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Constrained control and approximation properties of a rational interpolating curve

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Abstract

This paper deals with the convexity control and the strain energy control of interpolating curves using a rational cubic spline with linear denominator. The sufficient and necessary conditions for controlling the interpolating curve to be convex or concave are derived. When the function being interpolated is $f(t) \in C^{(3)}[t_0, t_n]$, the error estimation of the interpolating function and the boundedness of the optimal error coefficient and its double symmetry with regard to parameters are obtained.

Keywords: Rational interpolation; Convexity control; Constrained control; Approximation

1. Introduction

Design of high quality, manufacturable curves or surfaces, such as the outer shape of a ship, car or aeroplane, is an important yet challenging task to today's manufacturing industries. Many authors have studied various kinds of

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splines for curve and surface design and control [1–28]. In general, the common spline interpolations are fixed interpolations which means that the shape of the interpolating curve is fixed for the given interpolating data. If one wishes to modify the shape of the interpolating curve, the interpolating data need to be changed. An important question is how can the shape of the curve be modified under the condition that the given data are not changed? In [29], a rational interpolating spline with linear denominator was constructed, and it was used to control the shape of the interpolating curves, such as controlling the curves to be in a given region. Because the parameters in the interpolating function can be selected according to the control needs, the constrained control of the shape becomes possible.

In curve control, one of the most significant things is to control the convexity of the interpolating curves: to control the interpolating curves to be convex or concave, or even to change the convexity in a local area. On the other hand, the second-order derivative of an interpolating function has been used in estimating the strain energy and, consequently, smoothness of the interpolant. Smaller energy generally implies smoother shape. It is possible, however, that the overall energy of an interpolant is small yet great enough to generate abnormal shape at some points or even in some small intervals. Thus, both the convexity control and the energy control depend on the second-order derivative control. Because the spline has linear denominator, the derivation calculus is easier when compared to that with cubic or quadratic denominator, so, it is much easier to get the control conditions.

In Section 2 of this paper, rational cubic splines with linear denominator are restated briefly. Section 3 deals with the convexity and strain energy control. The sufficient and necessary conditions for the second-order derivative of the interpolating function to be constrained in the given interval $[N, M]$ are derived. Thus, the convexity control and the strain energy control can be carried out by constraining the second-order derivative of the interpolating function to be in the given interval $[0, M]$ or $[N, 0]$. Also the existence conditions for these interpolations are given in this section. Numerical examples are given in Section 4. Section 5 is about the approximation properties of the spline. When the function being interpolated is $f(t) \in C^{(3)}[t_0, t_n]$, the error estimation of the interpolating function is obtained and the boundedness of the optimal error coefficient and its double symmetry with regard to parameters are established.

2. Rational cubic interpolating spline with linear denominator

Given a dataset $\{(t_i, f_i, d_i), i = 0, 1, \ldots, n\}$, where f_i and d_i are the function value and the first-order derivative value of the function being interpolated, $f(t)$, defined at the knot, respectively, and $a = t_0 < t_1 < \cdots < t_n = b$ is the knot spacing. Let $h_i = t_{i+1} - t_i$, $\theta = (t - t_i)/h_i$, and let α_i and β_i be positive parameters. The C^1-continuous, piecewise, rational cubic spline with linear denominator is defined in the interpolatory interval $[a, b]$ as follows [29]:

$$P(t)|_{[t_i,t_{i+1}]} = \frac{p_i(t)}{q_i(t)}, \quad i = 0, 1, \ldots, n-1, \tag{1}$$

where

$$p_i(t) = (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1},$$
$$q_i(t) = (1-\theta)\alpha_i + \theta\beta_i$$

and

$$V_i = (2\alpha_i + \beta_i)f_i + \alpha_i h_i d_i,$$
$$W_i = (\alpha_i + 2\beta_i)f_{i+1} - \beta_i h_i d_{i+1}.$$

This rational cubic spline satisfies

$$P(t_i) = f_i, \quad P'(t_i) = d_i, \qquad i = 0, 1, \ldots, n.$$

Furthermore, let $P''(t_i+) = P''(t_i-)$ for $i = 1, 2, \ldots, n-1$. The conditions lead to the following tri-diagonal system of linear equations:

$$\begin{aligned} h_i\frac{\alpha_{i-1}}{\beta_{i-1}}d_{i-1} &+ \left(h_i\left(1+\frac{\alpha_{i-1}}{\beta_{i-1}}\right) + h_{i-1}\left(1+\frac{\beta_i}{\alpha_i}\right)\right)d_i + h_{i-1}\frac{\beta_i}{\alpha_i}d_{i+1} \\ &= h_{i-1}\left(1+2\frac{\beta_i}{\alpha_i}\right)\Delta_i + h_i\left(1+2\frac{\alpha_{i-1}}{\beta_{i-1}}\right)\Delta_{i-1}; \quad i = 1, 2, \ldots, n-1, \end{aligned} \tag{2}$$

where $\Delta_i = (f_{i+1} - f_i)/h_i$. It is easy to see that if the successive parameters $\{\alpha_i\}$ and $\{\beta_i\}$ satisfy (2), then $P(t) \in C^2[a, b]$.

