On stress concentration on nearly flat contacts

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Abstract: Fretting fatigue can severely damage components subjected to oscillatory tangential loads, leading to a dramatic reduction in fatigue life and causing catastrophic ruptures. A conservative approach that can be used when considering the effect of stress concentration induced by fretting is to ensure that the peak stress is smaller than the fatigue limit of the material. However, this depends on details of the geometry as well as loading conditions. In the present work, the contact problem of a flat rounded punch in contact with a half-plane is considered, where a dovetail joint contact geometry is approximated and the classical Hertzian contact is retrieved in the limit. Developing the analytical results given by Ciavarella, Hills and Monno, an approximate Hertzian equivalent solution using Cattaneo superposition is obtained, leading to a simple formula to estimate the maximum tangential stress as a function of the load parameter $Q/(fP)$ and geometric parameter a/b. The accuracy of the formula is checked numerically. The proposed formula gives a maximum error as low as 4 per cent in the case of zero bulk loads. For non-zero bulk loads an analytical solution is possible for the Hertzian case for moderate bulk. This leads to a second general formula containing the three dependencies (geometry, tangential load and bulk stress), which also gives a very good approximation for rounded flat and larger bulk loads, the error being generally well below 10 per cent.

Keywords: fretting fatigue, fatigue life, flat rounded punch, dovetail joint, maximum tangential stress

1 INTRODUCTION

Fretting fatigue is known to occur in mechanical devices where one assembly of two components is under oscillatory tangential load. The localized relative motion associated with fretting fatigue can cause localized damage and induce accelerated crack initiation, thus affecting the fatigue life of the mechanical device. Nowadays, fretting fatigue is of particular interest in the estimation of the fatigue life of aeronautical components such as discs and blades in turbine engines, which are connected via dovetail joints (Fig. 1a). Both theoretical and experimental studies on fretting fatigue have been mostly focused on simple geometries, such as the Hertzian problem, even though dovetail joints can be more accurately reproduced by flat rounded punches. The contact problem of a flat rounded punch in contact with an elastic half-plane has recently been solved by Ciavarella *et al.* [**1**]. In particular, they considered the problem shown in Fig. 1b, where a flat rounded punch is subjected to an oscillatory tangential load Q and pushed in contact with an elastic half-plane by a constant force P. The half-plane is loaded by a uniform remote bulk stress σ_{bulk}, cycling in phase with the tangential load (because generally the source of both components is the same).

An important geometric parameter in the flat rounded punch contact problem is the aspect ratio a/b between the semi-width of the flat region, a, and the semi-width of the whole contact region, b. Notice that the Hertzian punch problem (cylindrical profile) is retrieved for $a/b \to 0$, while the flat punch case is retrieved for $a/b \to 1$ (although not exactly, being the half-plane elasticity approximate in this case).

More recently, Giannakopoulos *et al.* [**2**] considered the contact problem of a flat rounded punch in contact with a finite thick elastic slab and introduced the concept of *notch analogue.* They compared the case of a rounded flat contact with the case of a rounded crack and the Creager–Paris equations for the latter case were modified to fit the stress field for the contact case. However, the notch analogue remains simply a stress concentration criterion for initiation in the simplest form, i.e. conservatively assuming that any crack initiated may propagate and ultimately induce failure. It is well known that, particularly for severe stress concentrations, initiation may only induce self-arrested cracks. Further, in the case of cracks, it is well known

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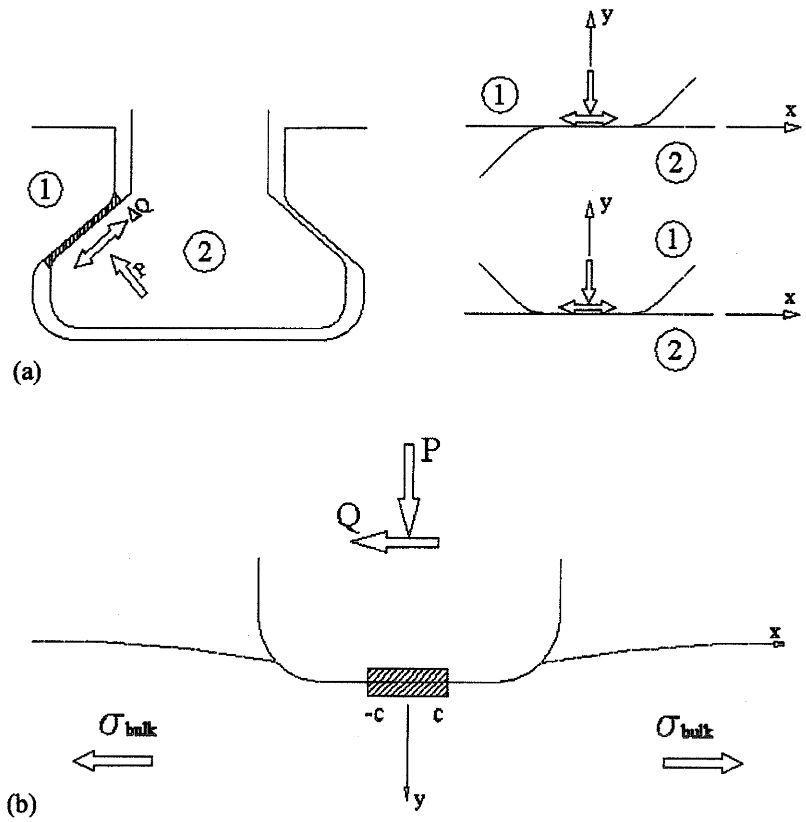


