Constrained control and approximation properties of a rational interpolating curve

Qi Duan a,b,c, K. Djidjeli b, W.G. Price b, E.H. Twizell c,*

a Department of Mathematics, Shandong University, Jinan 250100, PR China
b School of Engineering Sciences, Ship Science, University of Southampton, Southampton SO17 1BJ, UK
c Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex UB8 3PH, UK

Received 19 May 2001; received in revised form 19 May 2002; accepted 27 September 2002

Abstract

This paper deals with the convexity control and the strain energy control of interpolating curves using a rational cubic spline with linear denominator. The sufficient and necessary conditions for controlling the interpolating curve to be convex or concave are derived. When the function being interpolated is \( f(t) \in C^3([a, b]) \), the error estimation of the interpolating function and the boundedness of the optimal error coefficient and its double symmetry with regard to parameters are obtained.

© 2002 Elsevier Science Inc. All rights reserved.

Keywords: Rational interpolation; Convexity control; Constrained control; Approximation

1. Introduction

Design of high quality, manufacturable curves or surfaces, such as the outer shape of a ship, car or aeroplane, is an important yet challenging task to today's manufacturing industries. Many authors have studied various kinds of

*Corresponding author. Tel.: +44-1895-203274; fax: +44-1895-203303.
E-mail address: e.h.twizell@brunel.ac.uk (E.H. Twizell).
splines for curve and surface design and control [1–28]. In general, the common
spline interpolations are fixed interpolations which means that the shape of the
interpolating curve is fixed for the given interpolating data. If one wishes to
modify the shape of the interpolating curve, the interpolating data need to be
changed. An important question is how can the shape of the curve be modified
under the condition that the given data are not changed? In [29], a rational
interpolating spline with linear denominator was constructed, and it was used
to control the shape of the interpolating curves, such as controlling the curves
to be in a given region. Because the parameters in the interpolating function
can be selected according to the control needs, the constrained control of the
shape becomes possible.

In curve control, one of the most significant things is to control the con-
vexity of the interpolating curves: to control the interpolating curves to be
convex or concave, or even to change the convexity in a local area. On the
other hand, the second-order derivative of an interpolating function has been
used in estimating the strain energy and, consequently, smoothness of the
interpolant. Smaller energy generally implies smoother shape. It is possible,
however, that the overall energy of an interpolant is small yet great enough to
generate abnormal shape at some points or even in some small intervals. Thus,
both the convexity control and the energy control depend on the second-order
derivative control. Because the spline has linear denominator, the derivation
calculus is easier when compared to that with cubic or quadratic denominator,
so, it is much easier to get the control conditions.

In Section 2 of this paper, rational cubic splines with linear denominator are
restated briefly. Section 3 deals with the convexity and strain energy control.
The sufficient and necessary conditions for the second-order derivative of the
interpolating function to be constrained in the given interval \([N,M]\) are
derived. Thus, the convexity control and the strain energy control can be carried
out by constraining the second-order derivative of the interpolating function to
be in the given interval \([0,M]\) or \([N,0]\). Also the existence conditions for these
interpolations are given in this section. Numerical examples are given in Sec-
nion 4. Section 5 is about the approximation properties of the spline. When the
function being interpolated is \(f(t) \in C^0[t_0,t_n]\), the error estimation of the in-
terpolating function is obtained and the boundedness of the optimal error
coefficient and its double symmetry with regard to parameters are established.

2. Rational cubic interpolating spline with linear denominator

Given a dataset \(\{(t_i, f_i, d_i), i = 0, 1, \ldots, n\}\), where \(f_i\) and \(d_i\) are the function
value and the first-order derivative value of the function being interpolated,
\(f(t)\), defined at the knot, respectively, and \(a = t_0 < t_1 < \cdots < t_n = b\) is the knot
spacing. Let \(h_i = t_{i+1} - t_i\), \(\theta = (t - t_i)/h_i\), and let \(\alpha_i\) and \(\beta_i\) be positive para-
meters. The $C^1$-continuous, piecewise, rational cubic spline with linear denominator is defined in the interpolatory interval $[a, b]$ as follows [29]:

$$P(t)_{[a,t_{i+1}]} = \frac{p_i(t)}{q_i(t)}, \quad i = 0, 1, \ldots, n - 1,$$

where

$$p_i(t) = (1 - \theta)^2 x_i f_i + \theta (1 - \theta)^2 V_i + \theta^2 (1 - \theta) W_i + \theta^3 \beta_i f_{i+1},$$

$$q_i(t) = (1 - \theta) x_i + \theta \beta_i,$$

and

$$V_i = (2x_i + \beta_i) f_i + x_i h_i d_i,$$

$$W_i = (x_i + 2\beta_i) f_{i+1} - \beta_i h_i d_{i+1}.$$

This rational cubic spline satisfies

$$P(t_i) = f_i, \quad P'(t_i) = d_i, \quad i = 0, 1, \ldots, n.$$

Furthermore, let $P''(t_{i+}) = P''(t_{i-})$ for $i = 1, 2, \ldots, n - 1$. The conditions lead to the following tri-diagonal system of linear equations:

