Criticality of Damping in Multi-Degree-of-Freedom Systems

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The concept of criticality in multi-degree-of-freedom systems is discussed. Sufficient conditions for overdamping, critical damping, and underdamping are derived in terms of the matrices appearing in the modal coordinates. It is noted that results available in the literature for the case of overdamping and mixed damping are erroneous. This has been pointed out by Bhaskar (1991, 1992) for the cases of overdamping and mixed damping, and by Barkwell et al. (1992) for the case of overdamping. The error in the proof of the conditions for overdamping is brought out. A sufficient condition for overdamping is presented. Results obtained for the symmetric systems are then generalized to the symmetrizable systems. Theorems on eigenvalue bounds are applied to establish criticality.

1 Introduction

For a single-degree-of-freedom vibratory system, the damping ratio ζ determines the boundary of oscillatory and nonoscillatory damped free motion. The value of damping ratio at this boundary (also known as critical damping) is 1. It has been observed (see for example, Meirovitch, 1975) that, in case of critical damping, free response of the system approaches the equilibrium configuration fastest.

Consider the free damped motion of a multi-degree-of-freedom system, governed by the following matrix differential equation:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0, \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, stiffness, and damping matrices of order $n \times n$, respectively and $\mathbf{x}$ is the response vector of size $n \times 1$. In this section and the next, matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are assumed to be symmetric. In addition, it is also assumed that $\mathbf{M}$ and $\mathbf{C}$ are positive definite; while allowing for rigid-body modes, matrix $\mathbf{K}$ is assumed to be positive definite or positive semi-definite. The case of semi-definite damping matrix is not straightforward and is omitted from the present discussion. Statements regarding criticality, similar to those for single-degree-of-freedom systems, can be given to each of the decoupled modes, when damping is classical (i.e., when the equations of motion decouple in the modal coordinates). For a general case of damping, criticality is expressed in terms of latent roots of the λ-matrix $(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})$. Analogy suggests that, for the case of multi-degree-of-freedom systems, these conditions could be expressed in terms of definiteness properties of the matrices involved. It is surprising that this problem was not addressed until as late as 1955 (Duffin).

Nicholson (1978) defined a system to be underdamped when *all* the modes are underdamped with a sufficient condition $c_{max} \le 2k_{min}^{1/2}$, where c_{max} is the largest eigenvalue of $(\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2})$ and k_{min} is the smallest eigenvalue of $(\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2})$. Here $\mathbf{M}^{1/2}$ is the positive square root of $\mathbf{M}$. Muller (1979) gave a sufficient condition for a system to be underdamped as positive definiteness of $[4\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2} - (\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2})^2]$. This generalization was further improved by Inman and Andry (1980) with a sufficient condition of underdamping as positive definiteness of $[2(\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2})^{1/2} - (\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2})]$. They then demonstrated that Muller's condition is a special case of their's, when matrices $(\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2})$ and $(\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2})$ commute but, they did not notice that, this is not the only case when the two conditions are equivalent. In fact, whenever the smallest eigenvalue of $2(\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2})^{1/2}$ is greater than the greatest eigenvalue of $(\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2})$, all the three criteria for underdamping, viz. those of Nicholson's (1978), Muller's (1979), and Inman's (1980) are equivalent. To show this, the following property of real, symmetric, and positive definite matrices is required:

> If $a_1 \le a_2 \le \ldots \le a_n$ are the eigenvalues of a real, symmetric, and positive definite matrix $\mathbf{A}$ and $b_1 \le b_2 \le \ldots \le b_n$ are the eigenvalues of another real, symmetric, and positive definite matrix $\mathbf{B}$ and if $a_1 > b_n$, then $(\mathbf{A}^p - \mathbf{B}^p)$ is positive definite for any positive integer p.

This property can be readily proved using the min-max properties of Rayleigh quotients associated with the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}^p$, and $\mathbf{B}^p$ and noting that the eigenvalues of $\mathbf{A}^p$ are $a_1^p \le a_2^p \le \ldots \le a_n^p$ and those of $\mathbf{B}^p$ are $b_1^p \le b_2^p \le \ldots \le b_n^p$.

It is now clear that replacing $\mathbf{A}$ by $2(\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2})^{1/2}$ and $\mathbf{B}$ by $(\mathbf{M}^{-1/2}\mathbf{C}\mathbf{M}^{-1/2})$, when $a_{min} > b_{max}$ (Nicholson's criterion essentially), Inman's criterion follows for $p = 1$ and Muller's for $p = 2$. Of the three criteria, Nicholson's is undoubtedly the most conservative. Using a result (Bellman, 1968) that whenever $\mathbf{A} - \mathbf{B}$ is positive definite ($\mathbf{A}$, $\mathbf{B}$ non-negative), $(\mathbf{A}^{1/2} - \mathbf{B}^{1/2})$ is also positive definite (note that this implication is one way), we conclude that Inman's criterion is sharper than Muller's.

The three criteria presented in the literature involve matrices which appear in the governing equations expressed in the so-called pseudo-modal coordinates (coordinates obtained through the transformation $\mathbf{y} = \mathbf{M}^{-1/2}\mathbf{x}$, $\mathbf{y}$ being the vector of generalized displacements in the pseudo-modal coordinates). In this paper, we have chosen to express the conditions of criticality in the modal coordinates (coordinates in which inertia and stiffness terms decouple) so that the equations for damped free motion are given by (see Meirovitch, 1975; Newland, 1989)

$$\ddot{\mathbf{q}} + \tilde{\mathbf{C}}\dot{\mathbf{q}} + \Lambda\mathbf{q} = 0 \quad (2)$$

where $\tilde{\mathbf{C}} = \mathbf{U}^T\mathbf{C}\mathbf{U}$, $\mathbf{U}$ being the modal matrix (corresponding to the undamped problem). Define two λ-matrices as $\mathbf{Q}(\lambda) \stackrel{def}{=} (\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})$ and $\mathbf{R}(\lambda) \stackrel{def}{=} \mathbf{U}^T\mathbf{Q}(\lambda)\mathbf{U} = (\lambda^2\mathbf{I}_n + \lambda\tilde{\mathbf{C}} + \Lambda)$ whose $2n$ latent roots are given by $\det[\mathbf{Q}(\lambda)] = 0$ and

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$\det[R(\lambda)] = 0$, respectively. Since determinant of a product of matrices equals product of respective determinants, and since the rows and columns of the modal matrix U are linearly independent so U cannot be singular, we have $\det[R(\lambda)] = 0 \Leftrightarrow \det[Q(\lambda)] = 0$. This equivalence enables us to use matrices from the modal coordinates while arriving at conditions of criticality. These conditions are on the lines of those given by Inman et al. (1980), but the matrices taken in the present paper are coefficients from the equations of motion in the modal coordinates (in terms of Λ and $\tilde{C}$), instead of the pseudo-modal coordinates used by Inman et al. (1980). Definitions presented there draw incorrect conclusions for the case of overdamping and mixed damping and this will be discussed later.

