AIAA 2002-1534

AIAA-2002-1534

Statistical Analysis of the Forced Response of Mistuned Bladed Disks Using Stochastic Reduced Basis Methods

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Abstract

This paper is concerned with the forced response statistics of mistuned bladed disk assemblies subjected to a deterministic sinusoidal excitation. A stochastic reduced basis method (SRBM) is used to compute the statistics of the system component amplitudes. In this approach, the system response in the frequency domain is represented using a linear combination of stochastic basis vectors with undermined coefficients. The three terms of the second-order perturbation approximation (which span the stochastic Krylov subspace) are used as basis vectors and the undetermined coefficients are evaluated using stochastic variants of the Bubnov-Galerkin Scheme. This results in explicit expressions for the response quantities in terms of the random system parameters. The statistics of the system response can hence be efficiently computed in the post-processing stage. Numerical results are presented for a model problem to demonstrate that the stochastic reduced basis formulation gives highly accurate results for the response statistical moments.

1. Introduction

A bladed disk assembly represents a typical periodic structure where parameter uncertainties arising due to the stochastic nature of manufacturing processes and in-service degradation can lead to mistuning. The mistuning problem, which arises from the disruption of perfect periodicity, has received much attention in the literature [1-5]. A driving factor behind this has been the ever increasing need for efficient and accurate computational models to predict the existence of rogue blades that exhibit failure due to the excessive stress levels.

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Mistuning is known to have a potentially dramatic effect on the free vibration behavior of coupled blades, since it can lead to the spatial localization of energy around one or a few blades. In the case of the forced vibration, mistuning can lead to significant increases in the amplitudes and stresses of blades compared to their perfectly tuned counterparts [2]. Another characteristic is the increase in amplitude of the maximum-responding blade at any frequency, which could result in a significant decrease in fatigue life. Also, moderately weakly coupled systems are found to be more sensitive than strongly coupled ones through a greater increase in component response amplitudes. A detailed exposition of factors that influence the sensitivity of disordered periodic systems to mistuning can be found in the literature; see, for example, [3, 4, 5].

Statistical methods provide an enabling tool for dynamic analysis of mistuned bladed disk assembles. A direct but computationally expensive approach to accurately generate the response statistics remains the Monte Carlo simulation (MCS) method. Samples of the uncertain system parameters are generated in accordance with their probability density function (pdf), and the equations of motion are solved for each realization of these parameters. Subsequently, the statistical properties of the response amplitudes, stresses and life fatigue can be estimated. In practice, this statistical information about the system response can be employed to plan and interpret test results, and also to design systems that are more insensitive to mistuning.

A major disadvantage of simulation techniques is that the computational cost may become prohibitive, particularly for systems that are required to be analyzed using high-fidelity finite element models. This has motivated the development of reduced-order modeling techniques [4-7] to make simulation schemes more efficient. However, in this line of approach, a trade-off must be made between the accuracy of the reduced-order model and computational cost.

Another popular approach to mistuning analysis involves the application of perturbation techniques for analytically approximating the response statistics [3,8-12]. Since perturbation schemes are computationally very efficient, they can be readily applied to large-scale finite element models. Further, the resulting explicit expression for the response allows for the possibility of gaining physical insights into the dynamics of mistuned systems. Hence, these schemes, if accurate, can allow reliable statistical assessments during the turbomachinery design process. However, the accuracy of perturbation methods tends to deteriorate significantly for large coefficients of variation of the random system parameters and increasing frequency of excitation. In the particular case of mistuned bladed disks, the accuracy depends on the relative magnitudes of coupling strength, mistuning strength, and the material damping properties.

The inherent limitations of perturbation methods were illustrated in [3], where two perturbation approaches were presented for mistuning analysis. In the first approach, the forced response amplitude of each component system is obtained directly as a perturbation of the tuned system. It was found that for strongly coupled systems (i.e., when the ratio of mistuning strength to coupling is less than or equal to unity), the accuracy deteriorates significantly if the ratio of mistuning strength to damping ratio is of order greater than one. In the second approach, the modal properties of the free undamped mistuned system are first approximated using a perturbation method. Subsequently, a modal analysis is carried out to compute the forced response of each component system. This approach can be applied to strongly coupled systems with any damping but its accuracy depends on the ratio of mistuning strength to coupling strength. Also it was found that for weakly coupled systems (i.e., when the ratio of mistuning strength to coupling is greater than unity), only direct simulation techniques could provide accurate results for the response statistics.

More recently, Nair and Keane [13, 14] proposed a class of stochastic reduced basis methods (SRBMs) to solve random algebraic equations arising from discretization of linear stochastic partial differential equations in space, time, and the random dimension of the problem. This approach essentially involves approximating the random solution process using the terms of the preconditioned stochastic Krylov subspace as basis vectors. It was shown for a class of problems that SRBMs can be orders of magnitude more accurate than traditional perturbation methods. A more detailed exposition of the theoretical underpinnings of SRBMs can be found in [17].