$$h_i \frac{x_{i-1}}{\beta_{i-1}} d_{i-1} + \left(h_i \left(1 + \frac{x_{i-1}}{\beta_{i-1}}\right) + h_{i-1} \left(1 + \frac{\beta_{i-1}}{x_{i-1}}\right)\right) d_i + h_{i-1} \frac{\beta_{i-1}}{x_{i-1}} d_{i+1}$$

$$= h_{i-1} \left(1 + 2 \frac{\beta_{i-1}}{x_{i-1}}\right) A_i + h_i \left(1 + 2 \frac{x_{i-1}}{\beta_{i-1}}\right) A_{i-1}; \quad i = 1, 2, \ldots, n - 1,$$

where $A_i = (f_{i+} - f_i)/h_i$. It is easy to see that if the successive parameters $\{x_i\}$ and $\{\beta_i\}$ satisfy (2), then $P(t) \in C^2[a, b]$.

Note: The parameters $x_i$ and $\beta_i$ are used to keep the symmetry of the functions $p_i(t)$ and $q_i(t)$. In fact, by letting $a_i = x_i/\beta_i$, both $p_i(t)$ and $q_i(t)$ can be expressed in one independent parameter $a_i$ as

$$p_i(t) = (1 - \theta)^2 a_i f_i + \theta (1 - \theta)^2 V_i^* + \theta^2 (1 - \theta) W_i^* + \theta^3 f_{i+1},$$

$$q_i(t) = (1 - \theta) a_i + \theta$$

and

$$V_i^* = (2a_i + 1) f_i + a_i h_i d_i,$$

$$W_i^* = (a_i + 2) f_{i+1} - h_i d_{i+1}$$

and hence, one parameter can be used throughout the paper.
3. Constrained control of the convexity of the interpolating curves

The convexity control and strain energy control of the interpolating function depend on the control of its second-order derivative. If the second-order derivative of the interpolating function can be controlled to a given interval, then the convexity control and energy control can be carried out. Since the rational cubic splines defined by (1) have linear denominator, the second-order derivatives can be found easily. When \( t \in [t_i, t_{i+1}] \), from (1) it is easy to see that

\[
P''(t) = (h_i^2((1 - \theta) x_i + \theta \beta_i)^3 - ((1 - \theta) x_i + \theta \beta_i)^2(6(1 - \theta) x_i f_i \\
+ (6\theta - 4) V_i + (2 - 6\theta) W_i + 6\theta \beta_i f_{i+1}) - 2(\beta_i - x_i)((1 - \theta) x_i + \theta \beta_i) \\
\times (-3(1 - \theta)^2 x_i f_i + (1 - 4\theta + 3\theta^2) V_i + (2\theta - 3\theta^2) W_i + 3\theta^2 \beta_i f_{i+1}) \\
+ 2(\beta_i - x_i)^2((1 - \theta)^3 x_i f_i + \theta(1 - \theta)^2 V_i + \theta^2(1 - \theta) W_i + \theta^3 \beta_i f_{i+1})\}
\]

Let \( P''(t) \leq M \), then it is easy to show that

\[
Q(\theta) = MH_i^2((1 - \theta) x_i + \theta \beta_i)^3 + ((1 - \theta) x_i + \theta \beta_i)^2(6(1 - \theta) x_i f_i \\
+ (6\theta - 4) V_i + (2 - 6\theta) W_i + 6\theta \beta_i f_{i+1}) + 2(\beta_i - x_i)((1 - \theta) x_i + \theta \beta_i) \\
\times (-3(1 - \theta)^2 x_i f_i + (1 - 4\theta + 3\theta^2) V_i + (2\theta - 3\theta^2) W_i + 3\theta^2 \beta_i f_{i+1}) \\
- 2(\beta_i - x_i)^2((1 - \theta)^3 x_i f_i + \theta(1 - \theta)^2 V_i + \theta^2(1 - \theta) W_i + \theta^3 \beta_i f_{i+1}) \geq 0.
\]

Since

\[
Q'(\theta) = [(1 - \theta) x_i + \theta \beta_i]^2[3(\beta_i - x_i) MH_i^2 + 6(\alpha_i f_i - V_i + W_i - \beta_i f_{i+1})],
\]

\( Q(\theta) \) is monotone in \([0, 1]\). On the other hand,

\[
Q(0) = 2x_i^2 \beta_i (2f_i - 2f_{i+1} + h_i d_i + h_{id_{i+1}}) + a_i(MH_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i) \geq 0,
\]

\[
Q(1) = \beta_i^2(MH_i^2 - 2f_i + 2f_{i+1} - 2h_i d_{i+1}) + 2x_i \beta_i^2(4f_{i+1} - 4f_i - 2h_i d_{i+1} - 2h_i d_i) \geq 0.
\]

Therefore, the following theorem holds.

