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GENERAL THEOREMS AND GENERALIZED VARIATIONAL PRINCIPLES FOR NONLINEAR ELASTODYNAMICS

Jing-Tang Xing and W.G. Price

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General theorems and generalized variational principles for nonlinear elastodynamics

By Jing-Tang Xing[1,2] and W. G. Price[2]

[1] *Solid Mechanics Research Center, Beijing University of Aeronautics and Astronautics, Beijing 100083, People's Republic of China*

[2] *Department of Ship Science, University of Southampton, Highfield, Southampton SO9 5NH, U.K.*

Generalized variational principles with 11 - arguments, 9 - arguments, 5 - arguments, 3 - arguments and the variational principles of action of potential / complementary energy are developed to solve initial-value, final-value and two time boundary-value problems in nonlinear elastodynamic systems. The displacement gradient is decomposed into a symmetric part D_{ij} and a rotation part $W_{ij} = -e_{ijk}\omega_k$ which are variables in functionals.

The theoretical approach is illustrated by examining one-dimensional elastostatic and elastodynamic problems. In the former, it is shown that by solving for the displacement gradient $u_{i,j}$ as a function of the stress tensor σ_{ij} from the constraint equations of the variational principle of complementary energy, the functional of the variational principle of complementary energy can be expressed in a form involving the single argument σ_{ij} only.

An application of the variational principles is illustrated in an elastodynamic final-value problem. Complementing these examples is a discussion indicating how other generalized variational principles may be deduced and how numerical schemes of study may be enhanced through a matrix decomposition of the displacement gradient $u_{i,j}$.

Notations

Using tensor notation and summation convention (see, for example, Ogden 1984), e. g. δ_{ij} and e_{ijk} (i = 1,2,3 = j) to represent Kronecker delta and permutation symbol, respectively, we shall investigate the motion from time t_1 to time t_2 of an elastic body occupying a closed-bounded domain $\hat{\Omega}$ in a three dimensional Euclid space defined by lagrangian coordinates x_i. In order to do so effectively, the following notation definitions are usedand we assume that all the field variables are functions of x_i and t and they satisfy definitions of continuity and differentiation with respect to x_i and t as stated by Gurtin (1964a,b).

$\hat{f}_i$	Body force vector,
p_i	Momentum vector,
q_{ij}	Displacement gradient,
t	Time,
u_i	Displacement vector,
$\hat{u}_i$	Displacement prescribed on surface S_u,
v_i	Velocity vector,
x_i	Lagrangian coordinate,

E_{ij}	Green strain tensor,
$\tilde{()}$	Functionals defined in Section 5,
S	Surface of domain $\hat{\Omega}$, $S = S_u \cup S_T$, $S_u \cap S_T = void$,
S_u	Surface with prescribed displacement $\hat{u}_i$,
S_T	Surface with prescribed traction $\hat{T}$,
$\hat{T}$	Traction prescribed on the surface S_T,
ξ	Unit normal vector at time terminals in the time interval $[t_1, t_2]$, $\xi(t_1) = -1$, $\xi(t_2) = 1$,
ν_i	Unit normal vector to surface S, pointing out of the domain $\hat{\Omega}$,
σ_{ij}	Second Kirchhoff stress tensor,
τ_{ij}	Piola stress tensor,
Ω	Internal domain of $\hat{\Omega}$.

1. Introduction

The theory of elastostatics, through the development of general theorems and variational principles, has been extensively studied (see, for example, Love 1944, Green & Zerna 1954, Hu 1954 & 1981, Oden & Reddy 1976, Oden 1979, Chien 1979, 1980 & 1985, Gou 1980, Washizu 1982, Xing 1996). In its dynamics counterpart, i. e. elastodynamic theory, studies of variational principles have concentrated on modifications or extensions of Hamilton's principle in one form or another. Truesdell & Toupin (1960), Chen (1964), Yu (1964), Dean & Plass (1965), Barr (1966) adopted the displacement field as argument to define the configuration of an elastodynamic system and constrained variations of the displacement to vanish at the time instants t_1 and t_2. In this way, variational principles were developed to describe linear and non-linear elastodynamic systems and their forms relate closely to potential energy. Toupin (1952) and Crandall (1957) developed complementary principles to describe dynamic systems using momentum or velocity as the variable satisfying the dynamic equilibrium equations in association with conditions requiring their variation vanishing at the time instants t_1 and t_2. Oden and Reddy (1976) developed variational principles subject to the constraint that both the displacement and momentum vanish at the time instants t_1 and t_2; these variational principles also include constitutive principles (see, for example, Xing 1987a).

The requirement that the variations of both displacement and momentum vanish at the two time instants t_1 and t_2 is unnecessary and too restrictive a condition as demonstrated by Xing (1984, 1991a), Xing and Price (1992), Xing and Zheng (1992a). These linear elastodynamic studies discuss the two time boundary-value problem having four kinds of time terminal conditions. Variational principles for conservative and holonomical dynamical systems and linear elastodynamic systems are developed and it is shown that Hamilton's principle and Toupin's principle are special cases of the developed generalized variational principles. For initial - value problems of linear elastodynamics, Gurtin (1964a,b) developed some variational principles which include the initial conditions through a time convolution integral but are not Hamilton's forms. On the basis of a four dimensional space with time and boundary conditions prescribed at the time instants t_1 and t_2, Xing (1987b, 1989, 1990), Xing and Price (1996) developed variational principles for linear elastodynamic and conservative and holonomic systems to define and solve initial-value and final-value problems in addition to the two time boundary-value problem. In these variational approaches, the initial or final conditions are included in the stationary conditions of the derived functionals, allowing solution of the problems posed by Rudinger (1964) and Tiersten (1968).

This paper extends the previous discussions to *nonlinear elastodynamic problems* with the development of general theorems and generalized variational principles for three kinds of dynamical problems. That is *Problem A: an initial-value problem; Problem B: a final-value problem and Problem C: a two time boundary-value problem.* Generalized variational principles with 11-arguments (τ_{ij}, σ_{ij}, ξ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, E_{ij}, v_i, u_i), 9-arguments (τ_{ij}, σ_{ij}, ξ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, u_i), 5-arguments (σ_{ij}, p_i, E_{ij}, v_i, u_i), 3-arguments (σ_{ij}, p_i, u_i) and the variational principles of action of potential / complementary energy are given. It is shown that results derived by Chien, Lu & Wang (1989), Xing (1991b), Xing & Zheng (1992b), Zheng & Xing (1990), Zheng & Zhao (1992) are special cases of the generalized principles developed herein.

By way of example through a one dimensional problem, a discussion is included on the existence of a variational principle of complementary energy in one argument σ_{ij} to describe a nonlinear elastostatic problem analogous to the case found in linear elasticity theory. The approach proposed solves for the displacement gradient $u_{i,j}$ from the constraint equations applicable to the variational principle of complementary energy.

2. Governing equations

With the adoption of the notations, the equations describing the behaviour of nonlinear elastodynamic systems can be expressed in the following forms.

(i) Momentum equations of equilibrium:

$$\tau_{ij,j} + \hat{f}_i = p_{i,t}, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.1}$$

$$\tau_{ij} = (\delta_{ik} + q_{ik})\sigma_{kj}, \qquad (x_i, t) \in \Omega \times (t_1, t_2). \tag{2.2}$$

The Piola stress tensor τ_{ij} is not a symmetric tensor but can be decomposed into a symmetric part χ_{ij} and an antisymmetric part ψ_{ij} as follows:

$$\tau_{ij} = \chi_{ij} + \psi_{ij}, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.3}$$

$$\chi_{ij} = \frac{1}{2}(\tau_{ij} + \tau_{ji}), \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.4}$$

$$\psi_{ij} = \frac{1}{2}(\tau_{ij} - \tau_{ji}), \qquad (x_i, t) \in \Omega \times (t_1, t_2). \tag{2.5}$$

The antisymmetric component ψ_{ij} consists of three independent elements and can be represented by its corresponding vector b_i as follows

$$\psi_{ij} = -\frac{1}{2}e_{ijk}b_k, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.6}$$

$$b_k = -e_{kij}\psi_{ij} = -e_{kij}\tau_{ij}, \qquad (x_i, t) \in \Omega \times (t_1, t_2). \tag{2.7}$$

(ii) Strain-displacement relations and velocity-displacement relations:

$$E_{ij} = \frac{1}{2}(q_{ij} + q_{ji} + q_{ki}q_{kj}), \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.8}$$

$$q_{ij} = u_{i,j}, \qquad (x_i, t) \in \Omega \times (t_1, t_2); \tag{2.9}$$

$$v_i = u_{i,t}, \qquad (x_i, t) \in \Omega \times (t_1, t_2). \tag{2.10}$$

In general, the displacement gradient q_{ij} is also not a symmetric tensor. This again can be decomposed into a symmetric part D_{ij} and an antisymmetric part W_{ij} with associated vector ω_k, as follows:

$$q_{ij} = D_{ij} + W_{ij}, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.11}$$

$$D_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.12}$$

$$W_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) = -e_{ijk}\omega_k, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.13}$$

$$\omega_k = -\frac{1}{2}e_{kij}W_{ij} = -\frac{1}{2}e_{kij}u_{i,j}, \qquad (x_i, t) \in \Omega \times (t_1, t_2). \tag{2.14}$$

(iii) Stress-strain relations and momentum-velocity relations:

$$\sigma_{ij} = \frac{\partial A}{\partial E_{ij}}, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.15}$$

$$E_{ij} = \frac{\partial B}{\partial \sigma_{ij}}, \qquad (x_i, t) \in \Omega \times (t_1, t_2); \tag{2.16}$$

$$p_i = \frac{\partial T}{\partial v_i}, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.17}$$

$$v_i = \frac{\partial R}{\partial p_i}, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{2.18}$$

(iv) Boundary conditions:

$$\tau_{ij}\nu_j = \hat{T}_i, \qquad (x_i, t) \in S_T \times [t_1, t_2], \tag{2.19}$$

$$u_i = \hat{u}_i, \qquad (x_i, t) \in S_u \times [t_1, t_2]. \tag{2.20}$$

(v) Conditions at the time terminals:

$$p_i = \hat{p}_{ip}, \qquad (x_i, t) \in \hat{\Omega} \times \tilde{t}_p = \hat{\Omega}_p, \tag{2.21}$$

$$u_i = \hat{u}_{iu}, \qquad (x_i, t) \in \hat{\Omega} \times \tilde{t}_u = \hat{\Omega}_u. \tag{2.22}$$

Here $\tilde{t}_u$ and $\tilde{t}_p$ represent the set of time terminals at which the displacement $\hat{u}_{iu}$ and the momentum $\hat{p}_{ip}$ are prescribed (i.e. not allowed variation), respectively. Obviously, for each kind of dynamical problem, we have the following conditions at the time terminals.