The objective of this paper is to leverage SRBMs to develop a novel approach for statistical analysis of the forced response of mistuned bladed disks. In particular, our focus is on computing the response statistics in the frequency domain. The key idea here is to represent the response in the frequency domain by a linear combination of complex stochastic basis vectors with undermined coefficients. Motivated by the theoretical analysis in [15], we employ the terms of the preconditioned stochastic Krylov subspace as basis vectors. Note that for the choice of preconditioner used in the present investigation and the random parameterization of the system, the basis vectors become equivalent to the terms of the perturbation series. Subsequently, two variants of the stochastic Bubnov-Galerkin (BG) scheme are employed for computing the undetermined terms in the reduced basis representation - an exact and a zero-order BG scheme. It is shown that the present approach leads to an explicit expression for the response as a function of the random system parameters, which enables a complete statistical characterization of the system response in a computationally efficient fashion.

We present extensive numerical studies on a model problem to demonstrate that highly accurate results can be obtained for the first two statistical moments of the response and the mean of the maximum blade amplitude. The results obtained using SRBMs are compared with the classical second-order perturbation method and benchmark results computed using MCS. Our results clearly demonstrate that SRBMs can be up to orders of magnitude more accurate than the classical perturbation method. We conclude this paper with an outline of some directions for further research on stochastic reduced basis methods for dynamic analysis.

2. Preliminaries

Let **M**, **K** and **C** denote the system mass, stiffness and damping matrices respectively and *N* be the total number of degrees-of-freedom (dof). Let us assume that mistuning affects only the stiffness matrix **K**. Then the equations of motion in the frequency domain can be written as

$$\mathbf{A}(\boldsymbol{\theta})\mathbf{q}(\boldsymbol{\theta}) = \mathbf{F}, \tag{1}$$

where $\mathbf{q}(\boldsymbol{\theta})$ is the random displacement response and $\boldsymbol{\theta} = \{\theta_i\}$, $i = 1, \ldots, p$ is the vector of p uncertain system parameters. The uncertain parameters are assumed here to be uncorrelated zero-mean Gaussian random variables with standard deviation σ. $\mathbf{F}$ is the external excitation vector chosen to be the engine order excitation force. $\mathbf{A}(\boldsymbol{\theta}) = -\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}(\boldsymbol{\theta})$ is the random dynamic stiffness matrix, where ω is the external excitation frequency and $j = \sqrt{-1}$. Note that for simplicity of notation, we do not explicitly indicate the dependence of the dynamic stiffness matrix and the response on the excitation frequency ω.

If $\mathbf{K}_0$ is the stiffness matrix of the perfectly tuned system, then $\mathbf{K}(\boldsymbol{\theta})$ can be expanded as

$$\mathbf{K}(\boldsymbol{\theta}) = \mathbf{K}_0 + \delta\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{p} \mathbf{K}_i \theta_i , \qquad (2)$$

where $\delta\mathbf{K}$ is the deviation of the stiffness matrix due to mistuning and $\mathbf{K}_i$ is a deterministic matrix. Note that this representation is chosen here for the sake of convenience. When the random system parameters appear nonlinearly in the stiffness matrix, a similar expression can be derived by expanding $\mathbf{K}(\boldsymbol{\theta})$ in terms of orthogonal random polynomials; see, for example, Ghanem and Spanos [16].

Using Eqn. (2), the matrix $\mathbf{A}(\boldsymbol{\theta})$ can be written as

$$\begin{aligned} \mathbf{A}(\boldsymbol{\theta}) &= \mathbf{A}_0 + \delta\mathbf{A} \\ \mathbf{A}_0 &= -\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}_0 \\ \delta\mathbf{A} &= \delta\mathbf{K} \end{aligned} \qquad (3)$$

The fundamental idea of SRBMs is to approximate the solution of Eqn. (1) using a subspace spanned by a set of stochastic basis vectors. Nair [15] presented a theoretical justification for employing the terms of the preconditioned stochastic Krylov subspace as basis vectors. It was shown that the solution of a random algebraic system of equations can be approximated to an arbitrary degree of accuracy using this set of basis vectors. For the representation of the random stiffness matrix in Eqn. (2), and further by employing the matrix $\mathbf{A}_0^{-1}$ as a preconditioner, it can be shown that the terms of the preconditioned stochastic Krylov subspace coincides with the perturbation series expansion. This implies that the same results can be obtained by using the terms of the perturbation series as stochastic basis vectors.

In the present study, we use three basis vectors to represent the solution of Eqn. (1) as

$$\hat{\mathbf{q}}(\boldsymbol{\theta}) = \sum_{i=0}^{3} \xi_i(\boldsymbol{\theta})\psi_i(\boldsymbol{\theta}) = \boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}(\boldsymbol{\theta}), \qquad (4)$$

where $\boldsymbol{\Psi} = [\psi_0(\boldsymbol{\theta})\psi_1(\boldsymbol{\theta})\psi_2(\boldsymbol{\theta})] \in \mathbb{C}^{N\times 3}$ denotes the matrix of stochastic basis vectors. $\boldsymbol{\xi} = \{\xi_0(\boldsymbol{\theta}), \xi_1(\boldsymbol{\theta}), \xi_2(\boldsymbol{\theta})\}^T \in \mathbb{C}^{3\times 1}$ denotes the vector of undetermined coefficients in the reduced basis.