**Theorem 1.** For the rational cubic interpolating function \( P(t) \) defined by (1), the second-order derivative \( P''(t) \) is less than or equal to \( M \) in \([t_i, t_{i+1}]\) if and only if the positive parameters \( x_i, \beta_i \) satisfy the following inequality system

\[
\begin{align*}
2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_{id_{i+1}}) + a_i(MH_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i) &\geq 0, \\
\beta_i(MH_i^2 - 2f_i + 2f_{i+1} - 2h_i d_{i+1}) + 2a_i(2f_{i+1} - 2f_i - h_i d_{i+1} - h_i d_i) &\geq 0.
\end{align*}
\]

Let \( M = 0 \) in Theorem 1, then there is a corollary concerning the convexity control condition for the interpolating function \( P(t) \). It is

**Corollary 1.** The rational cubic interpolating function \( P(t) \) defined by (1) remains concave in \([t_i, t_{i+1}]\) if and only if the positive parameters \( x_i, \beta_i \) satisfy the following inequality system.
\[
\begin{aligned}
2\beta_i(2f_i - 2f_{i+1} + h_d i + h_d i + 1) + \alpha_i(2f_i - 2f_{i+1} + 2h_d i) \geq 0,
\beta_i(-2f_i + 2f_{i+1} - 2h_d i + 1) + 2\alpha_i(2f_{i+1} - 2f_i - h_d i + 1 - h_d i) \geq 0.
\end{aligned}
\]

In a similar way as for Theorem 1, the following theorem can be obtained.

**Theorem 2.** For the rational cubic interpolating function \( P(t) \) defined by (1), the second-order derivative \( P''(t) \) is greater than or equal to a given number \( N \) in \([t_i, t_{i+1}]\) if and only if the positive parameters \( \alpha_i, \beta_i \) satisfy the following inequality system
\[
\begin{aligned}
2\beta_i(2f_i - 2f_{i+1} + h_d i + h_d i + 1) + \alpha_i(Nh_i^2 + 2f_i - 2f_{i+1} + 2h_d i) \leq 0,
\beta_i(Nh_i^2 - 2f_i + 2f_{i+1} - 2h_d i + 1) + 2\alpha_i(2f_{i+1} - 2f_i - h_d i + 1 - h_d i) \leq 0.
\end{aligned}
\]

Let \( N = 0 \) in Theorem 2, then there is the Corollary 2 related to the convexity control condition for the interpolating function \( P(t) \). It is

**Corollary 2.** The rational cubic interpolating function \( P(t) \) defined by (1) is convex in \([t_i, t_{i+1}]\) if and only if the positive parameters \( \alpha_i, \beta_i \) satisfy the following inequality system
\[
\begin{aligned}
2\beta_i(2f_i - 2f_{i+1} + h_d i + h_d i + 1) + \alpha_i(2f_i - 2f_{i+1} + 2h_d i) \leq 0,
\beta_i(-2f_i + 2f_{i+1} - 2h_d i + 1) + 2\alpha_i(2f_{i+1} - 2f_i - h_d i + 1 - h_d i) \leq 0.
\end{aligned}
\]

(3)

From Theorems 1 and 2, the following theorem is obtained.

**Theorem 3.** For the rational cubic interpolating function \( P(t) \) defined by (1), the second-order derivative \( P''(t) \) on \([t_i, t_{i+1}]\) is in the given interval \([N, M]\) if and only if the positive parameters \( \alpha_i, \beta_i \) satisfy the following inequality system
\[
\begin{aligned}
2\beta_i(2f_i - 2f_{i+1} + h_d i + h_d i + 1) + \alpha_i(Mh_i^2 + 2f_i - 2f_{i+1} + 2h_d i) \geq 0,
\beta_i(Mh_i^2 - 2f_i + 2f_{i+1} - 2h_d i + 1) + 2\alpha_i(2f_{i+1} - 2f_i - h_d i + 1 - h_d i) \geq 0,
-2\beta_i(2f_i - 2f_{i+1} + h_d i + h_d i + 1) - \alpha_i(Nh_i^2 + 2f_i - 2f_{i+1} + 2h_d i) \geq 0,
-\beta_i(Nh_i^2 - 2f_i + 2f_{i+1} - 2h_d i + 1) - 2\alpha_i(2f_{i+1} - 2f_i - h_d i + 1 - h_d i) \geq 0.
\end{aligned}
\]

Considering both the convex interpolation and strain energy control simultaneously in shape design, it is necessary to constrain the second-order derivative to be bounded in a given interval \([0, M]\) or \([N, 0]\), where \( M > 0 \) and \( N < 0 \). These are described in the following corollaries.