2 Conditions of Criticality in Terms of Definiteness of the Matrices

Sufficient conditions for a system to be critically damped, underdamped, or overdamped are presented as follows:

Condition 1 (Critical damping): *If $\tilde{C} = 2\Lambda^{1/2}$, the system described by (2) must be critically damped.*

Condition 2 (Underdamping): *If $(2\Lambda^{1/2} - \tilde{C})$ is positive definite, the system described by (2) must be underdamped.*

Condition 3 (Overdamping): *If $(\tilde{C} - 2\Lambda_{\max}^{1/2} I_n)$ is positive definite, the system described by (2) must be overdamped.*

Here $\Lambda_{\max}$ denotes the maximum eigenvalue of Λ. The first two conditions are essentially the same as those of Inman and Andry (1980). Discussions on these two cases are presented here again for the sake of completeness and also to present a background in order to contrast the situation of overdamping with that of underdamping.

2.1 Critical Damping. The condition presented above for critical damping requires that the modal damping matrix $\tilde{C}$ be diagonal; i.e., damping must be classical. The ith equation can be written as

$$\ddot{q}_i + 2\Lambda_{ii}^{1/2}\dot{q}_i + \Lambda_{ii}q_i = 0. \tag{3}$$

Discriminant of the characteristic equation $(2\Lambda_{ii}^{1/2})^2 - 4\Lambda_{ii}$ is then equal to zero, and so latent roots of the λ-matrix $Q(\lambda)$ are repeated and real.

2.2 Underdamping. Positive definiteness of $(2\Lambda^{1/2} - \tilde{C})$, $\tilde{C}$ and 2Λ requires, for all nonzero vectors x, that $4(x^T\Lambda^{1/2}x)^2 > (x^T\tilde{C}x)^2$. Using Cauchy-Schwarz inequality for normalized vectors x, one obtains $(x^T\Lambda^{1/2}x)^2 \le x^T\Lambda x$. To look into the nature of latent roots of the λ-matrix $R(\lambda)$ we post multiply $R(\lambda)$ by its unit right latent vector x and premultiply by x^T so that the latent roots are given by

$$\lambda = [-x^T\tilde{C}x \pm \sqrt{(x^T\tilde{C}x)^2 - 4x^Txx^T\Lambda x}]. \tag{4}$$

It then follows that the latent roots occur in complex-conjugate pairs, which is necessary and sufficient for underdamping to be observed.

2.3 Overdamping. An overdamped system is defined as the one whose *all* modes are overdamped. This means that the latent roots of the λ-matrix $Q(\lambda)$ (or equivalently $R(\lambda)$) must all be real and negative. The criterion for overdamping given by Inman et al. (1980), when expressed in modal coordinates, states that *the system must be overdamped if $(\tilde{C} - 2\Lambda^{1/2})$ is positive definite*. This result has been accepted and/or used by many authors (see Ahmadian et al., 1984; Gray, 1982; Inman et al., 1982, 1982a, 1982b, 1983, 1987, 1989; Liang et al., 1988; Nicholson et al., 1983, 1987, 1987a; Ross et al., 1990; Ulsoy, 1989; etc., for example). In the following discussions it is shown that this is incorrect through a counter-example. This has been noted by Bhaskar (1991, 1992) and later by Barkwell et al. (1992).

- *A counter-example:* Consider the following matrices

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times 10^2 \text{ and } \tilde{C} = \begin{bmatrix} 22.0 & 1.0 & 1.0 \\ 1.0 & 32.0 & 1.0 \\ 1.0 & 1.0 & 36.0 \end{bmatrix}.$$

Both $\tilde{C}$ and $(\tilde{C} - C_c)$ are positive definite since

$$\mathrm{eig}(\tilde{C}) = \{21.844 \quad 31.818 \quad 36.338\} \quad \text{and}$$

$$\mathrm{eig}(\tilde{C} - C_c) = \{1.755 \quad 0.608 \quad 4.712\}$$

where $\mathrm{eig}(\cdot)$ represents the eigenvalue of $(\cdot)$. Thus the criterion of Inman et al. (1980), would predict that none of the modes oscillate. This could be checked by computing latent roots of the associated λ-matrix $R(\lambda)$, which can be shown to be equal to the eigenvalues of the constant matrix $\mathcal{A}$ defined as

$$\mathcal{A} = \begin{bmatrix} 0 & I_n \\ -\Lambda & -\tilde{C} \end{bmatrix}. \tag{5}$$

If positive definiteness of $(\tilde{C} - C_c)$ were a sufficient condition for overdamping, *all* the eigenvalues of $\mathcal{A}$ must be negative and real (a necessary and sufficient condition for overdamping). We observe that due to the presence of a complex conjugate pair of eigenvalues in

$$\mathrm{eig}(\mathcal{A}) = \{-6.259 \quad -25.448 \quad -8.826 \quad -14.185 \pm j1.040 \quad -21.097\},$$

one of the modes oscillates. Note that if $x^T\tilde{C}x \ge x^T(2\Lambda^{1/2})x$ for all x, the discriminant in the above equation need not necessarily be positive, since $(x^T\Lambda^{1/2}x)^2 \le x^T\Lambda x$. However, the difference $[(x^T\Lambda x) - (x^T\Lambda^{1/2}x)^2]$ is expected to be small (zero when x is an eigenvector of the matrices Λ or $\Lambda^{1/2}$) due to stationarity of Rayleigh-quotients around the eigenvectors. Thus if the eigenvector of the matrix Λ differs from the latent vector of the system by a small quantity δ (in the sense of an appropriate norm), the difference $[(x^T\Lambda x) - (x^T\Lambda^{1/2}x)^2]$ would be of the order of δ^2. In these cases, the approximation $(x^T\Lambda^{1/2}x)^2 \approx x^T\Lambda x$ would closely hold. Hence if the inequality $x^T\tilde{C}x > x^T(2\Lambda^{1/2})x$ is a strong one, where x is a unit latent vector of the system, it would outweigh in its favor as compared to the weak inequality $(x^T\Lambda^{1/2}x)^2 \le x^T\Lambda x$, so that the inequality $(x^T\tilde{C}x)^2 \ge 4x^T\Lambda x$ would hold, and hence overdamping would be correctly predicted. It is, therefore, not surprising that the sufficiency conditions for overdamping based on the positive definiteness of $(\tilde{C} - 2\Lambda^{1/2})$, although not strictly correct, have been in use for so long. The reason clearly is the fact that due to stationarity of the Rayleigh-quotients associated with the matrices Λ and $\Lambda^{1/2}$, counterexamples are hard to find. To study the behavior of latent roots of the system, while the difference $(\tilde{C} - 2\Lambda^{1/2})$ varies, consider the matrix $\tilde{C}(\epsilon) = 2\Lambda^{1/2} + \epsilon P$ where P is a constant positive definite matrix and ϵ is a positive scalar. Consider the scalar form $D(\epsilon) = [y^T(2\Lambda^{1/2} + \epsilon P)y]^2 - 4y^Tyy^T\Lambda y$ where $y(\epsilon)$ is a latent vector of the λ-matrix associated with the problem when $\tilde{C} = \tilde{C}(\epsilon)$. The discriminant $D(\epsilon, y(\epsilon)) > 0$ for any $\epsilon > 0$ if