We choose the three terms of the second-order perturbation method as basis vectors. As mentioned earlier, this is equivalent to employing the first three basis vectors spanning the preconditioned stochastic Krylov subspace. We assume that $\mathbf{q}(\boldsymbol{\theta})$ can be well approximated in the subspace spanned by $\psi_0, \psi_1(\boldsymbol{\theta})$ and $\psi_2(\boldsymbol{\theta})$. The first basis vector ψ_0 is obtained by solving for the frequency response of the tuned system, i.e.

$$\psi_0 = \mathbf{A}_0^{-1}\mathbf{F}. \qquad (5)$$

The other two basis vectors are given by

$$\psi_1(\boldsymbol{\theta}) = \sum_{i=1}^{p} \frac{\partial \mathbf{q}}{\partial \theta_i}\theta_i \qquad (6)$$

and

$$\psi_2(\boldsymbol{\theta}) = \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 \mathbf{q}}{\partial \theta_i \partial \theta_j}\theta_i\theta_j . \qquad (7)$$

The response sensitivities appearing in Eqns. (6, 7) can be computed as

$$\frac{\partial \mathbf{q}}{\partial \theta_i} = -\mathbf{A}_0^{-1}\frac{\partial \mathbf{K}}{\partial \theta_i}\psi_0 \qquad (8)$$

and

$$\frac{\partial^2 \mathbf{q}}{\partial \theta_i \partial \theta_j} = \mathbf{A}_0^{-1}\left(\frac{\partial \mathbf{K}}{\partial \theta_i}\mathbf{A}_0^{-1}\frac{\partial \mathbf{K}}{\partial \theta_j}\mathbf{q}_0 + \frac{\partial \mathbf{K}}{\partial \theta_j}\mathbf{A}_0^{-1}\frac{\partial \mathbf{K}}{\partial \theta_i}\mathbf{q}_0\right). \qquad (9)$$

It can be clearly seen from the preceding equations that sensitivity analysis of large-scale systems across a broad range of excitation frequencies will be computationally expensive. Further, at each excitation frequency point, we need to compute an independent set of stochastic basis vectors for obtaining the statistics of the response. Therefore the matrix $\mathbf{A}_0$ needs to be repeatedly inverted at each excitation frequency of interest. This may lead to a significant increase in computational cost particularly when the size of the system is large and/or the response at a large number of frequency points is to be computed.

The efficiency of the procedure described here can be improved by employing the eigenvectors of the tuned system to compute the sensitivities of $\mathbf{q}$ [17]. In the particular case of cyclic structures, $\mathbf{A}_0$ is a circulant matrix. Since these eigenvectors coincide with the eigenvectors of the Fourier matrix $\mathbf{E}$ that diagonalizes $\mathbf{A}_0$, the basis vectors can be computed more efficiently. When the system components have multiple dof, $\mathbf{A}_0$ is a block-circulant matrix that can be block-diagnolized using the transformation $\left(\mathbf{E}^* \otimes \mathbf{I}\right)\mathbf{A}_0\left(\mathbf{E}\otimes\mathbf{I}\right)$, where $*$, $\otimes$, and $\mathbf{I}$ denote the complex conjugate transpose of a matrix, the Kronecker product and an identity matrix of size equal to that of a block in $\mathbf{A}_0$ (i.e., of a blade-disk sector), respectively.

3. Stochastic Subspace Projection

To compute the undetermined coefficients in the stochastic reduced basis representation, we use stochastic variants of the Bubnov-Galerkin (BG) scheme [15]. This involves defining a stochastic residual error vector by substituting Eqn. (4) in (1), which gives

$$\mathbf{r}(\boldsymbol{\theta}) = \mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}(\boldsymbol{\theta}) - \mathbf{F}. \tag{10}$$

For simplicity of notation, the dependence of ψ_1 and ψ_2 on the random vector $\boldsymbol{\theta}$ will not be explicitly shown in the equations that follow. In the BG scheme, the undetermined coefficients are evaluated by enforcing the condition that $\mathbf{r}(\boldsymbol{\theta})$ is orthogonal to $\boldsymbol{\Psi}(\boldsymbol{\theta})$. Two variants of the BG scheme are presented next for the computation of $\boldsymbol{\xi}(\boldsymbol{\theta})$, which arises from the way the orthogonality condition for two random vectors is interpreted.

3.1. Zero-order BG Scheme

Here the undetermined coefficient vector $\boldsymbol{\xi}(\boldsymbol{\theta})$ is determined by enforcing that the stochastic residual $\mathbf{r}(\boldsymbol{\theta})$ is orthogonal to $\boldsymbol{\Psi}(\boldsymbol{\theta})$ in an approximate sense. By considering the inner product of two random vector functions in the Hilbert space of random variables, this involves enforcing the condition

$$\left\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{r}(\boldsymbol{\theta})\right\rangle = 0, \tag{11}$$

where <.> denotes the ensemble average and the superscript $*$ denotes the complex conjugate transpose. Since Eqn. (11) can be interpreted as a zero-order condition [15], this formulation is henceforth referred to as SRBM-BG$_0$. Eqn. (11) ultimately leads to the following 3×3 reduced deterministic system of equations for the coefficients ξ_0, ξ_1 and ξ_2

$$\left\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi} - \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{F}\right\rangle = 0. \tag{12}$$