**Corollary 3.** For the rational cubic interpolating function \( P(t) \) defined by (1), the second-order derivative \( P''(t) \) on \([t_i, t_{i+1}]\) is in the given interval \([0, M]\) with \( M > 0 \) if and only if the positive parameters \( \alpha_i, \beta_i \) satisfy the following inequality system
\[
\begin{aligned}
(a^* + b^*)\xi_i + (Mh_i^2 + a^*) \geq 0, \\
(Mh_i^2 - b^*)\xi_i - (a^* + b^*) \geq 0, \\
-(a^* + b^*)\xi_i - a^* \geq 0,
\end{aligned}
\]

(4)

(5)

(6)
\[ b^* \zeta_i + (a^* + b^*) \geq 0, \quad (7) \]

where \( \zeta_i = \beta_i / \alpha_i \), and
\[
\begin{align*}
  a^* &= 2(f_i - f_{i+1} + h_d_i), \\
b^* &= 2(f_i - f_{i+1} + h_d_{i+1}).
\end{align*}
\]

**Corollary 4.** For the rational cubic interpolating function \( P(t) \) defined by (1), the second-order derivative \( P''(t) \) on \([t_i, t_{i+1}]\) is in the given interval \([N, 0]\) with \( N < 0 \) if and only if the positive parameters \( \alpha_i, \beta_i \) satisfy the following inequality system

\[
\begin{align*}
  (a^* + b^*) \zeta_i + a^* &\geq 0, \\
- b^* \zeta_i - (a^* + b^*) &\geq 0, \\
- (a^* + b^*) \zeta_i - (a^* + Nh_i^2) &\geq 0, \\
( - Nh_i^2 + b^*) \zeta_i + (a^* + b^*) &\geq 0,
\end{align*}
\]

where \( \zeta_i, a^* \) and \( b^* \) are as defined for Corollary 3.

As discussed above, to ensure the second-order derivative of the interpolating function to be constrained in a given interval requires the existence of positive parameters \( \alpha_i \) and \( \beta_i \) which satisfy a corresponding constrained system of equations. The remaining of this section deals with the problem do these parameters always exist? and if so how to get them. The following theorems, Theorems 4 and 5, give the existence conditions for the parameters in Corollaries 2 and 3 under the conditions that the function being interpolated, \( f(t) \), is convex in the interpolating interval \([a, b]\).

**Theorem 4.** If the function being interpolated, \( f(t) \), is convex in the interpolating interval, for given \( \{(t_i, f_i, \alpha_i), i = 1, 2, \ldots, n\} \), there must exist parameters \( \alpha_i > 0, \beta_i > 0 \) such that the interpolating function \( P(t) \) defined by (1) is convex.

**Proof.** System (3) can be rewritten as
\[
\begin{align*}
- (a^* + b^*) \zeta_i - a^* &\geq 0, \\
b^* \zeta_i + (a^* + b^*) &\geq 0,
\end{align*}
\]

where \( \zeta_i, a^* \) and \( b^* \) are defined as in Corollary 3. Since \( f(t) \) is convex in the interpolating interval \([a, b]\), it is easy to get \( a^* < 0 \) and \( b^* > 0 \).

1. If \( a^* + b^* = 0 \), then for any \( \zeta_i > 0 \) both (8) and (9) hold.
2. If \( a^* + b^* > 0 \), it is obvious that for any \( \zeta_i > 0 \) (9) holds. Denote \( \zeta^*_i = -a^*/(a^* + b^*) \), then \( \zeta^*_i > 0 \), and when \( \zeta_i < \zeta^*_i \) (8) holds. Thus, when \( \zeta_i < \zeta^*_i \) both (8) and (9) hold.
3. If \( a^* + b^* < 0 \), for any \( \zeta_i > 0 \) (8) holds. Let \( \zeta^{**}_i = -(a^* + b^*)/b^* \), then \( \zeta^{**}_i > 0 \), and when \( \zeta_i > \zeta^{**}_i \) (9) holds. Thus, when \( \zeta_i > \zeta^{**}_i \) both (8) and (9) hold. Thus, the proof is complete. \( \square \)
The following theorem gives the conditions for the second-order derivative of the interpolating function to be constrained in the given interval [0, M].

**Theorem 5.** If the function being interpolated, \( f(t) \), is convex in the interpolating interval \([a, b]\), for given \( \{(t_i, f_i, d_i), i = 1, 2, \ldots, n\} \) and the given real number \( M > 0 \), when \( t \in [t_i, t_{i+1}] \), if one of the following conditions is satisfied, there must exist parameters \( \alpha_i > 0, \beta_i > 0 \) such that the second-order derivative values of the interpolating function \( P(t) \) defined by (1) are in the interval \([0, M]\).

1. \( a' + b' = 0, \quad Mh_i^2 + a' \geq 0 \) and \( mh_i^2 - b' \geq 0 \);
2. \( a' + b' > 0, \quad Mh_i^2 + a' \geq 0, \quad Mh_i^2 - b' > 0, \) and \((a' + b')^2 < a'(b' - Mh_i^2)\);
3. \( a' + b' < 0, \quad Mh_i^2 + a' > 0, \quad Mh_i^2 - b' \geq 0, \) and \((a' + b')^2 < b'(a' + Mh_i^2)\).

