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Elastic Multiscale Contact of Rough Surfaces: Archard's Model Revisited and Comparisons With Modern Fractal Models

M. Ciavarella
CNR-IRIS, str. Crocefisso 2/B, 70125 Bari, Italy
e-mail: M.Ciavarella@area.ba.cnr.it

G. Demelio
Dipartimento di Progettazione e Produzione Industriale, Politecnico di Bari, Viale Japigia 182, 70126 Bari, Italy
e-mail: Demelio@poliba.it

1 Introduction

Bowden and Tabor ([1] BT, in the following) state that friction is dictated by "adhesion (cold weld)" and "ploughing (inelastic deformation term)," between asperities. Amonton's law could easily be explained for the ploughing term, as the real area of contact would simply be $A = P/H$, where H is the hardness of the softer of the contacting bodies, and P is the applied load. However, for the elastic term, which in most cases would be the dominant one, Hertz' theory would not predict linearity with load. During the 1950s, several articles appeared in prestigious journals ([2–4]) where multiscale models were introduced to explain Amonton's and several connected well-known laws for friction, wear and electrical/thermal resistance, in terms of elastic deformations of multiscale, and rigorously ∞-scale model which we would now call a fractal ([4]), as depicted in Fig. 1—this is not the only possible choice, as the Archard model only takes into account of load redistribution and not of the actual geometry. These models found that the relation *real* contact area to *total* load for an ensemble of elastic asperities separated enough from each other to neglect interaction effects, is

$$A_n = K_n \left(\frac{W}{\tilde{E}} \right)^{\alpha_n} \tag{1}$$

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where $\tilde{E} = E/(1-\nu^2)$, and K_n is a coefficient which depends on the number of scales introduced n, and was computed by Archard for the first few scales only of his model. Archard's main finding was that α_n tends rapidly to one as n is increased. No particular attention was, vice versa, paid by Archard to the coefficient K_n, which will be here recomputed in general and will be specialized for a fractal geometry.

Independently from these multiscale models, and actually extending these results, Greenwood and Williamson ([5], GW model, in the following) showed that statistical distribution of heights asperities leads (at least approximately) to linearity between τ and p *independently on the exact law relating locally* τ *with* p, i.e., including any arbitrary local elastoplastic constitutive and frictional laws. Only recent experiments at very small scales with just one asperity in contact under very carefully controlled conditions are having some success in explaining the intrinsic properties of friction (see [6]). It has been found in particular that the friction coefficient is a function of the size of the asperity, and varies from very high values close to the elastic moduli of the materials (around $G/30$, in particular, where G is the shear modulus of the material), for smallest size to Peierls stress values, comparable to yield limits at larger sizes. When this knowledge will be completed, the way towards quantitative predictions of the actual "averaged" friction coefficient will depend on the actual precise determination of the distribution of contact sizes. This in turn will need an accurate modelling of real surfaces. As recently proposed, fractal models seem to have a promising role in concisely describing the apparent self-affinity of roughness, i.e., with features repeating themselves at different scales ([7,8]) but early attempts to use measurements of real surfaces and modelling of contact ([9,10]) were somehow unsatisfactory because they only considered a geometrical method for computing the contact area from a "bearing area" assumption. Borri-Brunetto et al. [11], vice versa, created a finite numerical realization of a surface with appropriate fractal properties and then used a numerical method to solve the resulting elastic contact problem at various levels of spatial discretization, suggesting that in the fractal limit the contact may consist of an infinite number of infinitesimal contact areas of total area zero. In other words, the actual contact area appears to be a fractal with dimension below two. This originated a discussion between the authors during the process of writing a review paper ([12]), and then to develop a rigorous analysis, specialized to the plane contact for a Weierstrass profile ([13], CDBJ model in the following), demonstrating that extended regions of contact are not possible with this model.