Note: The parameters α_i and β_i are used to keep the symmetry of the functions $p_i(t)$ and $q_i(t)$. In fact, by letting $a_i = \alpha_i/\beta_i$, both $p_i(t)$ and $q_i(t)$ can be expressed in one independent parameter a_i as

$$p_i(t) = (1-\theta)^3 a_i f_i + \theta(1-\theta)^2 V_i^* + \theta^2(1-\theta)W_i^* + \theta^3 f_{i+1},$$
$$q_i(t) = (1-\theta)a_i + \theta$$

and

$$V_i^* = (2a_i + 1)f_i + a_i h_i d_i,$$
$$W_i^* = (a_i + 2)f_{i+1} - h_i d_{i+1}$$

and hence, one parameter can be used throughout the paper.

3. Constrained control of the convexity of the interpolating curves

The convexity control and strain energy control of the interpolating function depend on the control of its second-order derivative. If the second-order derivative of the interpolating function can be controlled to a given interval, then the convexity control and energy control can be carried out. Since the rational cubic splines defined by (1) have linear denominator, the second-order derivatives can be found easily. When $t \in [t_i, t_{i+1}]$, from (1) it is easy to see that

$$\begin{aligned} P''(t) = {} & (h_i^2((1-\theta)\alpha_i + \theta\beta_i)^3)^{-1}\{((1-\theta)\alpha_i + \theta\beta_i)^2(6(1-\theta)\alpha_i f_i \\ & + (6\theta - 4)V_i + (2 - 6\theta)W_i + 6\theta\beta_i f_{i+1}) - 2(\beta_i - \alpha_i)((1-\theta)\alpha_i + \theta\beta_i) \\ & \times (-3(1-\theta)^2\alpha_i f_i + (1 - 4\theta + 3\theta^2)V_i + (2\theta - 3\theta^2)W_i + 3\theta^2\beta_i f_{i+1}) \\ & + 2(\beta_i - \alpha_i)^2((1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1})\}. \end{aligned}$$

Let $P''(t) \leqslant M$, then it is easy to show that

$$\begin{aligned} Q(\theta) = {} & Mh_i^2((1-\theta)\alpha_i + \theta\beta_i)^3 + \{-((1-\theta)\alpha_i + \theta\beta_i)^2(6(1-\theta)\alpha_i f_i \\ & + (6\theta - 4)V_i + (2 - 6\theta)W_i + 6\theta\beta_i f_{i+1}) + 2(\beta_i - \alpha_i)((1-\theta)\alpha_i + \theta\beta_i) \\ & \times (-3(1-\theta)^2\alpha_i f_i + (1 - 4\theta + 3\theta^2)V_i + (2\theta - 3\theta^2)W_i + 3\theta^2\beta_i f_{i+1}) \\ & - 2(\beta_i - \alpha_i)^2((1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1})\} \geqslant 0. \end{aligned}$$

Since

$$Q'(\theta) = [(1-\theta)\alpha_i + \theta\beta_i]^2[3(\beta_i - \alpha_i)Mh_i^2 + 6(\alpha_i f_i - V_i + W_i - \beta_i f_{i+1})],$$

$Q(\theta)$ is monotone in $[0, 1]$. On the other hand,

$$\begin{aligned} Q(0) &= 2\alpha_i^2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_i d_{i+1}) + \alpha_i^3(Mh_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i) \geqslant 0, \\ Q(1) &= \beta_i^3(Mh_i^2 - 2f_i + 2f_{i+1} - 2h_i d_{i+1}) + 2\alpha_i\beta_i^2(4f_{i+1} - 4f_i - 2h_i d_{i+1} - 2h_i d_i). \end{aligned}$$

Therefore, the following theorem holds.