[a] $\partial D(\epsilon, y = y_0)/\partial\epsilon > 0$ and

[b] $D(\epsilon = 0, y = y_0) \ge 0$

or,

[c] $\partial D[\epsilon, y(\epsilon)]/\partial\epsilon > 0$ and

[d] $D(\epsilon = 0, y = y(0)) \ge 0$.

In the first set of these conditions (i.e., [a] and [b]), the discriminant is a function of the parameter ϵ alone while the vector $\mathbf{y}$ is held constant equal to $\mathbf{y}_0$. These conditions, *if correct*, assure that the discriminant is positive for all positive values of ϵ when $\mathbf{y}$ equals any arbitrary constant vector $\mathbf{y}_0$ (thus it is also positive when this arbitrary vector is a latent vector of the system). The second set of conditions (i.e., [c] and [d]), in contrast, treat the vector $\mathbf{y}$ as a variable equal to the latent vector of the system while the parameter ϵ changes (the latent vector of the λ-matrix describing the system is a function of the matrix $\tilde{\mathbf{C}}(\epsilon)$, which in turn is a continuous function of ϵ due to Ostrowski's theorem on the continuity of eigenvalues; Wilkinson, 1965). *If* conditions [c] and [d] were correct, it would then follow, that the discriminant is always positive for all positive values of ϵ when vector $\mathbf{y}$ assumes a value equal to a latent vector of the system, as ϵ varies.

Validity of conditions [a] and [d] could be shown as follows. The left-hand side of [a] equals $4\mathbf{y}_0^T\Lambda^{1/2}\mathbf{y}_0\mathbf{y}_0^T\mathbf{P}\mathbf{y}_0 + 2\epsilon(\mathbf{y}_0^T\mathbf{P}\mathbf{y}_0)^2$ which is positive since the first term is a product of two quadratic forms of the positive definite matrices Λ and $\mathbf{P}$, and the second term is square of a positive definite quadratic form. When $\mathbf{y}(\epsilon)$ is treated as a variable equal to the latent vector of the system corresponding to the variable value of ϵ, the left-hand side of [d] equals zero since at $\epsilon = 0$ and $\mathbf{y} = \mathbf{y}(0)$, $\tilde{\mathbf{C}} = 2\Lambda^{1/2}$ and the latent vectors coincide with the eigenvectors of Λ (Caughey, 1965) (in fact, in the present formulation using modal coordinates, the matrix Λ is a diagonal matrix so that ith eigenvector is the ith basis vector $\mathbf{e}_i$). To examine the validity of [b], when the vector $\mathbf{y}$ is kept constant, the left-hand side of [b] could be shown to be equal to

$$D(\epsilon = 0, \mathbf{y} = \mathbf{y}_0) = 4(\mathbf{y}_0^T\Lambda^{1/2}\mathbf{y}_0)^2 - 4\mathbf{y}_0^T\mathbf{y}_0\mathbf{y}_0^T\Lambda\mathbf{y}_0$$

which is negative due to Cauchy-Schwarz inequality. Thus inequality [b] does not hold. Again it could be shown that the left-hand side of [c] *may not be* positive since the latent vector $\mathbf{y}(\epsilon)$ is no longer constant but varies with ϵ. Note that the left-hand side of [c] is different from that of [a], since derivatives of $\mathbf{y}(\epsilon)$ with respect to the parameter ϵ appear in the expression of [c]. Hence inequality [c] also does not hold.

In the reference (Inman et al., 1980) conditions [a] (the case when the vector $\mathbf{y}$ has been held constant during differentiation) and [d] (the case when $\mathbf{y}(\epsilon)$ is a variable depending on the value of ϵ), have been taken as sufficient for positive definiteness of the discriminant, which is incorrect. The fallacy in the proof presented by Inman et al. (1980) lies in the fact that, while calculating rate of change of the scalar form, vector $\mathbf{y}$ has been kept constant whereas for calculating its value at $\epsilon = 0$ it has been treated as a variable.

The variation of the qualitative behavior of the latent roots, while ϵ varies, is best illustrated through an example. The numerical values of Λ (hence $\mathbf{C}_c$) and $\mathbf{P} = [\tilde{\mathbf{C}}(\epsilon = 1) - \mathbf{C}_c]$ are taken to be the same as those in the counter-example presented in this section earlier. New values of the matrix $\tilde{\mathbf{C}}(\epsilon)$ are now generated by varying the scalar ϵ. The difference $\tilde{\mathbf{C}}(\epsilon) - 2\Lambda^{1/2} = \epsilon\mathbf{P}$ must now be positive definite *for all* positive values of ϵ, since ϵ is a positive scalar and $\mathbf{P}$ is a positive definite matrix. Therefore, if the sufficient condition of overdamping presented in the reference (Inman et al., 1980) were correct, the overall system must always remain overdamped, no matter what the value of ϵ be (so long as it is positive). This is not true for the present set of numerical values since there exist complex branches in the trajectory of the latent roots (Fig. 1, in which $\epsilon \in [0, 2]$). Imaginary part of the latent roots is plotted as a function of ϵ in Fig. 2. Note that there exist intervals of values of ϵ for which latent roots possess a complex-conjugate pair. The numerical values of Λ and $\mathbf{P}$ matrices chosen to generate these trajectories are such that the counter-example presented in the beginning of this section corresponds to $\epsilon = 1$.