After this effort, we moved back to the original Archard's work and recognized that, although its surface is not a fractal of well-known characteristics, and although the contact mechanics is not

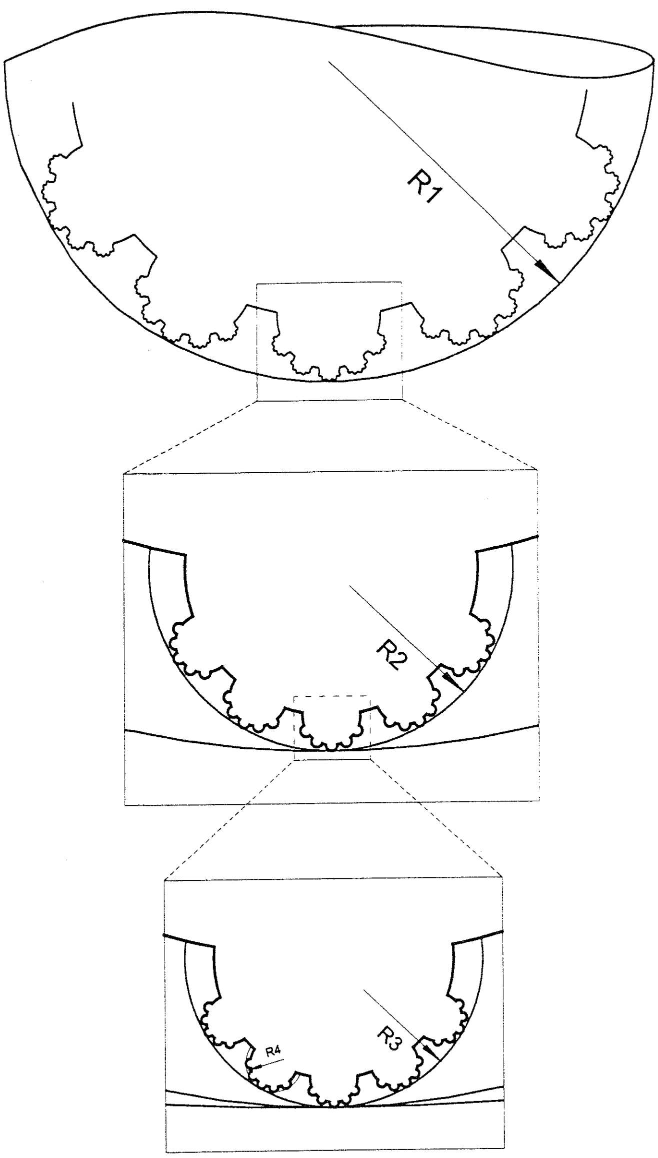

Fig. 1 Example of geometry for the Archard model

as rigorous as it can be with the Weierstrass profile, the result of fractal contact area can be reached very simply by extending the calculations of the original Archard paper ([4]) a little further, with the appropriate assumptions on the asperities geometry. The results, as shown in the present paper, are surprisingly similar to the Weierstrass profile CDBJ model, in many respects. A discussion of possible implications of these findings for friction theories follows.

2 The Revisited Archard's Model

Archard's model is based on the assumption that each asperity at scale n is replaced by many asperities at the higher scale. This permits an "uncoupling" of scales in the calculations of the redistribution of the pressure from one scale to the next, smaller one. Therefore, a very simple, analytical recursive argument can be developed. In order to consider "interaction of scales" one would need to consider the effect of smaller asperities in the calculation of the compliance of the larger ones, as Greenwood and Tripp [14] have done in the context of the statistical models. It is instructive to consider, however, that while the GT model seems to introduce some degree of multiscale features, the decision of having just two scales is somehow arbitrary, except for the case where the macroscopic roundness of the sphere is evident. Also, the GT model only obtains in most cases a minor modification of the pressure distribution at the largest scale, due to the small compliance effects, but loses the Archard feature of showing the pressure levels at smaller and smaller scales. It is clear that in principle a numerical model could take into account of all these effects simultaneously, but it remains a problem to build a multiscale model from a measured spectrum of the profile.