Theorem 1. *For the rational cubic interpolating function $P(t)$ defined by (1), the second-order derivative $P''(t)$ is less than or equal to M in $[t_i, t_{i+1}]$ if and only if the positive parameters α_i, β_i satisfy the following inequality system*

$$\begin{cases} 2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_i d_{i+1}) + \alpha_i(Mh_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i) \geqslant 0, \\ \beta_i(Mh_i^2 - 2f_i + 2f_{i+1} - 2h_i d_{i+1}) + 2\alpha_i(2f_{i+1} - 2f_i - h_i d_{i+1} - h_i d_i) \geqslant 0. \end{cases}$$

Let $M = 0$ in Theorem 1, then there is a corollary concerning the convexity control condition for the interpolating function $P(t)$. It is

Corollary 1. *The rational cubic interpolating function $P(t)$ defined by (1) remains concave in $[t_i, t_{i+1}]$ if and only if the positive parameters α_i, β_i satisfy the following inequality system*

$$\begin{cases} 2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_i d_{i+1}) + \alpha_i(2f_i - 2f_{i+1} + 2h_i d_i) \geqslant 0, \\ \beta_i(-2f_i + 2f_{i+1} - 2h_i d_{i+1}) + 2\alpha_i(2f_{i+1} - 2f_i - h_i d_{i+1} - h_i d_i) \geqslant 0. \end{cases}$$

In a similar way as for Theorem 1, *the following theorem can be obtained.*

Theorem 2. *For the rational cubic interpolating function $P(t)$ defined by* (1), *the second-order derivative $P''(t)$ is greater than or equal to a given number N in $[t_i, t_{i+1}]$ if and only if the positive parameters α_i, β_i satisfy the following inequality system*

$$\begin{cases} 2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_i d_{i+1}) + \alpha_i(Nh_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i) \leqslant 0, \\ \beta_i(Nh_i^2 - 2f_i + 2f_{i+1} - 2h_i d_{i+1}) + 2\alpha_i(2f_{i+1} - 2f_i - h_i d_{i+1} - h_i d_i) \leqslant 0. \end{cases}$$

Let $N = 0$ in Theorem 2, *then there is the Corollary* 2 *related to the convexity control condition for the interpolating function $P(t)$. It is*

Corollary 2. *The rational cubic interpolating function $P(t)$ defined by* (1) *is convex in $[t_i, t_{i+1}]$ if and only if the positive parameters α_i, β_i satisfy the following inequality system*

$$\begin{cases} 2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_i d_{i+1}) + \alpha_i(2f_i - 2f_{i+1} + 2h_i d_i) \leqslant 0, \\ \beta_i(-2f_i + 2f_{i+1} - 2h_i d_{i+1}) + 2\alpha_i(2f_{i+1} - 2f_i - h_i d_{i+1} - h_i d_i) \leqslant 0. \end{cases} \tag{3}$$

From Theorems 1 and 2, the following theorem is obtained.

Theorem 3. *For the rational cubic interpolating function $P(t)$ defined by* (1), *the second-order derivative $P''(t)$ on $[t_i, t_{i+1}]$ is in the given interval $[N, M]$ if and only if the positive parameters α_i, β_i satisfy the following inequality system*

$$\begin{cases} 2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_i d_{i+1}) + \alpha_i(Mh_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i) \geqslant 0, \\ \beta_i(Mh_i^2 - 2f_i + 2f_{i+1} - 2h_i d_{i+1}) + 2\alpha_i(2f_{i+1} - 2f_i - h_i d_{i+1} - h_i d_i) \geqslant 0, \\ -2\beta_i(2f_i - 2f_{i+1} + h_i d_i + h_i d_{i+1}) - \alpha_i(Nh_i^2 + 2f_i - 2f_{i+1} + 2h_i d_i) \geqslant 0, \\ -\beta_i(Nh_i^2 - 2f_i + 2f_{i+1} - 2h_i d_{i+1}) - 2\alpha_i(2f_{i+1} - 2f_i - h_i d_{i+1} - h_i d_i) \geqslant 0. \end{cases}$$

Considering both the convex interpolation and strain energy control simultaneously in shape design, it is necessary to constrain the second-order derivative to be bounded in a given interval $[0, M]$ or $[N, 0]$, where $M > 0$ and $N < 0$. These are described in the following corollaries.

Corollary 3. *For the rational cubic interpolating function $P(t)$ defined by* (1), *the second-order derivative $P''(t)$ on $[t_i, t_{i+1}]$ is in the given interval $[0, M]$ with $M > 0$ if and only if the positive parameters α_i, β_i satisfy the following inequality system*

$$(a^* + b^*)\zeta_i + (Mh_i^2 + a^*) \geqslant 0, \tag{4}$$

$$(Mh_i^2 - b^*)\zeta_i - (a^* + b^*) \geqslant 0, \tag{5}$$

$$-(a^* + b^*)\zeta_i - a^* \geqslant 0, \tag{6}$$

$$b^*\zeta_i + (a^* + b^*) \geqslant 0, \tag{7}$$

where $\zeta_i = \beta_i/\alpha_i$, *and*

$$a^* = 2(f_i - f_{i+1} + h_i d_i),$$
$$b^* = 2(f_i - f_{i+1} + h_i d_{i+1}).$$

Corollary 4. *For the rational cubic interpolating function* $P(t)$ *defined by* (1), *the second-order derivative* $P''(t)$ *on* $[t_i, t_{i+1}]$ *is in the given interval* $[N, 0]$ *with* $N < 0$ *if and only if the positive parameters* α_i, β_i *satisfy the following inequality system*