The deterministic system of equations to be solved for the vector of undetermined coefficients $\boldsymbol{\xi}$ can be written in a compact form as

$$\mathbf{A}_{SRBM-BG_0}\boldsymbol{\xi} = \mathbf{F}_{SRBM-BG_0}, \tag{13}$$

where $\mathbf{A}_{SRBM-BG_0} = \left\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Psi}(\boldsymbol{\theta})\right\rangle$ and $\mathbf{F}_{SRBM-BG_0} = \left\langle \boldsymbol{\Psi}^*(\boldsymbol{\theta})\mathbf{F}\right\rangle$ denote the reduced dynamic stiffness matrix and the force vector, respectively. Explicit expressions for their elements are given in Appendix A for the case when the elements of $\boldsymbol{\theta}$ are uncorrelated zero-mean Gaussian random variables.

Once the coefficients ξ_0, ξ_1 and ξ_2 are computed by solving the deterministic reduced-order problem in Eqn. (13), the mean $(\mu_{\hat{\mathbf{q}}})$ and covariance $\left(\Sigma_{\hat{\mathbf{q}}}\right)$ of the system response at each excitation frequency can be computed as

$$(\mu_{\hat{\mathbf{q}}}) = \langle \hat{\mathbf{q}} \rangle = \langle \xi_0\psi_0 + \xi_1\psi_1 + \xi_2\psi_2 \rangle \tag{14}$$

and

$$\begin{aligned}\left(\Sigma_{\hat{\mathbf{q}}}\right) &= \left\langle \hat{\mathbf{q}}(\boldsymbol{\theta})\hat{\mathbf{q}}^*(\boldsymbol{\theta})\right\rangle = \left\langle \boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi}\boldsymbol{\xi}^*\boldsymbol{\Psi}(\boldsymbol{\theta})^*\right\rangle \\ &= \sum_{i=0}^{p}\sum_{j=0}^{p}\xi_i\xi_j^*\left\langle \psi_i(\boldsymbol{\theta})\psi_j^*(\boldsymbol{\theta})\right\rangle\end{aligned} \tag{15}$$

The final expressions for the mean and covariance matrix are given in Appendix B.

3.2. Exact BG Scheme

By specifying that the stochastic residual error is orthogonal to the approximating space of basis vectors with probability one, an alternative formulation can be derived, henceforth referred to as SRBM-BG. In contrast to the SRBM-BG$_0$ formulation, this projection scheme leads to random function models for the undetermined coefficients since the following reduced–order *random* system of equations has to be solved

$$\mathbf{A}_{SRBM-BG}\xi = \mathbf{F}_{SRBM-BG}, \tag{16}$$

where $\mathbf{A}_{SRBM-BG} = \mathbf{\Psi}^*(\mathbf{\theta})\mathbf{A}(\mathbf{\theta})\mathbf{\Psi}(\mathbf{\theta})$ and $\mathbf{F}_{SRBM-BG} = \mathbf{\Psi}^*(\mathbf{\theta})\mathbf{F}$ are the reduced-order random matrix and the force vector, respectively.

It can be seen that explicit computation of a random function description of the undetermined coefficients will involve the symbolic inversion of $\mathbf{A}_{SRBM-BG}$. Note that this is readily possible here since only three vectors are used in the reduced basis representation of the random displacement response $\mathbf{q}(\mathbf{\theta})$. Hence, an explicit expression for the system response as a function of the random variables can be achieved. However, since the resulting approximation is a highly nonlinear function of the random system parameters, analytical characterization of the response statistics is no longer readily possible. Fortunately, Monte Carlo simulation schemes can be applied to efficiently compute the response statistics by sampling the stochastic reduced basis representation with random function models for the undetermined coefficients.

4. A Note on Theoretical Aspects

For the sake of completeness, we cite some important theoretical properties of SRBMs, which were derived in [17]. The first result mentioned in the previous section states that the solution of a linear random algebraic system of equations with a non-singular coefficient matrix always lie in the stochastic Krylov subspace. This guarantees that nearly exact results can be computed provided a sufficient number of stochastic basis vectors are deployed in the response representation. However, the computational cost and memory requirements increase significantly when the higher-order basis vectors are used. Fortunately, for many problems of practical interest, three basis vectors are sufficient to achieve highly accurate results; see also Section 5.

A desirable feature of any stochastic subspace projection scheme is that some measure of the error in the approximated solution must converge when the number of basis vectors is increased. In [17] it was proved for SRBM-BG$_0$ that the A-norm of the error is mean square convergent. For the exact BG scheme, it was conjectured that the A-norm of the error converges in probability. However, these results hold only for Hermitian positive definite matrices.

In the context of frequency response analysis of linear stochastic structural systems, the coefficient matrix $\mathbf{A}(\mathbf{\theta})$ turns out to be complex symmetric. For such non-Hermitian matrices, convergence results can be established for the L_2 norm of the residual only if an oblique stochastic subspace projection scheme is used. This involves incorporating the Petrov-Galerkin condition that the residual error is orthogonal to the stochastic subspace $\mathbf{A}(\mathbf{\theta})\,\mathbf{\Psi}(\mathbf{\theta})$. Equations similar to those presented earlier in Appendix A can be readily derived for the reduced-order terms when this oblique stochastic subspace projection scheme is employed; see Appendix A of [15] for details.