Problem A. The initial-value problem

$$\tilde{t}_u = \tilde{t}_p = \{t_1\}, \tag{2.23}$$

where the conditions at the initial time t_1 are prescribed, and a description of the subsequent motion is required.

Problems B. The final-value problem

$$\tilde{t}_u = \tilde{t}_p = \{t_2\}, \tag{2.24}$$

where the conditions at the final time t_2 are prescribed, but the initial conditions and the motion before time t_2 are unknown and their solutions are sought.

For convenience, we shall take $\tilde{t}_m$ to represent the time instant set at which the displacement and momentum are prescribed whereas $\tilde{t}_n$ denotes the time instant set at which both displacement and momentum are unknown and therefore subject to variations. The sets $\tilde{t}_m$ and $\tilde{t}_n$ are such that $\tilde{t}_m \cup \tilde{t}_n = \{t_1, t_2\}$ and $\tilde{t}_m \cap \tilde{t}_n = void$. Furthermore, we define $\hat{\Omega}_m = \hat{\Omega} \times \tilde{t}_m$ and $\hat{\Omega}_n = \hat{\Omega} \times \tilde{t}_n$. For example, in Problem A, we have

$$\tilde{t}_m = \{t_1\}, \qquad \tilde{t}_n = \{t_2\}, \tag{2.25}$$

$$\hat{\Omega}_m = \hat{\Omega} \times \{t_1\}, \qquad \hat{\Omega}_n = \hat{\Omega} \times \{t_2\}. \tag{2.26}$$

Thus, in Problems A and B, the conditions at time terminals expressed in equations (2.21) and (2.22) can be replaced by the following more convenient forms:

$$u_i = \hat{u}_{iu}, \qquad (x_i, t) \in \hat{\Omega} \times \tilde{t}_m = \hat{\Omega}_m, \tag{2.27}$$

$$p_i = \hat{p}_{ip}, \qquad (x_i, t) \in \hat{\Omega} \times \tilde{t}_m = \hat{\Omega}_m. \tag{2.28}$$

Problem C. The two time boundary-value problem

$$\tilde{t}_u \cup \tilde{t}_p = \{t_1, t_2\}, \tag{2.29}$$

$$\tilde{t}_u \cap \tilde{t}_p = void, \tag{2.30}$$

which represents the following four cases of time terminal conditions.

(1) $\tilde{t}_u = \{t_1, t_2\}$, $\tilde{t}_p = void$: The displacement fields at time t_1 and t_2 are prescribed, which correspond to the case of Hamilton's principle (see, for example, Green & Zerna 1954) .

(2) $\tilde{t}_u = void$, $\tilde{t}_p = \{t_1, t_2\}$: The momentum fields at time t_1 and t_2 are prescribed, which correspond to the studies of Toupin (1952) and Crandall (1957).

(3) $\tilde{t}_u = \{t_1\}$, $\tilde{t}_p = \{t_2\}$: The displacement field at time t_1 and the momentum field at time t_2 are prescribed.

(4) $\tilde{t}_u = \{t_2\}$, $\tilde{t}_p = \{t_1\}$: The momentum field at time t_1 and the displacement field at time t_2 are prescribed.

3. Definitions

In order to construct a mathematical framework in which to develop variational principles, it is desirable to introduce the following definitions.

3.1. Action

If the variable $f(t)$ is dependent on time t, the integral

$$I = \int_{t_1}^{t_2} f(\tau) d\tau, \qquad (t_2 > t_1), \tag{3.1}$$

defines the action of the variable $f(t)$ in the time interval $[t_1, t_2]$. For example, if $f(t)$ represents a force, then the integral I will be the impulse of this force. We do not constrain this definition to this case only, for if, $f(t)$ denotes a potential energy, then the integral I will define the action of the potential energy. The term *action* has its foundation in lagrangian mechanics (see, for example, Whittaker 1917). In the problems discussed herein, it is assumed that the integral described in equation (3.1) always exists.

3.2. Momentum at time t_τ

Let us consider an elastic body in its static undeformation reference configuration and, for the present, let $\hat{p}_i(x_j, t_\tau)$ represent an impulsive body force applied to the elastic body at time t_τ, $t_1 \le t_\tau \le t_2$. Thus the body force at time $t \epsilon (t_1, t_2)$ is given by

$$f_i(x_j, t) = \hat{p}_i(x_j, t_\tau) \Delta(t - t_\tau), \tag{3.2}$$

where $\Delta(t - t_\tau)$ is the Dirac delta function. The substitution of this body force into the dynamic equation (2.1) and the integration of these equations over the range from $t_{\tau-}$ to $t_{\tau+}$ defines the momentum at t_τ as

$$p_i(x_j, t_\tau) = \hat{p}_i(x_j, t_\tau), \tag{3.3}$$

in the limit as $t_{\tau-} \rightarrow t_{\tau+}$. This implies that the body force given in equation (3.2) corresponds to the momentum $p_i(x_j, t_\tau)$ and

$$f_i(x_j, t) = p_i(x_j, t_\tau)\Delta(t - t_\tau). \tag{3.4}$$

Furthermore, prescribing the momentum $p_i(x_j, t_\tau)$ at the instant t_τ implies that the corresponding impulsive body force $\hat{p}_i(x_j, t_\tau)$ as given in equation (3.2) acts on the elastic body.

3.3. Functions of potential energy and complementary energy

Let us assume that the elastic body under investigation is a hyperelastic body and all forces applied to it are potential forces. Therefore, there exists functions of potential energy and functions of complementary energy. These are determined by summing (i.e. integrating) all the changes of state or configurations between the reference and present configuration. They are defined as follows.

(i) Functions of strain energy density and complementary relation:

$$A(E_{ij}) = \int_0^{E_{ij}} \sigma_{ij} dE_{ij}, \qquad B(\sigma_{ij}) = \int_0^{\sigma_{ij}} E_{ij} d\sigma_{ij}; \tag{3.5}$$

(ii) Functions of kinetic energy density and complementary relation:

$$T(v_i) = \int_0^{v_i} p_i dv_i, \qquad R(p_i) = \int_0^{p_i} v_i dp_i; \tag{3.6}$$

(iii) Potential of body force and complementary relation:

$$G(u_i) = \int_0^{u_i} -f_i du_i, \qquad G^*(f_i) = \int_0^{f_i} -u_i df_i; \tag{3.7}$$

(iv) Potential of traction on the surface of the body and complementary relation:

$$Q(u_i) = \int_0^{u_i} -T_i du_i, \qquad Q^*(T_i) = \int_0^{T_i} -u_i dT_i; \tag{3.8}$$

Also, using the same reasoning as Courant and Hilbert (1962), we assume that there exists the following transformation relations between the potential functions and their complements:

$$A(E_{ij}) + B(\sigma_{ij}) = \sigma_{ij} E_{ij}, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{3.9}$$

$$T(v_i) + R(p_i) = p_i v_i, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{3.10}$$

$$G(u_i) + G^*(f_i) = -f_i u_i, \qquad (x_i, t) \in \Omega \times (t_1, t_2), \tag{3.11}$$

$$Q(u_i) + Q^*(T_i) = -T_i u_i, \qquad (x_i, t) \in \Omega \times (t_1, t_2). \tag{3.12}$$

It is necessary to point out that the potentials of body force and traction as well as their complementary relations are dependent not only on the elastic body under consideration but also bodies which exert forces on the elastic body. For example, for dead loads, i. e.

their amplitude and direction are not changed as the elastic body moves, $f_i = \hat{f}_i$, and

$$G(u_i) = -\hat{f}_i u_i, \qquad G^*(\hat{f}_i) = 0, \tag{3.13}$$

$$Q(u_i) = -\hat{T}_i u_i, \qquad Q^*(\hat{T}_i) = 0. \tag{3.14}$$

3.4. Action of the potential of momentum at t_τ and its complementary relation

Using equations (3.2), (3.3), (3.4) and (3.7) with the definition given in equation (3.1), we can define the action of the potential of momentum $p_i(x_j, t_\tau)$ at the time t_τ and the action of its complementary function in the following manner:

$$g_\tau(u_i) = \int_{t_1}^{t_2} \int_0^{u_i} -p_i(x_j, t_\tau)\Delta(t - t_\tau) du_i dt = \int_0^{u_i} -p_i(x_j, t_\tau) du_i(t_\tau), \tag{3.15}$$

and

$$g_\tau^*(p_i) = \int_{t_1}^{t_2} \int_0^{p_i} -u_i dp_i(x_j, t_\tau)\Delta(t - t_\tau) dt = \int_0^{p_i} -u_i(t_\tau) dp_i(x_j, t_\tau). \tag{3.16}$$

Physically, $g_\tau(u_i)$ and $g_\tau^*(p_i)$ are dependent on the relation between the amplitude $p_i(t_\tau)$ of the impulsive force and the displacement $u_i(t_\tau)$ of the elastic body. For the hyperelastic body under consideration, we assume this relation exists. Geometrically, this may be illustrated schematically as shown in figure 1 (t_m in stead of t_τ), where $-g_m(u_i), -g_m^*(p_i)$ denote the areas within the closed continuous lines.

