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State space elastostatics of prismatic structures

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Abstract

This paper provides an exposition of the problem of a prismatic elastic rod or beam subject to static end loading only, using a state space formulation of the linear theory of elasticity. The approach, which employs the machinery of eigenanalysis, provides a logical and complete resolution of the transmission (Saint-Venant's) problem for arbitrary cross-section, subject to determination of the Saint-Venant torsion and flexure functions which are cross-section specific. For the decay problem (Saint-Venant's principle), the approach is applied to the plane stress elastic strip, but in the transverse rather than the axial direction, leading to the well-known Papkovitch–Fadle eigenequations, which determine the decay rates of self-equilibrated loading; however, extension to other cross-sections appears unlikely. It is shown that only a repeating zero eigenvalue can lead to a non-trivial Jordan block; thus degenerate decay modes cannot exist for a prismatic structure.

Keywords: Linear elasticity; Hamiltonian; Saint-Venant; State space; Degenerate modes

1. Introduction

The present study was motivated by a series of papers in the Chinese literature [1–4] concerned with the methods of Hamiltonian mechanics applied to problems of the linear mathematical theory of elasticity, rather than the more usual stress function (for example, Airy

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Nomenclature

a, b, c	rigid-body displacements, element in matrix Eq. (9)
c	semi-depth of elastic strip
$\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$	partitions of matrix $\mathbf{H}$
$\mathbf{C}$	matrix of participation coefficients, C
$\mathbf{d}$	vector of displacement components
e, e, E	dilatation, base of Napierian logarithm, Young's modulus
G	shear modulus
$\mathbf{H}$	system matrix
$\mathbf{I}$	identity matrix
$\mathbf{J}$	Jordan block, metric matrix
k	eigenvalue
L	length, left
M, $\mathbf{M}$	moment, element of, matrix exponential
n	direction cosine
$\mathbf{p}$	vector of stress components
Q	shearing force
R, $\mathbf{R}$	right, element of, resolvent matrix
s, S, $\mathbf{s}$	Laplace variable, surface, state (eigen)vector
t	traction
u, v, w	displacements in the x-, y- and z-direction, respectively
U	strain energy
$\mathbf{V}$	transformation matrix of eigen- and principal vectors
x, y, z	Cartesian coordinate system
α	angle of rotation
β	constant
σ	direct stress
τ	shear stress
λ	Lame′ constant
v	Poisson's ratio
ϕ	Saint-Venant flexure function
ψ	Saint-Venant warping function

and Papkovitch-Neuber) or semi-inverse methods described in most, if not all, of the well-known texts on elasticity, see, for example, [5]. The Hamiltonian approach is standard within the study of both rigid body and quantum mechanics, and results in a first-order matrix differential equation. This, in itself, is not the distinctive feature, as one can always trade the order of a single differential equation with the size of the system matrix; rather, it treats position and momentum as independent variables on an equal footing, and these extra degrees of freedom aid the search for canonical transformations (which preserve Hamiltonian structure) for which the set of differential equations are either fully or partially solved. For application to the elasticity of a beam-like structure, a state space vector consisting of cross-sectional displacement and stress components naturally takes the place of position and momentum, and the governing equations then describe how these evolve spatially (rather than temporally) as one moves along the beam.

In general, an elasticity solution must satisfy the Hooke's law, the boundary conditions and the force equilibrium equations; if the latter are expressed in terms of stress then one must also employ strain compatibility in one form or another. However, if the equilibrium equations are expressed in terms of displacement (the Navier equations), then strain compatibility equations are not required; one immediate advantage of the state space approach is that it treats displacement and stress components on equal terms, so compatibility requirements are satisfied naturally.

A second advantage is that it unifies the two (usually) separate problems which bear the name of Saint-Venant–the transmission of load applied to the ends only, according to a prescribed distribution (Saint-Venant's problem), and the decay of self-equilibrated end loading (Saint-Venant's principle)–through the machinery of an eigenvalue problem. Indeed, Yao and Xu [6] have recently employed the state-space Hamiltonian approach, using a polar coordinate system, in a re-examination of the so-called *wedge paradox*, and show that previous difficulties can be attributed to incompleteness of the eigenspace associated with multiple eigenvalues–one needs both eigenvectors and coupled principal vectors to span the solution space for the critical wedge angle of approximately 257°; the coincidence of eigenvalues for the (usually) separate problems of decay of self-equilibrated loading and transmission of bending moment has been noted previously by Stephen and Wang [7]; the state space formulation provides the ideal setting in which to resolve this degeneracy.

Xu et al. [8] considered a prismatic elastic beam and showed how the governing equations can be derived from a Lagrangian function and use of the calculus of variations. But since the governing equations are already well known, their re-derivation ultimately serves a pedagogic purpose only; all that is really necessary is the casting of known equations into the requisite state variable form. They considered the transmission modes in detail, but appeared to regard the decay modes as an auxiliary problem, referring to the Papkovitch–Neuber solution provided by Stephen and Wang [9].

