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The General 3D Hertzian Contact Problem for Anisotropic Materials

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Abstract. This paper presents a general method for solving the 3D frictionless contact problem between generally anisotropic materials with any second order surface geometry. The method uses the Stroh formalism to find the Green's Functions (GF) of the materials with an efficient numerical integration process. The GFs are then expanded in Fourier series in order to solve the Hertzian contact problem between the two bodies as a perturbation to the first order, "equivalent isotropic", solution to the problem. The latter permits to define an "equivalent indentation modulus of the contact" which is a single parameter computed from the first terms of the Fourier expansion of the two GFs (i.e. the average values) and permits to use the standard Hertzian solution: this gives a good approximation to the contact area (at most elliptical in any case) which is approximated as a circle for axi-symmetrical geometry, and for the approach of remote points in the two bodies. The "equivalent indentation modulus", which depends on materials and orientation, is computed for a set of composite materials of practical interest.

Introduction

Composite materials have been in use for structural applications for years. They offer the advantage of high stiffness and strength combined with low weight. One of the concerns when using composite materials in a structural application, is their poor resistance to impact loading. In order to study this phenomenon, one needs to know the solution to the contact problem in the general case of two anisotropic materials. This solution is also useful in the analysis and interpretation of experimental results from indentation testing, a technique currently used extensively for measuring the elastic properties of anisotropic materials.

The anisotropic Hertzian contact problem has been solved previously through use of double Fourier transforms [1]. Here we obtain a simple, new formulation of the Hertzian contact problem as a perturbation to the first order solution, which is in essence an equivalent isotropic solution. This approach requires the Fourier expansion of the Green's for the frictionless normal contact problem, and this is found numerically using the efficient Barnett-Lothe method for finding the Green's function for an anisotropic half space.

The Green's function for an anisotropic half space

A convenient expression for the Green's function has been derived by Barnett and Lothe [2] using a technique based on the Stroh formalism for anisotropic materials. Consider a coordinate system (x_1 ,x_2, x_3). Let **t** be a unit vector in an infinite anisotropic medium, and **m** and **n** be two orthogonal unit vectors perpendicular to **t**, so that (**m, n, t**) form a right-hand Cartesian coordinate system. Let φ be the angle between **m** and some fixed datum in the plane perpendicular to **t** (see Fig 1.a).

Repeated indices imply a summation over the repeated index from 1 to 3. We define the matrices **B** and **S** by

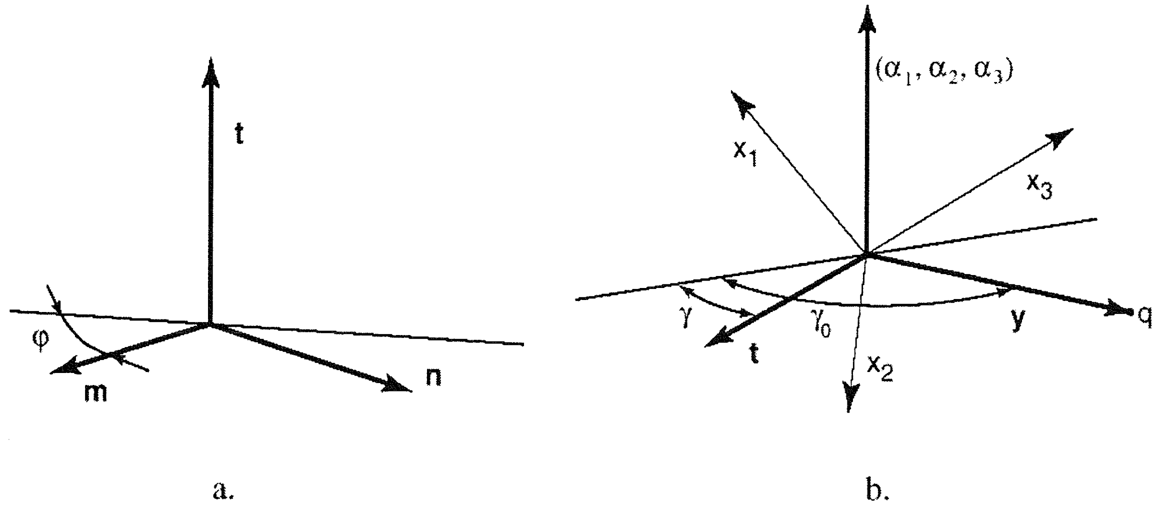

Figure 1 (a) Schematic representation of the unit vectors used to define the matrices **B(t)** and **S(t),** for an elastically anisotropic half space. (b) Schematic representation of the vectors involved in the definition of the Green's function for an anisotropic half space. The vector $(\alpha_1,\alpha_2,\alpha_3)$ is perpendicular to the surface of the half space, vectors **t** and **y** lie in the surface.