In the present study, we present results only for the orthogonal BG projection scheme. We show that, even though this scheme is not provably optimal for non-Hermitian matrices, highly accurate results can be obtained for the response statistics.

5. Results and Discussion

In this section, we present numerical studies conducted using a simplified discrete model [3] of a weakly coupled 10-blade assembly, although the SRBM schemes can be applied to more sophisticated models. One advantage of this simple model is the attendant ease of implementation. Further, this problem has the essential characteristics of real-life problems, and can help in gaining a preliminary understanding of the computational properties of SRBMs applied to mistuning analysis of periodic systems.

The model used here consists of a cyclic chain of masses (each has one grounded spring and damper) interconnected by identical linear springs representing the interblade coupling. Hence the total number of random variables p equals the number of dof N. The mass and damping of each blade is considered to be identical and represented by m and c respectively. The modal stiffness of the i-th blade is represented by k_i, where $k_i = k_0(1+\sigma\theta_i)$ and θ_i is the random variation in the i-th blade stiffness. θ_i are assumed to be uncorrelated zero-mean Gaussian random variables with standard deviation σ. The values of the model parameters are taken from reference [18] to be m=0.0114 kg, k_0 =430000 N/m (the nominal blade natural frequency ω_0=6141.6rad/s). The viscous damping ratio is fixed as ζ=0.01.

The external excitation vector chosen to be the engine order excitation force (see, for example, reference [7]), which can be written as $\mathbf{F} = F_m\left\{e^{j\phi_i}\right\}^T$, where $\phi_i = 2\pi n(i-1)/N$, $i=1,\ldots,N$, F_m is the amplitude of the excitation force, ϕ_i is the phase angle of force for the i-th blade component, and n is the engine order.

Numerical studies were conducted for the case when the standard deviation of the mistuning, σ=0.05. Three different values of the non-dimensional coupling strength parameter are considered: weak interblade coupling R=0.1, moderately weak interblade coupling R=0.325 and strong interblade coupling R=0.5. $R^2 = k_c/k_0$ is defined as a non-dimensional interblade coupling parameter.

Results are obtained using the exact and zero-order BG schemes. We show here the mean and variance of the frequency response of a typical degree of freedom, i.e., the first blade component. The response statistics computed using SRBMs are compared to the classical second-order (PM2) perturbation method and benchmark results generated by applying MCS. Note that in all the figures presented, the statistical moments are plotted as functions of the frequency of the first engine order excitation.

Figures 1 and 2 shows the mean and standard deviation of the first component computed using SRBM-BG, SRBM-BG$_0$, PM2 and MCS for $R=0.1$. It can be seen that SRBM-BG provides highly accurate results despite the fact that the coupling is weak and the damping is low. SRBM-BG$_0$ misses the higher amplitudes within the region of clustered resonant frequencies. PM2 drastically fails to predict these amplitudes and therefore cannot be applied to systems with such parameters. Figure 3 displays the mean of the maximum amplitude among the blades. It is shown that while PM2 clearly over predicts MCS, SRBM-BG gives excellent agreement with the benchmark results.

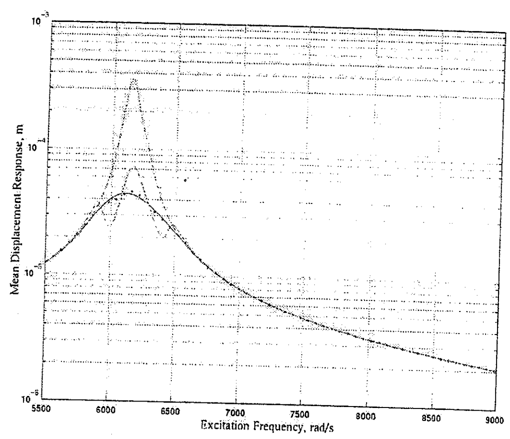

Figure 1: Mean of the first component amplitude for $R=0.1$. The solid line (—) represents exact results obtained by MCS, the dots represent SRBM-BG, the dashed line (—) line represents SRBM-BG$_0$ and the dash-dotted line (–.) represents PM2 results.

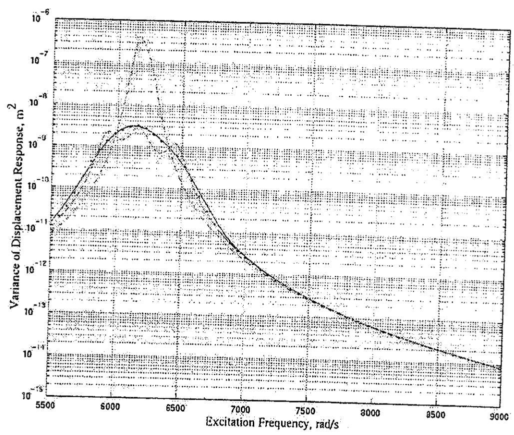

Figure 2: Variance of the first component amplitude as a function of excitation frequency for $R=0.1$. Same legend as in Figure 1.