Fig. 1 (a) Schematic of dovetail joints. (b) A flat rounded punch in contact with an elastic half-plane

that the appropriate condition becomes the threshold on the stress intensity range amplitude, rather than the peak stress.

Supposing, however, that the stress concentration is not too large, so that a peak stress criterion is realistically a design methodology, clearly it is the peak stress only, induced by the contact, that matters, and Giannakopoulos *et al.* [**2**] have already given a simple formula for a limited range of conditions, namely a nearly flat punch under full sliding conditions (when bulk stress effects on shear traction can be neglected, as well as, in turn, tensile surface stresses). The maximum surface tangential stress σ_{xx} would also be corrected for finite thickness on one of the contacting bodies, but in general for the partial slip condition, it depends on the exact loading condition.

Turning to stating the problem in mathematical terms, the σ_{xx} peak is localized at the trailing edge of the punch, i.e. the point (b,0) in Fig. 1b. For elastically similar half-planes in contact, $p(x)$ and $q(x)$, i.e. the normal and tangential tractions respectively, are uncoupled, and using the classical integrals of the point force solution, as in references [**3**] and [**4**],

$$\sigma_{xx}(b, 0) = \frac{2}{\pi}\int_{-b}^{b} \frac{q(t)}{b-t}\mathrm{d}t + \sigma_{\mathrm{bulk}} + \sigma_{xx}^{\mathrm{P}} \qquad (1)$$

where the notation in reference [**2**] has been used. In particular, regarding the σ_{xx}^{P} term, the correction due to the finite thickness of the slab over which the punch slides will receive no further attention in the present paper. The Giannakopoulus *et al.* [**2**] result, for the case of full sliding, i.e. $q(x) = f\,p(x)$, reads

$$\sigma_{xx}(b, 0) = \frac{1.926}{\sqrt{b-a}}\frac{Q}{\sqrt{\pi b}} + \sigma_{\mathrm{bulk}} + \sigma_{xx}^{\mathrm{P}} \qquad (2)$$

This formula is exact only under full sliding conditions and by direct integration of equation (1) can be found to overestimate significantly the maximum surface tangential stress, up to 15 per cent, when $Q < fP$. The aim of this work is to introduce simple and more accurate formulae for estimating the maximum surface tangential stress, and thus to estimate the non-linear effect of bulk stress into the fretting-induced stress concentration. Moreover, in order to judge the accuracy of the various formulae, approximated analytical results are compared with numerical results derived by integrating the general expression (1) by means of Chebyshev polynomials of the second type.

2 AN ALTERNATIVE EXPRESSION FOR THE CONTACT PRESSURE DISTRIBUTION OF A FLAT ROUNDED PUNCH

The contact pressure distribution at the interface between a flat rounded punch and an elastic half-plane has been given by Ciavarella *et al.* [1]. However, in this work, an alternative expression is used as presented below:

$$x(\varphi) = b\,\sin(\varphi)$$

$$p(\varphi)\frac{b}{P}=\frac{1}{\pi}\,\frac{2mb\left(\dfrac{\pi}{2}-\varphi_0\right)\cos(\varphi)+mb\ln\left|\dfrac{\sin(\varphi+\varphi_0)}{\sin(\varphi-\varphi_0)}\right|+D\ln\left|\dfrac{\cos(\varphi)-\cos(\varphi_0)}{\cos(\varphi)-\cos(\varphi_0)}\right|}{2D\,\cos(\varphi_0)-\dfrac{mb}{2}\sin(2\varphi_0)+mb\left(\varphi_0-\dfrac{\pi}{2}\right)} \tag{3}$$

where

$\varphi_0 = \arcsin(a/b)$
R = radius of the rounded edge
$m = -1/R$
$D = -a/R$
P = normal load

Equilibrium along the normal direction implies that

$$P=\frac{b}{A}\left[2D\cos(\varphi_0)-\frac{mb}{2}\sin(2\varphi_0)+mb\left(\varphi_0-\frac{\pi}{2}\right)\right] \tag{4}$$

where

A = composite modulus [for the same materials in plane strain $A = 4(1-\nu^2)/E$]
ν = Poisson's ratio
E = Young's modulus