While several recent applications by Zhong and his co-workers have drawn attention to the advantage of these modern (control) system state variable methods, it appears that Bahar [10] was the first to note that classical elasticity is well suited to the state space approach, and applied it to the problem of the plane stress elastic strip; he also noted the relationship to the method of *initial function*, or *parameter*, as developed earlier by Vlasov [11]. Sosa and Bahar [12] represented the fourth-order biharmonic (Airy) stress function approach as a first-order 4×4 matrix mixed state variable problem, and indicated advantages in the treatment of the Flamant problem. Earlier, Johnson and Little [13] had also considered the strip problem, but introduced an auxiliary function in addition to the three stress components, rather than the mixture of cross-sectional stress and displacement components employed by later authors.

More recently, Ustinov and co-workers have applied broadly equivalent spectral operator theory to the prism [14] and the so-called *pseudocylinder*, which encompasses pre-twisted structures such as a turbine blade, drill bit and a helical spring [15,16], as well as a prismatic structure for which the material displays helical anisotropy [17]. Their very concise treatment is more accessible to researchers with a mathematical background; here the author seeks to describe the approach as simply as possible, so that these powerful methods can be accessible to the widest possible readership.

In the present paper, the well-known governing equations of linear elasticity are first cast into a form suitable for the state space approach, applied to the general problem of a rod or beam of general cross-section. Before tackling the resulting eigenproblem, restrictions on purely imaginary eigenvalues are noted, and a proof is provided that degenerate decay modes do not exist for the prismatic structure—the only repeating eigenvalue that can give rise to a non-trivial Jordan block is the zero eigenvalue. The relationship between the reciprocal theorem and symplectic orthogonality is also noted. The twelve transmission eigen- and principal vectors associated with the multiple zero eigenvalue are determined, and we show how a division of the eigenspace allows one to transform the 6×6 system matrix into Jordan canonical form. It is seen that the state space approach can completely resolve the transmission problem for the rod, subject to determination of the Saint-Venant torsion and flexure functions which are cross-section specific. The decay problem is then considered for the specific two-dimensional problem of the plane stress elastic strip; the state space approach provides a very simple means of determining the well-known Papkovitch–Fadle eigenequations, which govern the rate of decay of self-equilibrated loading. Complete solution of the decay problem for the strip requires state space analysis in the axial and also, remarkably, in the transverse direction, suggesting that extension of the approach to other cross-sections is unlikely.

Finally, note that a broadly equivalent state space eigenanalysis of a repetitive discrete (pin-jointed) structure was presented in [18]; there, a transfer matrix **G** derived from the stiffness matrix **K** of a single cell, took the place of the present system matrix **H**. Unity eigenvalues pertain to the transmission modes, allowing one to determine equivalent continuum beam properties from their associated eigen- and principal vectors; non-unity eigenvalues occur as reciprocals and pertain to the decay of self-equilibrated loading, as anticipated by Saint-Venant's principle.

2. State space formulation

The governing equations of the linear theory of elasticity consist of the three stress equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0,$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0, \qquad \text{(1a, b, c)}$$

the Hooke's laws

$$\sigma_x = \lambda e + 2G\frac{\partial u}{\partial x}, \quad \sigma_y = \lambda e + 2G\frac{\partial v}{\partial y}, \quad \sigma_z = \lambda e + 2G\frac{\partial w}{\partial z},$$
$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{yz} = G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad \tau_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \qquad \text{(2a, b, c, d, e, f)}$$

where λ and G are the Lame′ constants, and the dilatation is $e = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z$. The boundary conditions are the surface tractions

$$t_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z,\ t_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z,\ t_z = \tau_{xz} n_x + \tau_{yz} n_y + n_z \sigma_z, \tag{3a,b,c}$$

where $n_x = \cos(n, x)$, $n_y = \cos(n, y)$, $n_z = \cos(n, z)$ are the direction cosines, and n is the outward normal to the surface. The z-coordinate is taken as directed along the axis of the rod, while its cross-section is defined by an equation of the form $f(x, y) = 0$.