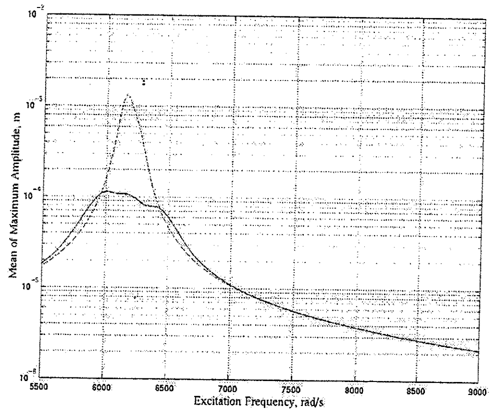

Figure 3: Mean of the maximum amplitude among the blades across the frequency region of interest, for $R = 0.1$. The solid line (—) represents exact results obtained by MCS, the dots represent SRBM-BG and the dashed line (—) represents PM2 results.

It is well known that in the case of weak coupling (i.e., $R = 0.1$), significant errors result when the classical perturbation method is used. To circumvent this problem, Wei and Pierre [3] proposed to use a modified perturbation method. In their approach, the coupling parameter is used as perturbation rather than the mistuning parameter. Furthermore, for systems where the ratio of mistuning to coupling $\sigma/R^2 > O(1)$, it was concluded that only MCS could be effectively applied. Figure 3 shows that under the same conditions of coupling and mistuning strengths, SRBM-BG gives highly accurate results and at a lower cost.

Figures 4 and 5 display the mean and standard deviation of the first component system amplitudes when the interblade coupling is moderately weak, i.e., $R = 0.325$. Here SRBM-BG and SRBM-BG$_0$ match very well with MCS results. PM2 misses the peak mean amplitude but is accurate elsewhere. However, it fails to predict the second moment at different excitation frequencies, see Figure 5.

When the interblade coupling is strong, i.e., $R = 0.5$, SRBM-BG and SRBM-BG$_0$ gives accurate results even when the damping ratio is low ($\zeta = 1\%$). This is illustrated in Figures 6 and 7. In contrast to the stochastic reduced basis approach, the traditional PM2 (which is based on perturbation of the tuned system) is valid only for ($\sigma/R^2 \leq O(1)$ and $\sigma/\zeta \leq O(1)$.

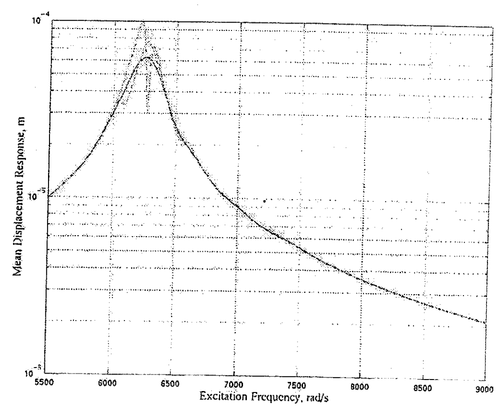

Figure 4: Mean of the first component amplitude for $R = 0.325$. Same legend as in Figure 1.

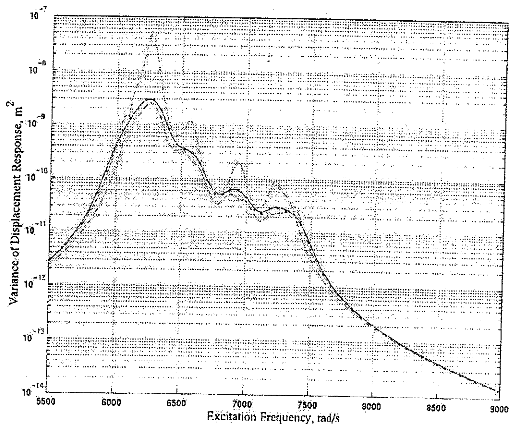

Figure 5: Variance of the first component amplitude as a function of excitation frequency for $R = 0.325$. Same legend as in Figure 1.

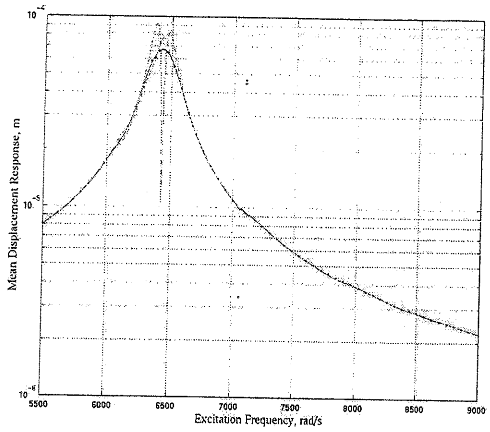

Figure 6: Mean of the first component amplitude for $R = 0.5$. Same legend as in Figure 1.

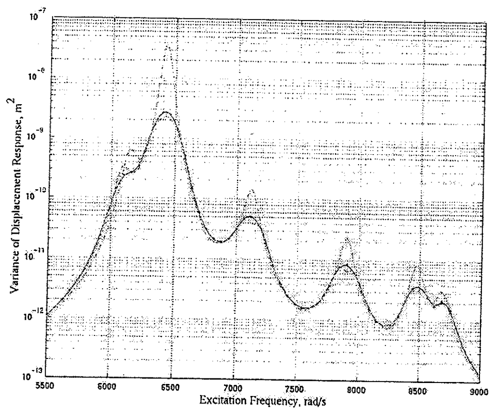

Figure 7: Variance of the first component amplitude for $R = 0.5$. Same legend as in Figure 1.