Substituting m and φ_0 as defined above in equation (4) and rearranging, a simple relation between the normal load P and the contact width $2b$ is then derived:

$$\frac{PAR}{b^2}=\frac{\pi}{2}-\frac{a}{b}\sqrt{1-\left(\frac{a}{b}\right)^2}-\arcsin\left(\frac{a}{b}\right) \tag{5}$$

Also, recalling that $\varphi = \arcsin(x/b)$ and $\varphi_0 = \arcsin(a/b)$, from equation (3) it follows that

$$p(x)\frac{\pi^2 b}{2P}=\left[2\arcsin\left(\frac{a}{b}\right)-\pi\right]\frac{\sqrt{b^2-x^2}}{b}$$
$$-\ln\left(\left|\frac{x\sqrt{b^2-a^2}+a\sqrt{b^2-x^2}}{x\sqrt{b^2-a^2}-a\sqrt{b^2-x^2}}\right|^{(x/b)}\left|\frac{\sqrt{b^2-x^2}+\sqrt{b^2-a^2}}{\sqrt{b^2-x^2}-\sqrt{b^2-a^2}}\right|^{(a/b)}\right) \tag{6}$$

This relation can be rearranged as the sum of a Hertzian contribution $p_{her}(x)$ and a corrective solution $p_{cor}(x)$, i.e. $p_{cor}(x) = p_{her}(x) + p_{cor}(x)$, where the Hertzian term is given by

$$p_{her}(x)=-\frac{b}{AR}\left[1-\frac{2}{\pi}\arcsin\left(\frac{a}{b}\right)\right]\sqrt{1-\left(\frac{x}{b}\right)^2}$$
$$=-\frac{b}{AR^*}\sqrt{1-\left(\frac{x}{b}\right)^2} \tag{7}$$

It is worth noticing that for $a = 0$ (the Hertzian problem) the load relation gives $2P/(\pi b) = b/(AR)$, whereas in general R^* is an equivalent Hertzian radius defined as

$$R^*=\frac{R}{1-(2/\pi)\arcsin(a/b)} \tag{8}$$

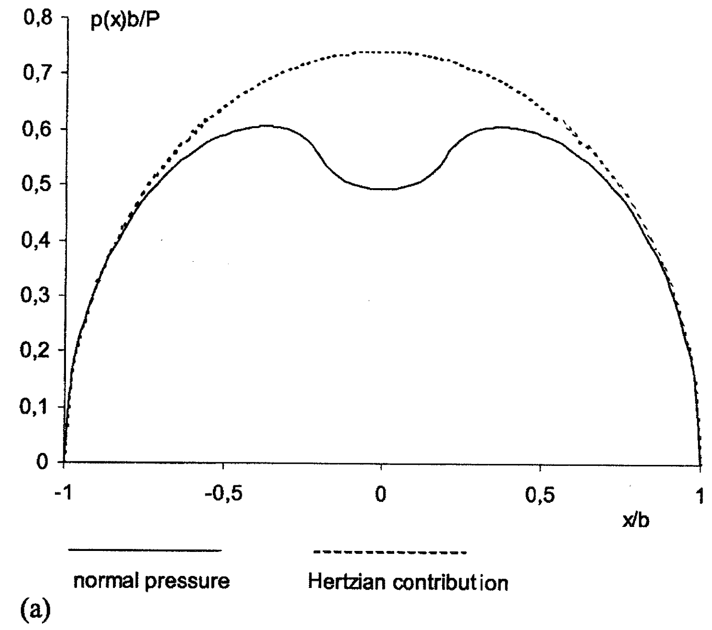
In Fig. 2 the pressure distribution for two flat rounded punches with different aspect ratios a/b is plotted together with the Hertzian pressure contribution. It is clearly shown that the pressure distribution close to the rounded edge is basically given by the Hertzian equivalent term, whereas the Hertzian contribution alone overestimates the contact pressure as the centre of the contact path is approached. Thus, the larger the ratio a/b the bigger the difference between the actual and Hertzian pressure contributions, and in the limit for $a/b = 0$ the pure Hertzian problem is recovered.