The cross-sectional state vector $\mathbf{s} = [\mathbf{d}^{\mathrm{T}} \quad \mathbf{p}^{\mathrm{T}}]^{\mathrm{T}}$ consists of displacement components $\mathbf{d} = [u\ v\ w]^{\mathrm{T}}$ and stress components $\mathbf{p} = [\tau_{xz}\ \tau_{yz}\ \sigma_z]^{\mathrm{T}}$. In order to cast these equations into the requisite state variable form, first rearrange the Hooke's law, Eqs. (2e, f), as

$$\frac{\partial u}{\partial z} = \frac{\tau_{xz}}{G} - \frac{\partial w}{\partial x},\ \frac{\partial v}{\partial z} = \frac{\tau_{yz}}{G} - \frac{\partial w}{\partial y}; \tag{4a,b}$$

these are the 1st and 2nd rows of matrix Eq. (9). The Hooke's law, Eq. (2c), can be rearranged as

$$\frac{\partial w}{\partial z} = \frac{1}{\lambda + 2G}\sigma_z - \frac{\lambda}{\lambda + 2G}\frac{\partial u}{\partial x} - \frac{\lambda}{\lambda + 2G}\frac{\partial v}{\partial y} \tag{5}$$

and this is the 3rd row.

The equilibrium Eqs. (1a, b) are rearranged as

$$\begin{aligned} \frac{\partial \tau_{xz}}{\partial z} &= -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{xy}}{\partial y}, \\ \frac{\partial \tau_{yz}}{\partial z} &= -\frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \sigma_y}{\partial y}, \\ \frac{\partial \sigma_z}{\partial z} &= -\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y}; \end{aligned} \tag{6a, b, c}$$

the third of these is in the required form, involving only the state variables and a derivative with respect to z, and can be written immediately as the 6th row of matrix Eq. (9). On the other hand, the first two involve stress components that are not state variables, and these must be re-expressed as follows. Rearrange (6a) as

$$\frac{\partial \tau_{xz}}{\partial z} = -G\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left((\lambda + 2G)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} + \lambda\frac{\partial w}{\partial z}\right); \tag{7}$$

now substitute for $\partial w/\partial z$ from Eq. (5), to give the 4th row of Eq. (9). Last, rearrange Eq. (6b) as

$$\frac{\partial \tau_{yz}}{\partial z} = -G\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial y}\left(\lambda\frac{\partial u}{\partial x} + (\lambda + 2G)\frac{\partial v}{\partial y} + \lambda\frac{\partial w}{\partial z}\right) \tag{8}$$

and again substitute for $\partial w/\partial z$ from Eq. (5), to give the 5th row of Eq. (9).

Thus, manipulation of the known governing equations leads to the formulation

$$\frac{\partial}{\partial z}\begin{bmatrix} u \\ v \\ w \\ \tau_{xz} \\ \tau_{yz} \\ \sigma_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\partial/\partial x & G^{-1} & 0 & 0 \\ 0 & 0 & -\partial/\partial y & 0 & G^{-1} & 0 \\ -\lambda(\lambda+2G)^{-1}\partial/\partial x & -\lambda(\lambda+2G)^{-1}\partial/\partial y & 0 & 0 & 0 & (\lambda+2G)^{-1} \\ a & b & 0 & 0 & 0 & -\lambda(\lambda+2G)^{-1}\partial/\partial x \\ b & c & 0 & 0 & 0 & -\lambda(\lambda+2G)^{-1}\partial/\partial y \\ 0 & 0 & 0 & -\partial/\partial x & -\partial/\partial y & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ \tau_{xz} \\ \tau_{yz} \\ \sigma_z \end{bmatrix}, \tag{9}$$

where

$$a = -G\frac{\partial^2}{\partial y^2} - \frac{4G(\lambda+G)}{\lambda+2G}\frac{\partial^2}{\partial x^2} = -G\frac{\partial^2}{\partial y^2} - \frac{E}{1-\nu^2}\frac{\partial^2}{\partial x^2}$$

$$b = -\frac{G(3\lambda+2G)}{\lambda+2G}\frac{\partial^2}{\partial x\partial y} = -\frac{E}{2(1-\nu)}\frac{\partial^2}{\partial x\partial y}$$

and

$$c = -G\frac{\partial^2}{\partial x^2} - \frac{4G(\lambda+G)}{\lambda+2G}\frac{\partial^2}{\partial y^2} = -G\frac{\partial^2}{\partial x^2} - \frac{E}{1-\nu^2}\frac{\partial^2}{\partial y^2}.$$

Other elements may be simplified by noting that

$$\frac{\lambda}{\lambda+2G} = \frac{\nu}{1-\nu} \quad \text{and} \quad \frac{1}{\lambda+2G} = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)},$$

where E and ν are the Young's modulus and Poisson's ratio, respectively.