With the adoption of the notation for the subscripts m and n in equation (2.25), the action of potential of momentum $\hat{p}_{ip}$ at time $t \epsilon \tilde{t}_m$ and the action of its complementary function can be expressed as

$$g_m(u_i) = \int_0^{u_i} -p_i(t) du_i(t) = -\eta \hat{p}_{ip} u_i(t), \qquad t \epsilon \tilde{t}_m, \tag{3.17}$$

$$g_m^*(p_i) = \int_0^{p_i} -u_i(t) dp_i(t) = -\eta^* \hat{u}_{iu} p_i(t), \qquad t \epsilon \tilde{t}_m. \tag{3.18}$$

Here, for example, $-g_m(u_i)$ represents an area beneath the curve L. The momentum $\hat{p}_{ip}$ is known but $u_i(t)$ remains unknown and is acceptable to variations. The area of the rectangle $\hat{p}_{ip} u_i(t)$ is weighted by the parameter η such that equation (3.17) holds. Alternatively, if $\hat{u}_{iu}$ is known and $p_i(t)$ unknown, then their combination is weighted by η^* such that equation (3.18) is valid. As deduced from figure 1, the two parameters must satisfy the condition

$$\eta + \eta^* = 1. \tag{3.19}$$

This allows the relation

$$g_m(u_i) + g_m^*(p_i) = -\hat{u}_{iu} \hat{p}_{ip}, \tag{3.20}$$

to be valid for the conditions in equations (2.27) and (2.28) of Problem A and Problem B as shown in figure 1.

The parameters η and η^* describe the relationship between $p_i(t)$ and $u_i(t)$. For example, if $\eta = \eta^* = 1/2$, this relationship is linear and the line L in figure 1 is a straight line through the origin. If $\eta = 1$ and $\eta^* = 0$, the relationship represents a constant momentum and the line L in figure 1 is parallel to the axis $u_i(t_\tau)$, whereas the relationship $\eta = 0$

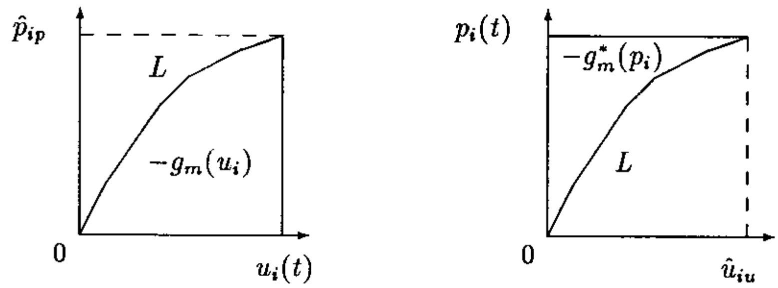

Figure 1. Geometrical representations of $g_m(u_i)$ and $g_m^*(p_i)$ at $t \epsilon \tilde{t}_m$.

and $\eta^* = 1$ represents a constant displacement and the line L in figure 1 is a straight line parallel to $p_i(t_\tau)$.

3.5. Admissible function spaces

Suppose FS is a *function space* consisting of all smoothing functions $F(x_i, t), (x_i, t) \epsilon \Omega \times (t_1, t_2)$ required to formulate the present problem. We define the following admissible function spaces.

(i) Admissible displacement field space ADF:

$$ADF = \{u_i : u_i \epsilon FS, satisfy\ (2.8), (2.9), (2.10), (2.20), (2.22)\}. \tag{3.21}$$

(ii) Admissible stress-momentum field space ASF:

$$ASF = \{(\tau_{ij}, p_i) : (\tau_{ij}, p_i) \epsilon FS, satisfy\ (2.1), (2.2), (2.19), (2.21)\}. \tag{3.22}$$

4. General theorems

4.1. Principle of admissible work action

The principle of admissible work action is stated as follows: *For an arbitrary admissible displacement field $u_i^* \epsilon ADF$ and an arbitrary admissible stress-momentum field $(\tau_{ij}^*, p_i^*) \epsilon AFS$, there exists the following integral relation*

$$\int_{t_1}^{t_2} \int_\Omega (\tau_{ij}^* u_{i,j}^* - p_i^* v_i^*) d\Omega dt = \int_{t_1}^{t_2} \{ \int_{S_T} \hat{T}_i u_i^* ds + \int_{S_u} T_i^* \hat{u}_i ds + \int_\Omega \hat{f}_i u_i^* d\Omega \} dt - \int_\Omega [(p_i^* u_i^*)_{t_2} - (p_i^* u_i^*)_{t_1}] d\Omega, \tag{4.1}$$

where the prescribed values of p_i^* and u_i^* at times t_1 and t_2 are determined by the time terminal conditions in equations (2.21) and (2.22).

It is not difficult to prove that this relation is valid using Green's theorem and the constraint conditions stated for ADF and ASF in equations (3.21) and (3.22).

4.2. Principle of virtual displacement

In equation (4.1), the replacement of the actual stress-momentum field (τ_{ij}, p_i) for the admissible quantities (τ_{ij}^*, p_i^*); the sum of the actual displacement field u_i and the virtual displacement field $\delta u_i \epsilon ADF$ represented by $u_i + \delta u_i$ for the admissible variable u_i^* in addition to the constraint conditions for ADF and ASF, allows the principle of virtual displacement to be expressed as

$$\int_{t_1}^{t_2} \int_\Omega (\tau_{ij}\delta u_{i,j} - p_i \delta v_i) d\Omega dt = \int_{t_1}^{t_2} \{ \int_{S_T} \hat{T}_i \delta u_i ds + \int_\Omega \hat{f}_i \delta u_i d\Omega \} dt - \int_\Omega [(p_i \delta u_i)_{t_2} - (p_i \delta u_i)_{t_1}] d\Omega, \tag{4.2}$$

where δu_i at times t_1 and t_2 belong to the set $\tilde{t}_u$ in equation (2.22) and is zero.

4.3. Principle of virtual stress-momentum

On reversing the approach to the one described in section 4.2, we find that the principle of virtual stress-momentum can be derived from equation (4.1). That is, the substitution of the actual displacement field u_i for the admissible variable u_i^*; $(\tau_{ij} + \delta\tau_{ij}, p_i + \delta p_i)$ representing the actual stress-momentum fields (τ_{ij}, p_i) and the virtual stress-momentum fields $(\delta\tau_{ij}, \delta p_i)$ for the admissible quantity (τ_{ij}^*, p_i^*) as well as using the constrain conditions for ADF and ASF gives

$$\int_{t_1}^{t_2} \int_\Omega (u_{i,j}\delta\tau_{ij} - v_i \delta p_i) d\Omega dt = \int_{t_1}^{t_2} \int_{S_u} \hat{u}_i \delta\tau_{ij}\nu_j ds dt - \int_\Omega [(u_i \delta p_i)_{t_2} - (u_i \delta p_i)_{t_1}] d\Omega, \tag{4.3}$$

where δp_i at times t_1 and t_2 belongs to the set $\tilde{t}_p$ in equation (2.21) and is zero.

4.4. Theorem of action of strain-kinetic energy

On substituting the actual displacement field u_i for the admissible displacement field u_i^* and the actual stress-momentum fields (τ_{ij}, p_i) for the admissible quantities (τ_{ij}^*, p_i^*) in equation (4.1), as well as using the relation

$$\int_\Omega \tau_{ij} u_{i,j} d\Omega = \int_\Omega [\sigma_{ij} E_{ij} + \frac{1}{2} u_{k,i} u_{k,j} \sigma_{ij}] d\Omega, \tag{4.4}$$

we find that the theorem of action of strain-kinetic energy can be expressed as follows:

$$\int_{t_1}^{t_2} \int_\Omega [A(E_{ij}) + B(\sigma_{ij}) - T(v_i) - R(p_i) + \frac{1}{2} u_{k,i} u_{k,j} \sigma_{ij}] d\Omega dt$$
$$= \int_{t_1}^{t_2} \{ \int_{S_T} \hat{T}_i u_i ds + \int_{S_u} \tau_{ij}\nu_j \hat{u}_i ds + \int_\Omega \hat{f}_i u_i d\Omega \} dt - \int_\Omega [(p_i u_i)_{t_2} - (p_i u_i)_{t_1}] d\Omega, \tag{4.5}$$

where p_i and u_i at times t_1 and t_2 are determined from equations (2.21) and (2.22). Although this theorem corresponds to the theorem of strain energy in static elasticity, it does not represent the relation of energy conservation in elastodynamics but now the action of strain- kinetic energy..

5. Variational principles for initial/final-value problems (Problems A and B)

5.1. Variational principle of action of potential energy

It was found that amongst all the admissible displacement fields u_i satisfying the strain-displacement relations in equations (2.8) and (2.9), the velocity-displacement relations in equation (2.10) and the displacement boundary condition in equation (2.20) as well as the corresponding momentum p_i, the actual displacement field satisfying the governing equations in equations (2.1) associated with (2.2), (2.19) and the initial/final-value conditions in equations (2.27) and (2.28) make the functional

$$\hat{H}_2[u_i, p_i] = \tilde{H}_1 + \hat{H}, \tag{5.1}$$

$$\begin{aligned} \tilde{H}_1 &= \int_{t_1}^{t_2} \{ \int_\Omega [A(E_{ij}) - T(v_i) + G(u_i)] d\Omega + \int_{S_T} Q(u_i) ds \} dt, \\ \hat{H} &= \int_{\hat{\Omega}_m} [\eta \xi \hat{p}_{ip} u_i + \eta^* \xi p_i (u_i - \hat{u}_{iu})] d\Omega - \int_{\hat{\Omega}_n} \xi g_n(u_i) d\Omega, \end{aligned} \tag{5.2}$$

stationary, if the conditions expressed in equations (2.15), (2.17), (3.15) and (3.19) are satisfied.