Proof of sufficiency of condition 3: Condition 3 for overdamping presented here requires that $\mathbf{x}^T(\tilde{\mathbf{C}} - 2\Lambda_{\max}^{1/2}\mathbf{I}_n)\mathbf{x} > 0$ for all arbitrary vectors $\mathbf{x}$. Separating the terms, one obtains

$$\min_{\mathbf{x}}(\mathbf{x}^T\tilde{\mathbf{C}}\mathbf{x}/\mathbf{x}^T\mathbf{x}) > (\mathbf{x}^T 2\Lambda_{\max}^{1/2}\mathbf{x}/\mathbf{x}^T\mathbf{x}) = 2\Lambda_{\max}^{1/2}. \quad (6)$$

Since both sides of the inequality are positive, the quantities on both sides can be squared without changing direction of the inequality. Thus for an arbitrary unit vector $\mathbf{x}$,

$$\min[(\mathbf{x}^T\tilde{\mathbf{C}}\mathbf{x})^2] = [\min(\mathbf{x}^T\tilde{\mathbf{C}}\mathbf{x})]^2$$

$$> 4\Lambda_{\max} = 4\max(\mathbf{x}^T\Lambda\mathbf{x}). \quad (7)$$

This inequality implies overdamping since the discriminant in Eq. (4) is always positive.

2.4 Mixed Damping. A system is said to possess mixed damping if, and only if, in the damped free response, at least one mode oscillates and at least one does not. For this case the criterion of Inman et al. (1980) demands that the matrix $(2\Lambda^{1/2} - \tilde{\mathbf{C}})$ must be *indefinite*. It should be emphasized here that indefiniteness does not imply that the matrix could be either positive definite or negative definite. Rather it means that it must not be either. It follows then that at least one eigenvalue of $(2\Lambda^{1/2} - \tilde{\mathbf{C}})$ must be negative and at least one must be positive. It so turns out that indefiniteness of $(2\Lambda^{1/2} - \tilde{\mathbf{C}})$ is neither a necessary nor a sufficient condition for mixed damping (this has been discussed by Bhaskar (1991, 1992)). Necessity is violated by the counter-example presented in Section 2.3, since mixed damping is observed, although $(\tilde{\mathbf{C}} - 2\Lambda^{1/2})$ is positive definite for the example chosen there. The incorrectness of sufficiency is demonstrated through the following two counter-examples.

- *Counter-examples:* Consider the following modal damping matrix $\tilde{\mathbf{C}}$ and the critical damping matrix $\mathbf{C}_c$

$$\tilde{\mathbf{C}} = \begin{bmatrix} 19.5 & 1 & 1 \\ 1 & 25 & 1 \\ 1 & 1 & 30 \end{bmatrix} \quad \text{and}$$

$$\mathbf{C}_c = \begin{bmatrix} 20.0 & 0 & 0 \\ 0 & 28.284 & 0 \\ 0 & 0 & 34.641 \end{bmatrix}.$$

Eigenvalues of $(\tilde{\mathbf{C}} - \mathbf{C}_c)$ and those of $\tilde{\mathbf{C}}$ are then given by

$$\mathrm{eig}(\tilde{\mathbf{C}} - \mathbf{C}_c) = \{-3.37 \quad -5.21 \quad +0.16\},$$

$$\mathrm{eig}(\tilde{\mathbf{C}}) = \{19.26 \quad 24.92 \quad 30.32\}.$$

Thus matrix $(\tilde{\mathbf{C}} - \mathbf{C}_c)$ is indefinite and $\tilde{\mathbf{C}}$ is positive definite. The condition of Inman et al. (1980) predicts that damping must be of *mixed* type and that at least one mode must oscillate and at least one must not. However, we note that *all* the modes oscillate, since eigenvalues of the λ-matrix are given by

$$\mathrm{eig}(\mathcal{A}) = \{-10.377 \pm j1.264,$$

$$-12.401 \pm j6.359, \quad -14.472 \pm j8.558\}$$

For the sake of completeness, the following example illustrates that the condition of Inman et al. (1980) predicts mixed damping although overdamping is actually observed. Consider

$$\Lambda = 10^3 \times \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{C}} = \begin{bmatrix} 63.246 & 1 \\ 1 & 89.443 \end{bmatrix}.$$

Clearly, $(\mathbf{C} - \mathbf{C}_c)$ is indefinite since $\mathrm{eig}(\mathbf{C} - \mathbf{C}_c) = \{\pm 1\}$. Again conditions of sufficiency of mixed damping presented by Inman et al. (1980) would predict that one of the modes oscillates and one of them does not. Eigenvalues of the associated

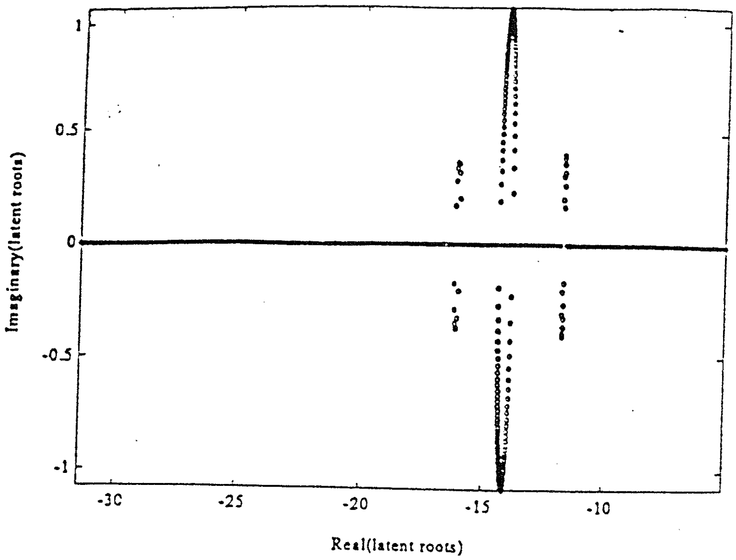

Fig. 1 Trajectory in the complex plane for the six latent roots of the system described in the counter-example of Section 2.3. The variable ϵ is the implicit parameter which varies along the trajectories.

matrix $\mathcal{A}$ indicates overdamping, since eig($\mathcal{A}$) = {−29.655, −35.443, −39.902, −47.690}.

3 Generalization to a Class of Nonconservative Systems

In the previous section, matrices M, C, and K were assumed to be symmetric while M and C were assumed to be positive definite. In this section, results of the previous section are generalized to a class of systems known as the *symmetrizable systems.* With a suitable transformation, the equations of motion for this class of systems can be cast in terms of symmetric matrices and are discussed by Inman (1983). Results of Inman et al. (1980) are generalized by Ahmadian et al. (1984) and are again incorrect for the cases of overdamping and mixed damping. In the present study, these results are modified appropriately.