Turning back to the Archard model, at scale 0 Hertzian relations ([15], 4.2a) give the contact area as (1) with $K_0=\pi(3R_0/4)^{2/3}$, $\alpha_0=2/3$, and R_0 being the radius of the sphere. The pressure distribution is $p(r)=2\tilde{E}/\pi R\sqrt{r_0^2-r^2}$. We now assume that in the area A_0 there is a uniform distribution of asperities having density m_1 (number of asperities per unit area). Extending Archard's procedure for a contact of always higher number of scales, we obtain the following general expressions of the contact area at the nth scale as a function of load:

$$A_n=\pi\left(\frac{3}{4}R_0\frac{W}{\tilde{E}}\right)^{(1-1/3^{n+1})}$$

$$\times\prod_{k=1}^{n}\left[\frac{2}{3-3^{-k}}m_k^{1/3^{n+1-k}}\left(\frac{3}{2}\frac{R_k}{R_{k-1}}\right)^{1-1/3^{n+1-k}}\right]. \tag{2}$$

Notice that for $n->\infty$ the dependence on load becomes linear, independently on the assumptions on the geometrical quantities m_n, R_n. If, on the other hand, the ratios m_k/m_{k-1} and R_k/R_{k-1} are kept constant, we get

$$\frac{A_n}{A_{n-1}}=\frac{2}{3-3^{-n}}(A_0m_0)^{3^{-n}}\left(\frac{3}{2}\frac{R_n}{R_{n-1}}\sqrt{\frac{m_n}{m_{n-1}}}\right)^{1-3^{-n}}. \tag{3}$$

If we now *assume* for the spacing λ_n between asperities and for the radii R_n a power-law function, i.e., $\lambda_{n-1}/\lambda_n=\gamma$ and $R_{n-1}/R_n=\gamma^D$, the density of asperities is obviously $m_n=1/\lambda_n^2$, i.e., $m_{n-1}/m_n=(\lambda_{n+1}/\lambda_n)^2=1/\gamma^2$. Then, the ratio between contact area at subsequent scales is found from (3) to be

$$\frac{A_n}{A_{n-1}}=\frac{2}{3-3^{-n}}\left(\frac{A_0}{\lambda_0^2}\right)^{3^{-n}}\left(\frac{3}{2}\gamma^{1-D}\right)^{1-3^{-n}}. \tag{4}$$

The limit for $n\rightarrow\infty$ is

$$\frac{A_n}{A_{n-1}}=\left(\frac{1}{\gamma}\right)^{D-1}. \tag{5}$$

The tendency to power-law (5) indicates that the contact area tends to a fractal set, whereas the fractal dimension computed with the box-counting method (see CDBJ) is

$$d_A=-\frac{\ln\frac{N_{n+1}}{N_n}}{\ln\frac{\lambda_{n+1}}{\lambda_n}}=-\frac{\ln\left(\frac{A_n}{A_{n-1}}\frac{m_{n+1}}{m_n}\right)}{\ln\frac{1}{\gamma}}$$

$$=-\frac{\ln\left[\left(\frac{1}{\gamma}\right)^{D-1}\left(\frac{1}{\gamma}\right)^{-2}\right]}{\ln\frac{1}{\gamma}} \tag{6}$$

indicating that the *limiting* fractal dimension is

$$d_A=3-D. \tag{7}$$

Considering that $D=1-2$ by analogy to the Weierstrass case (see CDBJ), the contact area has dimension between 1 and 2, analogously to what found in CDBJ.