By taking the variation of the functional $\hat{H}_2$ and using equations (2.15), (2.17), (3.15) and (3.19) together with the constraints given in equations (2.8), (2.9), (2.10) and (2.20), we can prove that the functional $\hat{H}_2$ is stationary. This is achieved by using Green's theorem and we obtain the following variational result

$$\delta \hat{H}_2 = \int_{t_1}^{t_2} \{ \int_\Omega [(\delta_{ik} + u_{i,k}) \sigma_{kj} \delta u_{i,j} - p_i \delta u_{i,t} - \hat{f}_i \delta u_i] d\Omega - \int_{S_T} \hat{T}_i \delta u_i ds \} dt + \delta \hat{H}, \tag{5.3}$$

$$\delta \hat{H} = \int_{\hat{\Omega}_m} [\eta \xi \hat{p}_{ip} \delta u_i + \eta^* \xi \delta p_i (u_i - \hat{u}_{iu}) + \eta^* \xi p_i \delta u_i] d\Omega + \int_{\hat{\Omega}_n} \xi p_i \delta u_i d\Omega. \tag{5.4}$$

A rearrangement of the terms gives

$$\begin{aligned} \delta \hat{H}_2 = &\int_{t_1}^{t_2} \{ \int_\Omega [-((\delta_{ik} + u_{i,k}) \sigma_{kj})_{,j} + p_{i,t} - \hat{f}_i] \delta u_i d\Omega + \int_{S_T} [(\delta_{ik} + u_{i,k}) \sigma_{kj} \nu_j - \hat{T}_i] \delta u_i ds \} dt \\ &+ \int_{\hat{\Omega}_m} [-\eta \xi (p_i - \hat{p}_{ip}) \delta u_i + \eta^* \xi (u_i - \hat{u}_{iu}) \delta p_i] d\Omega. \end{aligned} \tag{5.5}$$

Now because of the independence of the variations δu_i in $\Omega \times (t_1, t_2)$ and $S_T \times [t_1, t_2]$ as well as the variations δp_i and δu_i at $\hat{\Omega}_m$, equations (2.1) with (2.2), (2.19), (2.27) and (2.28) result when $\delta \hat{H}_2 = 0$, and vice versa.

5.2. Variational principle of action of complementary potential energy

In this case, it was found that amongst all the admissible stress-momentum fields σ_{ij} and p_i with corresponding body force f_i as well as the displacement $u_i \epsilon FS$ satisfying the dynamic equations (2.1), (2.2) and the traction boundary condition (2.19), the actual stress-momentum fields satisfying the strain-displacement relations (2.8) with (2.9), the velocity-displacement relation (2.10), the displacement boundary condition (2.20) and the initial/final-value conditions in equations (2.27) and (2.28) make the functional

$$\hat{\Pi}_3[\sigma_{ij}, p_i, u_i] = \tilde{\Pi}_2 + \hat{\Pi}, \tag{5.6}$$

$$\tilde{\Pi}_2 = \int_{t_1}^{t_2} \{ \int_\Omega [B(\sigma_{ij}) - R(p_i) + \frac{1}{2} \sigma_{ij} u_{k,i} u_{k,j} + G^*(\hat{f}_i)] d\Omega$$

$$-\int_{S_u}\hat{u}_i(\delta_{ik}+u_{i,k})\sigma_{kj}\nu_j ds+\int_{S_T}Q^*(\hat{T}_i)ds\}dt+\hat{\Pi},$$

$$\hat{\Pi}=\int_{\hat{\Omega}_m}[\eta^*\xi\hat{u}_{iu}p_i+\eta\xi u_i(p_i-\hat{p}_{ip})]d\Omega-\int_{\hat{\Omega}_n}\xi g_n^*(p_i)d\Omega \tag{5.7}$$

stationary, if the conditions expressed in equations (2.16), (2.18), (3.16), and (3.19) are satisfied.

This statement can be shown valid by taking the variation of this functional and using equations (2.16), (2.18), (3.16) and (3.19). By this means, we obtain the variational result

$$\delta\hat{\Pi}_3=\int_{t_1}^{t_2}\{\int_\Omega[E_{ij}\delta\sigma_{ij}-v_i\delta p_i+\sigma_{ij}u_{k,j}\delta u_{k,i}+\frac{1}{2}\delta\sigma_{ij}u_{k,i}u_{k,j}-u_i\delta\hat{f}_i]d\Omega$$
$$-\int_{S_u}\hat{u}_i\delta[(\delta_{ik}+u_{i,k})\sigma_{kj}]\nu_j ds-\int_{S_T}u_i\delta\hat{T}_i ds\}dt+\delta\hat{\Pi}, \tag{5.8}$$

$$\delta\hat{\Pi}=\int_{\hat{\Omega}_m}[\eta^*\xi\hat{u}_{iu}\delta p_i+\eta\xi\delta u_i(p_i-\hat{p}_{ip})+\eta\xi u_i\delta p_i]d\Omega+\int_{\hat{\Omega}_n}\xi u_i\delta p_i d\Omega. \tag{5.9}$$

The substitution of the variational forms of the constraint equations (2.1), (2.2) and (2.19), i.e.

$$\delta\tau_{ij,j}+\delta\hat{f}_i=\delta p_{i,t},\qquad \delta\tau_{ij}=\delta\sigma_{ij}+\delta u_{k,i}\sigma_{ij},\qquad \delta\tau_{ij}\nu_j=\delta\hat{T}_i, \tag{5.10}$$

into the previous equations and using Green's theorem together with a rearrangement of terms gives

$$\delta\hat{\Pi}_3=\int_{t_1}^{t_2}\{\int_\Omega[(E_{ij}-\frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}))\delta\sigma_{ij}-(v_i-u_{i,t})\delta p_i]d\Omega$$
$$+\int_{S_u}(u_i-\hat{u}_i)\delta\tau_{ij}\nu_j ds\}dt+\int_{\hat{\Omega}_m}[\eta\xi(p_i-\hat{p}_{ip})\delta u_i-\eta^*\xi(u_i-\hat{u}_{iu})\delta p_i]d\Omega. \tag{5.11}$$

Now because of the independence of the variations $\delta\sigma_{ij}$ and δp_i in $\Omega\times(t_1,t_2)$ and the variation $\delta\tau_{ij}\nu_j$ on $S_u\times[t_1,t_2]$ as well as the variations δp_i and δu_i at the $\hat{\Omega}_m$, equations (2.8) with (2.9), (2.20), (2.27) and (2.28) result when $\delta\hat{\Pi}_3=0$, and vice versa.

5.3. Generalized variational principle of action of potential and complementary energy with 5-arguments $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i

In equation (5.2), the functional $\hat{H}_2$ is dependent on the variables u_i and p_i. This functional may be modified to a generalized form dependent on the five variables $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i. This is achieved by relaxing the constraints described in equations (2.8) associated with (2.9), (2.10) and (2.20) through the lagrangian multiplier method (for example, see, Courant & Hilbert 1962). By this means, we obtain the following functional of the generalized variational principle of action of potential energy with 5-arguments $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i:

$$\hat{H}_{5g}[\sigma_{ij},p_i,u_i,E_{ij},v_i]=\tilde{H}_{5g}+\hat{H}, \tag{5.12}$$

$$\tilde{H}_{5g}=\int_{t_1}^{t_2}\{\int_\Omega[A(E_{ij})-T(v_i)+G(u_i)+p_i(v_i-u_{i,t})-\sigma_{ij}(E_{ij}$$
$$-\frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}))]d\Omega+\int_{S_T}Q(u_i)ds-\int_{S_u}(u_i-\hat{u}_i)(\delta_{ik}+u_{i,k})\sigma_{kj}\nu_j ds\}dt,$$

where $\hat{H}$ is given in equation (5.2).

By replacing $B(\sigma_{ij})$ by $A(E_{ij})$ and $R(p_i)$ by $T(v_i)$ through equations (3.9) and (3.10) respectively, and using the lagrangian multiplier method to relax the constraints expressed in equations (2.1) with (2.2) and (2.19), it follows from the functional $\hat{\Pi}_3$ in equation (5.6) that we can obtain another functional with 5-arguments $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i. That is, the generalized variational principle of action of complementary energy

$$\hat{\Pi}_{5g}[\sigma_{ij}, p_i, u_i, E_{ij}, v_i] = \tilde{\Pi}_{5g} + \hat{\Pi}, \tag{5.13}$$

$$\begin{aligned}\tilde{\Pi}_{5g} = \int_{t_1}^{t_2} \{ \int_{\Omega} [&\sigma_{ij} E_{ij} - A(E_{ij}) - p_i v_i + T(v_i) + u_i([(\delta_{ik} + u_{i,k})\sigma_{kj}]_{,j} + \hat{f}_i - p_{i,t}) \\ &+ G^*(\hat{f}_i) + \frac{1}{2}\sigma_{ij} u_{k,i} u_{k,j}] d\Omega - \int_{S_u} \hat{u}_i(\delta_{ik} + u_{i,k})\sigma_{kj}\nu_j ds \\ &+ \int_{S_T} Q^*(\hat{T}_i) ds - \int_{S_T} [(\delta_{ik} + u_{i,k})\sigma_{kj}\nu_j - \hat{T}_i] u_i ds \} dt,\end{aligned}$$

where $\hat{\Pi}$ is given by equation (5.7).