Assuming that the mass matrix is nonsingular, equations of motion can be expressed as

$$\mathbf{I}_n\ddot{\mathbf{x}} + \bar{\mathbf{C}}\dot{\mathbf{x}} + \bar{\mathbf{K}}\mathbf{x} = 0 \qquad (8)$$

where $\bar{\mathbf{C}} = \mathbf{M}^{-1}\mathbf{C}$ and $\bar{\mathbf{K}} = \mathbf{M}^{-1}\mathbf{K}$. No matrix is assumed to be symmetric at this stage which is why premultiplication by the inverse of mass matrix has been carried out, since preserving the symmetry is not the idea any more. It is now assumed that the matrices $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ are symmetrizable so that factorizations $\bar{\mathbf{C}} = \mathbf{S}_1\mathbf{S}_2$ and $\bar{\mathbf{K}} = \mathbf{T}_1\mathbf{T}_2$ are permissible (Inman, 1983), where matrices $\mathbf{S}_1$ and $\mathbf{T}_1$ are real, symmetric, and positive definite, while $\mathbf{S}_2$ and $\mathbf{T}_2$ need only be symmetric. The condition for symmetrizability demands that at least one factor in the above factorization be common and so it is assumed that $\mathbf{S}_1 = \mathbf{T}_1$. Transforming the coordinates according to $\mathbf{x}(t) = \mathbf{S}_1^{1/2}\mathbf{y}(t)$ and

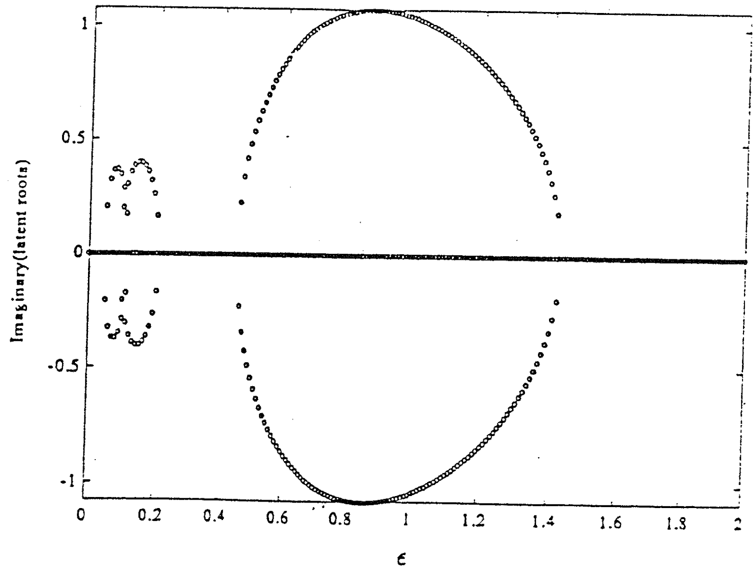

Fig. 2 Imaginary part of the latent roots as a function of ϵ. Note the existence of intervals of ϵ in which the imaginary part is nonzero indicating the presence of underdamped modes.

premultiplying by $\mathbf{S}^{-1/2}$, we recast the governing equations of motion (8) in terms of symmetric coefficient matrices as

$$\mathbf{I}_n\ddot{\mathbf{y}} + \mathbf{S}_1^{1/2}\mathbf{S}_2\mathbf{S}_1^{1/2}\dot{\mathbf{y}} + \mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2}\mathbf{y} = 0. \qquad (9)$$

Extension of the results obtained in the previous section is now straightforward.

Condition 4 (Critical damping): *If* $2(\mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2})^{1/2} = \mathbf{S}_1^{1/2}\mathbf{S}_2\mathbf{S}_1^{1/2}$, *then the system must be critically damped.*

Condition 5 (Underdamping): *If* $2(\mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2})^{1/2} - (\mathbf{S}_1^{1/2}\mathbf{S}_2\mathbf{S}_1^{1/2})$ *is positive definite then the system must be underdamped.*

Condition 6 (Overdamping): *If* $(\mathbf{S}_1^{1/2}\mathbf{S}_2\mathbf{S}_1^{1/2}) - 2(\mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2})^{1/2}_{\max}\mathbf{I}_n$ *is positive definite then the system must be overdamped.*

Here $(\mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2})_{\max}$ represents the largest eigenvalue of the matrix $(\mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2})$. The first two of these conditions are identical to those presented by Ahmadian et al. (1984). The third condition presented there for overdamping states that, if $\mathbf{S}_1^{1/2}\mathbf{S}_2\mathbf{S}_1^{1/2} - 2(\mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2})^{1/2}$ is positive definite, then the system must be overdamped, which is incorrect. The fourth condition presented there for mixed damping based on indefiniteness of $2(\mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2})^{1/2} - (\mathbf{S}_1^{1/2}\mathbf{S}_2\mathbf{S}_1^{1/2})$ is also incorrect.

3.1 Conditions in Terms of Matrices in the Physical Coordinates. In this section, conditions in terms of matrices in the physical coordinates are derived. Unlike symmetric matrices, the quadratic form is not defined for a general case of real square arrays. Hence instead of definiteness of the matrices, conditions in terms of eigenvalues of the matrices are now presented. Substitution from the factorizations for $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$, $\mathbf{S}_2 = \mathbf{S}_1^{-1}\bar{\mathbf{C}}$ and $\mathbf{T}_2 = \mathbf{S}_1^{-1}\bar{\mathbf{K}}$, into the expression $(\mathbf{S}_1^{1/2}\mathbf{S}_2\mathbf{S}_1^{1/2}) - 2(\mathbf{S}_1^{1/2}\mathbf{T}_2\mathbf{S}_1^{1/2})^{1/2}$ results in $\mathbf{S}_1^{-1/2}(\bar{\mathbf{C}} - 2\bar{\mathbf{K}}^{1/2})\mathbf{S}_1^{1/2}$, which is a similarity transform on $(\bar{\mathbf{C}} - 2\bar{\mathbf{K}}^{1/2})$. Since similarity transforms preserve eigenvalues, definiteness properties can be expressed in terms of eigenvalues of matrices involving $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$. Conditions 4 to 6 can now be expressed as follows:

(1) If $2(\mathbf{M}^{-1}\mathbf{K})^{1/2} = \mathbf{M}^{-1}\mathbf{C}$, the system must be critically damped.

(2) If $[2(\mathbf{M}^{-1}\mathbf{K})^{1/2} - \mathbf{M}^{-1}\mathbf{C}]$ has its eigenvalues all positive then the system must be underdamped.

(3) If $[\mathbf{M}^{-1}\mathbf{C} - 2(\mathbf{M}^{-1}\mathbf{K})^{1/2}_{\max}\mathbf{I}_n]$ has its eigenvalues all positive then the system must be overdamped.