3 AN APPROXIMATED EXPRESSION FOR THE TANGENTIAL TRACTIONS DISTRIBUTION OF A FLAT ROUNDED PUNCH

If the bulk stress is zero, the tangential tractions $q(x)$ can be approximated as the sum of two elliptic distributions of opposite signs, as proposed above for the contact pressure distribution:

$$q(x)=f\frac{b}{R^*A}\sqrt{1-\left(\frac{x}{b}\right)^2}-f\frac{c}{R^*A}\sqrt{1-\left(\frac{x}{c}\right)^2} \tag{9}$$

where c is the semi-width of the stick zone and R^* is given by equation (8). Invoking equilibrium along the x direction and integrating equation (9) over the contact area, it follows that



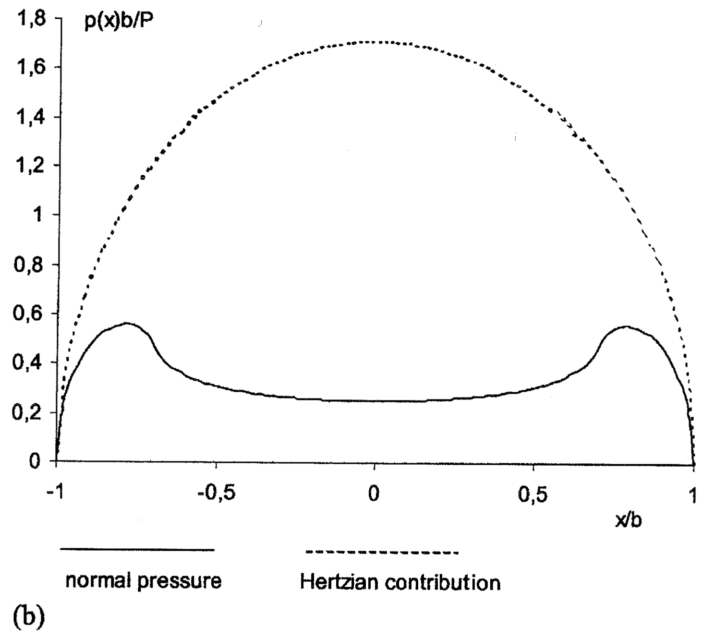


Fig. 2 Actual normal pressure (solid line) and Hertzian contribution (dashed line) for (a) $a/b = 0.2$ and (b) $a/b = 0.7$

$$Q = \frac{f\pi}{2R^*A}(b^2 - c^2) \tag{10}$$

from which the unknown c can be derived. In Fig. 3, the analytical distribution $q(x)$ proposed by Ciavarella *et al.* [**1**] (solid lines) is compared with the approximated distribution (dashed lines) given in equation (9) for several different values of the ratio a/b and under different load conditions $Q/(fP)$.

It can be seen that the approximated solution gives an accurate estimation of the actual tangential tractions and that the smaller the values a/b and $Q/(fP)$ the larger is the accuracy of the approximated solution. The simple formula (9) can readily be employed to estimate the maximum value of the tangential surface stress σ_{xx}, as described in the next section.

4 THE PEAK VALUE OF φ_{xx}: ZERO BULK STRESS

Using the expression in formula (9) and the formulae reported in classical contact mechanics handbooks [**3, 4**], the peak tangential stress σ_{xx} in the case of zero bulk stress is given by