The functionals $\hat{H}_{5g}$ and $\hat{\Pi}_{5g}$ have the same 5-arguments $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i as well as the same variational stationary conditions. These stationary conditions are the governing equations for the initial/final -value problems in elastodynamics expressed in equations (2.1) with (2.2), (2.8) with (2.9), (2.10), (2.15), (2.17), (2.19), (2.20), (2.27) and (2.28). It thus follows that the two functionals satisfy the following transformation relation

$$\hat{H}_{5g}[\sigma_{ij}, p_i, u_i, E_{ij}, v_i] + \hat{\Pi}_{5g}[\sigma_{ij}, p_i, u_i, E_{ij}, v_i] = 0. \tag{5.14}$$

5.4. Generalized variational principle of action of potential and complementary potential energy with 3-arguments σ_{ij}, p_i and u_i

The replacement of $A(E_{ij})$ by $B(\sigma_{ij})$ and $T(v_i)$ by $R(p_i)$ in the functional $\hat{H}_{5g}$ expressed in equation (5.14) using equations (3.9) and (3.10), allows the following functional with 3-arguments σ_{ij}, p_i and u_i to be derived. That is

$$\hat{H}_{3g}[\sigma_{ij}, p_i, u_i] = \tilde{H}_{3g} + \hat{H}, \tag{5.15}$$

$$\begin{aligned}\tilde{H}_{3g} = \int_{t_1}^{t_2} \{ \int_{\Omega} [&R(p_i) - B(\sigma_{ij}) + G(u_i) + \frac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \\ &- p_i u_{i,t}] d\Omega + \int_{S_T} Q(u_i) ds - \int_{S_u} (u_i - \hat{u}_i)(\delta_{ik} + u_{i,k})\sigma_{kj}\nu_j ds \} dt,\end{aligned}$$

where $\hat{H}$ is given in equation (5.2).

By the reverse process, the functional $\hat{\Pi}_{5g}$ in equation (5.15) can be transformed into the functional $\hat{\Pi}_{3g}$ with 3-arguments σ_{ij}, p_i and u_i. That is,

$$\hat{\Pi}_{3g}[\sigma_{ij}, p_i, u_i] = \tilde{\Pi}_{3g} + \hat{\Pi}, \tag{5.16}$$

$$\begin{aligned}\tilde{\Pi}_{3g} = \int_{t_1}^{t_2} \{ \int_{\Omega} [&B(\sigma_{ij}) - R(p_i) + G^*(\hat{f}_i) + u_i([(\delta_{ik} + u_{i,k})\sigma_{kj}]_{,j} + \hat{f}_i - p_{i,t}) + \frac{1}{2}\sigma_{ij} u_{k,i} u_{k,j}] d\Omega \\ &- \int_{S_u} \hat{u}_i(\delta_{ik} + u_{i,k})\sigma_{kj}\nu_j ds - \int_{S_T} [(\delta_{ik} + u_{i,k})\sigma_{kj}\nu_j - \hat{T}_i] u_i ds + \int_{S_T} Q^*(\hat{T}_i) ds \} dt,\end{aligned}$$

where $\hat{\Pi}$ is expressed in equation (5.7).

The functionals $\hat{H}_{3g}$ and $\hat{\Pi}_{3g}$ have the same 3-arguments σ_{ij}, p_i and u_i as well as the same variational stationary conditions. These conditions consist of the governing equations describing initial/final-value problems in elastodynamics expressed by equations (2.1) with (2.2), (2.19), (2.20), (2.27), (2.28) in addition to,

$$\frac{\partial B}{\partial \sigma_{ij}} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}), \quad \frac{\partial R}{\partial p_i} = u_{i,t}. \tag{5.17}$$

Thus, the two functionals satisfy the following transformation relation

$$\hat{H}_{3g}[\sigma_{ij}, p_i, u_i] + \hat{\Pi}_{3g}[\sigma_{ij}, p_i, u_i] = 0. \tag{5.18}$$

5.5. Generalized variational principle of action of potential and complementary energy with 11-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, E_{ij}, v_i and u_i

In deriving the functional $\hat{H}_{5g}$, equations (2.9), (2.11), (2.12), (2.13) and (2.14) are assumed automatically valid. If we replace $u_{i,j}$ by q_{ij} through equation (2.9) and $(\delta_{ik} + u_{i,k})\sigma_{kj}$ by $\chi_{ij} - \frac{1}{2}b_k e_{kij}$ through equation (2.3) with (2.6) as well as releasing the relations expressed in equations (2.11) with (2.13), (2.12) and (2.14) by using the lagrangian multiplier method from the functional $\hat{H}_{5g}$ in equation (5.12), the generalized variational principle of action of potential energy with 11-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, E_{ij}, v_i and u_i is defined by the functional

$$\hat{H}_{11g}[\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i, E_{ij}, v_i, u_i] = \tilde{H}_{11g} + \hat{H}, \tag{5.19}$$

$$\begin{aligned}\tilde{H}_{11g} = \int_{t_1}^{t_2} \{ \int_\Omega [&A(E_{ij}) - T(v_i) + G(u_i) + p_i(v_i - u_{i,t}) - b_k(\omega_k + \frac{1}{2}e_{kij}u_{i,j}) \\ &-\sigma_{ij}(E_{ij} - \frac{1}{2}(q_{ij} + q_{ji} + q_{ki}q_{kj})) - \tau_{ij}(q_{ij} - D_{ij} + e_{ijk}\omega_k) - \chi_{ij}(D_{ij} \\ &-\frac{1}{2}(u_{i,j} + u_{j,i}))]d\Omega - \int_{S_u} (u_i - \hat{u}_i)(\chi_{ij} - \frac{1}{2}b_k e_{kij})\nu_j ds + \int_{S_T} Q(u_i)ds\}dt,\end{aligned}$$

where $\hat{H}$ is given in equation (5.2).

By replacing $(\delta_{ik} + u_{i,k})\sigma_{kj}$ by $\chi_{ij} - \frac{1}{2}b_k e_{kij}$ through equation (2.3) with (2.6) and releasing the relations in equations (2.2), (2.4) and (2.7) by use of the lagrangian multiplier method from the functional in equation (5.13), allows the generalized variational principle of action of complementary energy with 11-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, E_{ij}, v_i and u_i to be defined as,

$$\hat{\Pi}_{11g}[\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i, E_{ij}, v_i, u_i] = \tilde{\Pi}_{11g} + \hat{\Pi}, \tag{5.20}$$

$$\begin{aligned}\tilde{\Pi}_{11g} = \int_{t_1}^{t_2} \{ \int_\Omega [&\sigma_{ij}E_{ij} - A(E_{ij}) - p_i v_i + T(v_i) + G^*(\hat{f}_i) + u_i[(\chi_{ij} - \frac{1}{2}b_k e_{kij})_{,j} + \hat{f}_i - p_{i,t}] \\ &+ \frac{1}{2}\sigma_{ij}u_{k,i}u_{k,j} + (\tau_{ij} - (\delta_{ik} + q_{ik})\sigma_{kj})q_{ij} + (\chi_{ij} - \frac{1}{2}(\tau_{ij} + \tau_{ji}))D_{ij} + (b_k + e_{kij}\tau_{ij})\omega_k]d\Omega \\ &- \int_{S_u} \hat{u}_i(\chi_{ij} - \frac{1}{2}b_k e_{kij})\nu_j ds - \int_{S_T} [(\chi_{ij} - \frac{1}{2}b_k e_{kij})\nu_j - \hat{T}_i]u_i ds + \int_{S_T} Q^*(\hat{T}_i)ds\}dt,\end{aligned}$$

where $\hat{\Pi}$ is expressed in equation (5.7).

The functionals $\hat{H}_{11g}$ and $\hat{\Pi}_{11g}$ have the same 11-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, E_{ij}, v_i and u_i as well as the same variational stationary conditions. These stationary conditions are the governing equations for initial/final-value problems in elastodynamics as expressed in equations (2.1) with (2.3) and (2.6), (2.2), (2.4), (2.7), (2.8), (2.11) with (2.13), (2.10), (2.12), (2.14), (2.15), (2.17), (2.19), (2.20), (2.27) and (2.28). Therefore, these two functionals satisfy the following transformation relation

$$\hat{H}_{11g} + \hat{\Pi}_{11g} = 0. \tag{5.21}$$

5.6. Generalized variational principle of action of potential and complementary energy with 9-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, and u_i

By replacing $A(E_{ij})$ with $B(\sigma_{ij})$ and $T(v_i)$ with $R(p_i)$ in the functional $\hat{H}_{11g}$ expressed in equation (5.19) through equations (3.9) and (3.10), we can derive the functional describing the generalized variational principle of action of potential energy with 9-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i and u_i as follows

$$\hat{H}_{9g}[\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i, u_i] = \tilde{H}_{9g} + \hat{H}, \tag{5.22}$$

$$\begin{aligned}\tilde{H}_{9g} = \int_{t_1}^{t_2} \{ \int_\Omega [R(p_i) - B(\sigma_{ij}) + G(u_i) - p_i u_{i,t} - b_k(\omega_k + \frac{1}{2} e_{kij} u_{i,j}) \\ + \frac{1}{2}\sigma_{ij}(q_{ij} + q_{ji} + q_{ki}q_{kj})) - \tau_{ij}(q_{ij} - D_{ij} + e_{ijk}\omega_k) - \chi_{ij}(D_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}))]d\Omega \\ - \int_{S_u} (u_i - \hat{u}_i)(\chi_{ij} - \frac{1}{2} b_k e_{kij})\nu_j ds + \int_{S_T} Q(u_i) ds\} dt,\end{aligned}$$

where $\hat{H}$ is given in equation (5.2).