**Proof.** Since \( f(t) \) is convex in \([a, b]\), then \( a' < 0 \) and \( b' > 0 \).

1. When \( a' + b' = 0, \quad Mh_i^2 + a' \geq 0 \) and \( mh_i^2 - b' \geq 0 \), it is obvious that (4)–(7) hold.
2. When \( a' + b' > 0 \), denote \( \zeta_i = -a'/(a' + b') \); as shown in the proof process of Theorem 4, when \( \zeta_i < \zeta_i' \) both (6) and (7) hold; when \( Mh_i^2 + a' \geq 0 \) for any \( \zeta_i > 0, \) (4) holds; only when \( Mh_i^2 - b' > 0 \) and \( \zeta_i \geq (a' + b')/(Mh_i^2 - b') \) (5) holds. Let \( \zeta_i' > (a' + b')/(Mh_i^2 - b') \), namely, \((a' + b')^2 < a'(b' - Mh_i^2)\), all of (4)–(7) hold simultaneously.
3. Similar as the proof process for (2), when the given real number satisfies the condition (3), all of (4)–(7) hold.

The proof is complete. \( \square \)

Theorems 4 and 5 give not only the conditions of the second-order derivative of the interpolating function \( P(t) \) defined by (1) to be less than or equal to a given real number \( M \) or in the given interval \([0, M]\), but also the method to find the parameters \( \alpha_i > 0 \) and \( \beta_i > 0 \) as shown in the proof process.

In a similar way, the existence conditions of the positive parameters for Corollaries 1 and 4 can be found.

### 4. Numerical examples

**Example 1.** Let \( f(t) = e^t, \quad 0 \leq t \leq 4 \), and the knots be 0, 1, 2, 3, 4. In what follows, three kinds of interpolating function of \( f(t) \) are given:

1. \( P(t) \) is the \( C^3 \)-continuous interpolating function defined by (1). Let \( a_i = \alpha_i/\beta_i \), and choose \( a_0 = 1.0 \); by (2)

\[
a_i = \frac{2\Delta_i - d_i - d_{i+1}}{a_{i-1}(d_{i-1} + d_i - 2\Delta_{i-1}) + 2d_i - \Delta_i - \Delta_{i-1}}.
\]
It can be checked that \( \{a_i\} \), for \( i = 0, 1, 2, 3 \), satisfy convexity interpolation condition (3), so \( P(t) \in C^2[0, 4] \) and is convex in \([0, 4]\).

(2) \( P_1(t) \) is the \( C^1 \)-continuous interpolating function defined by (1). For any subinterval \([t_i, t_{i+1}]\), choose \( a_i = \alpha_i/\beta_i = 1.2 \). It is easy to check that \( \{a_i\} \), for \( i = 0, 1, 2, 3 \), satisfy convexity interpolation condition (3), so \( P_1(t) \) is convex too. However, \( \{a_i\} \) does not satisfy (2), so \( P_1(t) \) is only \( C^1 \)-continuous in the interpolating interval.

(3) \( H(t) \) is the standard piecewise cubic Hermite interpolating function.

Table 1 gives the values of \( P(t) \), \( P_1(t) \), \( H(t) \) and \( f(t) \) for \( t \in [0, 4] \). It can be seen that the values of \( P(t) \) and \( P_1(t) \) are much closer to the values of \( f(t) \) than those of \( H(t) \), and that \( P(t) \in C^2 \) is smoother than \( P_1(t) \in C^1 \). All of these illustrate that the interpolation in which the parameters \( \alpha_i, \beta_i, i = 1, 2, \ldots, n-1 \) satisfy (2), not only is smooth interpolation, but also has good approximation to the function being interpolated.

