponding modes in Eq. (37).

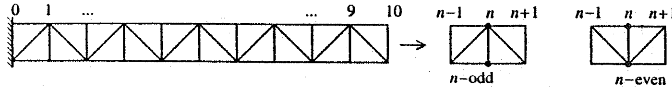

Fig. 3. Associate cells of a double-bay truss.

6.3. For the Warren truss, we have $R=4$ and $q=1$, i.e. the size and rank of the block $\mathbf{K}_1$ are 4 and 3, respectively. As a result, the Jordan form has only $2(R-q)=6$ non-zero columns (see the appendix). Using formulas (32) and (9), we then find six polynomial and $2q=2$ localised modes. The matrix of characteristic solutions (33) shows these modes as the following:

$$\mathbf{Y}(n)=\begin{pmatrix} 0 & 0 & 1 & \sqrt{3}(4n-1) & 0 & -\sqrt{3} & -\sqrt{3}(n-1) & -3\sqrt{3}(7-6n+2n^2) \\ 0 & \delta_{n,10} & 0 & -1 & 1 & 4n-1 & -3n+2n^2 & 3+22n-30n^2+8n^3 \\ 0 & 0 & 1 & \sqrt{3}(4n-3) & 0 & \sqrt{3} & \sqrt{3}(n-1) & 3\sqrt{3}(7-6n+2n^2) \\ \delta_{n,0} & 0 & 0 & 1 & 1 & 4n-3 & 2-5n+2n^2 & -15+58n-42n^2+8n^3 \end{pmatrix}. \tag{43}$$

The singular modes describe the blockage of self-equilibrated loads depicted in Fig. 2h, and the six transfer solutions can be conventionally related to the deformation due to other basic types of the end loads. For example, considering the tensile (38a) and bending (38b) problems with this lattice, we obtain the corresponding vectors (36), describing the modal contributions, and displacement solutions (34) as the following:

$$\begin{aligned} &\mathbf{C}=(a/4)(0\ 0\ 2\sqrt{3}\ 1\ 2\ 1\ 0\ 0)^{\mathrm{T}}, \quad \mathbf{d}(n)=an(\sqrt{3}\ 1\ \sqrt{3}\ 1)^{\mathrm{T}}, \quad a=(1/2\sqrt{3})\times 10^{-4}P, \\ &\mathbf{C}=-a(0\ 0\ 0\ 0\ 1\ 1\ 1\ 0)^{\mathrm{T}}, \quad \mathbf{d}(n)=an(\sqrt{3}\ \ -1-2n\ \ -\sqrt{3}\ \ 1-2n)^{\mathrm{T}}. \end{aligned} \tag{44}$$

Here, due to the geometry of this structure, the tensile load also invokes a rigid-body displacement along the y-axis and a rotation in the xy-plane—in contrast with the case (39a).

6.4. For the two-layered truss shown in Fig. 2d, we obtain $R=6$ and $q=2$. Then the use of Eqs. (9) and (26) allows one to construct four singular and eight transfer modes (two exponential and six polynomial), which are similarly meaningful, as those in the above examples. Remarkably, the structure has the unobvious pattern of the external load blockage, depicted in Fig. 2i. Such a load invokes one of the singular modes to give the nodal displacements as

$$\mathbf{d}(n)=10^{-4}P\delta_{n,10}(0\ \ -\sqrt{2}\ \ 1\ \ 0\ \ 0\ \ \sqrt{2})^{\mathrm{T}} \tag{45}$$

that implies undesirable localisation of stresses in lattice members to be avoided in practice.

6.5. In the analysis of double-bay trusses, the approach varies as follows. Since there exist two different associate cells—for even and odd n (an example is depicted in Fig. 3)—one has to construct two transfer matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ for each of them, employing Eq. (5) or Eq. (11). The blocks $\mathbf{K}_1$ are not identical in Eq. (1) for this case; therefore, another notation for $\mathbf{K}_1^{\mathrm{T}}$ should be used in Eqs. (5) and (10). The successive values of state vector (4) then can be related as $\mathbf{x}(n+2)=\mathbf{H}_1\mathbf{H}_2\mathbf{x}(n)$ to give the solution to Eq. (10):

$$\mathbf{x}(n)=\mathbf{H}^{n/2}\mathbf{x}(0), \quad n=2,4,\ldots, \quad \mathbf{H}=\mathbf{H}_1\mathbf{H}_2. \tag{46}$$

In result, the canonical modes for Eq. (46) can be obtained conventionally by accomplishing the Jordan decomposition of the matrix $\mathbf{H}=\mathbf{H}_1\mathbf{H}_2$ and employing the final formula (26), where value n is replaced with $n/2$ on the right side.