By applying the reverse process for the functional $\hat{\Pi}_{11g}$ in equation (5.20), we can also obtain the functional of the generalized variational principle of action of complementary energy with 9-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, and u_i. That is,

$$\hat{\Pi}_{9g}[\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i, u_i] = \tilde{\Pi}_{9g} + \hat{\Pi}, \tag{5.23}$$

$$\begin{aligned}\tilde{\Pi}_{9g} = \int_{t_1}^{t_2} \{ \int_\Omega [B(\sigma_{ij}) - R(p_i) + \frac{1}{2}\sigma_{ij} u_{k,i} u_{k,j} + G^*(\hat{f}_i) + u_i[(\chi_{ij} - \frac{1}{2} b_k e_{kij})_{,j} + \hat{f}_i - p_{i,t}] \\ + (\tau_{ij} - (\delta_{ik} + q_{ik})\sigma_{kj})q_{ij} + (\chi_{ij} - \frac{1}{2}(\tau_{ij} + \tau_{ji}))D_{ij} + (b_k + e_{kij}\tau_{ij})\omega_k]d\Omega \\ - \int_{S_u} \hat{u}_i(\chi_{ij} - \frac{1}{2} b_k e_{kij})\nu_j ds - \int_{S_T} [(\chi_{ij} - \frac{1}{2} b_k e_{kij})\nu_j - \hat{T}_i] u_i ds + \int_{S_T} Q^*(\hat{T}_i) ds\} dt,\end{aligned}$$

where $\hat{\Pi}$ is expressed in equation (5.7).

The functionals $\hat{H}_{9g}$ and $\hat{\Pi}_{9g}$ have the same 9-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i and u_i as well as the same variational stationary conditions. These stationary conditions include the governing equations for the initial/final-value problems in elastodynamics as expressed in equations (2.1) with (2.3) and (2.6), (2.2), (2.4), (2.7), (2.11) with (2.13),(2.12), (2.14), (2.19), (2.20), (2.27), (2.28), (5.17). Also, these two functionals satisfy the following transformation relation

$$\hat{H}_{9g} + \hat{\Pi}_{9g} = 0. \tag{5.24}$$

6. Variational principles for two time boundary-value problems(Problem C)

6.1. Variational principle of action of potential energy

From the principle of virtual displacement expressed in equation (4.2) and using equations (2.15) and (2.17), we can easily obtain the functional of the variational principle of action of potential energy for the two time boundary-value problems as follows.

$$H_1[u_i] = \tilde{H}_1 + H_p, \qquad H_p = \int_{\hat{\Omega}_p} \xi \hat{p}_{ip} u_i d\Omega. \tag{6.1}$$

The constraint conditions applied to this functional are given in equations (2.8) with (2.9), (2.10), (2.20) and (2.22) and its stationary conditions are expressed in equations (2.1) with (2.2), (2.19) and (2.21). The functional in equation (6.1) is suitable for the four kinds of time terminal conditions as described in equations (2.29) and (2.30). For example, if the time terminal conditions are defined by the case $\tilde{t}_u = \{t_1, t_2\}$, $\tilde{t}_p = void$, so that the set $\hat{\Omega}_p = void$, the last integral over $\hat{\Omega}_p$ in equation (6.1) vanishes and Hamilton's principle results as a special case of the functional expressed here.

6.2. Variational principle of action of complementary potential energy

From the principle of virtual stress-momentum expressed in equation (4.3) and using equations (2.16) and (2.18), we can obtain the functional of the variational principle of action of complementary energy for the two time boundary-value problems in the form

$$\Pi_2[\sigma_{ij}, p_i] = \tilde{\Pi}_2 + \Pi_u, \qquad \Pi_u = \int_{\hat{\Omega}_u} \xi \hat{u}_{iu} p_i d\Omega. \tag{6.2}$$

The constraint conditions applied to this functional are given in equations (2.1) with (2.2), (2.19), and (2.21) and its stationary conditions are expressed in equations (2.8) with (2.9), (2.10), (2.20) and (2.22). Also, the functional in equation (6.2) is suitable for the four kinds of time terminal conditions described in equations (2.29) and (2.30). For example, if the time terminal conditions are $\tilde{t}_p = \{t_1, t_2\}$, $\tilde{t}_u = void$ so that the set $\hat{\Omega}_u = void$, the last integral over $\hat{\Omega}_u$ in equation (6.2) vanishes and Toupin's principle results as a special case of this functional representation.

6.3. Generalized variational principle of action of potential and complementary energy with 5-arguments $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i

The functional H_1 in equation (6.1) is dependent on the variable u_i. This functional may be modified to a generalized form dependent on the five variables $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i by relaxing the constraints described in equations (2.8) with (2.9), (2.10), (2.20) and (2.22) through the lagrangian multiplier method. This functional of the generalized variational principle of action of potential energy with 5-arguments is expressed in the form

$$H_{5g}[\sigma_{ij}, p_i, u_i, E_{ij}, v_i] = \tilde{H}_{5g} + H_p + H_u, \qquad H_u = \int_{\hat{\Omega}_u} \xi (u_i - \hat{u}_{iu}) p_i d\Omega. \tag{6.3}$$

By the replacement of $B(\sigma_{ij})$ by $A(E_{ij})$ and $R(p_i)$ by $T(v_i)$ through equations (3.9) and (3.10), respectively, and using the lagrangian multiplier method to relax the constraints expressed in equations (2.1) with (2.2), (2.19) and (2.21), it follows that the functional Π_3 in equation (6.2) can be modified into a functional with 5-arguments $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i in the form

$$\Pi_{5g}[\sigma_{ij}, p_i, u_i, E_{ij}, v_i] = \tilde{\Pi}_{5g} + \Pi_u + \Pi_p, \qquad \Pi_p = \int_{\hat{\Omega}_p} \xi(p_i - \hat{p}_{ip}) u_i d\Omega. \tag{6.4}$$

The functionals H_{5g} and Π_{5g} have the same 5-arguments $\sigma_{ij}, p_i, u_i, E_{ij}$ and v_i as well as the same variational stationary conditions. These stationary conditions consist of the governing equations describing two time boundary-value problems in elastodynamics as expressed in equations (2.1) with (2.2), (2.8) with (2.9), (2.10), (2.15), (2.17), (2.19), (2.20), (2.21) and (2.22). Thus, the functionals H_{5g} and Π_{5g} satisfy the transformation relation

$$H_{5g}[\sigma_{ij}, p_i, u_i, E_{ij}, v_i] + \Pi_{5g}[\sigma_{ij}, p_i, u_i, E_{ij}, v_i] = 0. \tag{6.5}$$

6.4. Generalized variational principle of action of potential and complementary potential energy with 3-arguments σ_{ij}, p_i and u_i

By replacing $A(E_{ij})$ with $B(\sigma_{ij})$ and $T(v_i)$ with $R(p_i)$ in the functional H_{5g} expressed in equation (6.3) through equations (3.9) and (3.10), the following functional with 3-arguments σ_{ij}, p_i and u_i can be derived:

$$H_{3g}[\sigma_{ij}, p_i, u_i] = \tilde{H}_{3g} + H_p + H_u. \tag{6.6}$$

By the reverse process the functional Π_{5g} in equation (6.4) can be modified to create the functional Π_{3g} with 3-arguments σ_{ij}, p_i and u_i. That is,

$$\Pi_{3g}[\sigma_{ij}, p_i, u_i] = \tilde{\Pi}_{3g} + \Pi_u + \Pi_p. \tag{6.7}$$

The functionals H_{3g} and Π_{3g} have the same 3-arguments σ_{ij}, p_i and u_i as well as the same variational stationary conditions. These stationary conditions are the governing equations describing two time boundary-value problems in elastodynamics as expressed in equations (2.1) with (2.2), (2.19), (2.20), (2.21), (2.22), (5.17). Therefore, these two functionals satisfy the transformation relation

$$H_{3g}[\sigma_{ij}, p_i, u_i] + \Pi_{3g}[\sigma_{ij}, p_i, u_i] = 0. \tag{6.8}$$

6.5. Generalized variational principle of action of potential and complementary energy with 11-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, E_{ij}, v_i and u_i

By using of the same methods as adopted to derive the functionals $\hat{H}_{11g}$ in equation (5.22) and the functional $\hat{\Pi}_{11g}$ in equation (5.23), we can obtain two functionals dependent on the 11-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, E_{ij}, v_i and u_i from the functional H_{5g} in equation (6.3) and the functional Π_{5g} in equation (6.4). These are

$$H_{11g}[\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i, E_{ij}, v_i, u_i] = \tilde{H}_{11g} + H_p + H_u, \tag{6.9}$$

$$\Pi_{11g}[\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i, E_{ij}, v_i, u_i] = \tilde{\Pi}_{11g} + \Pi_u + \Pi_p. \tag{6.10}$$

The functionals H_{11g} and Π_{11g} have the same 11-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, E_{ij}, v_i and u_i as well as the same variational stationary conditions. These stationary conditions contain the governing equations describing two time boundary -value problems in nonlinear elastodynamics as expressed in equations (2.1) with (2.3) and (2.6), (2.2), (2.4), (2.7), (2.8), (2.9) with (2.13), (2.10), (2.12), (2.14), (2.15), (2.17), (2.19), (2.20), (2.21) and (2.22). These two functionals satisfy the transformation relation

$$H_{11g} + \Pi_{11g} = 0. \tag{6.11}$$

6.6. Generalized variational principle of action of potential and complementary energy with 9-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i, and u_i