Symbolically, this may be written as

$$\frac{\partial \mathbf{s}}{\partial z} = \mathbf{Hs}, \tag{10}$$

where the system matrix $\mathbf{H}$ is defined accordingly. Partition matrix $\mathbf{H}$ into 3×3 blocks as

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{D} & -\mathbf{A}^{\mathrm{T}} \end{bmatrix}, \tag{11}$$

where $\mathbf{A}^{\mathrm{T}}$ denotes the adjoint of $\mathbf{A}$ which for a matrix differential operator is its transposition with the sign changed for all odd order differentials; for a constant matrix, the adjoint is the same as the transpose. Writing $\mathbf{s}(x,y,z) = \bar{\mathbf{s}}(x,y)\mathrm{e}^{kz}$ leads to the eigenvalue problem

$$\mathbf{H}\bar{\mathbf{s}} = k\bar{\mathbf{s}}, \quad \text{or} \quad [\mathbf{H} - k\mathbf{I}]\bar{\mathbf{s}} = \mathbf{0}, \tag{12a, b}$$

where **I** is the 6×6 identity matrix. The formal solution is then

$$\mathbf{s}(x,y,z) = \mathrm{e}^{\mathbf{H}z}\mathbf{s}(0), \tag{13}$$

where $\mathbf{s}(0)$ is the state vector on the end $z=0$, and describes the cross-sectional (x,y) field of both displacement and stress components on that end, while $\mathrm{e}^{\mathbf{H}z}$ is the matrix exponential, which describes how that field evolves as one moves along the rod. Now for a 6×6 matrix having constant elements, there are obviously just 6 possible eigenvalues; however, the operator matrix **H**, can represent a multitude of 6×6 matrices, depending on the x- and y-dependence of the state vectors, each having 6 possible eigenvalues. Thus, one might expect an infinite number of eigenvalues of such a matrix. Employing a computer algebra program, such as MAPLE, one may easily determine that

$$\det[\mathbf{H} - k\mathbf{I}] = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)^3 = 0 \tag{14}$$

from which one may conclude that for state vectors in which the x- and y-dependence is harmonic, the eigenvalue $k=0$ has a multiplicity of at least 6; in fact, it will be seen that the zero eigenvalue has multiplicity 12, with the others associated with vectors containing harmonic terms, including the Saint-Venant warping (ψ) and flexure (ϕ) functions, and terms such as $(x^2 - y^2)$, for which the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ yields a zero.

3. Hamiltonian nature of the system matrix

The $(2n \times 2n)$ system matrix **H** satisfies the relationship

$$\mathbf{H}^{\mathrm{T}} = \mathbf{J}_m\mathbf{H}\mathbf{J}_m, \tag{15}$$

where $\mathbf{J}_m$ is the metric matrix

$$\mathbf{J}_m = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

with $\mathbf{J}_m^{\mathrm{T}} = \mathbf{J}_m^{-1} = -\mathbf{J}_m$, and **I** is now the $(n \times n)$ identity matrix. Post-multiply Eq. (15) by $\mathbf{J}_m^{-1} = -\mathbf{J}_m$ to give $\mathbf{J}_m\,\mathbf{H} = -\mathbf{H}^{\mathrm{T}}\mathbf{J}_m$, and further post-multiply by $\bar{\mathbf{s}}$, noting Eq. (12a), to give

$$\mathbf{H}^{\mathrm{T}}(\mathbf{J}_m\bar{\mathbf{s}}) = -k(\mathbf{J}_m\bar{\mathbf{s}}). \tag{16}$$

Since a matrix has the same eigenvalues as its transpose, one concludes that eigenvalues occur as equal and opposite pairs. One might have anticipated this result: the trace of a matrix is an invariant under a similarity transformation, and the trace of the system matrix **H**, from Eq. (9), is clearly zero.

4. Restrictions on imaginary eigenvalues

Synge [19] considered Saint-Venant's problem for a prismatic homogenous elastic cylinder, and for the decay modes considered stress varying as $\sigma(x,y)\mathrm{e}^{kz}$. According to Synge, *A purely* *imaginary k implies a periodic distribution of displacement and stress. Consider the energy in a length of cylinder equal to this period. It is equal to the work done by the terminal stress in passing from the natural state to the strained state. But from the periodicity, this is zero. Hence, the energy of a strained state is zero, which is contrary to a basic postulate of elasticity. Hence there can be no purely imaginary eigenvalue k. It should be added that we could not assert this if (Poisson's ratio) v is arbitrary. It is necessarily only true if strain energy is positive definite, i.e. if* $-1<v<1/2$, indicating that this *simple and ingenious argument* was originally put forward by Dougall [20]. One should add to the above the remark that imaginary eigenvalues *can* occur for beam-like structures having a pre-twisted form Ref. [15], as in a turbine blade.