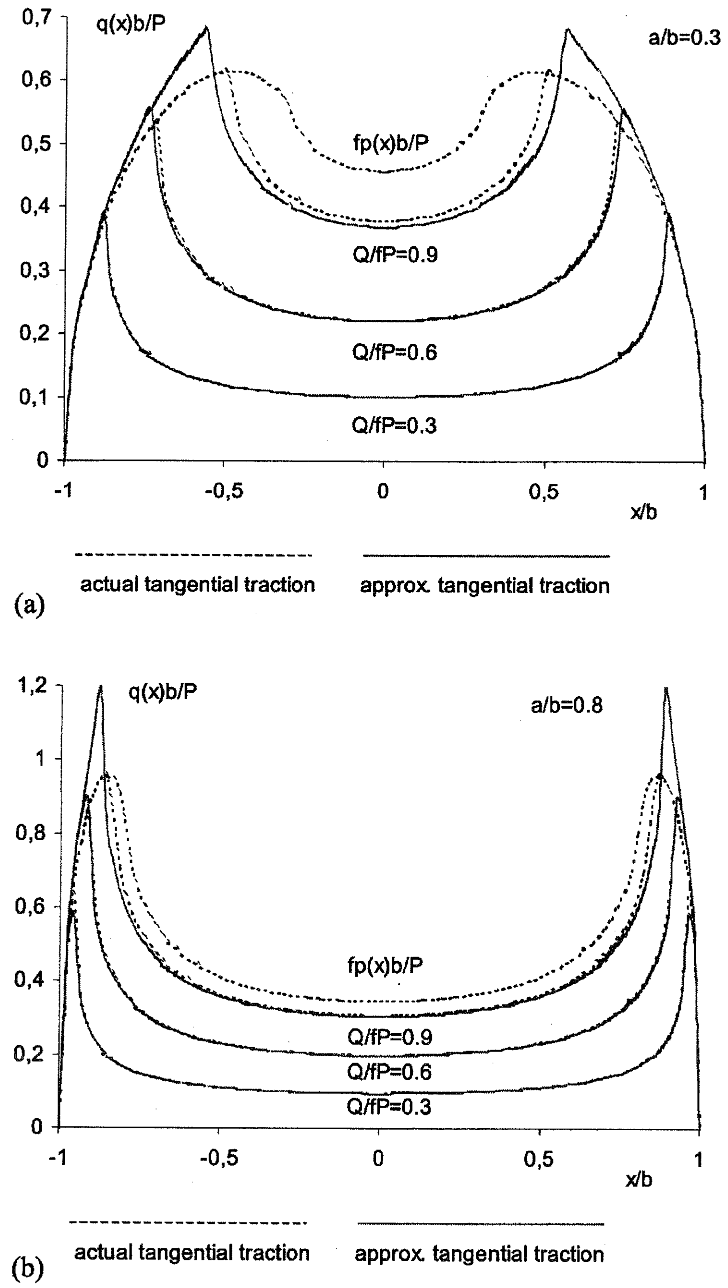


Fig. 3 Actual and approximated tangential tractions $q(x)$: (a) for $a/b = 0.3$; (b) for $a/b = 0.8$

$$\sigma_{xx}(b, 0) = 2f\frac{b}{R^*A}\sqrt{1-\left(\frac{c}{b}\right)^2} \tag{11}$$

and the maximum point is localized at the trailing edge of the contact path ($x = b$). Thus, considering the relations in equations (5) and (10) and introducing the corrective stress σ^{P}_{xx} due to the finite thickness of the slab, it follows that

$$\sigma_{xx}(b, 0) = 2f p_{\max} k\sqrt{\frac{Q}{fP}} + \sigma^{\mathrm{P}}_{xx} \tag{12}$$

with $p_{\max} = 2P/(\pi b)$ (note that it is not the maximum pressure, except for the Hertzian case) and the constant k given by

$$k = \sqrt{\frac{1-(2/\pi)\mathrm{asin}\,(a/b)}{1-(2/\pi)\mathrm{asin}(a/b)-(2/\pi)(a/b)\sqrt{1-(a/b)^2}}} \tag{13}$$

A comparison between this approximated formulation and the exact solution given by equation (1) with $(\sigma_{xx})_{\mathrm{bulk}} = 0$ is presented in Fig. 4, where the percentage error between the two solutions is plotted as a function of the ratio a/b and for different ratios $Q/(fP)$. The plot is related to the case $\sigma^{\mathrm{P}}_{xx} = 0$; thus the indenter is in contact

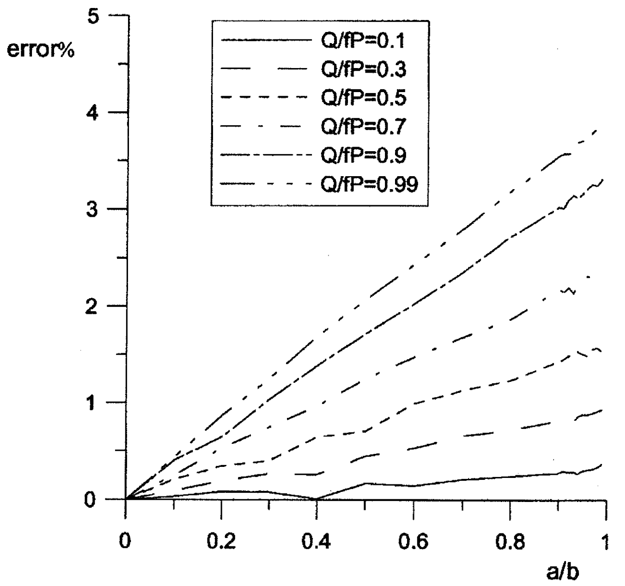


Fig. 4 Percentage error for the maximum value of σ_{xx} as a function of the ratio a/b for different values of $Q/(fP)$

with a half-plane. Clearly, the percentage error is always smaller than 4 per cent, thus showing the good accuracy of the approximated formula (11).

5 THE PEAK VALUE OF φ_{xx}: NON-ZERO BULK STRESS

The case of non-zero bulk stress is now considered. It is worth noticing that the bulk stress σ_{bulk} affects the tangential tractions distribution $q(x)$; thus its effect on the tangential stress σ_{xx} given in equation (1) is non-linear. However, as a first approximation, suppose that it can be neglected:

$$\sigma_{xx}(b, 0) = 2f p_{\max} k\sqrt{\frac{Q}{fP}} + \sigma_{\text{bulk}} + \sigma_{xx}^{\text{P}} \tag{14}$$