The replacement of $A(E_{ij})$ with $B(\sigma_{ij})$ and $T(v_i)$ with $R(p_i)$ in the functional H_{11g} expressed in equation (6.9) through equations (3.9) and (3.10) allows the following functional with 9-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i and u_i to be defined:

$$H_{9g}[\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i, u_i] = \tilde{H}_{9g} + H_p + H_u. \tag{6.12}$$

By the reverse process, it follows from the functional Π_{11g} in equation (6.10) that another functional with 9-arguments $\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i$, and u_i can be derived. This has the form

$$\Pi_{9g}[\tau_{ij}, \sigma_{ij}, \chi_{ij}, b_i, p_i, q_{ij}, D_{ij}, \omega_i, u_i] = \tilde{\Pi}_{9g} + \Pi_p + \Pi_u. \tag{6.13}$$

The functionals H_{9g} and Π_{9g} have the same 9-arguments τ_{ij}, σ_{ij}, χ_{ij}, b_i, p_i, q_{ij}, D_{ij}, ω_i and u_i as well as the same variational stationary conditions. These stationary conditions include the governing equations associated with two time boundary -value problems in elastodynamics as expressed in equations (2.1) with (2.3) and (2.6), (2.2), (2.4), (2.7), (2.9) with (2.13),(2.12), (2.14), (2.19), (2.20), (2.21), (2.22), (5.17). Therefore, these two functionals satisfy the following transformation relation

$$H_{9g} + \Pi_{9g} = 0. \tag{6.14}$$

7. A note about the principle of complementary energy in nonlinear elasticity

For problems of *static nonlinear elasticity* involving *dead loads*, the functional of action of complementary potential energy expressed in equation (6.2) reduces to the following functional of complementary energy

$$\Pi_1[\sigma_{ij}] = \int_\Omega [B(\sigma_{ij}) + \frac{1}{2}\sigma_{ij} u_{k,i} u_{k,j}] d\Omega - \int_{S_u} \hat{u}_i(\delta_{ik} + u_{i,k})\sigma_{kj}\nu_j ds. \tag{7.1}$$

This functional is subject to the constraint conditions

$$[(\delta_{ik} + u_{i,k})\sigma_{kj}]_{,j} + \hat{f}_i = 0, \quad x_i \epsilon \Omega; \qquad (\delta_{ik} + u_{i,k})\sigma_{kj}\nu_j = \hat{T}, \quad x_i \epsilon S_T, \tag{7.2}$$

and Euler equations

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}), \quad x_i \epsilon \Omega; \qquad u_i = \hat{u}_i, \quad x_i \epsilon S_u. \tag{7.3}$$

The functional in equation (7.1) reduces further to

$$\tilde{\Pi}_1[\sigma_{ij}] = \int_\Omega B(\sigma_{ij}) d\Omega - \int_{S_u} \hat{u}_i \sigma_{ij} \nu_j ds, \tag{7.4}$$

for the case of small displacement problems in linear elasticity, which in turn is the variational principle of complementary energy. In this simplified functional, there is only one argument σ_{ij}, but in the functional in equation (7.1) for the finite displacement case, the displacement gradient $u_{i,j}$ is also contained in the integral in addition to the stress σ_{ij}.

Many authors (see, for example, Guo 1980) have sought to determine whether finite displacement problems in nonlinear elasticity can also be described by a variational principle of complementary energy involving only one argument σ_{ij} analogous to the description of small displacement problems in linear elasticity. The problem remains mathematically unresolved. However, it is interesting to note that when the variational constraint equations in equations (7.2) are rewritten in the forms:

$$\sigma_{kj}u_{i,kj} + \sigma_{kj,j}u_{i,k} = -\sigma_{ij,j} - \hat{f}, \quad x_i \epsilon \Omega; \qquad u_{i,k}\sigma_{kj}\nu_j = \hat{T} - \sigma_{ij}\nu_j, \quad x_i \epsilon S_T, \tag{7.5}$$

they represent a system of first order differential equations involving the displacement gradient $u_{i,k}$ and stress σ_{ij}. Therefore, in principle the displacement gradient $u_{i,k}$ may be solved in terms of the stress σ_{ij} and on substituting this result into equation (7.1), the functional of complementary energy reduces to a form dependent only on the stress variable σ_{ij}. This discussion clearly indicates that the functional for finite displacement can also be expressed in terms of a single variable, i.e. the stress σ_{ij}.

As an illustration to this discussion, let us consider the simple example shown in figure 2. Here a straight rod of length $L = 1$ and of constant section area $B = 1$ is firmly held fixed at one end (i.e. the left end). It is loaded by a concentrated force $\hat{T}$ at the other end together with a distributed force $\hat{f}(x)$ acting along the length of the rod, i.e. $0 \leq x \leq 1$. The stress-strain relation of material in the rod is assumed given by

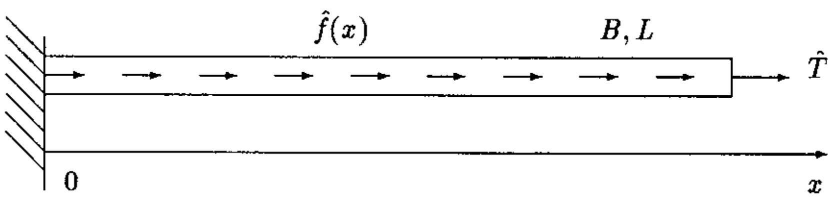

Figure 2. One-dimensional straight rod of length $L = 1$ and constant section area $B = 1$ loaded by a concentrated force $\hat{T}$ and a distributed body force $\hat{f}(x)$.

$$E = \begin{cases} \sigma^2 & \text{for extension} \\ -\sigma^2 & \text{for compression} \end{cases}, \tag{7.6}$$

where E is the Green strain and σ is the second Kirchhoff stress. The governing equations of the problem are

$$[(1 + u_{,x})\sigma]_{,x} = -\hat{f}, \qquad x \epsilon (0,1), \tag{7.7}$$

$$E = u_{,x} + \frac{1}{2}(u_{,x})^2, \qquad x \epsilon (0,1), \tag{7.8}$$

$$u = 0, \qquad x = 0, \tag{7.9}$$

$$(1 + u_{,x})\sigma = \hat{T}, \qquad x = 1, \tag{7.10}$$

together with equation (7.6). The function of density of complementary energy is

$$B(\sigma)=\begin{cases}\int_0^\sigma \sigma^2 d\sigma=\frac{1}{3}\sigma^3, & \text{for extension,}\\ \int_0^\sigma -\sigma^2 d\sigma=-\frac{1}{3}\sigma^3, & \text{for compression.}\end{cases} \tag{7.11}$$

The corresponding functional of complementary energy from equation (7.1) can be written as

$$\Pi_1[\sigma]=\int_0^1 [B(\sigma)+\frac{1}{2}\sigma(u_{,x})^2]dx, \tag{7.12}$$

subject to the constraint conditions in equations (7.7) and (7.10). From these latter equations we obtain the result

$$(1+u_{,x})\sigma=\hat{T}+\int_x^1 \hat{f}(x)dx=T(x), \qquad x \in (0,1), \tag{7.13}$$

allowing three possible conditions to exist. Namely,

(i) $T(x)>0$

$$u_{,x}=\frac{T(x)}{\sigma}-1, \qquad B(\sigma)=\frac{1}{3}\sigma^3; \tag{7.14}$$

(ii) $T(x)<0$

$$u_{,x}=\frac{T(x)}{\sigma}-1, \qquad B(\sigma)=-\frac{1}{3}\sigma^3; \tag{7.15}$$

(iii) $T(x)=0$

$$u_{,x}=\text{arbitrary}, \qquad \sigma=0. \tag{7.16}$$

For convenience let us assume

$$\hat{f}(x) \text{ and } T(x):\begin{cases}>0, & x_1<x\le 1\\ =0, & 0\le x\le x_1\end{cases}, \qquad \int_{x_1}^1 \hat{f}(x)dx+\hat{T}=0. \tag{7.17}$$

Substituting $\hat{f}(x)$ as expressed in equation (7.17), and the corresponding $T(x)$, $B(\sigma)$ and $u_{,x}$ given respectively in equations (7.14), (7.15-16) into equation (7.12), we obtain the functional of complementary energy principle. This is expressed in the form

$$\Pi_1[\sigma]=\int_{x_1}^1 (\frac{1}{3}\sigma^3+\frac{1}{2}\sigma+\frac{T^2}{2\sigma}-T)dx, \tag{7.18}$$

and depends only on the one variable σ. By examining the variation of this functional, the solution for stress is derived from the equation

$$\sigma^2-\frac{T(x)^2}{2\sigma^2}+\frac{1}{2}=0, \tag{7.19}$$

giving,

$$\sigma^2=\frac{1}{4}(\sqrt{1+8T(x)^2}-1), \quad x\epsilon[x_1,1]; \qquad \sigma=0, \quad x\epsilon[0,x_1], \tag{7.20}$$

and displacement

$$u(x)=\int_{x_1}^x \sqrt{1+\frac{1}{2}(\sqrt{1+8T(y)^2}-1)}dy, \quad x\epsilon[x_1,1]; \qquad u(x)=0, \quad x\epsilon[0,x_1]. \tag{7.21}$$

8. Discussion and an example of application

8.1. *Sum decomposition of displacement gradients* $u_{i,j}$

The displacement gradient $q_{ij} = u_{i,j}$ is not generally symmetric (that is, $u_{i,j} \neq u_{j,i}$) and in equations (2.11)-(2.14) this has been expressed as the sum of a symmetric tensor D_{ij} and a dual vector ω_i of an antisymmetric tensor W_{ij}. The generalized variational principles corresponding to expressions of gradient are given in equations (5.22), (5,23), (5.25), (5.26), (6.9), (6.12) and (6.13). They allow the introduction of simplifications into the mathematical model and hence create reductions in the subsequent numerical analyses. For example, from equations (2.8), (2.11) and (2.13), the Green strain tensor E_{ij} can be expressed as

$$E_{ij} = D_{ij} + \frac{1}{2}(D_{ki} - e_{kil}\omega_l)(D_{kj} - e_{kjr}\omega_r). \tag{8.1}$$

In the case of deformation of very flexible bodies, such as plates and shells, the quantities D_{ij} may frequently be assumed to be infinitesimal of first order but the components ω_i may be much larger (see, Novozhilov 1953). These assumptions allow the approximation

$$E_{ij} = D_{ij} + \frac{1}{2}(\delta_{ij}\omega_r\omega_r + \omega_i\omega_j). \tag{8.2}$$

Furthermore, if both D_{ij} and ω_i are inifinitesmals of first order and we neglect therefore their products and squares in comparison with their first order power, equation (8.2) reduces to

$$E_{ij} = D_{ij}. \tag{8.3}$$

Notice that in this case if $D_{ij} = 0$ (rigid motion), equation (8.3) becomes

$$E_{ij} = 0, \tag{8.4}$$

and

$$du_i = u_{i,j}dx_j = e_{ikj}\omega_k dx_j = \boldsymbol{\omega} \times \boldsymbol{dx}. \tag{8.5}$$

Thus, we obtain an infinitesimal displacement without strain by a rigid rotation of the line elements $\boldsymbol{dx}$.