It is easy to test that \( f(t) = \epsilon \) is convex in \([0, 4]\), the given data and the chosen \( a_i = \alpha_i/\beta_i \) satisfy the condition of convexity interpolation, so \( P(t) \) and \( P_1(t) \) are convex functions in \([0, 4]\). Fig. 1 is the graph of \( P(t) \) in \([0, 4]\). Since the graphs of \( P(t) \) and \( f(t) \) are so close they appear to be coincident.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( P(t) )</th>
<th>( P_1(t) )</th>
<th>( H(t) )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>0.2000</td>
<td>1.21972</td>
<td>1.22096</td>
<td>1.21972</td>
<td>1.22140</td>
</tr>
<tr>
<td>0.4000</td>
<td>1.48788</td>
<td>1.49078</td>
<td>1.48788</td>
<td>1.49182</td>
</tr>
<tr>
<td>0.6000</td>
<td>1.81801</td>
<td>1.82102</td>
<td>1.81801</td>
<td>1.82212</td>
</tr>
<tr>
<td>0.8000</td>
<td>2.22634</td>
<td>2.22503</td>
<td>2.22364</td>
<td>2.22554</td>
</tr>
<tr>
<td>1.0000</td>
<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
</tr>
<tr>
<td>1.4000</td>
<td>4.05204</td>
<td>4.05235</td>
<td>4.04448</td>
<td>4.05520</td>
</tr>
<tr>
<td>1.6000</td>
<td>4.94779</td>
<td>4.95004</td>
<td>4.94187</td>
<td>4.95303</td>
</tr>
<tr>
<td>2.0000</td>
<td>7.38906</td>
<td>7.38906</td>
<td>7.38906</td>
<td>7.38906</td>
</tr>
<tr>
<td>2.4000</td>
<td>11.01676</td>
<td>11.01544</td>
<td>11.09403</td>
<td>11.02318</td>
</tr>
<tr>
<td>3.0000</td>
<td>20.08554</td>
<td>20.08554</td>
<td>20.08554</td>
<td>20.08554</td>
</tr>
<tr>
<td>3.4000</td>
<td>29.95679</td>
<td>29.94308</td>
<td>29.88487</td>
<td>29.96410</td>
</tr>
<tr>
<td>3.6000</td>
<td>36.59103</td>
<td>36.57615</td>
<td>36.51579</td>
<td>36.59824</td>
</tr>
<tr>
<td>3.8000</td>
<td>44.69806</td>
<td>44.69086</td>
<td>44.66301</td>
<td>44.70118</td>
</tr>
<tr>
<td>4.0000</td>
<td>54.59815</td>
<td>54.59815</td>
<td>54.59815</td>
<td>54.59815</td>
</tr>
</tbody>
</table>
Example 2. The second example consists of points given by \((t_i, f_i)\), where 
\[ f(t) = -\sqrt{1 - (t - 1)^2} + 3/2 \] (semicircle of radius unity) [10], and the knots 
\[ t_i = \frac{3}{5}(i + 1), \quad i = 0, 1, \ldots, 4. \] The \(C^1\)-continuous interpolating function \(P_i(t)\) for the convex data \((t_i, f_i), i = 0, 1, \ldots, 4\), for \(a_i = 1.2\) and \(t\) in \([2/6, 10/6]\) is given in Table 2. For these values of \(a_i\), it can be easily checked that the convexity condition (3) is satisfied (other values of \(a_i\) which satisfy (3), can also be chosen easily). Fig. 2 shows the graph of the \(C^1\)-continuous interpolating function \(P_i(t)\) in \([2/6, 10/6]\). As the graphs of \(P_i(t)\) and \(f(t)\) are so close they appear to be coincident. To achieve the \(C^2\)-continuous interpolating function \((P(t))\), the values of \(a_i, i = 0, 1, \ldots, 4\) need to be chosen such that they satisfy both conditions (2) and (3). In this case, choosing \(a_0 = a_0/\beta_0 < 0.88\), it can be checked numerically that the values of \(a_i, i = 1, \ldots, 4\) obtained from Eq. (10) satisfy the convexity condition (3). The \(C^2\)-continuous interpolating function \(P(t)\) for \(a_0 = 0.5\) and \(t\) in \([2/6, 10/6]\) is given in Table 2. From this table, it can be seen that the values of \(P_i(t)\) and \(P(t)\) are closer to the values of \(f(t)\), and that the values of \(P(t)\) are slightly better than \(P_i(t)\).

In curve design, there is a need in which the shape of the designed curves need to be modified locally from convexity to concavity or from concavity to convexity. This is not usually easy. The following example shows that even if the function being interpolated, \(f(t)\), is not a convex function in the interpolating interval then, as long as the parameters \(a_i = a_i/\beta_i\) satisfy (3), \(P(t)\) can be constrained to be convex.
Table 2
The values of $p_1(t)$, $P_1(t)$, $f(t)$, $|P_1(t) - f(t)|$ and $|P_1(t) - f(t)|$ when $f(t) = -\sqrt{1 - (t-1)^2 + 3/2}$