5. Eigenvectors associated with $k=0$ do no work

The strain energy is equal to the work done by loading over the surface S, and may be written as

$$U=\frac{1}{2}\iint [(u\sigma_x+v\tau_{xy}+w\tau_{xz})n_x+(u\tau_{xy}+v\sigma_y+w\tau_{yz})n_y+(u\tau_{xz}+v\tau_{yz}+w\sigma_z)n_z]\,\mathrm{d}S. \tag{17}$$

For the cylinder with traction-free surface generators this reduces to

$$U=\frac{1}{2}\iint (u\tau_{xz}+v\tau_{yz}+w\sigma_z)n_z\mathrm{d}S. \tag{18}$$

Now calculate the strain energy within a slice of rod of length $\mathrm{d}z$: denote the state (eigen)vector on the left-hand side as $\mathbf{s}_L=\bar{\mathbf{s}}\,\mathrm{e}^{kz}$, then on the right-hand side one has

$$\mathbf{s}_R=\mathbf{s}_L+\frac{\partial\mathbf{s}_L}{\partial z}\mathrm{d}z=\mathbf{s}_L(1+k\mathrm{d}z). \tag{19}$$

Moreover, one has

$$\mathbf{s}_L=\begin{bmatrix}\mathbf{d}_L\\ \mathbf{p}_L\end{bmatrix} \quad\text{and}\quad \mathbf{s}_R=\begin{bmatrix}\mathbf{d}_R\\ \mathbf{p}_R\end{bmatrix}=\begin{bmatrix}(1+k\,\mathrm{d}z)\mathbf{d}_L\\ (1+k\,\mathrm{d}z)\mathbf{p}_L\end{bmatrix} \tag{20a, b}$$

and the strain energy becomes

$$U=\frac{1}{2}\iint (\mathbf{d}_R^{\mathrm{T}}\mathbf{p}_R-\mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L)\,\mathrm{d}x\,\mathrm{d}y \tag{21}$$

since n_z is equal to $+1$ on the right-hand face, but -1 on the left. Substitute from Eq. (20b) to give

$$U=\frac{1}{2}\iint [(1+k\,\mathrm{d}z)^2\mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L-\mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L]\,\mathrm{d}x\,\mathrm{d}y=k\iint \mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L\,\mathrm{d}x\,\mathrm{d}y, \tag{22}$$

when terms involving $(\mathrm{d}z)^2$ are neglected; thus the strain energy per unit length is $k\iint\mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L\,\mathrm{d}x\,\mathrm{d}y$. This is zero when k is equal to zero, which is in agreement with experience: the eigenvectors associated with the zero eigenvalue are the rigid-body displacements and rotation, for which the strain energy is zero.

6. Work associated with a principal vector

Now consider a principal vector (or generalized eigenvector) coupled to the eigenvector in a Jordan chain according to

$$\mathbf{H}\bar{\mathbf{s}}^p = k\bar{\mathbf{s}}^p + \bar{\mathbf{s}}, \tag{23}$$

where the superscript p denotes the principal vector. The state vector on the left-hand side of the slice is $\mathbf{s}_L = (\bar{\mathbf{s}}^p + z\bar{\mathbf{s}})\mathrm{e}^{kz}$, and on the right-hand side one has

$$\mathbf{s}_R = \mathbf{s}_L + \frac{\partial \mathbf{s}_L}{\partial z}\,\mathrm{d}z = \mathbf{s}_L + (k\bar{\mathbf{s}}^p + (1+kz)\,\bar{\mathbf{s}})\mathrm{e}^{kz}\,\mathrm{d}z, \tag{24}$$

or more fully

$$\mathbf{s}_L = \begin{bmatrix} \mathbf{d}_L^p + z\mathbf{d}_L \\ \mathbf{p}_L^p + z\mathbf{p}_L \end{bmatrix} \quad \text{and} \quad \mathbf{s}_R = \begin{bmatrix} \mathbf{d}_L^p + z\mathbf{d}_L + (k\mathbf{d}_L^p + (1+kz)\mathbf{d}_L)\,\mathrm{d}z \\ \mathbf{p}_L^p + z\mathbf{p}_L + (k\mathbf{p}_L^p + (1+kz)\mathbf{p}_L)\,\mathrm{d}z \end{bmatrix}. \tag{25}$$

Next, calculate the strain energy according to Eq. (21) to give

$$U = \frac{1}{2}\mathrm{d}z \int_{\Gamma} [2k\mathbf{d}_L^{p\mathrm{T}}\mathbf{p}_L^p + (1+2kz)(\mathbf{d}_L^{p\mathrm{T}}\mathbf{p}_L + \mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L^p) + 2z(1+kz)\mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L]\,\mathrm{d}x\,\mathrm{d}y; \tag{26}$$

again this in accord with experience: only the repeating zero eigenvalue is known to give rise to a non-trivial Jordan block, when the above reduces to

$$U = \frac{1}{2}\,\mathrm{d}z \int_{\Gamma} [(\mathbf{d}_L^{p\mathrm{T}}\mathbf{p}_L + \mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L^p) + 2z\mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L]\,\mathrm{d}x\,\mathrm{d}y. \tag{27}$$

As an example, consider the coupling of the principal vector for tension with the eigenvector for rigid-body displacement in the axial direction, which has $\mathbf{p}_L = \mathbf{0}$, and the above further reduces to

$$U = \frac{1}{2}\,\mathrm{d}z \int_{\Gamma} \mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L^p\,\mathrm{d}x\,\mathrm{d}y \tag{28}$$

so the strain energy is one-half of the product of the extension and the tensile force.