$$\begin{cases} (a^* + b^*)\zeta_i + a^* \geqslant 0, \\ -b^*\zeta_i - (a^* + b^*) \geqslant 0, \\ -(a^* + b^*)\zeta_i - (a^* + Nh_i^2) \geqslant 0, \\ (-Nh_i^2 + b^*)\zeta_i + (a^* + b^*) \geqslant 0, \end{cases}$$

where ζ_i, a^* and b^* are as defined for Corollary 3.

As discussed above, to ensure the second-order derivative of the interpolating function to be constrained in a given interval requires the existence of positive parameters α_i and β_i which satisfy a corresponding constrained system of equations. The remaining of this section deals with the problem do these parameters always exist? and if so how to get them. The following theorems, Theorems 4 and 5, give the existence conditions for the parameters in Corollaries 2 and 3 under the conditions that the function being interpolated, $f(t)$, is convex in the interpolating interval $[a, b]$.

Theorem 4. *If the function being interpolated,* $f(t)$, *is convex in the interpolating interval, for given* $\{(t_i, f_i, d_i), i = 1, 2, \dots, n\}$, *there must exist parameters* $\alpha_i > 0$, $\beta_i > 0$ *such that the interpolating function* $P(t)$ *defined by* (1) *is convex.*

Proof. System (3) can be rewritten as

$$-(a^* + b^*)\zeta_i - a^* \geqslant 0, \tag{8}$$
$$b^*\zeta_i + (a^* + b^*) \geqslant 0, \tag{9}$$

where ζ_i, a^* and b^* are defined as in Corollary 3. Since $f(t)$ is convex in the interpolating interval $[a, b]$, it is easy to get $a^* < 0$ and $b^* > 0$.

(1) If $a^* + b^* = 0$, then for any $\zeta_i > 0$ both (8) and (9) hold.

(2) If $a^* + b^* > 0$, it is obvious that for any $\zeta_i > 0$ (9) holds. Denote $\zeta_i^* = -a^*/(a^* + b^*)$, then $\zeta_i^* > 0$, and when $\zeta_i < \zeta_i^*$ (8) holds. Thus, when $\zeta_i < \zeta_i^*$ both (8) and (9) hold.

(3) If $a^* + b^* < 0$, for any $\zeta_i > 0$ (8) holds. Let $\zeta_i^{**} = -(a^* + b^*)/b^*$, then $\zeta_i^{**} > 0$, and when $\zeta_i > \zeta_i^{**}$ (9) holds. Thus, when $\zeta_i > \zeta_i^{**}$ both (8) and (9) hold.

Thus, the proof is complete. $\square$

The following theorem gives the conditions for the second-order derivative of the interpolating function to be constrained in the given interval $[0, M]$.

Theorem 5. *If the function being interpolated, $f(t)$, is convex in the interpolating interval $[a, b]$, for given $\{(t_i, f_i, d_i), i = 1, 2, \ldots, n\}$ and the given real number $M > 0$, when $t \in [t_i, t_{i+1}]$, if one of the following conditions is satisfied, there must exist parameters $\alpha_i > 0$, $\beta_i > 0$ such that the second-order derivative values of the interpolating function $P(t)$ defined by (1) are in the interval $[0, M]$.*
(1) $a^* + b^* = 0$, $Mh_i^2 + a^* \geqslant 0$ and $Mh_i^2 - b^* \geqslant 0$;
(2) $a^* + b^* > 0$, $Mh_i^2 + a^* \geqslant 0$, $Mh_i^2 - b^* > 0$, and $(a^* + b^*)^2 < a^*(b^* - Mh_i^2)$;
(3) $a^* + b^* < 0$, $Mh_i^2 + a^* > 0$, $Mh_i^2 - b^* \geqslant 0$, and $(a^* + b^*)^2 < b^*(a^* + Mh_i^2)$.

