Second-Order Theories for Extensional Vibrations of Piezoelectric Crystal Plates and Strips

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***Abstract*—An infinite system of two-dimensional (2-D) equations for piezoelectric plates with general symmetry and faces in contact with vacuum is derived from the 3-D equations of linear piezoelectricity in a manner similar to that of previous work, in which an infinite system of 2-D equations for plates with electroded faces was derived.**

By using a new truncation procedure, second-order equations for piezoelectric plates with faces in contact with either vacuums or electrodes are extracted from the aforementioned infinite systems of equations, respectively. The second-order equations for plates with or without electrodes are shown to predict accurate dispersion curves by comparing to the corresponding curves from the 3-D equations in a range up to the cut-off frequencies of the first symmetric thickness-stretch and the second symmetric thickness-shear modes without introducing any correction factors. Furthermore, a system of 1-D second-order equations for strips with rectangular cross section is deduced from the 2-D second-order equations by averaging variables across the narrow width of the plate. The present 1-D equations are used to study the extensional vibrations of barium titanate strips of finite length and narrow rectangular cross section. Predicted frequency spectra are compared with previously calculated results and experimental data.

I. Introduction

In a previous paper [1], a new term representing the in-plane displacements that vary linearly through the thickness (x_2 direction) was included with the infinite series expansion of cosine functions of the thickness coordinate. This term, induced by the gradients of the transverse displacement, improves the dispersion curve for the flexure mode of piezoelectric plates, especially at low frequencies. The first-order equations, including thickness-shear, flexure, and face-shear varying in the x_1 direction, and thickness-twist and face-shear varying in the x_3 direction predict frequency spectra that agrees closely with experimental results by Koga and Fukuyo [2] and Nakazawa *et al.* [3]. In the present paper, the same expansion is adopted for the mechanical displacements. The electric potential is expanded in a series of cosine functions that easily accommodates the specified electrical charge at electroded faces, and a sine series expansion of potential was chosen for plates with electroded faces in [1].

The truncation procedure used for extracting a finite number of equations from the infinite system directly affects the accuracy of the extracted equations in predicting dispersion relations or the frequency-wave number relations of straight-crested waves in an infinite plate. A new truncation procedure for extracting the second-order equations has been used by setting the next two higher-order stresses $\overline{T}_{2j}^{(n)}$, $n = 3, 4$ and $j = 1, 2, 3$, to zero, which allows the free development of the corresponding displacements $u_j^{(n)}$, $n = 3, 4$ and $j = 1, 2, 3$, and does not require them equal to zero. By this new procedure, 2-D second-order equations for plates with faces in contact with either vacuum or electrodes are extracted from the presently derived infinite system of equations and from those derived in [1], respectively. These second-order plate equations are shown to predict accurate dispersion curves for plates with or without electrodes by comparison with those from the 3-D equations for frequencies up to 2.5 times the cut-off frequency of the fundamental thickness-shear mode, without introducing any correction factors. In previous governing equations of vibrations of plates, such as those by Mindlin and Medick [4] and by Lee *et al.* [5], correction factors were introduced in order to improve the accuracy of predicted dispersion relations.

Furthermore, a system of 1-D second-order equations for the vibrations of a piezoelectric strip with narrow rectangular cross section is deduced from the 2-D second-order equations by averaging the plate variables across the narrow width of the plate as in [6]. Closed form solutions are obtained from the 1-D equations for the vibrations of in-plane symmetric modes in barium titanate strips of finite length and narrow cross section. Distribution of displacements for predominantly extensional (Et), edge (Eg), and thickness-stretch (TSt) modes are computed and plotted. Resonance frequency vs. length-to-width ratio of the strips are predicted and compared with the experimental data by Onoe and Pao [7] and calculated results by Medick and Pao [8].

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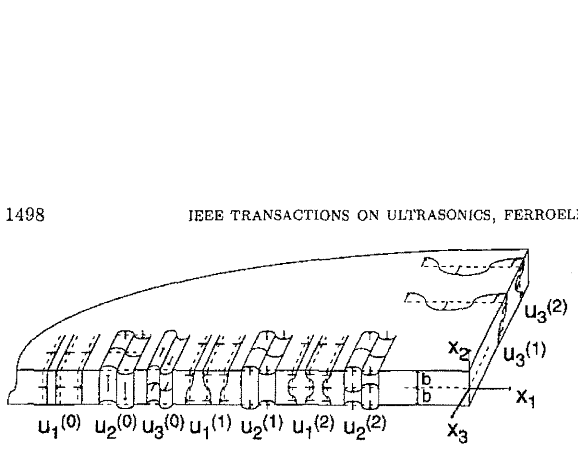

Fig. 1. Uniform plate and the first nine components of displacement of straight-crested waves in the x_1 direction.

II. 2-D Equations for Plates without Electrodes

A plate, referred to the rectangular coordinates x_i, as shown in Fig. 1, has faces of area A at $x_2 = \pm b$ and x_1 and x_3 coordinates of the middle plane.

The 3-D linear equations of piezoelectricity comprise the stress equations of motion and charge equation of electrostatics:

$$T_{ij,i} = \rho \ddot{u}_j, \quad D_{i,i} = 0, \tag{1}$$

the constitutive relations for linear piezoelectric materials:

$$\begin{aligned} T_{ij} &= c_{ijkl} S_{kl} - e_{kij} E_k, \\ D_i &= e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \end{aligned} \tag{2}$$

and the strain-displacement and field-potential relations:

$$\begin{aligned} S_{ij} &= \tfrac{1}{2}\left(u_{j,i} + u_{i,j}\right), \\ E_i &= -\varphi_{,i}. \end{aligned} \tag{3}$$

In the above, T_{ij}, S_{ij}, u_i, D_i, and E_i are the components of stress, strain, mechanical displacement, electric displacement, and electric field, respectively, and φ is the electric potential. The material properties are the elastic stiffness coefficients c_{ijkl}, the piezoelectric strain constants e_{ijk}, the dielectric permittivity ε_{ij}, and the mass density ρ.

For a piezoelectric crystal of volume V bounded by a surface S, the divergence in (1) can be replaced by the variational equation:

$$\int_{t_0}^{t_1} dt \int_V \left[(T_{ij,i} - \rho \ddot{u}_j)\,\delta u_j + D_{i,i}\delta\varphi\right] dV = 0, \tag{4}$$

which comes directly from the variational principle [9].

To derive the 2-D equations, the components of mechanical displacement and electric potential are expanded in an infinite series of trigonometrical functions of the thickness coordinate as follows:

$$u_j(x_1, x_2, x_3, t) = -u^{(0)}_{2,j} x_2 + \sum_{n=0}^{\infty} u^{(n)}_j \cos\frac{n\pi}{2}(1-\eta), \tag{5a}$$

$$\varphi(x_1, x_2, x_3, t) = \sum_{n=0}^{\infty} \varphi^{(n)} \cos\frac{n\pi}{2}(1-\eta), \tag{5b}$$

where $u^{(n)}_j = u^{(n)}_j(x_1, x_3, t)$ and $\varphi^{(n)} = \varphi^{(n)}(x_1, x_3, t)$ are the 2-D nth-order components of displacement and potential, and $\eta = \frac{x_2}{b}$. The expansion (5a) were shown to produce accurate dispersion curves and frequency spectra in a previous paper [1]. In (5b) the cosine series allows for the easy satisfaction of charge-free faces [5].

We substitute the expansions (5) into the gradient equations (3) to obtain a series expansion of strain and electric field:

$$\begin{aligned} S_{ij} &= S^d_{ij}\eta + \sum_{n=0}^{\infty}\left[S^{(n)}_{ij}\cos\frac{n\pi}{2}(1-\eta) + \overline{S}^{(n)}_{ij}\sin\frac{n\pi}{2}(1-\eta)\right], \\ E_i &= \sum_{n=0}^{\infty}\left[E^{(n)}_i\cos\frac{n\pi}{2}(1-\eta) + \overline{E}^{(n)}_i\sin\frac{n\pi}{2}(1-\eta)\right], \end{aligned} \tag{6}$$

where the 2-D nth-order components of strain and electric field are given by:

$$S^d_{ij} = -b u^{(0)}_{2,ij}, \tag{7a}$$

$$S^{(n)}_{ij} = \tfrac{1}{2}\left[u^{(n)}_{i,j} + u^{(n)}_{j,i} - \delta_{n0}\left(\delta_{2i}u^{(0)}_{2,j} + \delta_{2j}u^{(0)}_{2,i}\right)\right], \tag{7b}$$

$$\overline{S}^{(n)}_{ij} = \frac{n\pi}{4b}\left(\delta_{2j}u^{(n)}_i + \delta_{2i}u^{(n)}_j\right), \tag{7c}$$

