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SHAPE FUNCTIONS OF THREE-DIMENSIONAL TIMOSHENKO BEAM ELEMENT

A. BAZOUNE AND Y. A. KHULIEF

Department of Mechanical Engineering, King Fahd University of Petroleum and Minerals, KFUPM BOX # 579, Dhahran 31261, Saudi Arabia. E-mail: abazoune@kfupm.edu.sa

AND

N. G. STEPHEN

School of Engineering Sciences, Mechanical Engineering, University of Southampton, Southampton SO17 1BJ, England

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1. INTRODUCTION

Beams represent fundamental structural components in many engineering applications, and shape functions are essential for the finite element discretization of such structures. Przemeniecki [1] derived explicit expressions for the shape functions of two-dimensional Timoshenko and three-dimensional Euler–Bernoulli (EB) beam elements. Note that for the three-dimensional EB element presented in reference [1], a change of sign is required in those entries of the third column of the shape function matrix which correspond to the twist terms. Since that pioneering work, there does not appear to have been any attempt to extend these results to a three-dimensional Timoshenko beam element, and it is the purpose of this note to fill this gap in the literature.

2. FINITE ELEMENT DISCRETIZATION

Consider a typical two-node three-dimensional beam element of length l, where each node has six degrees of freedom. The nodal displacement vector $\{\mathbf{e}\}$ defined with respect to the element axes is denoted by

$$\{\mathbf{e}\}_{12\times 1} = [u_1 \quad v_1 \quad w_1 \quad \theta_{x1} \quad \theta_{y1} \quad \theta_{z1} \quad u_2 \quad v_2 \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad \theta_{z2}]^{\mathrm{T}}, \tag{1}$$

where (u_1, u_2) are the nodal axial displacements in the x-direction, and (v_1, v_2) and (w_1, w_2) are the translational displacements in the y- and z-directions, respectively, $(\theta_{x1}, \theta_{x2})$ are the torsional displacements about the x-axis, and $(\theta_{y1}, \theta_{y2})$ and $(\theta_{z1}, \theta_{z2})$ are the rotational displacements in the (xz)- and (xy)-planes, respectively.

According to the standard finite element procedure, the elastic deformation of an arbitrary point of the beam element can be expressed as

$$\{\mathbf{d}\} = [\mathcal{N}]\{\mathbf{e}\}, \tag{2}$$

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where $\{\mathbf{d}\}$ represents the elastic deformation vector of the beam element and $[\mathcal{N}]$ is the matrix of shape functions used to model its deformations. Note that the shape functions are spatially dependent while the nodal displacement vector is time dependent. Equation (2) is quite general and is valid for any form of shape functions $[\mathcal{N}]$ used to model the beam elements. The shape functions used for translational and rotational bending deformation are the conventional cubic Hermitian polynomials that incorporate, in addition to the continuity and completeness conditions, shear deformation parameters that account for the effects of shear. The shape functions for torsional and axial deformation are linear, and are included for completeness.

3. THE DISPLACEMENT FIELD

Timoshenko beam theory (TBT) is applied when the cross-sectional dimensions of the beam are not small compared to its length and/or when higher bending modes are required. The kinematic relations for a three-dimensional beam undergoing axial, torsional and bending deformations in the (xy)- and (xz)-plane can be expressed as

$$\begin{aligned} U &= u - y\left(\frac{\partial v}{\partial x}\right) - z\left(\frac{\partial w}{\partial x}\right), \\ V &= -z\theta_x + v, \\ W &= y\theta_x + w, \end{aligned} \tag{3}$$

where the translations $(v,\ w)$ consist of contributions $(v_b,\ w_b)$ and $(v_s,\ w_s)$ due to bending and transverse shear, that is

$$v = v_b + v_s, \quad w = w_b + w_s. \tag{4,5}$$

The relationships between total slope, bending rotation and transverse shear are

$$\frac{\partial v}{\partial x} = \frac{\partial v_b}{\partial x} + \frac{\partial v_s}{\partial x} = \theta_z + \gamma_{xy}, \tag{6}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x} = -\theta_y + \gamma_{xz}, \tag{7}$$

where γ_{xy} and γ_{xz} are shear strains in the (xy)- and (xz)-planes, respectively. The two rotations $(\theta_y,\ \theta_z)$ are related to the bending deformations $(v_b,\ w_b)$ by the expressions

$$\theta_z = \frac{\partial v_b}{\partial x}, \quad \theta_y = -\frac{\partial w_b}{\partial x}. \tag{8,9}$$

Note that axial warping displacement during torsion is ignored.