Proof. Since $f(t)$ is convex in $[a, b]$, then $a^* < 0$ and $b^* > 0$.
(1) When $a^* + b^* = 0$, $Mh_i^2 + a^* \geqslant 0$ and $Mh_i^2 - b^* \geqslant 0$, it is obvious that (4)–(7) hold.
(2) When $a^* + b^* > 0$, denote $\zeta_i^* = -a^*/(a^* + b^*)$; as shown in the proof process of Theorem 4, when $\zeta_i < \zeta_i^*$ both (6) and (7) hold; when $Mh_i^2 + a^* \geqslant 0$ for any $\zeta_i > 0$ (4) holds; only when $Mh_i^2 - b^* > 0$ and $\zeta_i \geqslant (a^* + b^*)/(Mh_i^2 - b^*)$ (5) holds. Let $\zeta_i^* > (a^* + b^*)/(Mh_i^2 - b^*)$, namely, $(a^* + b^*)^2 < a^*(b^* - Mh_i^2)$, all of (4)–(7) hold simultaneously.
(3) Similar as the proof process for (2), when the given real number satisfies the condition (3), all of (4)–(7) hold.
The proof is complete. □

Theorems 4 and 5 give not only the conditions of the second-order derivative of the interpolating function $P(t)$ defined by (1) to be less than or equal to a given real number M or in the given interval $[0, M]$, but also the method to find the parameters $\alpha_i > 0$ and $\beta_i > 0$ as shown in the proof process.

In a similar way, the existence conditions of the positive parameters for Corollaries 1 and 4 can be found.

4. Numerical examples

Example 1. Let $f(t) = e^t$, $0 \leqslant t \leqslant 4$, and the knots be 0, 1, 2, 3, 4. In what follows, three kinds of interpolating function of $f(t)$ are given:
(1) $P(t)$ is the C^2-continuous interpolating function defined by (1). Let $a_i = \alpha_i/\beta_i$ and choose $a_0 = 1.0$; by (2)

$$a_i = \frac{2\Delta_i - d_i - d_{i+1}}{a_{i-1}(d_{i-1} + d_i - 2\Delta_{i-1}) + 2d_i - \Delta_i - \Delta_{i-1}}. \tag{10}$$

It can be checked that $\{a_i\}$, for $i = 0, 1, 2, 3$, satisfy convexity interpolation condition (3), so $P(t) \in C^2[0, 4]$ and is convex in $[0, 4]$.

(2) $P_1(t)$ is the C^1-continuous interpolating function defined by (1). For any subinterval $[t_i, t_{i+1}]$, choose $a_i = \alpha_i/\beta_i = 1.2$. It is easy to check that $\{a_i\}$, for $i = 0, 1, 2, 3$, satisfy convexity interpolation condition (3), so $P_1(t)$ is convex too. However, $\{a_i\}$ does not satisfy (2), so $P_1(t)$ is only C^1-continuous in the interpolating interval.

(3) $H(t)$ is the standard piecewise cubic Hermite interpolating function.

Table 1 gives the values of $P(t)$, $P_1(t)$, $H(t)$ and $f(t)$ for $t \in [0, 4]$. It can be seen that the values of $P(t)$ and $P_1(t)$ are much closer to the values of $f(t)$ than those of $H(t)$, and that $P(t) \in C^2$ is smoother than $P_1(t) \in C^1$. All of these illustrate that the interpolation in which the parameters $\alpha_i, \beta_i, i = 1, 2, \ldots, n-1$ satisfy (2), not only is smooth interpolation, but also has good approximation to the function being interpolated.

It is easy to test that $f(t) = e^t$ is convex in $[0, 4]$, the given data and the chosen $a_i = \alpha_i/\beta_i$ satisfy the condition of convexity interpolation, so $P(t)$ and $P_1(t)$ are convex functions in $[0, 4]$. Fig. 1 is the graph of $P(t)$ in $[0, 4]$. Since the graphs of $P(t)$ and $f(t)$ are so close they appear to be coincident.

Table 1
The values of $P(t)$, $P_1(t)$, $H(t)$ and $f(t)$ when $f(t) = e^t$

t	$P(t)$	$P_1(t)$	$H(t)$	$f(t)$
0.00000	1.00000	1.00000	1.00000	1.00000
0.20000	1.21972	1.22096	1.21972	1.22140
0.40000	1.48788	1.49078	1.48788	1.49182
0.60000	1.81801	1.82102	1.81801	1.82212
0.80000	2.22364	2.22503	2.22364	2.22554
1.00000	2.71828	2.71828	2.71828	2.71828
1.20000	3.31803	3.31891	3.31553	3.32012
1.40000	4.05024	4.05235	4.04448	4.05520
1.60000	4.94779	4.95004	4.94187	4.95303
1.80000	6.04718	6.04825	6.04448	6.04965
2.00000	7.38906	7.38906	7.38906	7.38906
2.20000	9.02228	9.02174	9.01255	9.02501
2.40000	11.01676	11.01544	10.99403	11.02318
2.60000	13.45703	13.45561	13.43341	13.46374
2.80000	16.44153	16.44085	16.43060	16.44465
3.00000	20.08554	20.08554	20.08554	20.08554
3.20000	24.52927	24.52363	24.49866	24.53253
3.40000	29.95679	29.94308	29.88487	29.96410
3.60000	36.59103	36.57615	36.51579	36.59824
3.80000	44.69806	44.69086	44.66301	44.70118
4.00000	54.59815	54.59815	54.59815	54.59815