$$E^{(n)}_i = -\varphi^{(n)}_{,i}, \tag{7d}$$

$$\overline{E}^{(n)}_i = -\delta_{2i}\frac{n\pi}{2b}\varphi^{(n)}, \tag{7e}$$

where δ_{ij} is the Kronecker delta. By substituting the expansions (5) into the variational (4), setting $dV = dx_2 dA$, and integrating through the thickness x_2 from $-$b to b in a manner similar to that of [1], we obtain:

$$\begin{aligned} & T^{(0)}_{ij,i} + \frac{1}{b}F^{(0)}_j = 2\rho\ddot{u}^{(0)}_j, \\ & T^{(n)}_{ij,i} - \frac{n\pi}{2b}\overline{T}^{(n)}_{2j} + \frac{1}{b}F^{(n)}_j = \rho\ddot{u}^{(n)}_j - \rho b c_n \ddot{u}^{(0)}_{2,j} \\ & \qquad n \geq 1, \\ & D^{(n)}_{i,i} - \frac{n\pi}{2b}\overline{D}^{(n)}_2 + \frac{1}{b}D^{(n)} = 0 \\ & \qquad n \geq 0, \end{aligned} \tag{8}$$

where components of nth-order stress, electric displacement, face traction, and face charge are defined, respectively, by:

$$T^{(n)}_{ij} = \int_{-1}^{1} T_{ij}\cos\frac{n\pi}{2}(1-\eta)d\eta, \tag{9a}$$

$$\overline{T}^{(n)}_{ij} = \int_{-1}^{1} T_{ij}\sin\frac{n\pi}{2}(1-\eta)d\eta, \tag{9b}$$

$$D^{(n)}_i = \int_{-1}^{1} D_i\cos\frac{n\pi}{2}(1-\eta)d\eta, \tag{9c}$$

$$\overline{D}^{(n)}_i = \int_{-1}^{1} D_i\sin\frac{n\pi}{2}(1-\eta)d\eta, \tag{9d}$$

$$F^{(n)}_j = T_{2j}(b) - (-1)^n T_{2j}(-b), \tag{9e}$$

$$D^{(n)} = D_2(b) - (-1)^n D_2(-b). \tag{9f}$$

The 2-D constitutive relations are obtained by inserting

(6) into (2) and then the result into (9a)–(9d):

$$T_{ij}^{(n)} = c_n c_{ijkl} S_{kl}^d + (1+\delta_{(n)})(c_{ijkl} S_{kl}^{(n)} - e_{kij} E_k^{(n)}) + \sum_{m=0}^{\infty} B_{mn}(c_{ijkl}\overline{S}_{kl}^{(m)} - e_{kij}\overline{E}_k^{(m)}), \quad (10a)$$

$$\overline{T}_{ij}^{(n)} = s_n c_{ijkl} S_{kl}^d + (1-\delta_{n0})(c_{ijkl}\overline{S}_{kl}^{(n)} - e_{kij}\overline{E}_k^{(n)}) + \sum_{m=0}^{\infty} B_{nm}(c_{ijkl} S_{kl}^{(m)} - e_{kij} E_k^{(m)}), \quad (10b)$$

$$D_i^{(n)} = c_n e_{ikl} S_{kl}^d + (1+\delta_{n0})(e_{ikl} S_{kl}^{(n)} + \varepsilon_{ik} E_k^{(n)}) + \sum_{m=0}^{\infty} B_{mn}(e_{ikl}\overline{S}_{kl}^{(m)} + \varepsilon_{ik}\overline{E}_k^{(m)}), \quad (10c)$$

$$\overline{D}_i^{(n)} = s_n e_{ikl} S_{kl}^d + (1-\delta_{n0})(e_{ikl}\overline{S}_{kl}^{(n)} + \varepsilon_{ik}\overline{E}_k^{(n)}) + \sum_{m=0}^{\infty} B_{nm}(e_{ikl} S_{kl}^{(m)} + \varepsilon_{ik} E_k^{(m)}), \quad (10d)$$

where the integration constants are:

$$B_{mn} = \int_{-1}^{1} \sin\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)d\psi = \begin{cases} \frac{4m}{(m^2-n^2)\pi} & m+n \text{ odd} \\ 0 & m+n \text{ even} \end{cases},$$

$$c_n = \int_{-1}^{1} \psi\cos\frac{n\pi}{2}(1-\psi)d\psi = \begin{cases} \frac{8}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}, \quad (11)$$

$$s_n = \int_{-1}^{1} \psi\sin\frac{n\pi}{2}(1-\psi)d\psi = \begin{cases} \frac{4}{n\pi} & n = 2, 4, 6, \ldots \\ 0 & n = 0, 1, 3, 5, \ldots \end{cases}.$$

The (7), (8), and (10) form an infinite system of 2-D equations replacing the 3-D (1)–(3).

III. Second-Order Equations for Plates without Electrodes

From the infinite system of Section II, a finite set of approximate equations are extracted to form the second-order equations for piezoelectric crystal plates in which the zero, first-, and second-order displacements and potentials are kept and the higher order displacements and potentials are eliminated by a new truncation procedure as follows. We let:

$$u_j^{(n)} = 0, \qquad n > 4 \quad (12a)$$

$$\varphi^{(n)} = 0, \qquad n > 2 \quad (12b)$$

$$\overline{T}_{j2}^{(n)} = 0, \quad \ddot{u}_j^{(n)} = 0, \text{ but } u_j^{(n)} \neq 0 \quad n = 3, 4 \quad (12c)$$

and disregard $T_{ij}^{(n)}$, $\overline{T}_{ij}^{(n)}$, $D_i^{(n)}$, $\overline{D}_i^{(n)}$, and $S_{ij}^{(n)}$ for $n > 2$. The idea is to allow the next two higher-order displacements $u_j^{(n)}$, $n = 3, 4$ to develop freely by setting the corresponding stress components $\overline{T}_{2j}^{(n)}$ to zero for $n = 3, 4$. By substituting (10b) into (12c) and solving for the six components of strain or displacement

$$\overline{S}_{2j}^{(n)} = \frac{n\pi}{2b}(1+\delta_{2j})u_j^{(n)}, \quad j = 1, 2, 3 \text{ and } n = 3, 4, \quad (13)$$

we have

$$\overline{S}_{2j}^{(3)} = (1+\delta_{2\underline{j}})\frac{2}{\pi}\left(-\frac{1}{3}\frac{c_{2jkl}}{c_{2\underline{j}2\underline{j}}}S_{kl}^{(0)} - \frac{3}{5}\frac{c_{2jkl}}{c_{2\underline{j}2\underline{j}}}S_{kl}^{(2)} + \frac{1}{3}\frac{e_{k2\underline{j}}}{c_{2\underline{j}2\underline{j}}}E_k^{(0)} + \frac{3}{5}\frac{e_{k2\underline{j}}}{c_{2\underline{j}2\underline{j}}}E_k^{(2)}\right), \quad (14a)$$

$$\overline{S}_{2j}^{(4)} = (1+\delta_{2\underline{j}})\frac{1}{\pi}\left(-\frac{1}{2}\frac{c_{2jkl}}{c_{2\underline{j}2\underline{j}}}S_{kl}^{d} - \frac{8}{15}\frac{c_{2jkl}}{c_{2\underline{j}2\underline{j}}}S_{kl}^{(1)} + \frac{8}{15}\frac{e_{k2\underline{j}}}{c_{2\underline{j}2\underline{j}}}E_k^{(1)}\right), \quad (14b)$$

where there is no sum over j. We note that, in solving the first three simultaneous equations for $\overline{S}_{2j}^{(3)}$ of (14a), we have approximated the terms $c_{2jkl}\overline{S}_{kl}^{(3)}$ by the terms $2c_{2121}\overline{S}_{21}^{(3)}$ for $j = 1$, $c_{2222}\overline{S}_{22}^{(3)}$ for $j = 2$, and $2c_{2323}\overline{S}_{23}^{(3)}$ for $j = 3$, i.e., the terms involving the product of the third-order strains with off-diagonal elastic stiffness are neglected, such as $c_{2122}\overline{S}_{22}^{(3)}$, $2c_{2123}\overline{S}_{23}^{(3)}$, etc. In solving the second three simultaneous equations for $\overline{S}_{2j}^{(4)}$ of (14b), a similar approximation is made for the terms $c_{2jkl}\overline{S}_{kl}^{(4)}$. By substituting equations (14) into the remaining relations in (10) and neglecting other third- and fourth-order strains, the constitutive relations for the second-order theory are obtained:

$$T_{ij}^{(0)} = 2c_{ijkl}^{(0)}S_{kl}^{(0)} + \frac{4}{\pi}c_{ijkl}\overline{S}_{kl}^{(1)} - c_{ijkl}^{(3)}S_{kl}^{(2)} - 2e_{kij}^{(0)}E_k^{(0)} - \frac{4}{\pi}e_{kij}\overline{E}_k^{(1)} + e_{kij}^{(3)}E_k^{(2)}, \quad (15a)$$