4. DERIVATION OF SHAPE FUNCTIONS

Shape function matrices for axial and torsional deformation, $[\mathcal{N}_a]$ and $[\mathcal{N}_{\theta_x}]$, can be found in any elementary text, and are given by

$$[\mathcal{N}_a(\xi)] = [\mathcal{N}_{\theta_x}(\xi)] = [(1-\xi)\ \xi], \tag{10}$$

where $\xi = x/l$ is the dimensionless axial co-ordinate. Shape functions for bending deformation in the (xy)-plane are derived as follows: the translational deformation $v(x)$ at an arbitrary location x is expressed as

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{11}$$

or in matrix form as

$$v(x) = [\chi]\{\mathbf{a}_j\}, \tag{12}$$

where

$$[\chi] = [1 \quad x \quad x^2 \quad x^3] \tag{13}$$

and

$$\{\mathbf{a}_j\} = [a_0 \quad a_1 \quad a_2 \quad a_3]^{\mathrm{T}}. \tag{14}$$

The shear strain is assumed to be independent of the element axial co-ordinate x, in accordance with reference [2], i.e., constant along the finite element

$$\gamma_{xy} = \text{const.} = \gamma_0. \tag{15}$$

The bending moment M_z and the shearing force Q_y are related by

$$\frac{\mathrm{d}M_z}{\mathrm{d}x} - Q_y = 0 \tag{16}$$

and the moment–curvature relationship is

$$M_z = -EI_{zz}\frac{\partial\theta_z}{\partial x}, \tag{17}$$

where I_{zz} is the second moment of area about the z-axis; the shear force is related to the transverse shear strain by

$$Q_y = \kappa_y \; G \; A\gamma_{xy}. \tag{18}$$

In the above, κ_y is the shear correction factor that accounts for the non-uniform distribution of the shear stress over the cross-section A, E is the modulus of elasticity, and G is the shear modulus. The slope due to bending can be obtained by using equations (6), (13) and (15), that is

$$\theta_z = a_1 + 2a_2x + 3a_3x^2 - \gamma_0. \tag{19}$$

Taking the derivative of θ_z with respect to x and substituting it into equation (17) yields

$$M_z = -EI_{zz}(2a_2 + 6a_3x). \tag{20}$$

Taking the derivative of M_z with respect to x and substituting into equation (16) along with equations (15) and (18) yields

$$-6EI_{zz}a_3 - \kappa_y GA\gamma_0 = 0, \tag{21}$$

from which

$$\gamma_0 = -6(\frac{EI_{zz}}{\kappa_y GA})a_3 = -6\Lambda_z \; a_3, \tag{22}$$

where

$$\Lambda_z = \frac{EI_{zz}}{\kappa_y GA}. \tag{23}$$

Substitute equation (23) into the expression for θ_z, to give

$$\theta_z = a_1 + 2a_2x + (3x^2 + 6\Lambda_z)a_3. \tag{24}$$

To express the coefficients a_j in terms of the bending deformations and slopes, the following boundary conditions must be satisfied:

$$\begin{aligned} v(0) &= v_1 \quad \text{and} \quad v(l) = v_2, \\ \theta_z(0) &= \theta_{z1} \quad \text{and} \quad \theta_z(l) = \theta_{z2} \end{aligned} \tag{25}$$

and applying these to equations (11) and (24) gives

$$\begin{aligned} v(0) &= a_0 &&= v_1, \\ \theta_z(0) &= a_1 + 6\Lambda_z a_3 &&= \theta_{z1}, \\ v(l) &= a_0 + a_1 l + a_2 l^2 + a_3 l^3 &&= v_2, \\ \theta_z(l) &= a_1 + 2a_2 l + (3l^2 + 6\Lambda_z)a_3 &&= \theta_{z2}. \end{aligned} \tag{26}$$