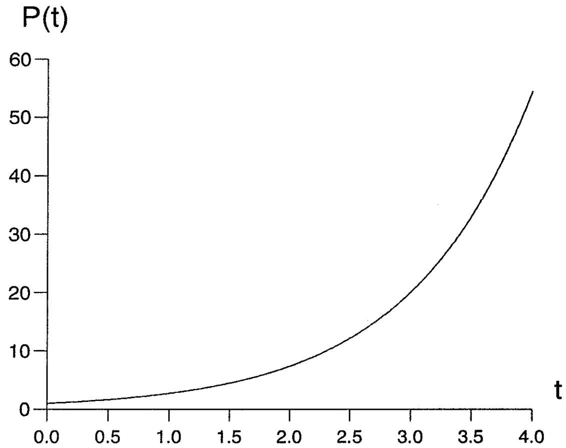

Fig. 1. The graph of $P(t)$ for Example 1.

Example 2. The second example consists of points given by (t_i, f_i), where $f(t) = -\sqrt{1-(t-1)^2} + 3/2$ (semicircle of radius unity) [10], and the knots $t_i = \frac{2}{6}(i+1)$, $i = 0, 1, \ldots, 4$. The C^1-continuous interpolating function $P_1(t)$ for the convex data $((t_i, f_i), i = 0, 1, \ldots, 4)$, for $a_i = 1.2$ and t in $[2/6, 10/6]$ is given in Table 2. For these values of a_i, it can be easily checked that the convexity condition (3) is satisfied (other values of a_i which satisfy (3), can also be chosen easily). Fig. 2 shows the graph of the C^1-continuous interpolating function $P_1(t)$ in $[2/6, 10/6]$. As the graphs of $P_1(t)$ and $f(t)$ are so close they appear to be coincident. To achieve the C^2-continuous interpolating function $(P(t))$, the values of a_i, $i = 0, 1, \ldots, 4$ need to be chosen such that they satisfy both conditions (2) and (3). In this case, choosing $a_0 = \alpha_0/\beta_0 < 0.88$, it can be checked numerically that the values of a_i, $i = 1, \ldots, 4$ obtained from Eq. (10) satisfy the convexity condition (3). The C^2-continuous interpolating function $P(t)$ for $a_0 = 0.5$ and t in $[2/6, 10/6]$ is given in Table 2. From this table, it can be seen that the values of $P_1(t)$ and $P(t)$ are closer to the values of $f(t)$, and that the values of $P(t)$ are slightly better than $P_1(t)$.

In curve design, there is a need in which the shape of the designed curves need to be modified locally from convexity to concavity or from concavity to convexity. This is not usually easy. The following example shows that even if the function being interpolated, $f(t)$, is not a convex function in the interpolating interval then, as long as the parameters $a_i = \alpha_i/\beta_i$ satisfy (3), $P(t)$ can be constrained to be convex.

Table 2
The values of $P_1(t)$, $P(t)$, $f(t)$, $|P_1(t)-f(t)|$ and $|P(t)-f(t)|$ when $f(t)=-\sqrt{1-(t-1)^2}+3/2$

t	$P_1(t)$	$P(t)$	$f(t)$	$\|P_1(t)-f(t)\|$	$\|P(t)-f(t)\|$
0.33333	0.75464	0.75464	0.75464	0.00000	0.00000
0.40000	0.69962	0.70017	0.70000	0.00038	0.00017
0.46667	0.65330	0.65439	0.65409	0.00079	0.00029
0.53333	0.61482	0.61580	0.61557	0.00075	0.00024
0.60000	0.58317	0.58357	0.58348	0.00032	0.00009
0.66667	0.55719	0.55719	0.55719	0.00000	0.00000
0.73333	0.53614	0.53626	0.53621	0.00007	0.00005
0.80000	0.52005	0.52027	0.52020	0.00015	0.00007
0.86667	0.50878	0.50897	0.50893	0.00015	0.00004
0.93333	0.50216	0.50224	0.50222	0.00007	0.00001
1.00000	0.50000	0.50000	0.50000	0.00000	0.00000
1.06667	0.50219	0.50224	0.50222	0.00003	0.00001
1.13333	0.50885	0.50897	0.50893	0.00008	0.00004
1.20000	0.52013	0.52027	0.52020	0.00008	0.00007
1.26667	0.53618	0.53626	0.53621	0.00004	0.00005
1.33333	0.55719	0.55719	0.55719	0.00000	0.00000
1.40000	0.58336	0.58357	0.58348	0.00012	0.00009
1.46667	0.61526	0.61580	0.61557	0.00031	0.00024
1.53333	0.65375	0.65439	0.65409	0.00035	0.00029
1.60000	0.69982	0.70017	0.70000	0.00018	0.00017
1.66667	0.75464	0.75464	0.75464	0.00000	0.00000