$$T_{ij}^{(1)} = \frac{8}{\pi^2}c_{ijkl}^{(d)}S_{kl}^d + c_{ijkl}^{(1)}S_{kl}^{(1)} + \frac{8}{3\pi}c_{ijkl}\overline{S}_{kl}^{(2)} - e_{kij}^{(1)}E_k^{(1)} - \frac{8}{3\pi}e_{kij}\overline{E}_k^{(2)}, \quad (15b)$$

$$T_{ij}^{(2)} = -c_{ijkl}^{(3)}S_{kl}^{(0)} - \frac{4}{3\pi}c_{ijkl}\overline{S}_{kl}^{(1)} + c_{ijkl}^{(2)}S_{kl}^{(2)} + e_{kij}^{(3)}E_k^{(0)} + \frac{4}{3\pi}e_{kij}\overline{E}_k^{(1)} - e_{kij}^{(2)}E_k^{(2)}, \quad (15c)$$

$$\begin{aligned}\overline{T}_{ij}^{(1)} &= \frac{4}{\pi}c_{ijkl}S_{kl}^{(0)} + c_{ijkl}\overline{S}_{kl}^{(1)} - \frac{4}{3\pi}c_{ijkl}S_{kl}^{(2)} \\ &\quad - \frac{4}{\pi}e_{kij}E_k^{(0)} - e_{kij}\overline{E}_k^{(1)} + \frac{4}{3\pi}e_{kij}E_k^{(2)},\end{aligned} \tag{15d}$$

$$\begin{aligned}\overline{T}_{ij}^{(2)} &= \frac{2}{\pi}c_{ijkl}S_{kl}^{d} + \frac{8}{3\pi}c_{ijkl}S_{kl}^{(1)} + 2c_{ijkl}\overline{S}_{kl}^{(2)} \\ &\quad - \frac{8}{3\pi}e_{kij}E_k^{(1)} - e_{kij}\overline{E}_k^{(2)},\end{aligned} \tag{15e}$$

$$\begin{aligned}D_i^{(0)} &= 2e_{ikl}^{(0)}S_{kl}^{(0)} + \frac{4}{\pi}e_{ikl}\overline{S}_{kl}^{(1)} - e_{ikl}^{(3)}S_{kl}^{(2)} \\ &\quad + 2\varepsilon_{ik}^{(0)}E_k^{(0)} + \frac{4}{\pi}\varepsilon_{ik}\overline{E}_k^{(1)} + \varepsilon_{ik}^{(3)}E_k^{(2)},\end{aligned} \tag{15f}$$

$$\begin{aligned}D_i^{(1)} &= \frac{8}{\pi^2}e_{ikl}^{(d)}S_{kl}^{d} + e_{ikl}^{(1)}S_{kl}^{(1)} + \frac{8}{3\pi}e_{ikl}\overline{S}_{kl}^{(2)} \\ &\quad + \varepsilon_{ik}^{(1)}E_k^{(1)} + \frac{8}{3\pi}\varepsilon_{ik}\overline{E}_k^{(2)},\end{aligned} \tag{15g}$$

$$\begin{aligned}D_i^{(2)} &= -e_{ikl}^{(3)}S_{kl}^{(0)} - \frac{4}{3\pi}e_{ikl}\overline{S}_{kl}^{(1)} + e_{ikl}^{(2)}S_{kl}^{(2)} \\ &\quad + \varepsilon_{ik}^{(3)}E_k^{(0)} - \frac{4}{3\pi}\varepsilon_{ik}\overline{E}_k^{(1)} + \varepsilon_{ik}^{(2)}E_k^{(2)},\end{aligned} \tag{15h}$$

$$\begin{aligned}\overline{D}_i^{(1)} &= \frac{4}{\pi}e_{ikl}S_{kl}^{(0)} + e_{ikl}\overline{S}_{kl}^{(1)} - \frac{4}{3\pi}e_{ikl}S_{kl}^{(2)} \\ &\quad + \frac{4}{\pi}\varepsilon_{ik}E_k^{(0)} + \varepsilon_{ik}\overline{E}_k^{(1)} - \frac{4}{3\pi}\varepsilon_{ik}E_k^{(2)},\end{aligned} \tag{15i}$$

$$\begin{aligned}\overline{D}_i^{(2)} &= \frac{2}{\pi}e_{ikl}S_{kl}^{d} + \frac{8}{3\pi}e_{ikl}S_{kl}^{(1)} + e_{ikl}\overline{S}_{kl}^{(2)} \\ &\quad + \frac{8}{3\pi}\varepsilon_{ik}E_k^{(1)} + \varepsilon_{ik}\overline{E}_k^{(2)},\end{aligned} \tag{15j}$$

where:

$$\begin{aligned}c_{ijkl}^{(d)} &= c_{ijkl} - \frac{\pi^2}{24}c_{ijkl}^{(3)}, \quad e_{kij}^{(d)} = -\frac{\pi^2}{24}e_{kij}^{(3)}, \\ c_{ijkl}^{(0)} &= c_{ijkl} - \frac{5}{18}c_{ijkl}^{(3)}, \quad e_{kij}^{(0)} = e_{kij} - \frac{5}{18}e_{kij}^{(3)}, \\ \varepsilon_{ik}^{(0)} &= \varepsilon_{ik} + \frac{5}{18}\varepsilon_{ik}^{(0)}, \\ c_{ijkl}^{(1)} &= c_{ijkl} - \frac{16}{45}c_{ijkl}^{(3)}, \quad e_{kij}^{(1)} = e_{kij} - \frac{16}{45}e_{kij}^{(3)}, \\ \varepsilon_{ik}^{(1)} &= \varepsilon_{ik} + \frac{16}{45}\varepsilon_{ik}^{(0)}, \\ c_{ijkl}^{(2)} &= c_{ijkl} - \frac{9}{5}c_{ijkl}^{(3)}, \quad e_{kij}^{(2)} = e_{kij} - \frac{9}{5}e_{kij}^{(3)}, \\ \varepsilon_{ik}^{(2)} &= \varepsilon_{ik} + \frac{9}{5}\varepsilon_{ik}^{(3)}, \\ c_{ijkl}^{(3)} &= \frac{16}{5\pi^2}\left(\frac{c_{ij21}c_{21kl}}{c_{2121}} + \frac{c_{ij22}c_{22kl}}{c_{2222}} + \frac{c_{ij23}c_{23kl}}{c_{2323}}\right), \\ e_{kij}^{(3)} &= \frac{16}{5\pi^2}\left(\frac{c_{ij21}e_{k21}}{c_{2121}} + \frac{c_{ij22}e_{k22}}{c_{2222}} + \frac{c_{ij23}e_{k23}}{c_{2323}}\right), \\ \varepsilon_{ik}^{(3)} &= \frac{16}{5\pi^2}\left(\frac{e_{i21}e_{k21}}{c_{2121}} + \frac{e_{i22}e_{k22}}{c_{2222}} + \frac{e_{i23}e_{k23}}{c_{2323}}\right).\end{aligned} \tag{16}$$

The stress equations of motion and charge equations of electrostatics are reduced from (8):

$$T_{ij,i}^{(0)} + \frac{1}{b}F_j^{(0)} = 2\rho\ddot{u}_j^{(0)}, \tag{17a}$$

$$T_{ij,i}^{(1)} - \frac{\pi}{2b}\overline{T}_{2j}^{(1)} + \frac{1}{b}F_j^{(1)} = \rho\ddot{u}_j^{(1)} - \frac{8}{\pi^2}\rho b\ddot{u}_{2,j}^{(0)}, \tag{17b}$$

$$T_{ij,i}^{(2)} - \frac{\pi}{b}\overline{T}_{2j}^{(2)} + \frac{1}{b}F_j^{(2)} = \rho\ddot{u}_j^{(2)}, \tag{17c}$$

$$D_{i,i}^{(0)} + \frac{1}{b}D^{(0)} = 0, \tag{17d}$$

$$D_{i,i}^{(1)} - \frac{\pi}{2b}\overline{D}_2^{(1)} + \frac{1}{b}D^{(1)} = 0, \tag{17e}$$

$$D_{i,i}^{(2)} - \frac{\pi}{b}\overline{D}_2^{(2)} + \frac{1}{b}D^{(2)} = 0. \tag{17f}$$

The strain-displacement relations are obtained from (7a)–(7c):

$$\begin{aligned}S_{ij}^{d} &= -bu_{2,ij}^{(0)}, \\ S_{ij}^{(0)} &= \frac{1}{2}\left(u_{i,j}^{(0)} + u_{j,i}^{(0)} - \delta_{2i}u_{2j}^{(0)} - \delta_{2,j}u_{2,i}^{(0)}\right), \\ S_{ij}^{(1)} &= \frac{1}{2}\left(u_{i,j}^{(1)} + u_{j,i}^{(1)}\right), \\ S_{ij}^{(2)} &= \frac{1}{2}\left(u_{i,j}^{(2)} + u_{j,i}^{(2)}\right), \\ \overline{S}_{ij}^{(1)} &= \frac{\pi}{4b}\left(\delta_{2j}u_i^{(1)} + \delta_{2i}u_j^{(1)}\right), \\ \overline{S}_{ij}^{(2)} &= \frac{\pi}{2b}\left(\delta_{2j}u_i^{(2)} + \delta_{2i}u_j^{(2)}\right).\end{aligned} \tag{18}$$