$$\mathbf{B}_{js}(\mathbf{t})=\mathbf{B}_{sj}(\mathbf{t})=\frac{1}{8\pi^2}\int_0^{2\pi}\left\{(\mathbf{mm})_{js}-(\mathbf{mn})_{jk}(\mathbf{nn})^{-1}_{kr}(\mathbf{nm})_{rs}\right\}d\varphi \tag{1}$$

$$\mathbf{S}_{sj}(\mathbf{t})=-\frac{1}{2\pi}\int_0^{2\pi}(\mathbf{nn})^{-1}_{sr}(\mathbf{nm})_{rj}\,d\varphi. \tag{2}$$

In Equations (1) and (2), the matrices (**ab**) are given by

$$(\mathbf{ab})_{jk}=a_i C_{ijkm} b_m\ , \tag{3}$$

where the C_{ijkm} are the elastic stiffness coefficients of the anisotropic material. The matrices **B** and **S** depend only on the elastic constants of the material and the direction t. Also, **B** is symmetric and positive definite [2] and $\mathbf{SB}^{-1}$ is antisymmetric [3].

Now, consider an anisotropic half space with its boundary through the origin of the coordinate system. The orientation of the boundary of the half space is arbitrary with respect of the coordinate system and is given by the direction cosines $(\alpha_1,\alpha_2,\alpha_3)$ of the normal to the boundary. The displacement in the x_k direction at a point q in the half space boundary due to a unit point load applied in the x_m direction at the origin of the coordinate system is then given by [2]

$$u_{km}(\mathbf{y})=\frac{1}{8\pi^2|\mathbf{y}|}\left[\mathbf{B}^{-1}_{km}\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)+\frac{1}{\pi}P\int_0^{\pi}\frac{\mathbf{B}^{-1}_{kj}(\mathbf{t})\mathbf{S}_{mj}(\mathbf{t})}{sin(\gamma-\gamma_0)}d\gamma\right], \tag{4}$$

where P indicates principal value, **y** is the position vector of q and **t** lies in the half space boundary. Also, γ and γ_0 are the angles between some fixed datum in the half space boundary and **t** and **y**, respectively (see Fig. 1b). If the concentrated unit load is perpendicular to the boundary of the half space, the displacement w(y) in the direction of the load is given by

$$w(\mathbf{y}) = \alpha_k u_{km}(\mathbf{y})\alpha_m = \frac{1}{8\pi^2|\mathbf{y}|}\left[\alpha_k \mathbf{B}_{km}^{-1}\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)\alpha_m + \frac{1}{\pi}P\int_0^{\pi}\frac{\alpha_k\alpha_m \mathbf{B}_{kj}^{-1}(\mathbf{t})\mathbf{S}_{mj}(\mathbf{t})}{sin(\gamma-\gamma_0)}d\gamma\right]. \tag{5}$$

Since $\mathbf{SB}^{-1}$ is antisymmetric, the last term in Eq. (5) vanishes and we obtain the following expression for the Green's function [4]:

$$w(\mathbf{y}) = \frac{1}{8\pi^2|\mathbf{y}|}\left[\alpha_k \mathbf{B}_{km}^{-1}\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)\alpha_m\right]. \tag{6}$$

The displacement of a point in the surface of a half space under influence of a point load at an other point in the surface, is inversely proportional to the distance to the point load. The angle-dependent part of the surface Green's function, h(ϑ), can be readily calculated through numerical integration as in Eq. (1).

Formulation of contact problem

The formulation of Hertz's problem between two anisotropic bodies is obtained as a perturbation of the case for two isotropic bodies with a quadratic gap functions, solved by Hertz in 1881 [5, 1]. Consider a coordinate system (x_{1a},x_{2a}) for body "a", so that the principal radii of curvature of body "a" coincide with the coordinate axes, and similarly for surface "b".