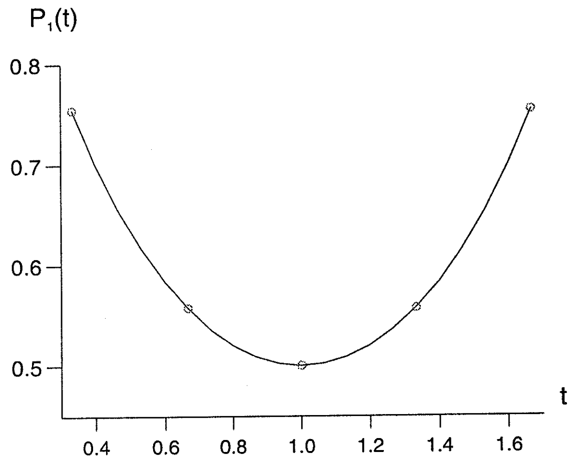

Fig. 2. The graph of $P_1(t)$ for Example 2.

Example 3. Let $f(t)=\cos^6(\pi t/3), 1.5 \leqslant t \leqslant 2.7$ with interpolating knots at $t=1.5, 2.1, 2.7$. It is obvious that $f(t)$ does not stay convex in the whole interval $[1.5, 2.7]$. As in Examples 1 and 2, denote the C^1-continuous interpo-

Table 3
The values of $P_1(t)$, $H(t)$ and $f(t)$ when $f(t) = \cos^6(\pi t/3)$

t	$P_1(t)$	$H(t)$	$f(t)$
1.50000	0.00000	0.00000	0.00000
1.56000	0.00002	−0.00077	0.00000
1.62000	0.00012	−0.00256	0.00000
1.68000	0.00045	−0.00457	0.00004
1.74000	0.00116	−0.00603	0.00024
1.80000	0.00248	−0.00613	0.00087
1.86000	0.00477	−0.00409	0.00249
1.92000	0.00856	0.00088	0.00596
1.98000	0.01470	0.00956	0.01250
2.04000	0.02467	0.02275	0.02367
2.10000	0.04124	0.04124	0.04124
2.16000	0.08061	0.06998	0.06708
2.22000	0.13510	0.11230	0.10290
2.28000	0.19625	0.16652	0.15006
2.34000	0.26223	0.23100	0.20926
2.40000	0.33241	0.30407	0.28038
2.46000	0.40650	0.38406	0.36230
2.52000	0.48438	0.46931	0.45283
2.58000	0.56595	0.55816	0.54877
2.64000	0.65117	0.64895	0.64606
2.70000	0.74001	0.74001	0.74001

lating function by $P_1(t)$. If $P_1(t)$ is required to be convex in [1.5,2.7], it could be done just by choosing suitable parameters. For example, let $a_1 = \alpha_1/\beta_1 = 3.2$ in [1.5,2.1], and $a_2 = \alpha_2/\beta_2 = 0.1$ in [2.1,2.7], then $P_1(t)$ is convex in [1.5,2.7]. Table 3 gives the values of $P_1(t)$, $H(t)$ and $f(t)$. Fig. 3 gives the graph of $P_1(t)$ in [1.5,2.7].

5. Approximation properties

When $f(t) \in C^2[t_0, t_n]$, the error estimation of the interpolating function defined by (1) is discussed in [30]. This section deals with the approximation properties when the function $f(t)$ being interpolated is $f(t) \in C^3[t_0, t_n]$ and will derive the boundedness of the optimal error coefficient and its double symmetry with regard to parameters. When $P(t)$ is the rational cubic interpolating function of $f(t)$ defined by (1) in $[t_i, t_{i+1}]$, it is known that the Peano–Kernel Theorem can be used [31], thus

$$R[f] = f(t) - P(t) = \frac{1}{2!}\int_{t_i}^{t_{i+1}} f^{(3)}(\tau)R_t[(t-\tau)_+^2]\,\mathrm{d}\tau, \tag{11}$$

where

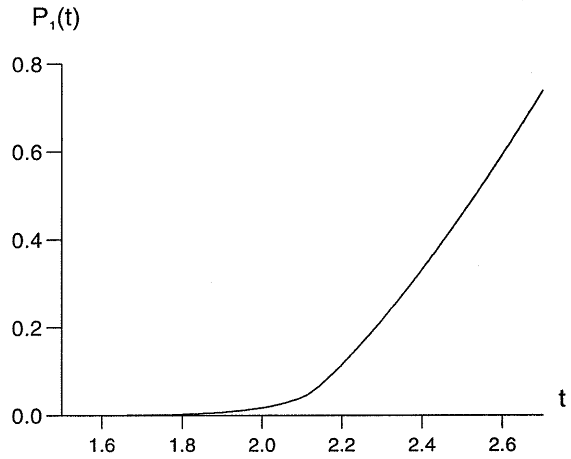

Fig. 3. The graph of $P_1(t)$ for Example 3.