| $t$    | $P_1(t)$ | $P_1(t)$ | $f(t)$ | $|P_1(t) - f(t)|$ | $|P_1(t) - f(t)|$ |
|--------|----------|----------|--------|------------------|------------------|
| 0.3333 | 0.75464  | 0.75464  | 0.75464| 0.00000          | 0.00000          |
| 0.4000 | 0.69962  | 0.70017  | 0.70000| 0.00038          | 0.00038          |
| 0.4666 | 0.65330  | 0.65439  | 0.65409| 0.00079          | 0.00029          |
| 0.5333 | 0.61482  | 0.61580  | 0.61557| 0.00075          | 0.00024          |
| 0.6000 | 0.58317  | 0.58357  | 0.58348| 0.00032          | 0.00009          |
| 0.6666 | 0.55719  | 0.55719  | 0.55719| 0.00000          | 0.00000          |
| 0.7333 | 0.53614  | 0.53626  | 0.53621| 0.00007          | 0.00005          |
| 0.8000 | 0.52005  | 0.52027  | 0.52020| 0.00015          | 0.00007          |
| 0.8666 | 0.50878  | 0.50897  | 0.50893| 0.00015          | 0.00004          |
| 0.9333 | 0.50216  | 0.50224  | 0.50222| 0.00007          | 0.00001          |
| 1.0000 | 0.50000  | 0.50000  | 0.50000| 0.00000          | 0.00000          |
| 1.0666 | 0.50219  | 0.50224  | 0.50222| 0.00003          | 0.00000          |
| 1.1333 | 0.50885  | 0.50897  | 0.50893| 0.00008          | 0.00004          |
| 1.2000 | 0.52013  | 0.52027  | 0.52020| 0.00008          | 0.00007          |
| 1.2666 | 0.53618  | 0.53626  | 0.53621| 0.00004          | 0.00005          |
| 1.3333 | 0.55719  | 0.55719  | 0.55719| 0.00000          | 0.00000          |
| 1.4000 | 0.58336  | 0.58357  | 0.58348| 0.00012          | 0.00009          |
| 1.4666 | 0.61526  | 0.61580  | 0.61537| 0.00031          | 0.00024          |
| 1.5333 | 0.65375  | 0.65439  | 0.65409| 0.00035          | 0.00029          |
| 1.6000 | 0.69982  | 0.70017  | 0.70000| 0.00018          | 0.00017          |
| 1.6666 | 0.75464  | 0.75464  | 0.75464| 0.00000          | 0.00000          |

Fig. 2. The graph of $P_1(t)$ for Example 2.

Example 3. Let $f(t) = \cos^6(\pi t/3)$, $1.5 \leq t \leq 2.7$ with interpolating knots at $t = 1.5, 2.1, 2.7$. It is obvious that $f(t)$ does not stay convex in the whole interval $[1.5, 2.7]$. As in Examples 1 and 2, denote the $C^1$-continuous interpo-
Table 3

The values of $P_i(t)$, $H(t)$ and $f(t)$ when $f(t) = \cos^6(\pi t/3)$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P_i(t)$</th>
<th>$H(t)$</th>
<th>$f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.50000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>1.56000</td>
<td>0.00002</td>
<td>-0.00077</td>
<td>0.00000</td>
</tr>
<tr>
<td>1.62000</td>
<td>0.00012</td>
<td>-0.00256</td>
<td>0.00000</td>
</tr>
<tr>
<td>1.68000</td>
<td>0.00045</td>
<td>-0.00457</td>
<td>0.00004</td>
</tr>
<tr>
<td>1.74000</td>
<td>0.00116</td>
<td>-0.00603</td>
<td>0.00024</td>
</tr>
<tr>
<td>1.80000</td>
<td>0.00248</td>
<td>-0.00613</td>
<td>0.00087</td>
</tr>
<tr>
<td>1.86000</td>
<td>0.00477</td>
<td>-0.00409</td>
<td>0.00249</td>
</tr>
<tr>
<td>1.92000</td>
<td>0.00856</td>
<td>0.00088</td>
<td>0.00596</td>
</tr>
<tr>
<td>1.98000</td>
<td>0.01470</td>
<td>0.00956</td>
<td>0.01250</td>
</tr>
<tr>
<td>2.04000</td>
<td>0.02467</td>
<td>0.02275</td>
<td>0.02367</td>
</tr>
<tr>
<td>2.10000</td>
<td>0.04124</td>
<td>0.04124</td>
<td>0.04124</td>
</tr>
<tr>
<td>2.16000</td>
<td>0.08061</td>
<td>0.06998</td>
<td>0.06708</td>
</tr>
<tr>
<td>2.22000</td>
<td>0.13510</td>
<td>0.11230</td>
<td>0.10290</td>
</tr>
<tr>
<td>2.28000</td>
<td>0.19625</td>
<td>0.16652</td>
<td>0.15006</td>
</tr>
<tr>
<td>2.34000</td>
<td>0.26223</td>
<td>0.23100</td>
<td>0.20926</td>
</tr>
<tr>
<td>2.40000</td>
<td>0.33241</td>
<td>0.30407</td>
<td>0.28038</td>
</tr>
<tr>
<td>2.46000</td>
<td>0.40650</td>
<td>0.38406</td>
<td>0.36230</td>
</tr>
<tr>
<td>2.52000</td>
<td>0.48438</td>
<td>0.46931</td>
<td>0.45283</td>
</tr>
<tr>
<td>2.58000</td>
<td>0.56595</td>
<td>0.55816</td>
<td>0.54877</td>
</tr>
<tr>
<td>2.64000</td>
<td>0.65117</td>
<td>0.64895</td>
<td>0.64606</td>
</tr>
<tr>
<td>2.70000</td>
<td>0.74001</td>
<td>0.74001</td>
<td>0.74001</td>
</tr>
</tbody>
</table>

If $P_i(t)$ is required to be convex in $[1.5,2.7]$, it could be done just by choosing suitable parameters. For example, let $a_1 = \alpha_1/\beta_1 = 3.2$ in $[1.5,2.1]$, and $a_2 = \alpha_2/\beta_2 = 0.1$ in $[2.1,2.7]$, then $P_i(t)$ is convex in $[1.5,2.7]$. Table 3 gives the values of $P_i(t)$, $H(t)$ and $f(t)$. Fig. 3 gives the graph of $P_i(t)$ in $[1.5,2.7]$.