Now, it is well known that if one adds an arbitrary multiple (say, β) of the generating eigenvector to a principal vector, then it is still a principal vector; on the other hand, one cannot in general add an arbitrary multiple of a principal vector of one grade to a principal vector of another grade. Thus suppose the left-hand state vector is now written as $\mathbf{s}_L = ((\bar{\mathbf{s}}^p + \beta\bar{\mathbf{s}}) + z\bar{\mathbf{s}})\mathrm{e}^{kz}$; the left- and right-hand side state vectors are, more fully

$$\mathbf{s}_L = \begin{bmatrix} \mathbf{d}_L^p + (\beta+z)\mathbf{d}_L \\ \mathbf{p}_L^p + (\beta+z)\mathbf{p}_L \end{bmatrix} \quad \text{and} \quad \mathbf{s}_R = \begin{bmatrix} \mathbf{d}_L^p + (\beta+z)\mathbf{d}_L + (k\mathbf{d}_L^p + (1+k(\beta+z))\mathbf{d}_L)\,\mathrm{d}z \\ \mathbf{p}_L^p + (\beta+z)\mathbf{p}_L + (k\mathbf{d}_L^p + (1+k(\beta+z))\mathbf{p}_L)\,\mathrm{d}z \end{bmatrix} \tag{29}$$

and the strain energy is

$$U = \frac{1}{2}\mathrm{d}z \int_{\Gamma} [2k\mathbf{d}_L^{p\mathrm{T}}\mathbf{p}_L^p + (1+2(\beta+z)k)(\mathbf{d}_L^{p\mathrm{T}}\mathbf{p}_L + \mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L^p) + 2(\beta+z)(1+(\beta+z)k)\mathbf{d}_L^{\mathrm{T}}\mathbf{p}_L]\,\mathrm{d}x\,\mathrm{d}y. \tag{30}$$

This degenerates to expression (26) when β is equal to zero, as it must.

If one now demands that the strain energy stored should be independent of the arbitrary β, then one requires, comparing with expression (26)

$$\iint [\beta k(\mathbf{d}_L^{pT}\mathbf{p}_L + \mathbf{d}_L^{T}\mathbf{p}_L^{p}) + (k\beta^2 + \beta(1+2kz))\mathbf{d}_L^{T}\mathbf{p}_L]\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{31}$$

or more compactly

$$A\beta^2 + B\beta = 0, \tag{32}$$

where

$$A = k\iint \mathbf{d}_L^{T}\mathbf{p}_L\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$B = k\iint (\mathbf{d}_L^{pT}\mathbf{p}_L + \mathbf{d}_L^{T}\mathbf{p}_L^{p})\,\mathrm{d}x\,\mathrm{d}y + (1+2kz)\iint \mathbf{d}_L^{T}\mathbf{p}_L\,\mathrm{d}x\,\mathrm{d}y.$$

Since β is quite arbitrary, one must have $A = B = 0$; for $A=0$, one must have either $k=0$, or $\iint \mathbf{d}_L^{T}\mathbf{p}_L\,\mathrm{d}x\,\mathrm{d}y = 0$, or perhaps both. If one chooses $k\neq 0$, then $\iint \mathbf{d}_L^{T}\mathbf{p}_L\,\mathrm{d}x\,\mathrm{d}y = 0$, and the requirement that $B=0$, leads to $\iint (\mathbf{d}_L^{pT}\mathbf{p}_L + \mathbf{d}_L^{T}\mathbf{p}_L^{p})\mathrm{d}x\,\mathrm{d}y = 0$. Thus one may conclude that either

$$\text{(a) } k = 0 \text{ or } \iint \mathbf{d}_L^{T}\mathbf{p}_L\mathrm{d}x\,\mathrm{d}y = 0 \text{ or perhaps both} \tag{33}$$

or

$$\text{(b) } k\neq 0 \text{ and both } \iint \mathbf{d}_L^{T}\mathbf{p}_L\mathrm{d}x\,\mathrm{d}y = 0, \iint (\mathbf{d}_L^{pT}\mathbf{p}_L + \mathbf{d}_L^{T}\mathbf{p}_L^{p})\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{34}$$

Case (b) is inconceivable as the first integral implies that the work done (and strain energy stored) within an entire semi-infinite rod for a decaying state of stress, is zero, which is again contrary to a basic postulate of elasticity; on the other hand, case (a) accords with experience—addition of an arbitrary rigid body displacement, for which $k=0$, has no effect on energy stored. Thus, one may conclude that the only possible repeating root giving rise to a secular term (within a Jordan block) is $k=0$, at least for a prismatic structure. Note that Getman and Ustinov [21,22] have considered the issue of repeating eigenvalues (termed resonances, or critical frequencies) in the context of energy flux during wave propagation.