In the principal coordinate system, the surfaces can be described by definition with no mixed term "x_1x_2":

$$x_{3a}(x_{1a},x_{2a}) = \frac{1}{2}B_a x_{1a}^2 + \frac{1}{2}C_a x_{2a}^2 \quad ; \quad x_{3b}(x_{1b},x_{2b}) = \frac{1}{2}B_b x_{1b}^2 + \frac{1}{2}C_a x_{2b}^2 . \tag{7}$$

The principal radii of curvature are then $R_a=1/B_a$ and $R'_a=1/C_a$ (and similarly for body "b"). Define the angle formed by the x_{1a}- and x_{1b}-direction as α. The total gap function is the difference between x_{3a} and x_{3b}. We now define the principal relative coordinate system, (x_{1p},x_{2p}), as the coordinate system for which the "x_{1p},x_{2p}" mixed term vanishes: In the isotropic case, the main axes of contact area ellipse axis are aligned with (x_{1p},x_{2p}) and the non-vanishing coefficients B, C are found from the following set of equations:

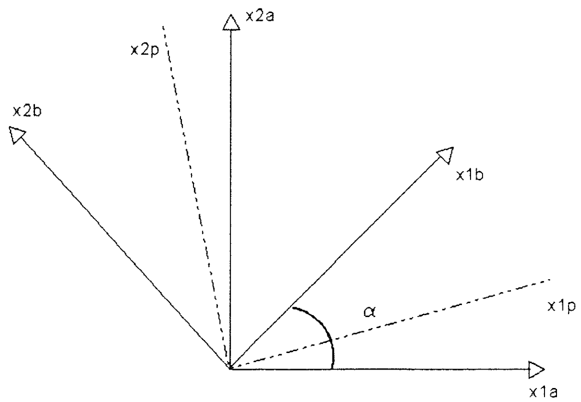

Figure 2 Representation of the principal coordinate systems for bodies "a" and "b" and the "relative" principal coordinate systems.

$$B + C = B_a + C'_a + B_b + C'_b$$
$$|C - B| = \left[(B_a - C'_a)^2 + (B_b - C'_b)^2 + 2(B_a - C'_a)(B_b - C'_b)\cos 2\alpha\right]^{1/2} \tag{8}$$

The "principal relative radii of curvature" of the two surfaces are then $R' = 1/B$ and $R'' = 1/C$. We now turn to the solution of the elastic problem: as we know from Willis' solution [1], in the general anisotropic case, the contact ellipse will not be aligned with the relative principal coordinate system, which we have defined as (x_{1p},x_{2p}), and we need to obtain the coefficients B, C, D of the generic gap function in the coordinate system (x_1,x_2) aligned with the contact ellipse, which is shifted over an unknown angle φ (see Fig.3).

We call $R_1(\varphi)$,$R_2(\varphi)$,$R_{12}(\varphi)$ the relative radii of curvature of the bodies in this system, which can be found from elementary geometrical considerations as:

$$\frac{1}{R_1(\varphi)} = \frac{\cos^2\varphi}{R'} + \frac{\sin^2\varphi}{R''},$$
$$\frac{1}{R_2(\varphi)} = \frac{\sin^2\varphi}{R'} + \frac{\cos^2\varphi}{R''}, \tag{9}$$
$$\frac{1}{R_{12}(\varphi)} = \left(-\frac{1}{R''} + \frac{1}{R'}\right)\sin(2\varphi),$$

where R_{12} vanishes for axysimmetric geometry . The gap function is:

$$g_0(\varphi) = \frac{1}{2R_1(\varphi)}x_1^2 + \frac{1}{2R_2(\varphi)}x_2^2 + \frac{1}{2R_{12}(\varphi)}xy = B(\varphi)x_1^2 + C(\varphi)x_2^2 + D(\varphi)x_1x_2. \tag{10}$$