5. Approximation properties

When $f(t) \in C^2[t_0, t_n]$, the error estimation of the interpolating function defined by (1) is discussed in [30]. This section deals with the approximation properties when the function $f(t)$ being interpolated is $f(t) \in C^3[t_0, t_n]$ and will derive the boundedness of the optimal error coefficient and its double symmetry with regard to parameters. When $P(t)$ is the rational cubic interpolating function of $f(t)$ defined by (1) in $[t_i, t_{i+1}]$, it is known that the Peano-Kernel Theorem can be used [31], thus

$$R[f] = f(t) - P(t) = \frac{1}{24} \int_{t_i}^{t_{i+1}} f^{(3)}(\tau)R_2[(\tau - t)^2]d\tau,$$

where
Following the analysis done in [31], it may be shown that

1. When \( \frac{\alpha_t}{(\alpha_t + \beta_t)} \leq \theta \leq 1 \),

\[
|R(f)| \leq |f(t) - P(t)| \leq \frac{h^3}{3!} \max_{\frac{\alpha_t}{(\alpha_t + \beta_t)} \leq \theta \leq 1} \omega_1(\alpha_t, \beta_t, \theta),
\]

where

\[
\omega_1(\alpha_t, \beta_t, \theta) = \frac{\theta^2(1 - \theta)^2[(\alpha_t^3 + \beta_t^3 - 2\alpha_t\beta_t^2)^2 - (6\alpha_t\beta_t^2)^2]}{(a_t - \beta_t)^2(\alpha_t + \beta_t)^2 + a_t^2\beta_t^2 + \alpha_t^3}.
\]

2. When \( 0 \leq \theta \leq \frac{\alpha_t}{(\alpha_t + \beta_t)} \),

\[
|R(f)| \leq |f(t) - P(t)| \leq \frac{h^3}{3!} \max_{0 \leq \theta \leq \frac{\alpha_t}{(\alpha_t + \beta_t)}} \omega_2(\alpha_t, \beta_t, \theta),
\]

where

\[
\omega_2(\alpha_t, \beta_t, \theta) = \frac{\theta^2(1 - \theta)^2[8\beta_t^2(1 - \theta) + (\alpha_t - \beta_t)((1 - \theta)\alpha_t + (2 - \theta)\beta_t)]^2}{((1 - \theta)\alpha_t + \theta\beta_t)((1 - \theta)\alpha_t + (2 - \theta)\beta_t)^2}.
\]
It is easy to prove that
\[ \omega_1(\alpha_i, \beta_i, \theta) = \omega_2(\beta_i, \alpha_i, 1 - \theta), \]
and it follows that \( \omega_1 \) and \( \omega_2 \) are symmetric about the parameters \( \alpha_i \) and \( \beta_i \) and anti-symmetric about \( \theta \) in \([t_i, t_{i+1}]\). To sum up, \( \omega_1 \) and \( \omega_2 \) can be called double symmetric about \( \alpha_i, \beta_i \) and \( \theta \). This leads to the following theorem about the approximation properties.

**Theorem 6.** If \( f(t) \in C^3[t_0, t_n] \), \( \Delta : t_0 < t_1 < \cdots < t_n \), \( P(t) \) is the corresponding rational cubic interpolating spline defined by (1), for the given \( \alpha_i, \beta_i, \) when \( t \in [t_i, t_{i+1}] \)
\[ \|R[f]\| = \|f(t) - P(t)\| \leq \frac{h_i^4}{3!} \|f^{(3)}(t)\| c_i, \]
where
\[ c_i = \max_{0 < \theta < 1} \omega_1(\alpha_i, \beta_i, \theta) = \max_{0 < \theta < 1} \omega_2(\alpha_i, \beta_i, \theta). \]
The following theorem gives the bounds of the optimal error constant \( c_i \) in Theorem 6.

**Theorem 7.** For any given \( \alpha_i > 0 \) and \( \beta_i > 0 \), the optimal error constant \( c_i \) in Theorem 6 is bounded with
\[ \frac{1}{16} \leq c_i \leq \frac{4}{27}. \]

The proof of Theorem 7 is straightforward and is omitted.

**Acknowledgements**

The support of The Royal Society of London and The Natural Science Foundation of China are gratefully acknowledged.

**References**