The field-potential relations are obtained from (7d) and (7e):

$$\begin{aligned}&E_i^{(0)} = -\varphi_{,i}^{(0)}, \quad E_i^{(1)} = -\varphi_{,i}^{(1)}, \quad E_i^{(2)} = -\varphi_{,i}^{(2)}, \\ &\overline{E}_2^{(1)} = -\frac{\pi}{2b}\varphi^{(1)}, \quad \overline{E}_2^{(2)} = -\frac{\pi}{b}\varphi^{(2)}.\end{aligned} \tag{19}$$

Eq. (15)–(19) are combined to form nine displacement equations of motion and three potential equations where 9 $u_j^{(n)}$ and 3 $\varphi^{(n)}$, $n = 0$, 1, 2 and $j = 1$, 2, 3, are 12 unknown functions of x_1, x_3, and t. The nine components of displacement of straight-crested waves propagating in the x_1 direction are depicted in Fig. 1.

IV. Second-Order Equations for Plates with Electrodes

The series expansion of potential (5b) is well suited to satisfy charge-free face conditions, but it is not appropriate for the satisfaction of an applied potential across the thickness. For the case in which the plate faces are covered by conducting electrodes, the expansion of potential used by Lee *et al.* [1] is adopted:

$$\varphi = A + B\eta + \sum_{n=0}^{\infty}\overline{\varphi}^{(n)}\sin\frac{n\pi}{2}(1-\eta). \tag{20}$$

The same truncation procedure (12) is used to extract a system of second-order equations for plates with electroded faces from the infinite set given in [1]. The constitutive relations (15), the modified elastic constants (16), and the strain-displacement relations (18) remain the same. The charge equations of electrostatics (17d)–(17f) are replaced by:

$$\begin{aligned}\overline{D}_{1,1}^{(1)} + \overline{D}_{3,3}^{(1)} + \frac{\pi}{2b}D_2^{(1)} &= 0,\\ \overline{D}_{1,1}^{(2)} + \overline{D}_{3,3}^{(2)} + \frac{\pi}{b}D_2^{(2)} &= 0,\end{aligned} \tag{21}$$

and the field-potential relations (19) are replaced by:

$$\begin{aligned}&E_2^{(0)} = -\frac{B}{b}, \quad E_i^{(1)} = \frac{\pi}{2b}\overline{\varphi}^{(1)}, \quad E_i^{(2)} = \frac{\pi}{b}\overline{\varphi}^{(2)},\\ &\overline{E}_i^{(1)} = -\overline{\varphi}_{,i}^{(1)}, \quad \overline{E}_i^{(2)} = -\overline{\varphi}_{,i}^{(2)}.\end{aligned} \tag{22}$$

V. Dispersion Relations

Expressions (15), (16), (17a)–(17c) (18), (21), and (22) are used to calculate the dispersion curves for x_1 varying straight-crested plane waves for plates of SC-cut quartz and barium titanate with electroded faces, and the curves (in solid lines) are compared with those (in dotted lines) from the 3-D solutions with close agreement as shown in Figs. 2 and 3, respectively, in which the normalized frequency Ω and wave number Z are defined by (36a) and (36b). We note that dispersion curves also were computed by Lin [10] from the 2-D, second-order equations for plates with electrodes that were obtained by using the truncation procedure of Mindlin and Medick [4]. The improvement of present predictions shown in Figs. 2 and 3 as compared with those in [10] is slight for quartz, but much more significant for barium titanate, especially for the TSh2 and TSt branches in Fig. 3. Dispersion curves for the x_1 varying straight-crested plane waves in a plate of barium titanate with charge-free faces also are calculated, from (15)–(19), and are compared with the exact ones with close agreement as shown in Fig. 4. The values of material constants for quartz crystal and barium titanate ceramic given in [4] and [11], respectively, are used for computation. The solution forms and dispersion relations from the present 2-D equations are similar to those in [10] and very lengthy, hence they are not included in this paper.

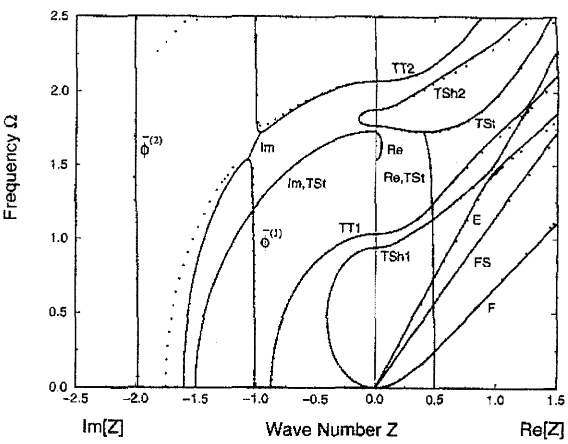

Fig. 2. Comparison of dispersion curves from the 2-D equations with those from the 3-D equations for a SC-cut of quartz with electroded faces.

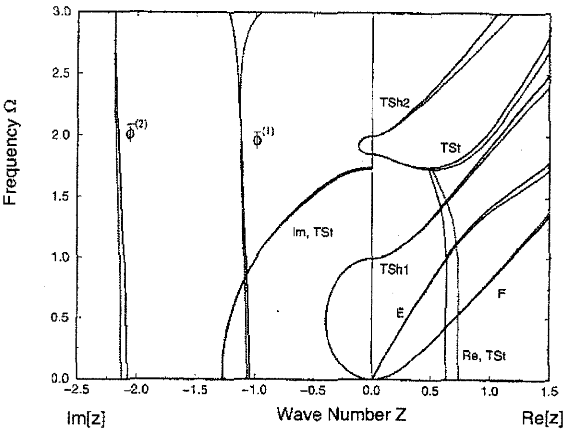

Fig. 3. Comparison of dispersion curves from the 2-D equations with those from the 3-D equations for a barium titanate ceramic plate with electroded faces.

VI. 1-D Equations for Extensional Vibrations of a Strip

Consider a strip of finite length and narrow rectangular cross section as shown in Fig. 5, in which the length is designated by $2a$, width by $2b$, and thickness by $2c$. When the thickness is small as compared to the wave length and other dimensions of the strip, a system of 1-D equations can be deduced from the 2-D equations given in Section III by averaging the variables and equations across the thickness in a manner similar to that of [6] and [8].

By integrating (17) with respect to x_3 from $-c$ to c and dividing by $2c$, we obtain the 1-D second-order stress equations of motion and electrostatic charge equations:

$$\begin{aligned}\sigma_{1j,1}^{(0)} + H_j^{(0)} &= 2\rho\ddot{v}_j^{(0)},\\ \sigma_{1j,1}^{(1)} - \frac{\pi}{2b}\overline{\sigma}_{2j}^{(1)} + H_j^{(1)} &= \rho\ddot{v}_j^{(1)},\\ \sigma_{1j,1}^{(2)} - \frac{\pi}{b}\overline{\sigma}_{2j}^{(2)} + H_j^{(2)} &= \rho\ddot{v}_j^{(2)},\\ d_{1,1}^{(0)} + d^{(0)} &= 0,\\ d_{1,1}^{(1)} - \frac{\pi}{2b}\overline{d}_2^{(1)} + d^{(1)} &= 0,\\ d_{1,1}^{(2)} - \frac{\pi}{b}\overline{d}_2^{(2)} + d^{(2)} &= 0,\end{aligned} \tag{23}$$

where the nth-order components of averaged displacement,

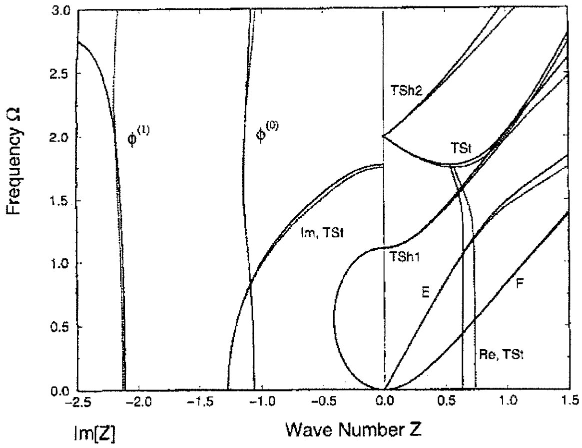

Fig. 4. Comparison of dispersion curves from the 2-D equations with those from the 3-D equations for a barium titanate ceramic plate without electrodes.