7. Reciprocal theorem and symplectic orthogonality

Zhong and Williams [23] have considered in detail the symplectic orthogonality of the eigenvectors, and show that it is a direct consequence of the reciprocal theorem of Betti–Maxwell: for two different load systems, denoted by subscripts 1 and 2, the work done by the stress components $\mathbf{p}_1$ acting through displacements $\mathbf{d}_2$ is equal to the work done by the stress components $\mathbf{p}_2$ acting through displacements $\mathbf{d}_1$. Again employ subscripts L and R to denote the left- and right-hand sides of a slice of length dz, then from Eq. (21)

$$\frac{1}{2}\int_A (\mathbf{d}_{R2}^{\mathrm{T}}\mathbf{p}_{R1} - \mathbf{d}_{L2}^{\mathrm{T}}\mathbf{p}_{L1})\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_A (\mathbf{d}_{R1}^{\mathrm{T}}\mathbf{p}_{R2} - \mathbf{d}_{L1}^{\mathrm{T}}\mathbf{p}_{L2})\,\mathrm{d}x\,\mathrm{d}y. \tag{35}$$

Assume the two load systems are different eigenvectors and writing $\mathbf{s}_{L1} = \bar{\mathbf{s}}_1\,\mathrm{e}^{k_1 z}$ and $\mathbf{s}_{L2} = \bar{\mathbf{s}}_2\,\mathrm{e}^{k_2 z}$, gives

$$\mathbf{s}_{L1} = \begin{bmatrix}\mathbf{d}_{L1}\\ \mathbf{p}_{L1}\end{bmatrix},\quad \mathbf{s}_{R1} = \begin{bmatrix}\mathbf{d}_{R1}\\ \mathbf{p}_{R1}\end{bmatrix} = \begin{bmatrix}(1+k_1\mathrm{d}z)\mathbf{d}_{L1}\\ (1+k_1\mathrm{d}z)\mathbf{p}_{L1}\end{bmatrix}$$

and

$$\mathbf{s}_{L2} = \begin{bmatrix}\mathbf{d}_{L2}\\ \mathbf{p}_{L2}\end{bmatrix},\quad \mathbf{s}_{R_2} = \begin{bmatrix}\mathbf{d}_{R2}\\ \mathbf{p}_{R2}\end{bmatrix} = \begin{bmatrix}(1+k_2\mathrm{d}z)\mathbf{d}_{L2}\\ (1+k_2\mathrm{d}z)\mathbf{p}_{L2}\end{bmatrix}. \tag{36}$$

Substitute into Eq. (35), and ignore terms involving $(\mathrm{d}z)^2$, leads to the expression

$$(k_1 + k_2)\,\mathrm{d}z\int_A (\mathbf{d}_{L2}^{\mathrm{T}}\mathbf{p}_{L1} - \mathbf{d}_{L1}^{\mathrm{T}}\mathbf{p}_{L2})\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{37}$$

Thus, for $k_1 \neq -k_2$, one has

$$\int_A (\mathbf{d}_{L2}^{\mathrm{T}}\mathbf{p}_{L1} - \mathbf{d}_{L1}^{\mathrm{T}}\mathbf{p}_{L2})\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{38}$$

and omitting the subscript L, the above may be expressed as

$$\int_A \mathbf{s}_2^{\mathrm{T}}\mathbf{J}_m\mathbf{s}_1\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{39}$$

which is the statement of symplectic orthogonality. In principle, this allows the expansion of an arbitrary end load, $\mathbf{s}(0)$, into its constituent transmission and decay modes; in practice, however, the decay modes would more often than not be neglected by invoking Saint-Venant's principle, and the magnitude of, say, tension or bending moment would be known in advance.

8. Eigen- and principal vectors associated with $k = 0$

A zero eigenvalue implies that the eigenvector satisfies the equation

$$\mathbf{H}\bar{\mathbf{s}} = \mathbf{0}; \tag{40}$$

there are four eigenvectors, and these are identified with the superscript (0):

(i) rigid-body displacement in the x-direction:

$$\bar{\mathbf{s}}_1^{(0)} = [a \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{41}$$

where a is a constant,

(ii) rigid-body displacement in the y-direction:

$$\bar{\mathbf{s}}_2^{(0)} = [0 \quad b \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{42}$$

where b is a constant,

(iii) rigid body displacement in the z-direction:

$$\bar{\mathbf{s}}_3^{(0)} = [0 \quad 0 \quad c \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{43}$$

where c is a constant,
(iv) rigid-body rotation about the z-axis:

$$\bar{\mathbf{s}}_4^{(0)} = [-y\alpha \quad x\alpha \quad 0 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{44}$$

where the constant α is the (small) angle of rotation.