$$R_t[(t-\tau)_+^2] = \begin{cases} (t-\tau)^2 - \frac{t_{i+1}-\tau}{(1-\theta)\alpha_i+\beta_i\theta}\{\theta^2(1-\theta)[(\alpha_i+2\beta_i)(t_{i+1}-\tau)-2\beta_i h_i] \\ \quad +\theta^3\beta_i(t_{i+1}-\tau)\}, \quad t_i<\tau<t; \\ -\frac{t_{i+1}-\tau}{(1-\theta)\alpha_i+\beta_i\theta}\{\theta^2(1-\theta)[(\alpha_i+2\beta_i)(t_{i+1}-\tau)-2\beta_i h_i]+\theta^3\beta_i(t_{i+1}-\tau)\}, \\ \quad t<\tau<t_{i+1}. \end{cases}$$

Following the analysis done in [31], it may be shown that

(1) When $\alpha_i/(\alpha_i+\beta_i)\leqslant\theta\leqslant 1$,

$$|R[f]|\leqslant\|f(t)-P(t)\|\leqslant\frac{h_i^3}{3!}\|f^{(3)}(t)\|\max_{\alpha_i/(\alpha_i+\beta_i)\leqslant\theta\leqslant 1}\omega_1(\alpha_i,\beta_i,\theta),$$

where

$$\omega_1(\alpha_i,\beta_i,\theta) = \frac{\theta^2(\theta-1)^2[(\alpha_i^3+\alpha_i^2\beta_i-\alpha_i\beta_i^2-\beta_i^3)\theta^2-(6\alpha_i^3+2\alpha_i\beta_i^2)\theta-\alpha_i^2\beta_i+\alpha_i^3]}{((\alpha_i-\beta_i)\theta-\alpha_i)((\alpha_i+\beta_i)\theta+\alpha_i)^2}.$$

(2) When $0\leqslant\theta\leqslant\alpha_i/(\alpha_i+\beta_i)$,

$$|R[f]|\leqslant\|f(t)-P(t)\|\leqslant\frac{h_i^3}{3!}\|f^{(3)}(t)\|\max_{0\leqslant\theta\leqslant\alpha_i/(\alpha_i+\beta_i)}\omega_2(\alpha_i,\beta_i,\theta),$$

where

$$\omega_2(\alpha_i,\beta_i,\theta) = \frac{\theta^2(\theta-1)^2[8\beta_i^3(1-\theta)+(\alpha_i-\beta_i)((1-\theta)\alpha_i+(2-\theta)\beta_i)^2]}{((1-\theta)\alpha_i+\theta\beta_i)((1-\theta)\alpha_i+(2-\theta)\beta_i)^2}.$$

It is easy to prove that

$$\omega_1(\alpha_i, \beta_i, \theta) = \omega_2(\beta_i, \alpha_i, 1 - \theta),$$

and it follows that ω_1 and ω_2 are symmetric about the parameters α_i and β_i and anti-symmetric about θ in $[t_i, t_{i+1}]$. To sum up, ω_1 and ω_2 can be called double symmetric about α_i, β_i and θ. This leads to the following theorem about the approximation properties.

Theorem 6. *If $f(t) \in C^3[t_0, t_n]$, $\Delta : t_0 < t_1 < \cdots < t_n$, $P(t)$ is the corresponding rational cubic interpolating spline defined by* (1), *for the given α_i, β_i, when $t \in [t_i, t_{i+1}]$*

$$\|R[f]\| = \|f(t) - P(t)\| \leqslant \frac{h_i^3}{3!} \|f^{(3)}(t)\| c_i,$$

where

$$c_i = \max_{0 \leqslant \theta \leqslant 1} \omega_1(\alpha_i, \beta_i, \theta) = \max_{0 \leqslant \theta \leqslant 1} \omega_2(\alpha_i, \beta_i, \theta).$$

The following theorem gives the bounds of the optimal error constant c_i in Theorem 6.

Theorem 7. *For any given $\alpha_i > 0$ and $\beta_i > 0$, the optimal error constant c_i in Theorem* 6 *is bounded with*

$$\frac{1}{16} \leqslant c_i \leqslant \frac{4}{27}.$$

The proof of Theorem 7 is straightforward and is omitted.

Acknowledgements

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