We can write the compatibility of displacements in the generic system (x_1,x_2) for a point lying inside the area of contact :

$$w_a + w_b + g_0 = \lambda_a + \lambda_b = \lambda\ , \tag{11}$$

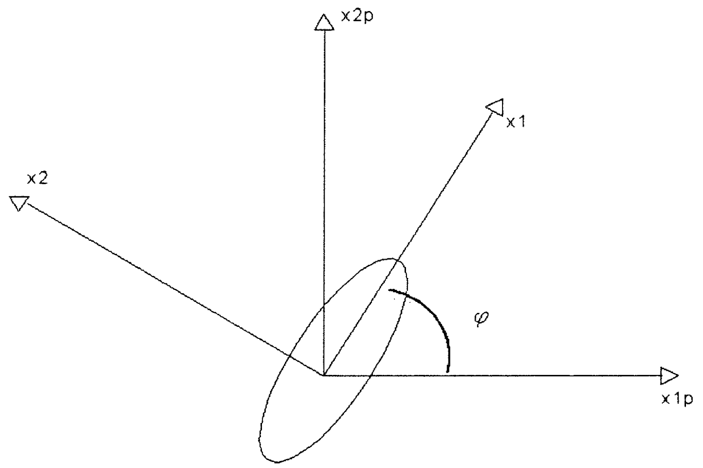

Figure 3 Representation of the coordinate systems (x_{1p},x_{2p}), (x_1,x_2).

where w_a is the displacement normal to the surface of body "a", w_b is the displacement of body "b" and λ_a and λ_b are displacements of two points remote from the contact area. If we substitute Eq. (10) into Eq. (11), we obtain:

$$w_a + w_b = \lambda - B(\varphi)x_1^{\,2} - C(\varphi)x_2^2 - D(\varphi)x_1x_2 \ . \tag{12}$$

In order to find the displacements on LHS of Eq. (12), we recollect Galin's theorem (and the Willis' solution) which states that the contact area is always elliptic and that the pressure distribution has the same ellipsoidal shape as the isotropic case

$$\sigma_{zz}(x_1, x_2, 0) = -p_0\left[1 - \left(\frac{x_1}{a}\right)^2 - \left(\frac{x_2}{b}\right)^2\right]^{1/2} \tag{13}$$

The normal displacement in body "a" as a result of this pressure distribution is given by [5]:

$$u_{za}(x_1, x_2, 0) = p_0\frac{\pi}{2}\left(b\alpha_a - \frac{1}{b}\beta_a x_1^2 - \frac{1}{b}\gamma_a x_2^2 - \frac{1}{b}\delta_a x_1 x_2\right), \tag{14}$$

inside the area of contact. The constants $(\alpha_a, \beta_a, \gamma_a, \delta_a)$ can be calculated using the following expressions:

$$\alpha_a = \int_0^{\pi}\frac{h_a(\theta)}{\left(1 - e^2\cos^2\vartheta\right)^{1/2}}d\vartheta,$$
$$\begin{Bmatrix}\beta_a\\ \gamma_a\\ \delta_a\end{Bmatrix} = \left(1 - e^2\right)\int_0^{\pi}\frac{h_a(\theta)}{\left(1 - e^2\cos^2\vartheta\right)^{3/2}}\begin{Bmatrix}sin^2\vartheta\\ \cos^2\vartheta\\ sin(2\vartheta)\end{Bmatrix}d\vartheta. \tag{15}$$

Here, "e" is the eccentricity of the contact ellipse:

$$e = \sqrt{1 - \frac{b^2}{a^2}} \ . \tag{16}$$

The displacement in body "b" can be calculated using the same equations, *mutatis mutandis*. The contact problem is then solved by substituting the appropriate expressions for u_{za} and u_{zb} into Eq. (12). Thus, we obtain the total displacement u_z=u_{za}+u_{zb}, and the corresponding total Green's function is $h(\vartheta)$=$h_a(\vartheta)$+$h_b(\vartheta)$. The following observations are useful:

- The axes of the ellipse (x_1,x_2) don't coincide with the relative principal coordinate system (x_{1p},x_{2p}), as in the isotropic case, but they are rotated over an unknown angle φ. In fact, since $h(\vartheta)$ is not constant in the anisotropic case, δ is not generally zero.
- The eccentricity depends not only on geometrical properties as in the isotropic case, but also on material properties.

- In Eq. (15), the angle ϑ is measured starting from the x_1 axis, which coincides with the ellipse axis. When we calculate $h_1(\vartheta)$, we use the elastic stiffness coefficients in the system that simplifies the definition of the tensor, which may be rotated over yet another angle, ζ.