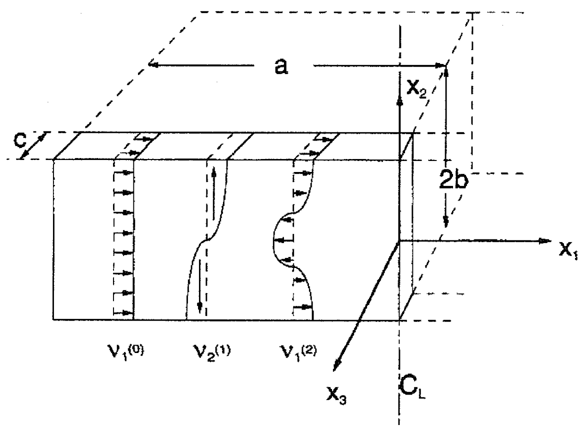

Fig. 5. A strip of length $2a$, width $2b$, and thickness $2c$ ($c/b \ll 1$) and the first three components of displacement for the in-plane symmetrical extensional vibrations.

stress, charge, and potential are defined, respectively, by:

$$v_j^{(n)}(x_1,t) = \frac{1}{2c}\int_{-c}^{c} u_j^{(n)}(x_1,x_3,t)dx_3, \tag{24a}$$

$$\sigma_{ij}^{(n)}(x_1,t) = \frac{1}{2c}\int_{-c}^{c} T_{ij}^{(n)}(x_1,x_3,t)dx_3, \tag{24b}$$

$$\overline{\sigma}_{ij}^{(n)}(x_1,t) = \frac{1}{2c}\int_{-c}^{c} \overline{T}_{ij}^{(n)}(x_1,x_3,t)dx_3, \tag{24c}$$

$$d_k^{(n)}(x_1,t) = \frac{1}{2c}\int_{-c}^{c} D_k^{(n)}(x_1,x_3,t)dx_3, \tag{24d}$$

$$\overline{d}_k^{(n)}(x_1,t) = \frac{1}{2c}\int_{-c}^{c} \overline{D}_k^{(n)}(x_1,x_3,t)dx_3, \tag{24e}$$

$$\psi^{(n)}(x_1,t) = \frac{1}{2c}\int_{-c}^{c} \varphi^{(n)}(x_1,x_3,t)dx_3, \tag{24f}$$

and the nth-order components of tractions and surface charge density on the faces at $x_3 = \pm c$ by:

$$\begin{aligned} H_j^{(n)} &= \frac{1}{2c}\left[T_{3j}^{(n)}(c) - T_{3j}^{(n)}(-c)\right], \\ d^{(n)} &= \frac{1}{2c}\left[D_3^{(n)}(c) - D_3^{(n)}(-c)\right]. \end{aligned} \tag{25}$$

In obtaining (23), the traction-free and charge-free edge conditions at $x_2 = \pm b$ have been taken into account by setting $F_j^{(n)} = D^{(n)} = 0$, $n = 0, 1, 2$ in (17).

For a thin strip in a state of generalized plane (x_1, x_2) stress, we assume:

$$T_{33} = 0 \quad \text{and} \tag{26a}$$

$$T_{31}\Big|_{x_3=\pm c} = T_{32}\Big|_{x_3=\pm c} = 0. \tag{26b}$$

Then, substituting (26a) into the definitions in (9a), we have:

$$T_{33}^{(0)} = 0, \tag{27a}$$

$$T_{33}^{(2)} = 0. \tag{27b}$$

For extensional motions in thin strips of piezoelectric ceramic with poling axis in x_3 direction, displacements $u_j(x_1, x_2, x_3, t)$ must be even functions with respect to the x_3 coordinate and symmetric with respect to the x_1–x_3 plane. Hence, only five of the nine components of the 2-D displacement $u_j^{(n)}$, $n = 0, 1, 2$ are taken into consideration and they satisfy:

$$\int_{-c}^{c}\left(u_1^{(0)}, u_1^{(2)}, u_2^{(1)}, u_3^{(0)}, u_3^{(2)}\right)_{,3} dx_3 = 0. \tag{28}$$

By substituting (15a) and (15c) into (27a) and (27b) and neglecting the coupling terms between $u_{1,1}^{(0)}$ and $u_{1,1}^{(2)}$, we obtain:

$$\begin{aligned} u_{3,3}^{(0)} &= -\frac{c_{31}^{(0)}}{c_{33}^{(0)}}u_{1,1}^{(0)} - \frac{1}{b}\frac{c_{32}}{c_{33}^{(0)}}u_2^{(1)} + \frac{1}{2}\frac{c_{31}^{(3)}}{c_{33}^{(0)}}u_{1,1}^{(2)} \\ &\quad + 2\frac{e_{33}^{(0)}}{c_{33}^{(0)}}\varphi_{,3}^{(0)} - \frac{e_{33}^{(3)}}{c_{33}^{(0)}}\varphi_{,3}^{(2)}, \\ u_{3,3}^{(2)} &= \frac{c_{31}^{(3)}}{c_{33}^{(2)}}u_{1,1}^{(0)} + \frac{2}{3b}\frac{c_{32}}{c_{33}^{(2)}}u_2^{(1)} - \frac{c_{31}^{(2)}}{c_{33}^{(2)}}u_{1,1}^{(2)} \\ &\quad - \frac{e_{33}^{(3)}}{c_{33}^{(2)}}\varphi_{,3}^{(0)} + \frac{e_{33}^{(2)}}{c_{33}^{(2)}}\varphi_{,3}^{(2)}. \end{aligned} \tag{29}$$

By substituting (29) into (15) and the result into (24b)–(24e) and using the relations (28), we have the averaged constitutive equations:

$$\sigma_1^{(0)} = 2\tilde{c}_{11}^0 \nu_{1,1}^{(0)} + \frac{2}{b}\tilde{c}_{12}^4 \nu_2^{(1)} - \tilde{c}_{11}^3 \nu_{1,1}^{(2)} + \frac{1}{c}\tilde{e}_{31}^0 \left[\varphi^{(0)}\right]_{-c}^{c} - \frac{1}{2c}\tilde{e}_{31}^3 \left[\varphi^{(2)}\right]_{-c}^{c}, \tag{30a}$$

$$\sigma_5^{(0)} = 2c_{55}^{(0)} \nu_{3,1}^{(0)} - c_{55}^{(3)} \nu_{3,1}^{(2)} + 2e_{15}^{(0)} \psi_{,1}^{(0)} - e_{15}^{(3)} \psi_{,1}^{(2)}, \tag{30b}$$

$$\sigma_6^{(1)} = c_{66}^{(1)} \nu_{2,1}^{(1)} + \frac{8}{3b} c_{66} \nu_1^{(2)}, \tag{30c}$$

$$\overline{\sigma}_2^{(1)} = \frac{4}{\pi}\tilde{c}_{12}^4 \nu_{1,1}^{(0)} + \frac{\pi}{2b}\tilde{c}_{22}^1 \nu_2^{(1)} - \frac{4}{3\pi}\tilde{c}_{21}^5 \nu_{1,1}^{(2)} + \frac{2}{c\pi}\tilde{e}_{32}^1 \left[\varphi^{(0)}\right]_{-c}^{c} - \frac{2}{3c\pi}\tilde{e}_{32}^4 \left[\varphi^{(2)}\right]_{-c}^{c}, \tag{30d}$$

$$\sigma_1^{(2)} = -\tilde{c}_{11}^3 \nu_{1,1}^{(0)} - \frac{2}{3b}\tilde{c}_{21}^5 \nu_2^{(1)} + \tilde{c}_{11}^2 \nu_{1,1}^{(2)} - \frac{1}{2c}\tilde{e}_{31}^5 \left[\varphi^{(0)}\right]_{-c}^{c} + \frac{1}{2c}\tilde{e}_{31}^2 \left[\varphi^{(2)}\right]_{-c}^{c}, \tag{30e}$$

$$\sigma_5^{(2)} = -c_{55}^{(3)} \nu_{3,1}^{(0)} + c_{55}^{(2)} \nu_{3,1}^{(2)} - e_{15}^{(3)} \psi_{,1}^{(0)} + e_{15}^{(2)} \psi_{,1}^{(2)}, \tag{30f}$$

$$\overline{\sigma}_4^{(2)} = \frac{\pi}{b} c_{44} \nu_3^{(2)} + \frac{\pi}{b} e_{24} \psi^{(2)}, \tag{30g}$$

$$\overline{\sigma}_6^{(2)} = \frac{8}{3\pi} c_{66} \nu_{2,1}^{(1)} + \frac{\pi}{b} c_{66} \nu_1^{(2)}, \tag{30h}$$

$$d_1^{(0)} = 2e_{15}^{(0)} \nu_{3,1}^{(0)} - e_{15}^{(3)} \nu_{3,1}^{(2)} - 2\varepsilon_{11}^{(0)} \psi_{,1}^{(0)} - \varepsilon_{11}^{(3)} \psi_{,1}^{(2)}, \tag{30i}$$