6.6. We finally note that the case of space trusses does not yield any new mathematical types of the solution modes. Each component of the displacement vector function $\mathbf{d}(n)$ still reads a combination of a Kronecker delta, exponent, polynomial and quasi-polynomial. The polynomial modes similarly represent the basic rigid-body motions and deflections, including also a torsion mode that describes the response to end torque loads, and which is normally given by first-order polynomials.

7. Conclusion

In this paper, the equilibrium states of free static deformation of regular pin-jointed beam trusses have been decomposed over a number of non-series functional modes. The total number of the modes is twice the number of degrees of freedom in one repeating nodal location. The modes have been classified to form two primary groups of transfer and localised solutions. For each truss, there exist a unique set of transfer canonical modes that have been subclassified as exponential, polynomial and quasi-polynomial. Although the case of quasi-polynomial modes is less common, the polynomial solutions exist for any repetitive beam-like truss, and the exponential type arise in studying statically indeterminate structures. The localised solutions may arise for both indeterminate and determinate lattices in case of a mathematically insufficient coupling between adjacent nodal locations.

The canonical and localised modes represent unique structural characteristics of a beam truss, being implied only by the intrinsic lattice properties. The polynomial and exponential solutions can often be related to the basic continuum-like types of the static response. However, lattice structures generally do not behave like ordinary beams, and sometimes feature essentially discontinuous characteristics, such as the patterns of boundary load blockage, described by the modes of localised type.

The full set of canonical and localised modes represents a system of fundamental solutions to the governing finite difference equation; therefore, it can be found once only and further utilised for solving a variety of specific practical problems on the structure static response.

In forthcoming papers, the method will be expanded to write elementary function solutions for a pin-jointed truss under distributed loads, and we will also discuss the characteristic modal expansion of the static response of geometrically imperfect trusses.

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Appendix

The Jordan canonical form **J** and transformation matrix **T** of the decomposition (18) for the X-braced-1 and Warren trusses are as follows:

$$\text{X-braced-1:}\quad \begin{pmatrix} -9.55217 & 0 & 0\,0\,0\,0\,0\,0 \\ 0 & -0.104688 & 0\,0\,0\,0\,0\,0 \\ 0 & 0 & 1\,1\,0\,0\,0\,0 \\ 0 & 0 & 0\,1\,0\,0\,0\,0 \\ 0 & 0 & 0\,0\,1\,1\,0\,0 \\ 0 & 0 & 0\,0\,0\,1\,1\,0 \\ 0 & 0 & 0\,0\,0\,0\,1\,1 \\ 0 & 0 & 0\,0\,0\,0\,0\,1 \end{pmatrix},$$

$$\left(\begin{array}{c|c|cc|cccc} 1.4565 & 1.4565 & 2 & -1 & 0 & -4 & 4 & -3-2\sqrt{2} \\ -6.8802 & 6.8802 & 0 & 1-\sqrt{2} & 8 & -4 & 2 & -1 \\ 1.4565 & 1.4565 & 2 & -1 & 0 & 4 & -4 & 3+2\sqrt{2} \\ 6.8802 & -6.8802 & 0 & \sqrt{2}-1 & 8 & -4 & 2 & -1 \\ -13.913 & -0.15248 & 2 & 1 & 0 & -4 & 0 & 1-2\sqrt{2} \\ 65.720 & -0.72027 & 0 & 1-\sqrt{2} & 8 & 4 & -2 & 1 \\ -13.913 & -0.15248 & 2 & 1 & 0 & 4 & 0 & -1+2\sqrt{2} \\ -65.720 & 0.72027 & 0 & \sqrt{2}-1 & 8 & 4 & -2 & 1 \end{array}\right),$$

$$\text{Warren:}\quad \begin{pmatrix} 0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1 \end{pmatrix}, \quad \left(\begin{array}{c|c|cc|cccc} 0 & 1 & 12 & -3 & 0 & -4\sqrt{3} & 4\sqrt{3} & -7\sqrt{3} \\ 0 & 0 & 0 & -\sqrt{3} & 16 & -4 & 0 & 1 \\ 0 & 0 & 12 & -9 & 0 & 4\sqrt{3} & -4\sqrt{3} & 7\sqrt{3} \\ 1 & 0 & 0 & \sqrt{3} & 16 & -12 & 8 & -5 \\ 0 & 0 & 12 & 9 & 0 & -4\sqrt{3} & 0 & -3\sqrt{3} \\ 0 & 0 & 0 & -\sqrt{3} & 16 & 12 & -4 & 1 \\ 0 & 0 & 12 & 3 & 0 & 4\sqrt{3} & 0 & 3\sqrt{3} \\ 0 & 0 & 0 & \sqrt{3} & 16 & 4 & -4 & 3 \end{array}\right).$$

Here, the columns of **T** have been rearranged and normalised, so that the generalised eigenvectors satisfy the Jordan chain (20) and orthogonality demand (21) with both ordinary eigenvectors for $\lambda = 1$ (the third and fifth columns).

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