The equivalent isotropic and the perturbation solutions

Let's consider the special case of a rigid, axisymmetric indenter on a half space of composite material. Only the angular part of the Green's function of the elastic half space, $h(\vartheta)$ in the (x_1,x_2), enters into the problem. Geometrical symmetry allows us to put ζ equal to zero, so that $h(\vartheta)$ has to be shifted only over φ. The method can, however, be readily extended to the more general case. We first develop $h(\vartheta)$ into a Fourier series (it is a periodic function with period π):

$$h(\vartheta)=h_0+\sum_{k=1}^{\infty}h_{ck}\cos(2k\vartheta)+\sum_{k=1}^{\infty}h_{sk}sin(2k\vartheta), \tag{17}$$

where h_0 is the average value; it defines an equivalent isotropic problem. The Fourier coefficients in the (x_1,x_2) coordinate system can then be written in terms of the Fourier coefficients in the principal coordinate system (x_{1p},x_{2p}) as follows:

$$\begin{aligned} h'_{ck}(\varphi)&=h_{ck}\cos(2k\varphi)+h_{sk}sin(2k\varphi)\\ h'_{sk}(\varphi)&=h_{sk}\cos(2k\varphi)-h_{ck}sin(2k\varphi)\end{aligned} \tag{18}$$

Substituting Eq. (17) into Eq. (15) and retaining the first two terms of the Fourier series, allows us to express the different integrals as functions of the elliptic integrals of the first and second kind. We obtain:

$$\alpha=\left\{2K(e)h_0-\frac{2}{e^2}\left[2E(e)+\left(e^2-2\right)K(e)\right]h'_{c1}(\varphi)+\frac{2}{3e^4}\left[8\left(e^2-2\right)E(e)+\left(e^2-4\right)\left(3e^2-4\right)K(e)\right]h'_{c2}(\varphi)+O[h_{c3}]\right\}$$

$$\beta=\left(1-e^2\right)\left\{\begin{array}{l}\frac{2}{e^2}\left[K(e)-E(e)\right]h_0-\frac{2}{e^4}\left[\left(4-e^2\right)E(e)+\left(3e^2-4\right)K(e)\right]h'_{c1}(\varphi)+\\ +\frac{2}{3e^6}\left[\left(e^2\left(56-3e^2\right)-64\right)E(e)+\left(64+e^2\left(27e^2-88\right)\right)K(e)\right]h'_{c2}(\varphi)+O[h_{c3}]\end{array}\right\}$$

$$\gamma=\left(1-e^2\right)\left\{\begin{array}{l}\frac{2}{e^2}\left[\frac{a^2}{b^2}E(e)-K(e)\right]h_0-\frac{2}{e^4\left(e^2-1\right)}\left[\left(4-3e^2\right)E(e)-\left(e^2-4\right)\left(e^2-1\right)K(e)\right]h'_{c1}(\varphi)+\\ +\frac{2}{3e^6\left(e^2-1\right)}\left[\left(e^2\left(72-11e^2\right)-64\right)E(e)-\left(e^2-1\right)\left(64+e^2\left(3e^2-40\right)\right)K(e)\right]h'_{c2}(\varphi)+O[h_{c3}]\end{array}\right\}$$

$$\delta=\left(1-e^2\right)\left\{-\frac{8}{e^4}\left[2E(e)+\left(e^2-2\right)K(e)\right]h'_{s1}(\varphi)+\frac{16}{3e^6}\left[8\left(e^2-2\right)E(e)+\left(e^2-4\right)\left(3e^2-4\right)K(e)\right]h'_{s2}(\varphi)+O[h_{s3}]\right\} \tag{19}$$

Note that the first terms in these equations correspond to the Hertz solution of the problem. By substituing (12) into (14) and using (9,10) we obtain:

$$
\begin{aligned}
& p_0 \frac{\pi}{2} b\alpha = \lambda \\
& p_0 \frac{\pi}{2}\frac{1}{b}\beta = \frac{1}{2R_1(\varphi)} \\
& p_0 \frac{\pi}{2}\frac{1}{b}\gamma = \frac{1}{2R_1(\varphi)} \\
& p_0 \frac{\pi}{2}\frac{1}{b}\delta = \frac{1}{2R_{12}(\varphi)}
\end{aligned}
\tag{20}
$$