Here $(\mathbf{M}^{-1}\mathbf{K})_{\max}$ represents the maximum eigenvalue of $(\mathbf{M}^{-1}\mathbf{K})$. Once again, the first two conditions presented here are identical to those by Ahmadian et al. (1984). The condition for overdamping presented there is based on definiteness of $[\mathbf{M}^{-1}\mathbf{C} - 2(\mathbf{M}^{-1}\mathbf{K})^{1/2}\mathbf{I}_n]$ which is incorrect and is consequently modified here. Again, counter-examples can be constructed for the cases of overdamping and mixed damping (Bhaskar 1992).

4 The Damping-Ratio Matrix

Analogy with a single-degree-of-freedom system suggests that the scalar quantity damping ratio could possibly be replaced by a matrix for a multi-degree-of-freedom system. An attempt of this can be found in Inman et al. (1987, 1989). A single matrix takes the role of damping ratios there and definiteness of the difference between this matrix and the identity matrix determines criticality for the system. However, we observe that the development of this matrix assumes the results of Inman et al. (1980), and consequently derives erroneous conclusions. In the following discussion, new results are presented in this light.

The damping ratio matrix $\mathbf{Z}$ is defined as

$$\mathbf{Z} = \mathbf{C}_c^{-1/2}\tilde{\mathbf{C}}\mathbf{C}_c^{-1/2} \qquad (10)$$

which is exactly the same as the definition presented by Inman (1989). The sufficient conditions of criticality, presented there, are in terms of definiteness of $(\mathbf{I}_n - \mathbf{Z})$. As expected, the conditions for overdamping and mixed damping are incorrect there. The sufficient conditions for criticality presented in the Section 2 and Section 3 can now be expressed in terms of the damping ratio matrix $\mathbf{Z}$ as follows:

(1) If $(\mathbf{I}_n - \mathbf{Z}) = 0$ then the system must be critically damped.

(2) If the matrix $(\mathbf{I}_n - \mathbf{Z})$ is positive definite then the system must be underdamped.

(3) If $\{\mathbf{Z} - \mathrm{diag}(\Lambda_{\max}/\Lambda_{ii})^{1/2}\}$ is positive definite then the system must be overdamped.

5 Application of the Theorems on Eigenvalue Bounds

Some useful information can be derived by mere inspection of the terms on the diagonal of the matrix $(2\mathbf{\Lambda}^{1/2} - \tilde{\mathbf{C}})$ and $(\tilde{\mathbf{C}} - 2\Lambda^{1/2}_{\max}\mathbf{I}_n)$. It turns out that if entries on the diagonal of $(2\mathbf{\Lambda}^{1/2} - \tilde{\mathbf{C}})$ are all positive and if, it is diagonally dominant, then the system must be underdamped. Similarly, if the entries on the diagonal of $(\tilde{\mathbf{C}} - 2\Lambda^{1/2}_{\max}\mathbf{I}_n)$ are all positive and if, it is diagonally dominant, then the system must be overdamped. These results follow immediately by applying the well-known Gerschgorin's theorems (1931). In this section, $\mathbf{A}$ denotes the matrix $(2\mathbf{\Lambda}^{1/2} - \tilde{\mathbf{C}})$ which is a real and symmetric matrix.

Given that the entries on the diagonal of the matrix $\mathbf{A}$ are all positive, it can be concluded that if the centers of the Gerschgorin discs fall on the positive real axis, and if $\mathbf{A}$ is diagonally dominant, then none of the disks fall in the left half of the complex plane. Therefore, all the eigenvalues of $\mathbf{A}$ must be in the right half of the complex plane. Hence the matrix $\mathbf{A}$ must be positive definite, which is sufficient for underdamping of the system. On similar lines it could be shown that if another matrix say $\mathbf{B} = (\tilde{\mathbf{C}} - 2\Lambda^{1/2}_{\max}\mathbf{I}_n)$ is diagonally dominant and if, all the entries on its diagonal are positive, then the system must be overdamped. It is noted that all of these conditions are only sufficient but not necessary.

Conditions of the previous theorem may become stringent at times and a further refinement is possible using the following theorem due to Brauer (1946, 1947).

Theorem 1 (Brauer). *Every eigenvalue of a matrix* $\mathbf{A}$ *lies in the interior or on the boundary of at least one of the following* $\frac{1}{2}n(n-1)$ *Cassini ovals on the complex z-plane*

$$|z - A_{ii}||z - A_{jj}| \le \sum_r{}' |A_{ir}| \sum_s{}' |A_{js}|, \quad i = j. \qquad (11)$$

The proof can be found in Brauer (1946, 1947). The above condition may appear to be complicated, but a simple extension of the conclusions reached earlier on the basis of Gerschgorin's theorem can be obtained. For a real matrix (which is the case), the ovals must be symmetric about the real axis and about the line $x = (A_{ii} + A_{jj})/2$. The ovals for this situation intersect the real axis and satisfy $(x - A_{ii})(x - A_{jj}) = \sigma_i\sigma_j$ at the points of intersection on the x-axis, where $\sigma_i = \sum_{r \ne i}^{n} |A_{ir}|$. Solving the quadratic, the following roots are obtained:

$$x_{1,2} = (A_{ii} + A_{jj}) \pm (1/2)\sqrt{(A_{ii} - A_{jj})^2 + 4\sigma_i\sigma_j}. \qquad (12)$$

Note that the discriminant is always positive, which is expected since the roots must remain real. In order that the ovals remain in the right half of the complex plane, both x_1 and x_2 must be positive, i.e. $(A_{ii} + A_{jj})^2 \ge (A_{ii} - A_{jj})^2 + 4\sigma_i\sigma_j$. Rearranging this inequality leads to

$$(A_{ii}/\sigma_i)(A_{jj}/\sigma_j) \ge 1, \quad \text{for all} \quad i, j. \qquad (13)$$

This result was obtained for general Hermitian matrices in Brauer (1947). It also follows from a more general theorem for complex matrices given in Brauer (1946). Here a simplified result is presented for the case of general real matrices (which may *not* necessarily be symmetric and hence not Hermitian). It should be noted that for asymmetric *real* matrices, Cassini ovals are symmetric about the real axis. The asymmetric formulation of Section 3 then allows us to apply these results to symmetrizable systems also.