$$d_1^{(2)} = -e_{15}^{(3)} \nu_{3,1}^{(0)} + e_{15}^{(2)} \nu_{3,1}^{(2)} - \varepsilon_{11}^{(3)} \psi_{,1}^{(0)} - \varepsilon_{11}^{(2)} \psi_{,1}^{(2)}, \tag{30j}$$

$$\overline{d}_2^{(2)} = \frac{\pi}{b} e_{24} \nu_3^{(2)} - \frac{\pi}{b} \varepsilon_{22} \psi^{(2)}, \tag{30k}$$

$$d^{(0)} = -2\tilde{\varepsilon}_{33}^0 \left[\varphi_{,3}^{(0)}(c) - \varphi_{,3}^{(0)}(-c)\right] - \tilde{\varepsilon}_{33}^3 \left[\varphi_{,3}^{(2)}(c) - \varphi_{,3}^{(2)}(-c)\right], \tag{30l}$$

$$d^{(2)} = -\tilde{\varepsilon}_{33}^3 \left[\varphi_{,3}^{(0)}(c) - \varphi_{,3}^{(0)}(-c)\right] - \tilde{\varepsilon}_{33}^2 \left[\varphi_{,3}^{(2)}(c) - \varphi_{,3}^{(2)}(-c)\right], \tag{30m}$$

where the modified material constants are:

$$\tilde{c}_{11}^0 = c_{11}^{(0)} - \frac{c_{13}^{(0)} c_{31}^{(0)}}{c_{33}^{(0)}} - \frac{1}{2}\frac{c_{13}^{(3)} c_{31}^{(3)}}{c_{33}^{(2)}}, \tag{31a}$$

$$\tilde{e}_{31}^0 = e_{31}^{(0)} + 2\frac{c_{13}^{(0)} e_{33}^{(0)}}{c_{33}^{(0)}} + \frac{1}{2}\frac{c_{13}^{(3)} e_{33}^{(3)}}{c_{33}^{(2)}}, \tag{31b}$$

$$\tilde{c}_{22}^1 = c_{22} - \frac{8}{\pi^2}\frac{c_{23}c_{32}}{c_{33}^{(0)}} - \frac{16}{9\pi^2}\frac{c_{23}c_{32}}{c_{33}^{(2)}}; \tag{31c}$$

$$\tilde{e}_{32}^1 = e_{32} + 2\frac{c_{23}e_{33}^{(0)}}{c_{33}^{(0)}} + \frac{1}{3}\frac{c_{23}e_{33}^{(3)}}{c_{33}^{(2)}}, \tag{31d}$$

$$\tilde{c}_{11}^2 = c_{11}^{(2)} - \frac{1}{2}\frac{c_{13}^{(3)} c_{31}^{(3)}}{c_{33}^{(0)}} - \frac{c_{13}^{(2)} c_{31}^{(2)}}{c_{33}^{(2)}}, \tag{31e}$$

$$\tilde{e}_{31}^2 = e_{31}^{(2)} + \frac{c_{13}^{(3)} e_{33}^{(3)}}{c_{33}^{(0)}} + \frac{c_{13}^{(2)} e_{33}^{(2)}}{c_{33}^{(2)}}, \tag{31f}$$

$$\tilde{c}_{11}^3 = c_{11}^{(3)} - \frac{c_{13}^{(0)} c_{31}^{(3)}}{c_{33}^{(0)}} - \frac{c_{13}^{(3)} c_{31}^{(2)}}{c_{33}^{(2)}}, \tag{31g}$$

$$\tilde{e}_{31}^3 = e_{31}^{(3)} + 2\frac{c_{13}^{(0)} e_{33}^{(3)}}{c_{33}^{(0)}} + \frac{c_{13}^{(3)} e_{33}^{(2)}}{c_{33}^{(2)}}, \tag{31h}$$

$$\tilde{c}_{12}^4 = c_{12} - \frac{c_{13}^{(0)} c_{32}}{c_{33}^{(0)}} - \frac{1}{3}\frac{c_{32}c_{13}^{(3)}}{c_{33}^{(2)}}, \tag{31i}$$

$$\tilde{e}_{32}^4 = e_{32} + 3\frac{c_{23}e_{33}^{(3)}}{c_{33}^{(0)}} + \frac{c_{23}e_{33}^{(2)}}{c_{33}^{(2)}}, \tag{31j}$$

$$\tilde{c}_{21}^5 = c_{21} - \frac{3}{2}\frac{c_{23}c_{31}^{(3)}}{c_{33}^{(0)}} - \frac{c_{23}c_{31}^{(2)}}{c_{33}^{(2)}}, \tag{31k}$$

$$\tilde{e}_{31}^5 = e_{31}^{(3)} + 2\frac{c_{13}^{(3)} e_{33}^{(0)}}{c_{33}^{(0)}} + \frac{c_{13}^{(2)} e_{33}^{(3)}}{c_{33}^{(2)}}, \tag{31l}$$

$$\tilde{\varepsilon}_{33}^0 = \varepsilon_{33}^{(0)} - 2\frac{e_{33}^{(0)} e_{33}^{(0)}}{c_{33}^{(0)}} - \frac{e_{33}^{(3)} e_{33}^{(3)}}{c_{33}^{(2)}}, \tag{31m}$$

$$\tilde{\varepsilon}_{33}^2 = \varepsilon_{33}^{(2)} - \frac{e_{33}^{(3)} e_{33}^{(3)}}{c_{33}^{(0)}} - \frac{e_{33}^{(3)} e_{33}^{(3)}}{c_{33}^{(2)}}, \tag{31n}$$

$$\tilde{\varepsilon}_{33}^3 = \varepsilon_{33}^{(3)} + 2\frac{e_{33}^{(0)} e_{33}^{(3)}}{c_{33}^{(0)}} + \frac{e_{33}^{(3)} e_{33}^{(2)}}{c_{33}^{(2)}}. \tag{31o}$$

By substituting the modified constitutive relations (30) into the stress and charge equations (23), we obtain the 1-D equations of displacements and potentials for the extensional vibrations of a strip for piezoelectric ceramics:

$$\tilde{c}_{11}^0 \nu_{1,11}^{(0)} + \frac{1}{b}\tilde{c}_{12}^4 \nu_{2,1}^{(1)} - \frac{1}{2}\tilde{c}_{11}^3 \nu_{1,11}^{(2)} + \frac{1}{2c}\tilde{e}_{31}^0 \left[\varphi_{,1}^{(0)}\right]_{-c}^{c} - \frac{1}{4c}\tilde{e}_{31}^3 \left[\varphi_{,1}^{(2)}\right]_{-c}^{c} = \rho\nu_1^{(0)}, \tag{32a}$$

$$-\frac{2}{b}\tilde{c}_{12}^4 \nu_{1,1}^{(0)} + c_{66}^{(1)} \nu_{2,11}^{(1)} - \frac{\pi^2}{4b^2}\tilde{c}_{22}^1 \nu_2^{(1)} + \frac{2}{3b}\left(4c_{66} + \tilde{c}_{21}^5\right)\nu_{1,1}^{(2)} - \frac{1}{bc}\tilde{e}_{32}^1 \left[\varphi^{(0)}\right]_{-c}^{c} + \frac{1}{3bc}\tilde{e}_{32}^4 \left[\varphi^{(2)}\right]_{-c}^{c} = \rho\ddot{\nu}_2^{(1)}, \tag{32b}$$

$$-\tilde{c}_{11}^3 \nu_{1,11}^{(0)} - \frac{2}{3b}\left(4c_{66} + \tilde{c}_{21}^5\right)\nu_{2,1}^{(1)} + \tilde{c}_{11}^2 \nu_{1,11}^{(2)} - \frac{\pi^2}{b^2} c_{66}\nu_1^{(2)} - \frac{1}{2c}\tilde{e}_{31}^5 \left[\varphi_{,1}^{(0)}\right]_{-c}^{c} + \frac{1}{2c}\tilde{e}_{31}^2 \left[\varphi_{,1}^{(2)}\right]_{-c}^{c} = \rho\ddot{\nu}_1^{(2)}, \tag{32c}$$

$$2e_{15}^{(0)} \nu_{3,11}^{(0)} - e_{15}^{(3)} \nu_{3,11}^{(2)} - 2\varepsilon_{11}^{(0)} \psi_{,11}^{(0)} - \varepsilon_{11}^{(3)} \psi_{,11}^{(2)} + d^{(0)} = 0, \tag{32d}$$

$$-e_{15}^{(3)} \nu_{3,11}^{(0)} + e_{15}^{(2)} \nu_{3,11}^{(2)} - \varepsilon_{11}^{(3)} \psi_{,11}^{(0)} - \varepsilon_{11}^{(2)} \psi_{,11}^{(2)} - \frac{\pi^2}{b^2} e_{24}\nu_3^{(2)} + \frac{\pi^2}{b^2}\varepsilon_{22}\psi^{(2)} + d^{(2)} = 0, \tag{32e}$$

$$c_{55}^{(0)} \nu_{3,11}^{(0)} - \frac{1}{2}c_{55}^{(3)} \nu_{3,11}^{(2)} + e_{15}^{(0)} \psi_{,11}^{(0)} - \frac{1}{2}e_{15}^{(3)} \psi_{,11}^{(2)} = \rho\ddot{\nu}_3^{(0)}, \tag{32f}$$

$$-c_{55}^{(3)} \nu_{3,11}^{(0)} + c_{55}^{(2)} \nu_{3,11}^{(2)} - \frac{\pi^2}{b^2} c_{44}\nu_3^{(2)} - e_{15}^{(3)} \psi_{,11}^{(0)} + e_{15}^{(2)} \psi_{,11}^{(2)} - \frac{\pi^2}{b^2} e_{24}\psi^{(2)} = \rho\ddot{\nu}_3^{(2)}. \tag{32g}$$