This set of equations (where we use eqt.19 for the terms on LHS) can be separated into 2 parts and solved for "e" and φ by setting the last equation to zero and by taking the ratio of the third and second equation equal to one, since the indenter is axisymmetric (R_1=R_2=R). We thus obtain a system of two non-linear equations for "e, φ". It is possible to further simplify the solution by expanding the elliptic integrals in a Taylor series and retain terms up to e^2

$$
K(e) = \frac{\pi}{2}\left[1 + \frac{e^2}{4} + ...\right], \qquad E(e) = \frac{\pi}{2}\left[1 - \frac{e^2}{4} - .....\right]. \tag{21}
$$

After substituting Eqs (21) into Eqs (19) we obtain the following expressions that define an approximate solution:

$$
\begin{aligned}
\alpha_{app} &= \left\{ \pi h_0 + e^2\pi\left(\frac{1}{4}h_0 + \frac{1}{8}h'_{c1}(\varphi)\right) + O[e^4, h_{c3}] \right\} \\
\beta_{app} &= \left\{ \frac{\pi}{4}(2h_0 - h'_{c1}(\varphi)) - e^2\pi\left(\frac{5}{16}h_0 - \frac{1}{4}h'_{c1}(\varphi) + \frac{3}{32}h'_{c2}(\varphi)\right) + O[e^4, h_{c3}] \right\} \\
\gamma_{app} &= \left\{ \frac{\pi}{4}(2h_0 + h'_{c1}(\varphi)) + e^2\pi\left(\frac{1}{16}h_0 + \frac{1}{8}h'_{c1}(\varphi) + \frac{3}{32}h'_{c2}(\varphi)\right) + O[e^4, h_{c3}] \right\} \\
\delta_{app} &= \left\{ \frac{1}{2}\pi h'_{s1}(\varphi) + e^2\pi\left(-\frac{1}{8}h'_{s1}(\varphi) + \frac{3}{16}h'_{s2}(\varphi)\right) + O[e^4, h_{s3}] \right\}
\end{aligned}
\tag{22}
$$

After some lengthy algebra, we obtain:

$$
e^2 \approx \frac{8h'_{c1}(\varphi)}{-6h_0 + 2h'_{c1}(\varphi) - 3h'_{c2}(\varphi)}, \tag{23}
$$

$$
\varphi \approx \frac{1}{2}\tan^{-1}\left(\frac{2h_0 h_{s1} - h_{s2}h_{c1} + h_{s1}h_{c2}}{h_{s1}h_{s2} + h_{c1}(h_{c2} - 2h_0)}\right) + n\frac{\pi}{2}, \tag{24}
$$

where n is taken to be either zero or one to ensure that the expression for e^2 in Eq. (23) is positive. Note that Eq. (24) allows for a straightforward calculation of the orientation of the contact ellipse with respect to the relative principal coordinate system and of the Fourier coefficients in the (x_1,x_2) coordinate system. Equation (22) can also be derived directly from Eq. (15) by developing the integrand as a Taylor series in "e" and retaining terms up to second order in "e". It is interesting to note that the equations thus derived only depend on Fourier coefficients up to h_{c2} and h_{s2}, even if the full series is used to evaluate the integrals. We can find "e" using Eq. (23) and substitute back into Eq. (19) for an approximate solution. For the exact solution, Eq. (23,24) are still useful as starting values for a better approximation to the full non-linear system using (19) or, even better, the more general system of equations retaining higher order Fourier terms.

Results

We studied six different unidirectional composites with the following properties [6], where we have taken x_1 and x_2 parallel and normal to the fiber direction, respectively.

	AS4/3501-6	Boron-Al	Kevlar-Epoxy	S-2glass/epoxy	SCS-6/Ti-15-3	T300/5208
E1 (Gpa)	148,0	227,0	76,8	43,5	221,0	132,0
E2=E3 (GPa)	10,5	139,0	5,5	11,5	145,0	10,8
nu12=nu13	0,3	0,2	0,3	0,3	0,3	0,2
nu23	0,6	0,4	0,3	0,4	0,4	0,6
G12=G13 (GPa)	5,6	57,6	2,1	3,5	53,2	5,7
G23 (GPa)	3,2	49,1	1,4	4,1	51,7	3,4

Table 1 Material properties of the composites

The values of Fourier coefficients obtained are:

	AS4/3501-6	Boron-Al	Kevlar-Epoxy	S-2glass/epoxy	SCS-6/Ti-15-3	T300/5208
h_0 (10^{-12}) [m^2 / N]	24.2723	2.1222	56.2836	26.7713	2.0263	23.6650
h_{c1}(10^{-12}) [m^2 / N]	6.167603	0.128934	9.184335	0.74277	0.08025	5.647247
h_{c2}(10^{-12}) [m^2 / N]	0.235427	-9.894691E-3	0.141886	-0.398512	-0.015526	-0.10052

Table 2 Coefficients of Fourier expansion

Note that h_{s1} and h_{s2} are zero. The angle-dependant part of the Green's function of these materials, normalized by h_0 is given in Fig 4. As expected, the major axis of the elliptical contact area is perpendicular to the direction of the fibers for all of the composites considered. Figure 5 shows a comparison between the contact areas for T300/5208 and the corresponding equivalent isotropic case.