Clearly, the condition in the inequality (13) is more relaxed than the one obtained through Gerschgorin's theorem, since whenever the latter is satisfied (i.e., diagonal dominance is observed), (13) is automatically satisfied. Inequality (13) allows the violation of dominance by at the most one row (or column). All that one needs to check is whether product of the smallest and the next smallest of the numbers (A_{ii}/σ_i) exceeds unity. When $\mathbf{A}$ is replaced by $(2\mathbf{\Lambda}^{1/2} - \bar{\mathbf{C}})$ in the previous discussion of this section, inequality (13) provides a sufficient condition for underdamping. Similarly, when $(\bar{\mathbf{C}} - 2\mathbf{\Lambda}^{1/2}_{\max}\mathbf{I}_n)$ replaces $\mathbf{A}$, a sufficient condition for overdamping is obtained.

6 Coupling of Single-Degree-of-Freedom Oscillators

Consider a collection of single-degree-of-freedom oscillators, each of which is critically damped. These oscillators are then coupled through dashpots such that the mass elements only are connected through these new dashpots while the spring and the dashpots of the originally uncoupled oscillators remain grounded. If the statements of Ahmadian (1984), Inman et al. (1980, 1989) regarding overdamping were correct, it would imply that, coupling these individual oscillators via additional dashpot elements would *always* result in an overdamped system for any positive damping. This follows from the fact that in the equations of motion for the coupled system, the mass and stiffness matrices are diagonal and the damping matrix is such that $(\bar{\mathbf{C}} - 2\mathbf{\Lambda}^{1/2})$ has terms on its diagonal greater than the sum of the absolute values of the terms off the diagonal. Applying Gerschgorin's theorem it is observed that none of the eigenvalues of $(\bar{\mathbf{C}} - 2\mathbf{\Lambda}^{1/2})$ fall in the left half of the complex plane. This, as seen earlier, does not guarantee overdamping. Thus a collection of *overdamped oscillators may, in fact, exhibit underdamped modes when coupled through additional dashpot elements!* However, the condition of overdamping presented in this paper, offers a class of overdamped oscillators, which when coupled through further dashpot elements remain overdamped. This is expressed through the following lemma.

Lemma 1 *If each of the oscillators in a collection of single-degree-of-freedom oscillators with mass, dashpot constant, and stiffness for the ith oscillator as m_i, c_i and k_i is overdamped in such a way, that $(c_i/m_i) \geq \max_r 2(k_r/m_r)^{1/2}$ for all i, then coupling the oscillators with further dashpots of any arbitrary values always produces a coupled system which is overdamped.*

The prove this, consider a collection of n single-degree-of-freedom oscillators with mass, stiffness and dashpot constants as m_i, k_i and c_i associated to the ith oscillator. Since each of these is overdamped in the manner described above, we have

$$(c_i/m_i) - 2 \max_r (k_r/m_r)^{1/2} \geq 0 \quad \text{for all} \quad i. \tag{14}$$

The dashpot connecting ith oscillator to the jth one is denoted by the dashpot constant equal to c_{ij}. Premultiplying by inverse of the mass matrix, elements of the matrix $(\bar{\mathbf{C}} - 2\bar{\mathbf{K}}^{1/2}_{\max}\mathbf{I}_n)$ on its diagonal are obtained as

$$\bar{C}_{ii} - 2\bar{K}^{1/2}_{\max} = [c_i + \sum_{j=1}^{n} c_{ij}/m_i] - \max_r 2(k_r/m_r)^{1/2} \tag{15}$$

and those off the diagonal on the ith row and jth column as

$$\bar{C}_{ij} = -c_{ij}/m_i \quad \text{when} \quad i \neq j. \tag{16}$$

Note that Eq. (16) is not symmetric in i and j so that $(\bar{\mathbf{C}} - 2\bar{\mathbf{K}}^{1/2}_{\max}\mathbf{I}_n)$ is not symmetric. The conditions of symmetrizability of Section 3 are satisfied by the present class of systems since each of the matrices $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ have a common symmetric factor, viz. inverse of the mass matrix (which is diagonal). Since $\mathbf{M}$ and $\mathbf{K}$ are diagonal matrices, $(\mathbf{M}^{-1}\mathbf{K})_{\max} = \max_r(k_r/m_r)$ is the maximum eigenvalue of $(\mathbf{M}^{-1}\mathbf{K})$. Using conditions of Eq. (14); Eqs. (15) and (16) imply that $(\bar{\mathbf{C}} - 2\bar{\mathbf{K}}^{1/2}_{\max}\mathbf{I}_n)$ is diagonally dominant. Using Gerschgorin's theorem and the conditions of overdamping stated in Section 3.1, it is concluded that the system must be overdamped. Two further special cases arise: *If a collection of overdamped oscillators is such that the ratio of stiffness to inertia or the ratio of dashpot constant to inertia is the same for each one of them, then coupling these oscillators with dashpots of any arbitrary value always results in an overdamped system.*

Application of Brauer's theorem, using Cassini-ovals as the basis for choosing regions of eigenvalue bounds, results in a more liberal condition. It could be shown after some algebra that the coupled system is overdamped if

$$(1 + \delta_i/\sigma_i)(1 + \delta_j/\sigma_j) \geq 1, \quad \text{for all} \quad i, j \tag{17}$$

where, $\delta_i = [c_i/m_i - \max_r 2(k_r/m_r)^{1/2}]$, and $\sigma_i = \sum_{j=1}^{n'} c_{ij}$. It is easy to see that, inequality (17) always holds if the condition of Lemma 1 holds since δ_i is always positive for all i when the latter is satisfied.

Consider coupling the *underdamped* oscillators next. On the lines of Eqs. (15) and (16), elements of $2\bar{\mathbf{K}}^{1/2} - \bar{\mathbf{C}}^{1/2}$ are given by

$$2\bar{K}^{1/2}_{ii} - \bar{C}_{ii} = 2(k_i/m_i)^{1/2} - [c_i + \sum_{j=1}^{n'} c_{ij}/m_i] \tag{18}$$

and

$$2\bar{K}^{1/2}_{ij} - \bar{C}_{ij} = c_{ij}/m_i, \quad i \neq j. \tag{19}$$

It could be seen in this instance that, if $2(k_i/m_i)^{1/2} - (c_i/m_i) > 2(\sum'_{j=1} c_{ij})/m_i$ then, the matrix $(2\bar{\mathbf{K}}^{1/2} - \bar{\mathbf{C}}^{1/2})$ is diagonally dominant and has positive numbers on its diagonal. Hence $(2\bar{\mathbf{K}}^{1/2} - \bar{\mathbf{C}}^{1/2})$ is positive definite. This leads to the following result.