In matrix form, this can be written as

$$\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6\Lambda_z \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & (3l^2 + 6\Lambda_z) \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{27}$$

or in more compact form

$$\{\mathbf{d}\} = [\mathsf{A}]\{\mathbf{a}_j\}, \tag{28}$$

from which

$$\{\mathbf{a}_j\} = [\mathsf{A}]^{-1}\{\mathbf{d}\}. \tag{29}$$

Solving for $\{\mathbf{a}_j\}$ gives

$$a_0 = v_1, \tag{30}$$

$$a_1 = \bar{\Phi}_z\left(-\frac{1}{l}\Phi_z v_1 + \left(1 + \frac{\Phi_z}{2}\right)\theta_{z1} + \frac{1}{l}\Phi_z v_2 - \frac{\Phi_z}{2}\theta_{z2}\right), \tag{31}$$

$$a_2 = \bar{\Phi}_z\left(-\frac{3v_1}{l^2} - \frac{1}{l}\left(2 + \frac{\Phi_z}{2}\right)\theta_{z1} + \frac{3v_2}{l^2} - \frac{1}{l}\left(1 - \frac{\Phi_z}{2}\right)\theta_{z2}\right), \tag{32}$$

$$a_3 = \bar{\Phi}_z\left(\frac{2v_1}{l^3} + \frac{\theta_{z1}}{l^2} - \frac{2v_2}{l^3} + \frac{\theta_{z2}}{l^2}\right), \tag{33}$$

where

$$\bar{\Phi}_z = \frac{1}{(1 + \Phi_z)} \tag{34}$$

and

$$\Phi_z = \frac{12\Lambda_z}{l^2} = \frac{12EI_{zz}}{\kappa_y G A l^2} \tag{35}$$

is the shear deformation parameter that represents the ratio between bending and shear stiffnesses. Substituting the values of a_j into the expression of $v(\xi = x/l)$ and simplifying, one obtains

$$v(\xi) = \bar{\Phi}_z(1 - 3\xi^2 + 2\xi^3 + \Phi_z(1-\xi))v_1 + l\bar{\Phi}_z\left(\xi - 2\xi^2 + \xi^3 + \frac{\Phi_z}{2}(\xi - \xi^2)\right)\theta_{z1} + \bar{\Phi}_z(3\xi^2 - 2\xi^3 + \Phi_z\xi)v_2 + l\bar{\Phi}_z\left(-\xi^2 + \xi^3 + \frac{\Phi_z}{2}(-\xi + \xi^2)\right)\theta_{z2}. \quad (36)$$

Hence, $v(\xi)$ can be written in the form

$$v(\xi) = \mathcal{N}_{v_1}v_1 + \mathcal{N}_{v_2}\mu_{z1} + \mathcal{N}_{v_3}v_2 + \mathcal{N}_{v_4}\mu_{z2}, \quad (37)$$

where

$$\begin{aligned}
\mathcal{N}_{v_1} &= \bar{\Phi}_z(1 - 3\xi^2 + 2\xi^3 + \Phi_z(1-\xi)),\\
\mathcal{N}_{v_2} &= l\bar{\Phi}_z\left(\xi - 2\xi^2 + \xi^3 + \frac{\Phi_z}{2}(\xi - \xi^2)\right),\\
\mathcal{N}_{v_3} &= \bar{\Phi}_z(3\xi^2 - 2\xi^3 + \Phi_z\xi),\\
\mathcal{N}_{v_4} &= l\bar{\Phi}_z\left(-\xi^2 + \xi^3 + \frac{\Phi_z}{2}(-\xi + \xi^2)\right).
\end{aligned} \quad (38)$$

Similarly, substitute a_j into the equation for $\theta_z(\xi)$ to get

$$\begin{aligned}
\theta_z(\xi) &= \frac{6\bar{\Phi}_z}{l}(-\xi + \xi^2)v_1 + \bar{\Phi}_z(1 - 4\xi + 3\xi^2 + \Phi_z(1-\xi))\theta_{z1}\\
&\quad + \frac{6\bar{\Phi}_z}{l}(\xi - \xi^2)v_2 + \bar{\Phi}_z(-2\xi + 3\xi^2 + \Phi_z\xi)\theta_{z2}.
\end{aligned} \quad (39)$$