We see that (32) are uncoupled into two sets of equations: (32a)–(32c) govern the in-plane extension $\nu_1^{(0)}$, width-stretch $\nu_2^{(1)}$, and second-order width-shear $\nu_1^{(2)}$ vibrations that can be piezoelectrically excited by an applied potential difference $\varphi(x_3 = \pm c) = \pm\varphi_0 e^{-i\omega t}$ across the faces of the strip; (32d)–(32g) govern the out-of-plane modes $\nu_3^{(0)}$ and $\nu_3^{(2)}$ and the potentials $\psi^{(0)}$ and $\psi^{(2)}$, which cannot be excited by the same applied potential because $d^{(0)} = d^{(2)} = 0$. The distribution of the displacements of the in-plane vibrations across the width of the strip is depicted in Fig. 5.

VII. Extension, Width-Stretch, and Width-Shear Vibrations

Expressions (32a)–(32c) are used to study the in-plane extensional vibrations of a piezoelectric ceramic strip of finite length. For shorted faces at $x_3 = \pm c$, the boundary terms on $\varphi^{(0)}$ and $\varphi^{(2)}$ are set to zero in (32a)–(32c) and, for traction-free end conditions we require, at $x_1 = \pm a$,

$$\sigma_1^{(0)} = \sigma_6^{(1)} = \sigma_1^{(2)} = 0. \tag{33}$$

By substituting the vibrational form:

$$\begin{aligned} \nu_1^{(0)} &= A_1 \sin \xi x_1 e^{-i\omega t}, \\ \nu_2^{(1)} &= A_2 \cos \xi x_1 e^{-i\omega t}, \\ \nu_1^{(2)} &= A_3 \sin \xi x_1 e^{-i\omega t}, \end{aligned} \tag{34}$$

into (32a)–(32c), we obtain:

$$\begin{bmatrix} 2\bar{c}^0_{11}Z^2 - 2\Omega^2 & \frac{4}{\pi}\bar{c}^4_{12}Z & -\bar{c}^3_{11}Z^2 \\ \frac{4}{\pi}\bar{c}^4_{12}Z & \bar{c}^{(1)}_{66}Z^2 + \bar{c}^1_{22} - \Omega^2 & -\frac{4}{3\pi}\left[4 + \bar{c}^5_{21}\right]Z \\ -\bar{c}^3_{11}Z^2 & -\frac{4}{3\pi}\left[4 + \bar{c}^5_{21}\right]Z & \bar{c}^2_{11}Z^2 + 4 - \Omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = 0, \tag{35}$$

where:

$$\Omega = \omega \frac{2b}{\pi}\sqrt{\frac{\rho}{c_{66}}}, \tag{36a}$$

$$Z = \xi \frac{2b}{\pi}, \tag{36b}$$

$$\bar{c}^n_{ij} = \frac{\tilde{c}^n_{ij}}{c_{66}}, \tag{36c}$$

$$\bar{c}^{(n)}_{ij} = \frac{c^{(n)}_{ij}}{c_{66}}, \tag{36d}$$

$$\bar{c}_{ij} = \frac{c_{ij}}{c_{66}}. \tag{36e}$$

The vanishing of the determinant of the coefficients matrix in (35) for a nontrivial solution yields the dispersion relation that is a bicubic equation in Ω and Z and is expressed by:

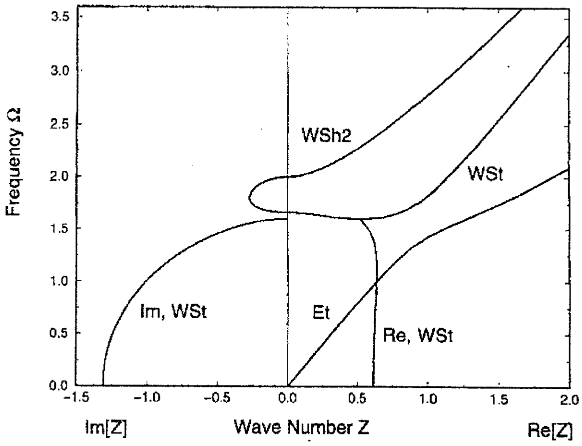

Fig. 6. Dispersion curves computed from (37) of 1-D equations of the in-plane vibrations of extensional (Et), width-stretch (WSt), and the second-order width-shear (WSh2) modes in an infinite strip of barium titanate.

$$D(\Omega, Z) = 0. \tag{37}$$

These roots or dispersion curves for an infinite strip of barium titanate are shown in Fig. 6.

For a given value of Ω, three roots Z_j, $j = 1, 2, 3$ are computed from (37), and by substituting each Z_j back into (35) the corresponding amplitude ratios:

$$\alpha_{1j} = 1, \quad \alpha_{2j} = \frac{A_{2j}}{A_{1j}}, \quad \alpha_{3j} = \frac{A_{3j}}{A_{1j}}, \tag{38}$$

are calculated.

The general solution for free vibrations is then formed by a linear combination of the three solutions:

$$\begin{aligned} \nu_1^{(0)} &= \sum_{j=1}^{3} \alpha_{1j} B_j \sin \frac{\pi}{2b} Z_j x_1 e^{-i\omega t}, \\ \nu_2^{(1)} &= \sum_{j=1}^{3} \alpha_{2j} B_j \cos \frac{\pi}{2b} Z_j x_1 e^{-i\omega t}, \\ \nu_1^{(2)} &= \sum_{j=1}^{3} \alpha_{3j} B_j \sin \frac{\pi}{2b} Z_j x_1 e^{-i\omega t}. \end{aligned} \tag{39}$$

By inserting (39) into (30a), (30c), and (30e) and imposing end conditions (33), we obtain three equations in the three unknown amplitudes B_j:

$$M_{ij} B_j = 0, \tag{40}$$

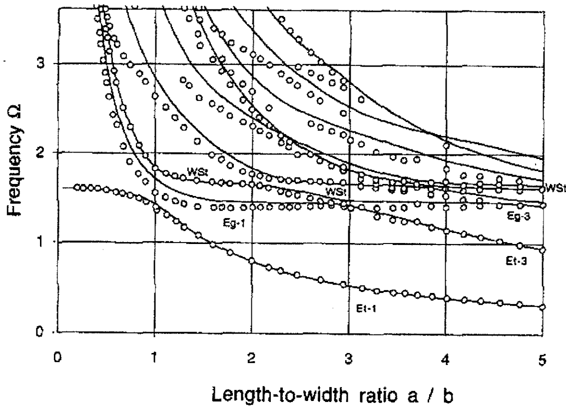

Fig. 7. Comparison of frequency spectrum predicted by (42) of 1-D equations of the Et-WSt-WSh2 vibrations for barium titanate strips with the experimental data by Onoe and Pao [7].

where

$$M_{1j} = \left(\pi\tilde{c}^0_{11}\alpha_{1j}Z_{\underline{j}} + 2\tilde{c}^4_{12}\alpha_{2\underline{j}} - \pi\tilde{c}^3_{11}\alpha_{3\underline{j}}Z_{\underline{j}}\right)\cos\frac{\pi}{2}\frac{a}{b}Z_{\underline{j}},$$
$$M_{2j} = \left(-\pi c^{(1)}_{66}\alpha_{2\underline{j}}Z_{\underline{j}} + \frac{8}{3}c_{66}\alpha_{3\underline{j}}\right)\sin\frac{\pi}{2}\frac{a}{b}Z_{\underline{j}},$$
$$M_{3j} = \left(-\pi\tilde{c}^3_{11}\alpha_{1\underline{j}}Z_{\underline{j}} - \frac{4}{3}\tilde{c}^5_{21}\alpha_{2\underline{j}} + \pi\tilde{c}^2_{11}\alpha_{3\underline{j}}Z_{\underline{j}}\right)\cos\frac{\pi}{2}\frac{a}{b}Z_{\underline{j}}, \qquad (41)$$

with no sum over j in (41). The vanishing of the determinant of the coefficient matrix of (40) gives the frequency equation for the finite strip, which can be expressed by:

$$F(\Omega, a/b) = 0. \qquad (42)$$

We note that (42) is independent of c/b, the thickness-to-width ratio, for the free vibrations. The thickness dimension $2c$ will appear in the particular solution of (32a)–(32c) for piezoelectrically forced vibrations.