3 Discussion

The Archard model leads to asymptotic fractal behavior, under assumptions on radii of curvature for the asperities similar to the Weierstrass CDBJ model, and in contrast with Majumdar and Bhushan [9,10], and their "bearing area" geometrical assumption, with resulting finite area of contact. Obviously, we don't expect the contact area to be really zero, as real surfaces will have a truncation at one point, with asperities of given minimum size. Even if this was not the case, at a certain scale, deformations would be so intense that plastic deformation, nonlinearities, and other effects not included in our model would appear. It is clear that, in the spirit of Archard's model, if one asperity yields, this does not affect the load redistribution at the other asperities, as equilibrium is already accounted for. Therefore, an idea of the pressure level increase with scale can be found from the total contact area variation with n, as can be obtained from equations given above. The resulting trends are very similar to the ones obtained for the CDBJ model. In particular, the fractal dimension is a constant depending only on geometry and not on load level, but at the first few scale, both higher and lower apparent fractal dimension can occur. Also, as proved with the CDBJ model results, although both the contact areas and the distance between asperities become smaller, the ratio between the two decreases, so that interaction effects become increasingly smaller, and the Hertzian approximation becomes valid in any conditions, i.e., even in cases where high loads predict full contact at macroscopic scale (in this case, the present model is poor whereas the CDBJ model correctly considers the Westergaard solution for predicting contact area size).

4 Conclusions

The most striking conclusion of the calculations is that with multiscale models the contact area generally (if the radius of asperities decreases fast enough) tends to zero, i.e., is a fractal. The reason why such an implication had escaped the attention of researchers for more than 40 years, particularly as the model is quite well known. A possible explanation is that the main issue at that time was to find the linearity of relation (1), i.e., that $\alpha_n \to 1$, whereas the coefficients K_n were never computed for more than 2–3 scales. The results confirm the conclusion reached numerically by Borri-Brunetto et al. [11] that the contact area is defined by a fractal set—i.e., that contact is restricted to an infinite set of infinitesimal contact segments in the limit $n \to \infty$; there are no contact segments of finite dimension and the total contact area tends regularly to zero. In addition, the deviation from simple power-law fractal behavior at low wave numbers provides an explanation of their observation that the apparent fractal dimension is load-dependent. Even at large n, the splitting of segments of the contact area does not follow a "simple" rule for successive scales. Therefore, at successive scales, even if yielding is reached at one location, contact splitting will continue at other location, until yielding is reached even there. Therefore, it now becomes clear that the Archard model is in the limit compatible in a sense to the old Bowden-Tabor simple idea of contact area size given by $A = P/H$. Therefore, Greenwood-Williamson's model predicts Amonton's law from just the effect of randomness of the asperity height distribution, independently on the constitutive law at microscopic scale, Archard's model explains it as just an effect of load redistribution for a deterministic geometry, and leads in the limit to the other possible explanation (the Bowden-Tabor's fully plastic one), this goes some way in explaining why Amonton's law is so well hidden and intrinsic in the contact of any surface.

However, as Bowden-Tabor's theories and experiments show that full yield (ploughing term of friction) is negligible with respect to "adhesive" elastic term, particularly for hard materials and repeated sliding (shakedown), we can infer that the real case has a combination of features from all of the above models, and that the normal contact problem is largely unsolved. Future predictions of global friction coefficient depend crucially on better solutions and understanding of this problem.

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Appendix

Plane and Oversimplified Archard Models. In two dimension, with the same assumption for the ratios m_{k+1}/m_k and R_{k+1}/R_k, we have

$$\frac{A_n}{A_{n-1}} = \left(\frac{A_0}{\lambda_0}\right)^{1/2^n} \frac{\sqrt{\pi}}{2^{2-1/2^n}} \frac{\Gamma\left(\frac{3}{2} - \frac{1}{2^{n+1}}\right)}{\Gamma\left(2 - \frac{1}{2^{n+1}}\right)} \left(\frac{8}{\pi}\gamma^{1-D}\right)^{1-2^{-n}} \tag{8}$$

which, in the limit for $n \to \infty$ reduces to γ^{1-D}. Evaluating the fractal dimension as in the three-dimensional case the limiting fractal dimension is

$$d_A = 2 - D. \tag{9}$$

An oversimplified model of a surface could be imagined as having at scale n a number γ^{2n} of equal asperities of radius R_n not necessarily equally distributed leads to a fractal dimension

$$d_A = \frac{2}{3}(4 - D) \tag{10}$$

which ranges from 4/3 to 2. However, the relation real contact area versus load is still Hertzian at all scales, which is contradicting Amonton's law.

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