The numerical results obtained for this example problem clearly depicts the accuracy of SRBMs. In particular, the response statistical moments computed using SRBM-BG can be orders of magnitude more accurate than the classical perturbation method.

6. Concluding Remarks

In this paper, we present a stochastic reduced basis approach for computing the forced response statistics of mistuned bladed disk assemblies. The fundamental idea is to approximate the frequency domain response using the subspace spanned by the first three terms of the stochastic Krylov subspace. For the model problem considered, this is shown to be equivalent to employing the terms of the perturbation series as stochastic basis vectors. Subsequently, two stochastic variants of the Bubnov-Galerkin scheme were presented for computing the undetermined coefficients in the reduced basis representation. It is shown that this allows us to arrive at explicit expressions for the system response as a function of the random system parameters. This in turn enables an efficient statistical characterization of the system response. Some theoretical properties of the Bubnov-Galerkin scheme are also outlined.

Extensive numerical studies on a model problem are presented to test the veracity of the present approach. The results computed using the stochastic reduced basis methods (SRBMs) have been compared with the classical second-order perturbation method and benchmark results generated using Monte Carlo simulation. It is shown that SRBMs give accurate results for the response statistics across a wide range of coupling strengths. In particular, the results clearly demonstrate that SRBMs can be orders of magnitude more accurate than the classical second-order perturbation method, particularly for the mean of the maximum blade displacement.

Even though, the results presented here are for a simple model problem, SRBMs can be readily applied to mistuning analysis of bladed disks analyzed using large-scale finite element models. It also remains to be seen whether employing the oblique stochastic subspace projection outlined in this paper can further improve the accuracy of the response statistics.

Acknowledgments

This work was supported by the University Technology Partnership for Design—a partnership between Rolls Royce Plc., BAE Systems and the Universities of Southampton, Cambridge and Sheffield.

Appendix A

Here, we present expressions for the elements of the reduced matrix $\mathbf{A}_{SRBM-BG_0}$ and vector $\mathbf{F}_{SRBM-BG_0}$ when the elements of $\boldsymbol{\theta}$ are uncorrelated zero-mean Gaussian random variables with standard deviation σ.

$$\mathbf{A}_{SRBM-BG_0}(1,1)=\psi_0^{*}\mathbf{A}_0\psi_0$$

$$\mathbf{A}_{SRBM-BG_0}(1,2)=\sigma^2\psi_0^{*}\sum_{i=1}^{p}\left\{\frac{\partial \mathbf{K}}{\partial\theta_i}\frac{\partial \mathbf{q}}{\partial\theta_i}\right\}$$

$$\mathbf{A}_{SRBM-BG_0}(1,3)=\sigma^2\psi_0^{*}\mathbf{A}_0\sum_{i=1}^{p}\frac{\partial^2 \mathbf{q}}{\partial\theta_i^2}$$

$$\mathbf{A}_{SRBM-BG_0}(2,1)=\sigma^2\sum_{i=1}^{p}\left\{\frac{\partial \mathbf{q}^*}{\partial\theta_i}\frac{\partial \mathbf{K}}{\partial\theta_i}\right\}\psi_0$$

$$\mathbf{A}_{SRBM-BG_0}(2,2)=\sigma^2\sum_{i=1}^{p}\left(\frac{\partial \mathbf{q}^*}{\partial\theta_i}\mathbf{A}_0\frac{\partial \mathbf{q}}{\partial\theta_i}\right)$$

$$\mathbf{A}_{SRBM-BG_0}(2,3)=\sum_{i,j,k,l=1}^{p}\left\langle\theta_i\theta_j\theta_k\theta_l\right\rangle\left(\frac{\partial \mathbf{q}^*}{\partial\theta_i}\frac{\partial \mathbf{K}}{\partial\theta_j}\frac{\partial^2 \mathbf{q}}{\partial\theta_k\partial\theta_l}\right)$$

where

$$\left\langle\theta_i\theta_j\theta_k\theta_l\right\rangle=\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk},$$

and $\delta_{ij}=\begin{cases}1, \text{ if } i=j\\ 0, \text{ if } i\neq j\end{cases}$

$$\mathbf{A}_{SRBM-BG_0}(3,1)=\sigma^2\left\{\sum_{i=1}^{p}\frac{\partial^2 \mathbf{q}^*}{\partial\theta_i^2}\right\}\mathbf{A}_0\boldsymbol{\Psi}_0$$

$$\mathbf{A}_{SRBM-BG_0}(3,2)=\sum_{i,j,k,l=1}^{p}\left\langle\theta_i\theta_j\theta_k\theta_l\right\rangle\left(\frac{\partial^2 \mathbf{q}^*}{\partial\theta_i\partial\theta_j}\frac{\partial \mathbf{K}}{\partial\theta_k}\frac{\partial \mathbf{q}}{\partial\theta_l}\right)$$

$$\mathbf{A}_{SRBM-BG_0}(3,3)=\sum_{i,j,k,l=1}^{p}\left\langle\theta_i\theta_j\theta_k\theta_l\right\rangle\left(\frac{\partial^2 \mathbf{q}^*}{\partial\theta_i\partial\theta_j}\mathbf{A}_0\frac{\partial^2 \mathbf{q}}{\partial\theta_k\partial\theta_l}\right)$$

$$\mathbf{F}_{SRBM-BG_0}=\left\{\boldsymbol{\Psi}_0\quad 0\quad \sigma^2\sum_{i=1}^{p}\frac{\partial^2 \mathbf{q}}{\partial\theta_i^2}\right\}^{*}\mathbf{F}$$