VIII. Frequency Spectrum and Modes for a Strip

Resonance frequencies Ω as a function of the length-to-width ratio a/b are computed from (42) for finite strips of barium titanate [12] with four edges free of traction and charge as shown in Fig. 7, in which the segments of frequency branches predominantly in extensional modes are denoted by Et-n, edge modes by Eg-n and fundamental width-stretch by WSt, and the integer $n = 1, 3, 5\ldots$ denotes the nth-anharmonic overtone. The experimental data by Onoe and Pao [7], which are extracted from Medick and Pao [8] are presented as circles in Fig. 7 for comparison. The present result in Fig. 7 also may be compared with previously calculated result in Fig. 3 of [8]. It may be seen that predicted resonance frequencies for the Et-1 and Et-3 modes are very accurate from both the present theory and that of [8], but the present theory gives slightly closer predictions for the Eg-1 and Eg-3 modes and for the WSt mode for $a/b = 1$ to 2. The frequency of the edge mode presently predicted at 1.464 is within 4% of the experimentally measured value of 1.411; the predicted value in [8] was within 8%.

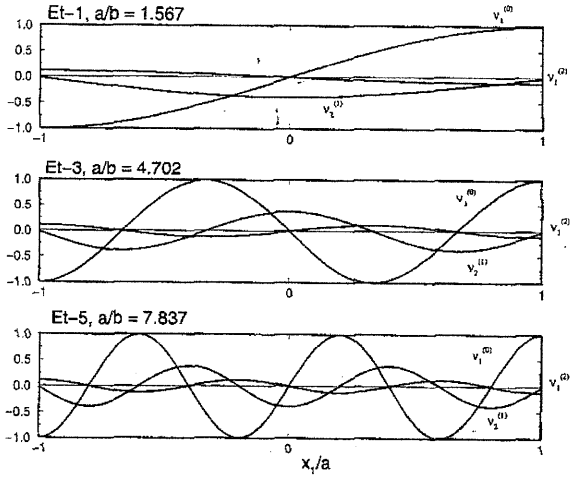

Fig. 8. Mode shapes of predominantly extensional vibrations, $Et-n$, $n = 1, 3, 5$, for barium titanate strips at $\Omega = 1.0$, and $n = 1, 3, 5$, $a/b = 1.567$, 4.702, and 7.837, respectively.

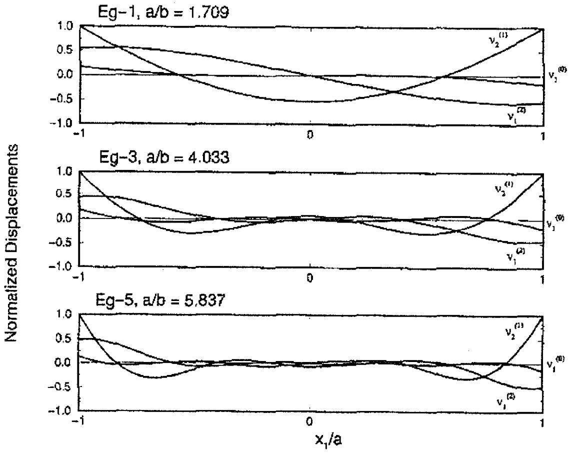

Fig. 9. Mode shapes of predominantly edge modes, $Eg-n$, $n = 1, 3, 5$, for strips at $\Omega = 1.464$, and $a/b = 1.709$, 4.033, and 5.837, respectively.

Mode shapes or displacement variation along the length of the strip at resonance are calculated from (39) and are shown in Figs. 8–10. In Fig. 8, the predominantly fundamental extensional mode Et-1 and overtones Et-3 and Et-5 are plotted for $\Omega = 1.0$ and $a/b = 1.567$, 4.702, and 7.837, respectively. In Fig. 9 the edge modes predominantly in width-stretch $\nu_2^{(1)}$, Eg-n, $n = 1, 3, 5$, are plotted for $\Omega = 1.464$ and $a/b = 1.709$, 4.033, and 5.837. In Fig. 10

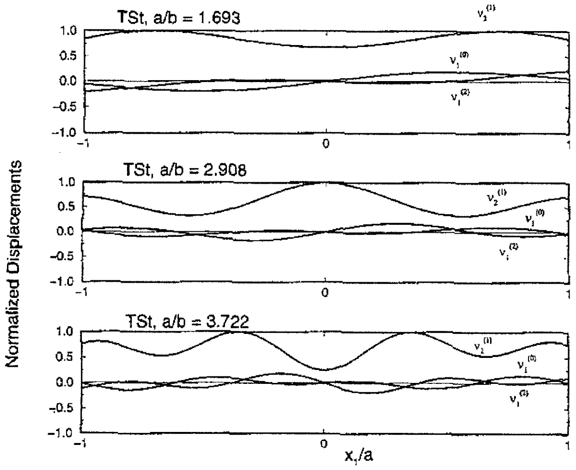

Fig. 10. Mode shapes of predominantly fundamental width-stretch, WSt, for strips at $\Omega = 1.664$, and $a/b = 1.693$, 2.908, and 3.722, respectively.

the fundamental width-stretch modes WSt are plotted for $\Omega = 1.664$ and $a/b = 1.693$, 2.908, and 3.722. It is seen in Fig. 10 that the mode of $\nu_2^{(1)}$ changes slightly for increasing values of a/b, but it may be regarded as essentially uniform throughout the length of the strip and, hence, identified as the fundamental width-stretch mode.

IX. Conclusions

In summary, 2-D, second-order equations of extensional vibrations for piezoelectric crystal plates, with or without electrodes, are obtained by a new truncation procedure and shown to predict accurate dispersion curves as compared with those from the 3-D equations, without using any correction factors. Furthermore, a system of 1-D, second-order equations of extensional vibrations for piezoelectric ceramic strips with narrow rectangular cross section is deduced and shown to predict accurately, without using any correction factors, the frequency spectrum for finite strips with all four edges free as compared with experimental data [7] and the previous prediction [8].

References

[1] P. C. Y. Lee, J. D. Yu, and W.-S. Lin, "A new two-dimensional theory for vibrations of piezoelectric crystal plates with electroded faces," *J. Appl. Phys.*, vol. 83, no. 3, p. 1213, 1998.

[2] I. Koga and H. Fukuyo, "Vibration of thin piezoelectric quartz plates," *J. Inst. Elec. Comm. Eng. Jpn.*, vol. 36, p. 59, 1953.

[3] M. Nakazawa, K. Horiuchi, and H. Ito, "Frequency mode charts of rectangular AT-cut quartz plates," in *Proc. IEEE Ultrason. Symp.*, 1990, p. 547.

[4] R. D. Mindlin and M. A. Medick, "Extensional vibrations of elastic plates," *J. Appl. Mech.*, vol. 26, p. 561, 1959.

[5] P. C. Y. Lee, S. Syngellakis, and J. P. Hou, "A two-dimensional theory for high-frequency vibrations of piezoelectric crystal plates with or without electrodes," *J. Appl. Phys.*, vol. 61, no. 4, p. 1249, 1987.

[6] P. C. Y. Lee, "Extensional, flexural, width-shear and width-stretch vibrations of thin rectangular resonators," *J. Appl. Phys.*, vol. 42, no. 11, p. 4139, 1971.

[7] M. Onoe and Y. H. Pao, "Edge mode of thin rectangular plate of barium titanate," *J. Acoust. Soc. Amer.*, vol. 33, p. 1628, 1961.

[8] M. Medick and Y. H. Pao, "Extensional vibrations of thin rectangular plates," *J. Acoust. Soc. Amer.*, vol. 37, p. 59, 1965.

[9] H. F. Tiersten, *Linear Piezoelectric Plate Vibrations.* New York: Plenum, 1969.

[10] W.-S. Lin, "A new two-dimensional theory for vibrations of piezoelectric crystal plates with electroded faces," Ph.D. dissertation, Department of Civil and Environmental Eng., Princeton University, Princeton, NJ, 1998.

[11] R. Bechman, "Elastic and piezoelectric constants of alpha-quartz," *Phys. Rev.*, vol. 110, no. 5, p. 1060, 1958.

[12] D. A. Berlincourt, D. R. Curran, and H. Jaffe, "Piezoelectric and piezomagnetic materials and their function in transducers," in *Physical Acoustics, Principles and Methods.* vol. 1, pt. A, W. P. Mason, Ed. New York: Academic Press, p. 202, 1964.

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