Hence, $\theta_z(\xi)$ can be written in the form

$$\mu_z(\xi) = \mathcal{N}_{\mu_{z1}}v_1 + \mathcal{N}_{\mu_{z2}}\mu_{z1} + \mathcal{N}_{\mu_{z3}}v_2 + \mathcal{N}_{\mu_{z4}}\mu_{z2}, \quad (40)$$

where

$$\begin{aligned}
\mathcal{N}_{\mu_{z1}} &= \frac{6\bar{\Phi}_z}{l}(-\xi + \xi^2),\\
\mathcal{N}_{\mu_{z2}} &= \bar{\Phi}_z(1 - 4\xi + 3\xi^2 + \Phi_z(1-\xi)),\\
\mathcal{N}_{\mu_{z3}} &= \frac{-6\bar{\Phi}_z}{l}(-\xi + \xi^2),\\
\mathcal{N}_{\mu_{z4}} &= \bar{\Phi}_z(-2\xi + 3\xi^2 + \Phi_z\xi).
\end{aligned} \quad (41)$$

Shape functions for bending in the (xz)-plane are obtained in a similar manner; the bending slope θ_y is given by equation (7) while the shear deformation parameter is

$$\Phi_y = \frac{12EI_{yy}}{\kappa_z GAl^2} \quad (42)$$

and

$$\bar{\Phi}_y = \frac{1}{(1+\Phi_y)}. \tag{43}$$

The shape functions corresponding to bending in the (xz)-plane can then be written as

$$\begin{aligned}
\mathcal{N}_{w_1} &= \bar{\Phi}_y(1-3\xi^2+2\xi^3+\Phi_y(1-\xi)),\\
\mathcal{N}_{w_2} &= -l\bar{\Phi}_y(\xi-2\xi^2+\xi^3+\frac{\Phi_y}{2}(\xi-\xi^2)),\\
\mathcal{N}_{w_3} &= \bar{\Phi}_y(3\xi^2-2\xi^3+\Phi_y\xi),\\
\mathcal{N}_{w_4} &= -l\bar{\Phi}_y(-\xi^2+\xi^3+\frac{\Phi_y}{2}(-\xi+\xi^2))
\end{aligned} \tag{44}$$

and the corresponding bending slope shape functions are

$$\begin{aligned}
\mathcal{N}_{\mu_{y1}} &= \frac{6\bar{\Phi}_y}{l}(-\xi+\xi^2),\\
\mathcal{N}_{\mu_{y2}} &= -\bar{\Phi}_y(1-4\xi+3\xi^2+\Phi_y(1-\xi)),\\
\mathcal{N}_{\mu_{y3}} &= \frac{-6\bar{\Phi}_y}{l}(-\xi+\xi^2),\\
\mathcal{N}_{\mu_{y4}} &= -\bar{\Phi}_y(-2\xi+3\xi^2+\Phi_y\xi).
\end{aligned} \tag{45}$$

By virtue of equations (10), (38), (41), (44) and (45), the kinematic relations given by equation (3) are now expressed as

$$\begin{aligned}
U = {} & (1-\xi)u_1 - 6\bar{\Phi}_z(-\xi+\xi^2)\eta v_1 - 6\bar{\Phi}_y(-\xi+\xi^2)\zeta w_1\\
&+ l\bar{\Phi}_y(1-4\xi+3\xi^2+\Phi_y(1-\xi))\zeta\theta_{y1} - l\bar{\Phi}_z(1-4\xi+3\xi^2+\Phi_z(1-\xi))\eta\theta_{z1}\\
&+ \xi u_2 - 6\bar{\Phi}_z(\xi-\xi^2)\eta v_2 - 6\bar{\Phi}_y(\xi-\xi^2)\zeta w_2\\
&+ l\bar{\Phi}_y(-2\xi+3\xi^2+\Phi_y\xi)\zeta\theta_{y2} - l\bar{\Phi}_z(-2\xi+3\xi^2+\Phi_z\xi)\eta\theta_{z2},
\end{aligned}$$

$$\begin{aligned}
V = {} & \bar{\Phi}_z(1-3\xi^2+2\xi^3+\Phi_z(1-\xi))v_1\\
&- l\zeta(1-\xi)\theta_{x1} + l\bar{\Phi}_z\left(\xi-2\xi^2+\xi^3+\frac{\Phi_z}{2}(\xi-\xi^2)\right)\theta_{z1}\\
&+ \bar{\Phi}_z(3\xi^2-2\xi^3+\Phi_z\xi)v_2 - l\zeta\xi\theta_{x2} + l\bar{\Phi}_z\left(-\xi^2+\xi^3+\frac{\Phi_z}{2}(-\xi+\xi^2)\right)\theta_{z2},
\end{aligned}$$