Appendix B

Here, we present expressions for the mean $\left(\mu_{\hat{\mathbf{q}}}\right)$ and covariance $\left(\Sigma_{\hat{\mathbf{q}}}\right)$ of the system response when the elements of $\boldsymbol{\theta}$ are uncorrelated zero-mean Gaussian random variables with standard deviation σ.

$$\left(\mu_{\hat{\mathbf{q}}}\right)=\left\langle\xi_0\psi_0+\xi_1\sum_{i=1}^{p}\frac{\partial \mathbf{q}}{\partial\theta_i}\theta_i+\xi_2\sum_{i=1}^{p}\sum_{j=1}^{p}\frac{\partial^2 \mathbf{q}}{\partial\theta_i\partial\theta_j}\theta_i\theta_j\right\rangle$$

$$\left(\mu_{\hat{\mathbf{q}}}\right)=\xi_0\psi_0+\xi_2\sigma^2\sum_{i=1}^{p}\frac{\partial^2 \mathbf{q}}{\partial\theta_i^2}$$

$$\left(\Sigma_{\hat{\mathbf{q}}}\right)=\left\langle\xi_0\xi_0^*\psi_0\psi_0^*+\xi_0\xi_1^*\psi_0\sum_{i=1}^{p}\frac{\partial \mathbf{q}^*}{\partial\theta_i}\theta_i+\mathbb{Z}\;\xi_0\xi_2^*\psi_0\sum_{i,j=1}^{p}\frac{\partial^2 \mathbf{q}^*}{\partial\theta_i\partial\theta_j}\theta_i\theta_j\right\rangle+$$

$$\left\langle\xi_1\xi_0^*\left(\sum_{i=1}^{p}\frac{\partial \mathbf{q}}{\partial\theta_i}\theta_i\right)\psi_0^*+\xi_1\xi_1^*\sum_{i=1}^{p}\frac{\partial \mathbf{q}}{\partial\theta_i}\theta_i\sum_{i=1}^{p}\frac{\partial \mathbf{q}^*}{\partial\theta_i}\theta_i+\xi_1\xi_2^*\sum_{i=1}^{p}\frac{\partial \mathbf{q}}{\partial\theta_i}\theta_i\sum_{i,j=1}^{p}\frac{\partial^2 \mathbf{q}^*}{\partial\theta_i\partial\theta_j}\theta_i\theta_j\right\rangle+$$

$$\left\langle\xi_2\xi_0^*\left(\sum_{i,j=1}^{p}\frac{\partial^2 \mathbf{q}}{\partial\theta_i\partial\theta_j}\theta_i\theta_j\right)\psi_0^*+\xi_2\xi_1^*\sum_{i,j=1}^{p}\frac{\partial^2 \mathbf{q}}{\partial\theta_i\partial\theta_j}\theta_i\theta_j\sum_{i=1}^{p}\frac{\partial \mathbf{q}^*}{\partial\theta_i}\theta_i\right\rangle+$$

$$\left\langle\xi_2\xi_2^*\sum_{i,j=1}^{p}\frac{\partial^2 \mathbf{q}}{\partial\theta_i\partial\theta_j}\theta_i\theta_j\sum_{i,j=1}^{p}\frac{\partial^2 \mathbf{q}^*}{\partial\theta_i\partial\theta_j}\theta_i\theta_j\right\rangle$$

The covariance matrix of the frequency response can be further simplified as

$$\left(\Sigma_{\hat{\mathbf{q}}}\right)=\xi_0\xi_0^*\psi_0\psi_0^*+\sigma^2\xi_0\xi_2^*\psi_0\sum_{i=1}^{p}\frac{\partial^2 \mathbf{q}^*}{\partial^2\theta_i}+\sigma^2\xi_1\xi_1^*\sum_{i=1}^{p}\frac{\partial \mathbf{q}}{\partial\theta_i}\frac{\partial \mathbf{q}^*}{\partial\theta_i}+\sigma^2\xi_2\xi_0^*\left\{\sum_{i=1}^{p}\frac{\partial^2 \mathbf{q}}{\partial^2\theta_i}\right\}\psi_0^*+\xi_2\xi_2^*\sum_{i,j,k,l=1}^{p}\left\langle\theta_i\theta_j\theta_k\theta_l\right\rangle\frac{\partial^2 \mathbf{q}}{\partial\theta_i\partial\theta_j}\frac{\partial^2 \mathbf{q}^*}{\partial\theta_k\partial\theta_l}.$$

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