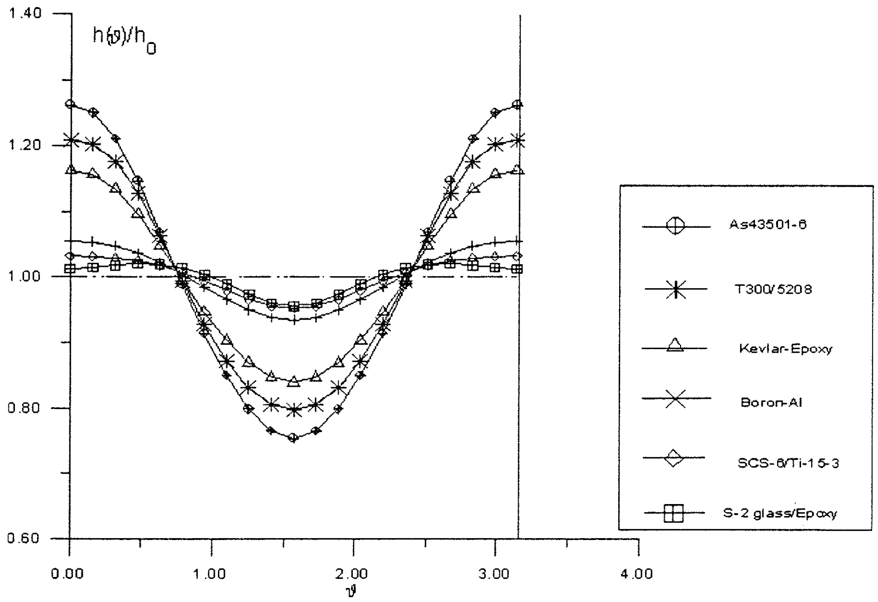

Figure 4. Representation of the Green's function for the composite materials under consideration

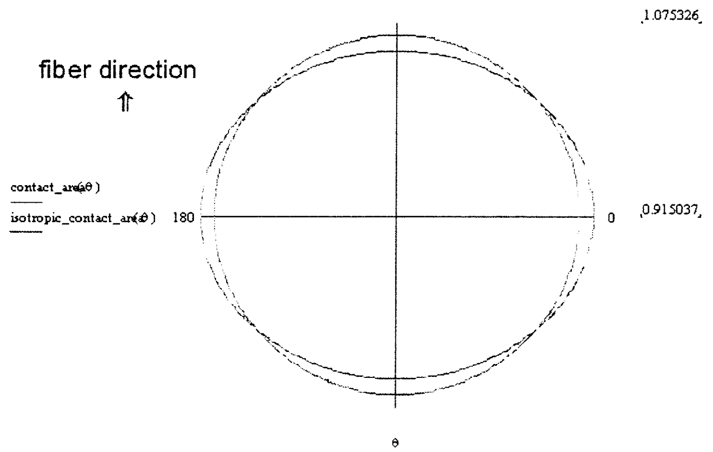

Figure5. This figure represents the contact area for T300/5208 normalized respect to the radius found for the equivalent isotropic case.

The eccentricity of the contact area for the material under consideration is given in the following table 3. Note that the error by using the approximate method is very small even for e=0.54, as it may be explained by considering that we only neglect terms of order e^4.