$$\begin{aligned}
W = {} & \bar{\Phi}_y(1-3\xi^2+2\xi^3+\Phi_y(1-\xi))w_1\\
&+ l\eta(1-\xi)\theta_{x1} - l\bar{\Phi}_y\left(\xi-2\xi^2+\xi^3+\frac{\Phi_y}{2}(\xi-\xi^2)\right)\theta_{y1}\\
&+ \bar{\Phi}_y(3\xi^2-2\xi^3+\Phi_y\xi)w_2 + l\eta\xi\theta_{x2} - l\bar{\Phi}_y\left(-\xi^2+\xi^3+\frac{\Phi_y}{2}(-\xi+\xi^2)\right)\theta_{y2},
\end{aligned} \tag{46}$$

where $\eta = y/l$ and $\zeta = z/l$ are dimensionless co-ordinates in the y- and z-directions respectively. In matrix form, this can be written as

$$\{\mathbf{d}\}_{3\times 1} = \begin{bmatrix} U & V & W \end{bmatrix}^{\mathrm{T}} = [\mathcal{N}]_{3\times 12}\{\mathbf{e}\}_{12\times 1} \tag{47}$$

These results are summarized in the matrix of the shape functions, $[\mathcal{N}]$, shown in Appendix A. If the shear deformation parameters Φ_y and Φ_z are neglected, then $[\mathcal{N}]$ reduces to the three-dimensional Euler–Bernoulli beam shape function derived in reference [1], where a sign change is required in the fourth and tenth entries of the third column of the shape function matrix, which correspond to the twist terms.

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APPENDIX A

Shape function matrix for the three-dimensional Timoshenko beam element:

$$
[\mathcal{N}]^{\mathrm{T}} = \begin{bmatrix}
(1-\xi) & 0 & 0 \\
6\bar{\Phi}_z(\xi-\xi^2)\eta & \bar{\Phi}_z(1-3\xi^2+2\xi^3+\Phi_z(1-\xi)) & 0 \\
6\bar{\Phi}_y(\xi-\xi^2)\zeta & 0 & \bar{\Phi}_y(1-3\xi^2+2\xi^3+\Phi_y(1-\xi)) \\
0 & -(1-\xi)l\zeta & (1-\xi)l\eta \\
l\bar{\Phi}_y(1-4\xi+3\xi^2+\Phi_y(1-\xi))\zeta & 0 & -l\bar{\Phi}_y\left(\xi-2\xi^2+\xi^3+\frac{\Phi_y}{2}(\xi-\xi^2)\right) \\
-l\bar{\Phi}_z(1-4\xi+3\xi^2+\Phi_z(1-\xi))\eta & l\bar{\Phi}_z\left(\xi-2\xi^2+\xi^3+\frac{\Phi_z}{2}(\xi-\xi^2)\right) & 0 \\
\xi & 0 & 0 \\
6\bar{\Phi}_z(-\xi+\xi^2)\eta & \bar{\Phi}_z(3\xi^2-2\xi^3+\Phi_z\xi) & 0 \\
6\bar{\Phi}_y(-\xi+\xi^2)\zeta & 0 & \bar{\Phi}_y(3\xi^2-2\xi^3+\Phi_y\xi) \\
0 & -\xi l\zeta & \xi l\eta \\
l\bar{\Phi}_y(-2\xi+3\xi^2+\Phi_y\xi)\zeta & 0 & -l\bar{\Phi}_y\left(-\xi^2+\xi^3-\frac{\Phi_y}{2}(\xi-\xi^2)\right) \\
-l\bar{\Phi}_z(-2\xi+3\xi^2+\Phi_z\xi)\eta & l\bar{\Phi}_z\left(-\xi^2+\xi^3-\frac{\Phi_z}{2}(\xi-\xi^2)\right) & 0
\end{bmatrix},
$$

where $\bar{\Phi}_y = 1/(1+\Phi_y)$ and $\bar{\Phi}_z = 1/(1+\Phi_z)$.