Lemma 2 *If a collection of underdamped single-degree-of-freedom oscillators having mass, stiffness, and dashpot constant associated with the ith oscillator equal to m_i, k_i and c_i respectively, is connected through additional dashpots of constant c_{ij} that connect the ith oscillator to the jth one and which satisfy $2(k_i/m_i)^{1/2}(c_i/m_i) > 2(\sum'_{j=1} c_{ij})m_i$ then connecting the oscillators through such additional dampers always produces an underdamped system.*

Cassini-ovals can still relax the condition but they yield to a cumbersome result and thus it is omitted from the present discussion.

7 Conclusions

A set of sufficient conditions was presented for criticality in terms of matrices involved in the governing equations of motion, when they are expressed in the modal coordinates. This was later extended to the case of general asymmetric (but symmetrizable) systems. Similar conditions available in the literature for the cases of overdamping and mixed damping were found to be incorrect. This was shown through some counterexamples. The theorems on eigenvalue bounds were applied to infer criticality of a multi-degree-of-freedom system. This may lead to computational saving in practical applications. Two cases when underdamped oscillators remain underdamped, when coupled through additional dampers, and when overdamped oscillators remain overdamped, when coupled through similar elements were discussed in the perspective of the sufficient conditions presented here. It is emphasized that all these conditions are only sufficient but not necessary. However, when the equations of motion decouple (i.e., when damping is classical), they become both necessary and sufficient.

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References

Ahmadian, M., and Inman, D. J., 1984, "On the Nature of Eigenvalues of General Nonconservative Systems," ASME JOURNAL OF APPLIED MECHANICS, Vol. 51, pp. 193–194.

Barkwell, L., and Lancaster, P., 1992, "Overdamped and Gyroscopic Vibrating Systems," ASME JOURNAL OF APPLIED MECHANICS, Vol. 59, pp. 176–181.

Bellman, R., 1968, "Some Inequalities for the Square Root of a Positive Definite Matrix," *Linear Algebra and Its Applications*, Vol. 1, pp. 321–324.

Bhaskar, A., 1991, Chapter 2 of Dissertation submitted to Trinity College, Cambridge, U.K., for the annual fellowship competition.

Bhaskar, A., 1992, "Damping in Mechanical Vibrations: New Methods of Analysis and Estimation," Chapter 2 of Ph.D. thesis submitted to Cambridge University, Cambridge, U.K.

Brauer, A., 1946, "Limits for the Characteristic Roots of a Matrix I," *Duke Mathematical Journal*, Vol. 13, pp. 387–395.

Brauer, A., 1947, "Limits for the Characteristic Roots of a Matrix II," *Duke Mathematical Journal*, Vol. 14, pp. 21–26.

Caughey, T. K., and O'Kelly, M. E. J., 1965, "Classical Normal Modes in Damped Linear Dynamic Systems," ASME JOURNAL OF APPLIED MECHANICS, Vol. 32, pp. 583–588.

Duffin, R. J., 1955, "A Minimax Theory for Overdamped Networks," *Journal of Rational Mechanics and Analysis*, Vol. 4, pp. 221–233.

Gerschgorin, S., 1931, "Über die Abgrenzung der Eigenwerte einer Matrix," *Izv. Akad. Nauk. SSSR, Ser. Fiz.-mat.*, Vol. 6, pp. 749–754.

Gray, J. A., and Andry, A. N., 1982, "A simple Calculation of Critical Damping Matrix of a Linear Multi-degree-of-freedom System," *Mechanics Research Communications*, Vol. 9, No. 6, pp. 379–380.

Inman, D. J., 1983, "Dynamics of Asymmetric Nonconservative Systems," ASME JOURNAL OF APPLIED MECHANICS, Vol. 50, pp. 199–203.

Inman, D. J., and Andry, A. N., 1980, "Some Results on the Nature of Eigenvalues of Discrete Damped Linear Systems," ASME JOURNAL OF APPLIED MECHANICS, Vol. 47, pp. 927–930.

Inman, D. J., 1989, *Vibration with Control, Measurement and Stability*, Prentice-Hall Int., Inc.

Inman, D. J., and Jiang, B. L., 1987, "On Damping Ratios for Multiple-degree-of-freedom Linear Systems," *International Journal of Analytical & Experimental Modal Analysis*, Vol. 2, pp. 38–42.

Inman, D. J., and Orabi, I., 1983, "An Efficient Method for Computing the Critical Damping Condition," ASME JOURNAL OF APPLIED MECHANICS, Vol. 50, pp. 679–682.

Inman, D. J., and Andry, A. N., 1982, "A Procedure for Designing Overdamped Lumped Parameter Systems," *Shock and Vibration Bulletin*, Vol. 52, No. 5, pp. 49–53.

Inman, D. J., 1982a, "Oscillatory Damped Distributed Systems," *Mechanics Research Communications*, Vol. 9, No. 2, pp. 101–107.

Inman, D. J., and Andry, A. N., 1982b, "The Nature of Temporal Solutions of Damped Distributed Systems with Classical Normal Modes," ASME JOURNAL OF APPLIED MECHANICS, Vol. 49, pp. 867–870.

Liang, Z., and Soong, T. T., 1988, "Bounds on Harmonically Forced Linear Systems," ASME JOURNAL OF APPLIED MECHANICS, Vol. 55, pp. 988–990.

Meirovitch, L., 1975, *Elements of Vibration Analysis*, McGraw-Hill, New York.

Muller, P. C., 1979, "Oscillatory Damped Systems," *Mechanics Research Communications*, Vol. 6, No. 2, pp. 81–85.

Newland, D. E., 1989, *Mechanical Vibration Analysis and Computation*, Longman, Harlow, and John Wiley, New York.

Nicholson, D. W., 1978, "Eigenvalue Bounds for Damped Linear Systems," *Mechanics Research Communications*, Vol. 5, No. 3, pp. 147–152.

Nicholson, D. W., 1983, "Overdamping of a Linear Mechanical System," *Mechanics Research Communications*, Vol. 10, No. 2, pp. 67–76.

Nicholson, D. W., 1987, "Eigenvalue Bounds for Linear Mechanical Systems with Nonmodal Damping," *Mechanics Research Communications*, Vol. 14, No. 2, pp. 115–122.

Nicholson, D. W., 1987a, "Response Bounds for Nonclassically Damped Mechanical Systems under Transient Loads," ASME JOURNAL OF APPLIED MECHANICS, Vol. 55, pp. 430–433.

Ross, A. D. S., and Inman, D. J., 1990, "Settling Time of Underdamped Linear Lumped Parameter Systems," *Journal of Sound and Vibration*, Vol. 140, No. 1, pp. 117–127.

Ulsoy, A. G., 1989, "Utilization of Control Effort Constraints in Linear Controller Design," ASME *Journal of Dynamic Systems, Measurement and Control*, Vol. 111, pp. 378–381.

Wilkinson, J. H., 1965, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, U.K.