This approximation will clearly be closer to correct when the tangential load is high and hence dominating over the bulk stress. In particular, it is exact when $Q/(fP) = 1$. At the other extreme, when the bulk stress is large but the tangential load is negligible, or indeed zero, the formula will give an underestimate of the stress concentration. Although some analytical results are possible for the case of bulk stress only, this would not be possible in general (such as in the case of combined loading or non-Hertzian geometry). Therefore, the correct tangential stress σ_{xx} will be computed numerically by approximating the tangential traction distribution as a linear combination of Chebyscheff polynomials of the second type. In this way the integral equation in the tangential direction is replaced by a linear algebraic system whose inversion gives the coefficient of the Chebyshev polynomial series and thus the actual traction distribution. Details of the technique are not given here, as they are described in the book of Hills *et al.* [**4**]. As an example result, in Fig. 5 the tangential tractions distribution is presented for $a/b = 0.6$ and $Q/(fP) = 0.6$, with $\sigma_{\text{bulk}}[b/(fP)] = 0.5$, determined by the numerical integration described above.

Before moving to the comparison between the approximated solution [equation (14)] and the numerical 'exact' solution, the case of bulk stress alone will be considered in detail, as it is where the approximate solution equation (14) is expected to give largest errors. The solution will also give a more accurate formula for the combined loading case.

6 BULK STRESS ONLY: ANALYTICAL SOLUTIONS FOR THE HERTZIAN PROBLEM

From equation (1) an analytical expression for the maximum tangential stress can be derived in the case of Hertzian contact, i.e. $a = 0$, and weak bulk stress:

$$\sigma_{\text{bulk}} \leqslant 4 f p_{\max}\left(1 - \sqrt{1 - \frac{|Q|}{fP}}\right)$$

Under this condition, in fact, the tangential tractions $q(x)$ are obtained as the sum of two elliptic distributions of opposite signs, as in equation (9), with the only difference being an offset between the two distribution centres (more details are given in reference [**4**]). The resulting peak stress has not been computed analytically before, to the best of the authors' knowledge, but it takes little effort to obtain it from standard Hertzian stress field solutions [**3**]:

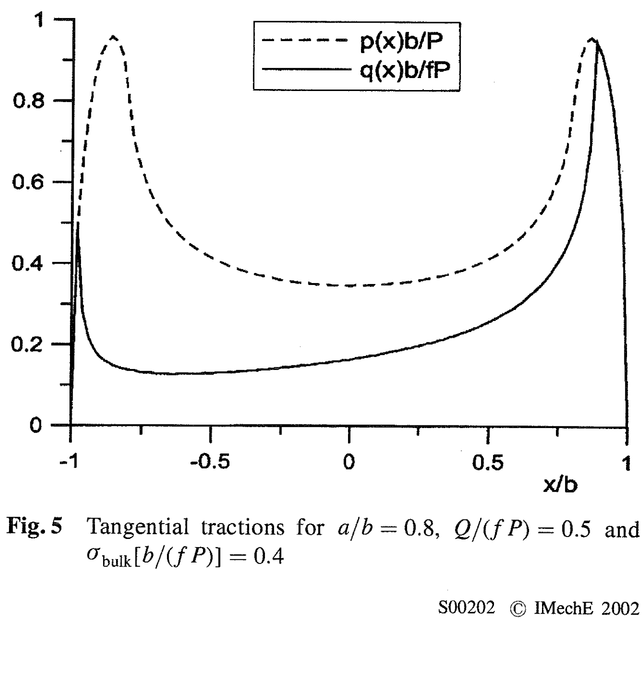


Fig. 5 Tangential tractions for $a/b = 0.8$, $Q/(fP) = 0.5$ and $\sigma_{\text{bulk}}[b/(fP)] = 0.4$

$$\sigma_{xx}(b,0)=2fp_{\max}\sqrt{\left(1+\frac{\sigma_{\text{bulk}}}{4fP_{\max}}\right)^2-\left(1-\frac{|Q|}{fP}\right)}+\frac{\sigma_{\text{bulk}}}{2} \quad (15)$$

Notice that for full sliding conditions $Q/(fP)=1$, equation (15) gives equation (14) exactly. On the other extreme, when $Q/(fP)=0$ (bulk stress only), the formula does not give an exact solution (for bulk stress only is a special case of large bulk), but for very large bulk, when the second term under parentheses can be neglected, it tends again to equation (14). Considering that equation (14) is correct for a shear distribution of full sliding, whereas large bulk stress induces antisymmetrical shear traction, it is clear that both equations (15) and (14) give wrong asymptotic behaviour. However, if the increasing bulk stress term is included, the total stress will tend to the correct value. Comparison with the 'exact' numerical solution in fact shows that the maximum error is limited to only 4 per cent (see Fig. 6), whereas if equation (14) is considered, the error would be extremely high as equation (14) only adds to the bulk stress, the term associated with the tangential load. For bulk stress only, the maximum error is simple to estimate, being as high as 100 per cent in the limit of nearly zero bulk stresses and remaining very high for quite large values of bulk stress [e.g. it is around 25 per cent at $\sigma_{\text{bulk}}/4fp_{\max}=1$].