8.2. *Other generalized variational principles deduced from the functionals presented*

By means of the lagrangian multiplier method to relax some constraints or to introduce additional constraints into the variational principles presented, other generalized variational principles may be obtained. For example, let us consider the generalized variational principle $\hat{H}_{11g}$ in equation (5.19). If the constraints in equations (2.8) with (2.11) and (2.13), (2.10), (2.20), (2.15) and (2.17) are included, a generalized variational principle with 6-arguments

$$\begin{aligned}\hat{H}_{6g}[\chi_{ij}, b_i, p_i, D_{ij}, \omega_i, u_i] = \int_{t_1}^{t_2} \{ \int_\Omega [A(D_{ij}, \omega_i) - T(v_i) + G(u_i) \\ - b_k(\omega_k + \frac{1}{2}e_{kij}u_{i,j}) - \chi_{ij}(D_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}))] d\Omega + \int_{S_T} Q(u_i) \} dt + \hat{H},\end{aligned} \tag{8.6}$$

can be derived. The stationary conditions applying to this functional are equations (2.12), (2.14), (2.1) with (2.3) and (2.6), (2.19 with (2.3) and (2.6), (2.27), (2.28). Further, if the relation in equation (2.12) is introduced into equation (8.6), we deduce a variational principle with 4 arguments of the form

$$\hat{H}_{4g}[b_i, p_i, \omega_i, u_i] = \int_{t_1}^{t_2} \{ \int_\Omega [A(\frac{1}{2}(u_{i,j} + u_{j,i}), \omega_i) - T(v_i) + G(u_i) \\ -b_k(\omega_k + \frac{1}{2}e_{kij}u_{i,j})]d\Omega + \int_{S_T} Q(u_i)\}dt + \hat{H}, \tag{8.7}$$

which reduces to the functional $\hat{H}_2$ corresponding to equation (5.1), if the constraint in equation (2.14) is introduced.

Furthermore from the formulations, sub-region or piecewise variational principles can be obtained if the time interval or the space domain or both are divided into elemental portions (see, Xing & Zheng 1992a). Based on these piecewise variational principles, a time element model or a four dimensional element model for the numerical analysis of nonlinear elastodynamic problems can be formulated.

This discussion shows that the generalized variational principles presented have a wide range of application and are adaptable for further development. They can be cast into forms providing frameworks in which effective and efficient numerical schemes of study can be devised.

8.3. An example of application

To illustrate the application of the variational principles developed in this paper, the one-dimensional straight rod shown in figure 2 is again considered by way of example. For simplicity, let us assume that the distributed force $\hat{f}(x)$ acting along the length of the rod is zero. Both the elastic modulus and the density of the rod's material equal 1. A final-value problem requiring that the displacement $\hat{u}_\alpha = x$ and the momentum $\hat{p}_\alpha = x$ of the rod at the final time $t_2 = t_\alpha$ is solved. The governing equations described in section 2 now take the following forms.

$$[(1 + u_{,x})\sigma]_{,x} = p_{,t}, \qquad (x,t) \in (0,1) \times (0,\alpha), \tag{8.8}$$

$$E = u_{,x} + \frac{1}{2}u_{,x}^2, \qquad v = u_{,t}, \qquad (x,t) \in (0,1) \times (0,\alpha), \tag{8.9}$$

$$\sigma = E, \quad p = v, \quad (x,t) \in (0,1) \times (0,\alpha), \tag{8.10}$$

$$(1 + u_{,x})\sigma = \hat{T}, \qquad x = 1,\ t \in [0,\alpha], \tag{8.11}$$

$$u = 0, \qquad x = 0,\ t \in [0,\alpha], \tag{8.12}$$

$$u = x = \hat{u}_\alpha, \qquad p = x = \hat{p}_\alpha, \qquad t = \alpha,\ x \in [0,1]. \tag{8.13}$$

Here, u_1 represents the displacement at the right end of the rod and the subscripts 0 and α represent the variables at the initial time $t_1 = 0$ and at the final time $t_2 = \alpha$, respectively. The functional described in equation (5.1) takes the form

$$\hat{H}_2[u,p] = \int_0^\alpha \{ \int_0^1 \frac{1}{2}(E^2 - v^2)dx - \hat{T}u_1\}dt \\ + \int_0^1 \frac{1}{2}[u_\alpha \hat{p}_\alpha + p_\alpha(u_\alpha - \hat{u}_\alpha)]dx + \int_0^1 g_0(u)dx. \tag{8.14}$$

The variation of this functional is represented as

$$\delta\hat{H}_2 = \int_0^\alpha \{\int_0^1 (\delta EE - \delta vv)dx - \delta u_1 \hat{T}\}dt + \int_0^1 \frac{1}{2}[\delta u_\alpha(p_\alpha + \hat{p}_\alpha) + \delta p_\alpha(u_\alpha - \hat{u}_\alpha)]dx - \int_0^1 \delta u_0 p_0 dx. \quad (8.15)$$

Now through the functional given in equation (8.14) or (8.15), a semi-analytical method (see, for example, Oden & Reddy 1976) can be developed to obtain an approximate solution. We shall assume that an approximate solution of this problem satisfying the constrain conditions of this functional, i.e. the strain-displacement relation the velocity-displacement relation in equation (8.9) and the displacement boundary condition in equation (8.12), is represented by

$$u = x\phi(t). \quad (8.16)$$

The associated variables corresponding to this assumed solution take the following forms:

$$E = \phi(t) + \frac{1}{2}\phi^2(t), \qquad p = v = x\phi'(t),$$
$$u_1 = \phi(t), \quad u_\alpha = x\phi(\alpha), \quad p_\alpha = x\phi'(\alpha), \quad u_0 = x\phi(0), \quad p_0 = x\phi'(0). \quad (8.17)$$

On substituting these equations into the functional described in equation (8.15), integrating x from 0 to 1 and through time integration by parts for the term of $\delta\phi'\phi'$, it follows that

$$\delta\hat{H}_2 = \int_0^\alpha \delta\phi[\phi(1 + \frac{3}{2}\phi + \frac{1}{2}\phi^2) + \frac{1}{3}\phi'' - \hat{T}]dt + \frac{1}{6}\delta\phi'(\alpha)[\phi(\alpha) - 1] - \frac{1}{6}\delta\phi(\alpha)[\phi'(\alpha) - 1]. \quad (8.18)$$

Furthermore, through the variation $\delta\hat{H}_2 = 0$, we derive the result that

$$\phi'' + 3\phi(1 + \frac{3}{2}\phi + \frac{1}{2}\phi^2) = 3\hat{T}, \qquad \phi(\alpha) = 1, \qquad \phi'(\alpha) = 1. \quad (8.19)$$

This equation is an ordinary differential equation derived from the partial differential governing equations describing nonlinear elastodynamic problems through the variational principles given in this paper. The solution of this equation for $\phi(t)$ provides an approximate solution of the final-value problem. The initial conditions required to reach the final-value conditions in equations (8.13) can be obtained from equations (8.17).

If a space-time element method (see, Xing & Price 1996) is adopted, by means of the functionals developed here a numerical matrix equation can be derived to obtain a numerical solution of nonlinear elastodynamic problems.

9. Conclusion

Our idea and approaches in the previous papers (Xing & Price 1992, 1996; Xing & Zheng 1992) have been successfully extended to the theory of nonlinear elastodynamics. General theorems and generalized variational principles are developed to solve initial-value, final-value and two time boundary-value dynamical problems in nonlinear elastodynamics .

By adopting the methods discussed and relaxing or adding constraints, we show that modified or additional functionals are created. Piecewise generalized variational principles to solve nonlinear elastodynamic problems can be generated from the presented mathematical models by dividing the time interval or space or both into smaller elements. The division of the displacement gradient into two parts, i.e. a symmetric part and a rotation vector corresponding to the antisymmetric part provides a means of simplifying the application of the mathematical model in nonlinear numerical analysis.

The examples presented show that the model developed is applicable to describe nonlinear elastostatic and elastodynamic problems. In the elastostatic problem, it is shown how a functional of the principle of complementary energy with one argument, i.e. the stress σ_{ij}, may be constructed. In a final-value elastodynamic problem, we illustrate how a semi-analytical method can be used to reduce the partial differential governing equations to an ordinary differential equation solvable using conventional methods.

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