	AS4/3501-6	Boron-Al	Kevlar-Epoxy	S-2glass/epoxy	SCS-6/Ti-15-3	T300/5208
E1/E2	12.6	1.6	14.8	4.6	1.5	12.3
e [exact method]	0.539219	0.279379	0.442838	0.190699	0.227151	0.525264
e [approx method]	0.55763	0.28210	0.45399	0.19216	0.22872	0.54236
Difference%	3.4	1.0	2.5	0.8	0.7	3.3

Table 3 Values of the eccentricity that we found for these materials

Having defined the "equivalent isotropic" contact problem as the contact having an isotropic material with Green's function given by h_0, we consider three methods for calculating λ:

1. The "exact" one (with Fourier terms for $h(\theta)$ up to h_{c2}, h_{s2} if we use eqt.(19)):

$$\lambda = \left(\frac{3}{4}F\right)^{\frac{2}{3}}\left[\left(1-e^2\right)\frac{\alpha^3}{\beta}\right]^{\frac{1}{3}}\left(\frac{1}{2R}\right)^{\frac{1}{3}} = \lambda_{\text{exact}}\left(\frac{F^2}{R}\right)^{\frac{1}{3}}; \quad b = \left[\frac{3}{4}F\left(1-e^2\right)^{\frac{1}{2}}\beta(2R)\right]^{1/3}. \tag{25}$$

2. The small-eccentricity approximation:

$$\lambda = \left(\frac{3}{4}F\right)^{\frac{2}{3}}\left[\left(1-e^2\right)\frac{\alpha_{app}^{3}}{\beta_{app}}\right]^{\frac{1}{3}}\left(\frac{1}{2R}\right)^{\frac{1}{3}} = \lambda_{\text{small-e}}\left(\frac{F^2}{R}\right)^{\frac{1}{3}}; \quad b_{app} = \left[\frac{3}{4}F\left(1-e^2\right)^{\frac{1}{2}}\beta_{app}(2R)\right]^{\frac{1}{3}} \tag{26}$$

3. The equivalent isotropic solution:

$$\lambda_H = \left(\frac{3}{4\frac{1}{\pi h_0}}F\right)^{\frac{2}{3}}\left(\frac{1}{R}\right)^{\frac{1}{3}} = \lambda_{\text{iso}}\left(\frac{F^2}{R}\right)^{\frac{1}{3}}; \quad r_H = \left(\frac{3}{4\frac{1}{\pi h_0}}FR\right)^{\frac{1}{3}} \tag{27}$$

Here, F is the applied force and R the radius of curvature of the rigid indenter, and (α,β,α_{app},β_{app}) are given by Eq. (19) and Eq. (22), respectively. Calculations of normalized indentation depths using three methods are listed in Table 4 for each of the composite materials considered in this study. The errors of the approximate methods with respect to the correct solution are shown as well.

	AS4/3501-6	Boron-Al	Kevlar-Epoxy	S-2glass/epoxy	SCS-6/Ti-15-3	T300/5208
λ_{exact}	1.4790E-7	2.9236E-8	2.5966E-7	1.5855E-7	2.8352E-8	1.4549E-7
$\lambda_{small-e}$	1.4541E-7	2.9206E-8	2.5783E-7	1.5846E-7	2.8339E-8	1.4331E-7
λ_{iso}	1.4844E-7	2.9242E-8	2.6005E-7	1.5846E-7	2.8354E-8	1.4595E-7
Error (small-e)	1.68%	0.10%	0.70%	0.06%	0.05%	1.52%
Error (iso)	-0.37%	-0.02%	-0.15%	0.06%	-0.01%	0.32%

Table 4 Normalized indentation depths and relative errors calculated using the methods listed in the text . The indentation depths are given in units of $m^{4/3}/N^{2/3}$.

The result in Table 4 clearly show that the indentation depth of an anisotropic contact is very close to that of an equivalent isotropic contact, even for high degrees of anisotropy. In other words, the indentation depth is determined mainly by the equivalent isotropic modulus h_0. Since in the limit of isotropic materials:

$$\frac{1}{\pi h_0} = E^* = \frac{E}{\left(1-\nu^2\right)} \tag{28}$$

Given the good accuracy of the equivalent isotropic solution in calculating λ, we can say that $1/(\pi h_0)$ has the physical meaning of the "indentation" modulus of the contact.

Conclusions

We have given a solution of the general Hertzian contact problem for anisotropic materials thorough a perturbation solution of the "equivalent isotropic" contact problem. Through this numerical study, we have verified that the equivalent isotropic case permits in most cases of practical interest a very good analytical, closed form approximate approach to the problem, and that 1 elastic parameter is sufficient to describe the contact problem instead of the initial 42 independent constants (21 for each body) in the most general anisotropic material case.

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