7 A GENERAL APPROXIMATE SOLUTION

It is clear from the previous sections that equation (15) is the most promising for Hertzian geometries under combined bulk and tangential load. It has been shown that by attention to that geometry, the largest error (occurring when only bulk stress is concerned) is limited to around 4 per cent. The situation becomes clearer from Fig. 7, where the prediction of equation (15) is shown for the entire range of tangential and bulk stress loadings. In particular, for zero bulk, equations (14) and (15) coincide, since in this case the tangential tractions are independent of the bulk. Therefore, equation (14), not shown in the figure for simplicity, would simply correspond to the horizontal line starting from the initial values of the lines in the figure, plotting equation (15).

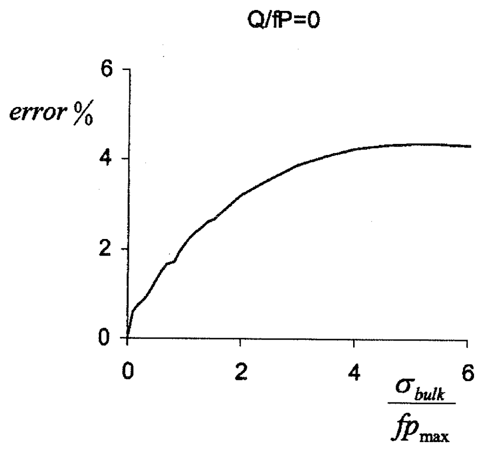


Fig. 6 Hertzian geometry and bulk stress only. Error using equation (15) with respect to the 'exact' numerical solution. Equation (15) turns out to be very accurate and conservative, as it overestimates stress concentration

For small values of bulk, or more precisely for moderate bulk, i.e.

$$\sigma_{\text{bulk}} \leqslant 4fp_{\max}\left(1-\sqrt{1-\frac{|Q|}{fP}}\right)$$

Equation (15) is exact. Therefore, except for the case of bulk stress only (when the limit bulk becomes zero), there is a range where equation (15) is exact. The locus of the limit between moderate to strong bulk is also included in the figure, with the line with symbols. Above the limit bulk, i.e. to the right of the locus, equation (15) is not exact, but the error is smaller than 4 per cent in absolute terms, as discussed in the previous section. However, since only the additional stresses due to shear traction are represented (i.e. the difference between the peak and bulk stresses), the error appears to be larger, as is made clear from the comparison with the included line of 'exact' solution for bulk stress only. In all other cases, the exact line (not represented) would depart from the initial value of the represented equation (15) [where equation (14) would also start and remain constant] and would make a small deviation for increasing bulk stress, although it would be smaller than for the case of bulk stress only.

Clearly, it can be concluded that there is no reason to use equation (14) instead of the correct equation (15) for Hertzian geometry and moderate or large bulk, as equation (15) is not significantly more complicated to use. Obviously, equation (14) is of interest because it is based on linear superposition, and it may be convenient to use for large enough tangential loads.

The next step would obviously be to generalize equation (15) for non-Hertzian geometries, and this can be accomplished by comparison with equation (14), inclusive of the coefficient k. In order to add the geometrical factor k into equation (15) the terms would need to be grouped such that the geometrical factor affects the entire term due to shear tractions, resulting in

$$\sigma_{xx}(b,0)=2fp_{\max}k\left[\sqrt{\left(1+\frac{\sigma_{\text{bulk}}}{4fp_{\max}}\right)^2-\left(1-\frac{|Q|}{fP}\right)}-\frac{\sigma_{\text{bulk}}}{df\,p_{\max}}\right]+\sigma_{\text{bulk}}+\sigma_{xx}^{P} \quad (16)$$

This is the final equation proposed, taking into account the

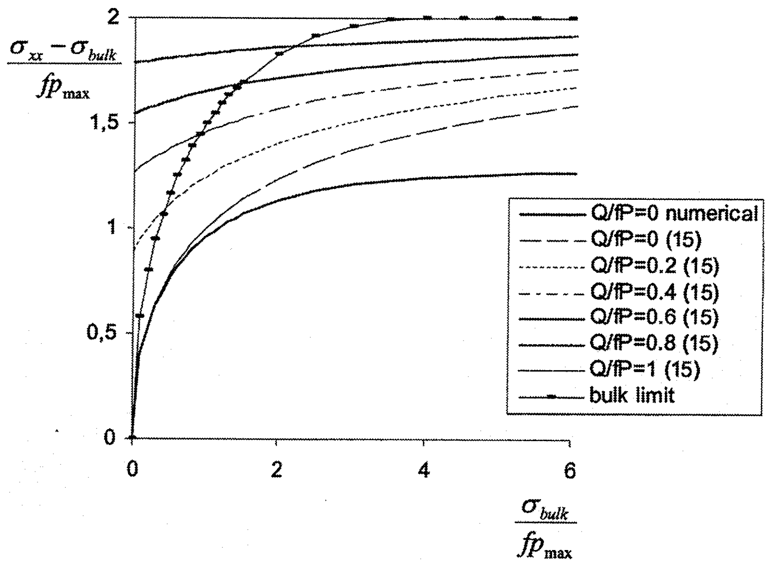


Fig. 7 Hertzian geometry and combined tangential plus bulk stress. Prediction using equation (15) of the stress induced by shear traction. Also shown are the 'exact' solution for bulk stress only and the locus of limit bulk stress dividing moderate bulk conditions (on the left of the locus line), where equation (15) is exact, from strong bulk conditions, where equation (15) is approximate

three factors (tangential load as well as bulk stress and the geometrical factor k). It will be recalled that the formula is exact for Hertzian contact under moderate bulk stress and contains a limited error for non-Hertzian geometry or large bulk. These errors have been shown to be within a few per cent when either one or the other effects are considered. In the case of combined effects, the error may be slightly greater, but, as shown by the range of conditions in Table 1, for very flat geometries, low $Q/(fP)$ and large bulk stress, the error may be near 10 per cent. However, for these cases even the numerical solution is difficult to obtain. Also included in the table are results for equation (14), which show considerably larger errors.

8 CONCLUSIONS

The contact problem of a flat rounded punch has been revised using and developing the analytical results given by Ciavarella *et al.* [1]. A simple and sufficiently accurate formula has been devised for estimating the peak value of

Table 1 The numerical and approximated peak values of σ_{xx} for different bulk stresses and aspect ratios a/b, with $Q/(fP) = 0.3, 0.6, 0.9$

	$a/b = 0.3$			$a/b = 0.6$			$a/b = 0.9$		
σ_{bulk} $[b/(fP)]$	Numerical value	Equation (14)	Equation (16)	Numerical value	Equation (14)	Equation (16)	Numerical value	Equation (14)	Equation (16)
$Q/(fP) = 0.3$									
0.1	0.939	0.893	0.935	1.167	1.104	1.158	2.202	2.033	2.137
0.5	1.475	1.293	1.453	1.782	1.504	1.708	3.147	2.433	2.825
1.0	2.07	1.793	2.043	2.427	2.004	2.321	4.06	2.933	3.543
1.5	2.61	2.293	2.603	2.992	2.504	2.897	4.631	3.433	4.188
2.0	3.132	2.793	3.146	3.51	3.004	3.452	5.143	3.933	4.794
$Q/(fP) = 0.6$									
0.1	1.238	1.221	1.237	1.539	1.52	1.540	2.905	2.833	2.871
0.5	1.697	1.621	1.686	2.057	1.92	2.002	3.662	3.233	3.390
1.0	2.251	2.121	2.227	2.624	2.42	2.555	4.254	3.733	3.993
1.5	2.769	2.621	2.757	3.136	2.92	3.093	4.763	4.233	4.565
2.0	3.274	3.121	3.280	3.645	3.42	3.621	5.266	4.733	5.120
$Q/(fP) = 0.9$									
0.1	1.462	1.473	1.476	1.826	1.839	1.843	3.442	3.447	3.455
0.5	1.88	1.873	1.886	2.246	2.239	2.255	3.867	3.847	3.878
1.0	2.386	2.373	2.395	2.752	2.739	2.767	4.374	4.347	4.400
1.5	2.893	2.873	2.901	3.251	3.239	3.275	4.872	4.847	4.916
2.0	3.391	3.373	3.406	3.752	3.739	3.782	5.37	5.347	5.429

the tangential stress σ_{bulk}. A unique formula equation (16) for both zero and non-zero bulk stresses has been proposed for Hertzian and non-Hertzian geometry. The formula is correct for Hertzian geometry and moderate bulk, whereas it is approximate for either non-Hertzian geometry or large bulk, but the error is limited to within a few per cent except in extreme cases.

REFERENCES

1 **Ciavarella, M., Hills, D. A.** and **Monno, G.** The influence of rounded edges on indentation by a flat punch. *Proc. Instn Mech. Engrs, Part C, Journal of Mechanical Engineering Science*, 1998, **212**(C4), 319–328.
2 **Giannakopoulos, A. E., Lindley, T. C.** and **Suresh, S.** Similarities of stress concentrations in contact at round punches and fatigue at notches: implications to fretting fatigue crack initiation. *Fatigue Fract. Engng Mater. Structs*, March 2000, **23**, 561–571.
3 **Johnson, K. L.** *Contact Mechanics*, 1985 (Cambridge University Press, Cambridge).
4 **Hills, D. A., Nowell, D.** and **Sackfield, A.** *Mechanics of Elastic Contacts*, 1993 (Butterworth-Heinemann, Oxford).