Coupled to these eigenvectors are four principal vectors of the first grade, according to equations of the form

$$\mathbf{H}\bar{\mathbf{s}}_i^{(1)} = \bar{\mathbf{s}}_i^{(0)}, \ i = 1, 2, 3, 4 \tag{45}$$

and these are identified with the superscript (1),
(v) rigid-body rotation about the y-axis:

$$\bar{\mathbf{s}}_1^{(1)} = [0 \quad 0 - ax \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{46}$$

(vi) rigid-body rotation about the x-axis:

$$\bar{\mathbf{s}}_2^{(1)} = [0 \quad 0 - by \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \tag{47}$$

(vii) tensile force in the z-direction:

$$\bar{\mathbf{s}}_3^{(1)} = [-cvx \quad -cvy \quad 0 \quad 0 \quad 0 \quad cE]^{\mathrm{T}}, \tag{48}$$

(viii) torsional moment about the z-axis:

$$\bar{\mathbf{s}}_4^{(1)} = \left[0 \quad 0 \quad \alpha\psi \quad G\alpha(\partial\psi/\partial x - y) \quad G\alpha(\partial\psi/\partial y + x) \quad 0\right]^{\mathrm{T}}, \tag{49}$$

where ψ is the Saint-Venant warping function which must satisfy $\nabla^2\psi = 0$.

The rigid-body rotations (v) and (vi) are coupled to two principal vectors of the second grade, superscript (2), according to

$$\mathbf{H}\bar{\mathbf{s}}_i^{(2)} = \bar{\mathbf{s}}_i^{(1)}, \ i = 1, 2. \tag{50}$$

These are:
(ix) bending moment about the y-axis:

$$\bar{\mathbf{s}}_1^{(2)} = \left[av(x^2 - y^2)/2 \quad avxy \quad 0 \quad 0 \quad 0 \quad -aEx\right]^{\mathrm{T}}, \tag{51}$$

(x) bending moment about the x-axis:

$$\bar{\mathbf{s}}_2^{(2)} = \left[bvxy \quad -bv(x^2 - y^2)/2 \quad 0 \quad 0 \quad 0 \quad -bEx\right]^{\mathrm{T}}. \tag{52}$$

Last, these bending moments are coupled to two principal vectors of the third grade, superscript (3), according to

$$\mathbf{H}\bar{\mathbf{s}}_i^{(3)} = \bar{\mathbf{s}}_i^{(2)}, \; i = 1, 2. \tag{53}$$

These are:
(xi) shearing force in the xz-plane:

$$\bar{\mathbf{s}}_1^{(3)} = \begin{bmatrix} 0 & 0 & a(\phi - vxy^2/2 + (2+v)x^3/6) & aG(\partial\phi/\partial x - vy^2 + (1+v)x^2) & aG\partial\phi/\partial y & 0 \end{bmatrix}^{\mathrm{T}}, \tag{54}$$

(xii) shearing force in the yz-plane:

$$\bar{\mathbf{s}}_2^{(3)} = \begin{bmatrix} 0 & 0 & b(\phi - vx^2y/2 + (2+v)y^3/6) & bG\partial\phi/\partial x & bG(\partial\phi/\partial y - vx^2 + (1+v)y^2) & 0 \end{bmatrix}^{\mathrm{T}}, \tag{55}$$

where ϕ is the Saint-Venant flexure function which, again, must satisfy $\nabla^2\phi = 0$.

We now see that there are 12 eigen- and principal vectors pertaining to the zero eigenvalue, and one may construct the rectangular (6×12) matrix of eigen- and principal vectors $\mathbf{V}$, where

$$\mathbf{V} = [\bar{\mathbf{s}}_1^{(0)} \; \bar{\mathbf{s}}_1^{(1)} \; \bar{\mathbf{s}}_1^{(2)} \; \bar{\mathbf{s}}_1^{(3)} \; \bar{\mathbf{s}}_2^{(0)} \; \bar{\mathbf{s}}_2^{(1)} \; \bar{\mathbf{s}}_2^{(2)} \; \bar{\mathbf{s}}_2^{(3)} \; \bar{\mathbf{s}}_3^{(0)} \; \bar{\mathbf{s}}_3^{(1)} \; \bar{\mathbf{s}}_4^{(0)} \; \bar{\mathbf{s}}_4^{(1)}] \tag{56}$$

such that

$$\mathbf{HV} = \mathbf{VJ}, \tag{57}$$

where $\mathbf{J}$ is the Jordan canonical form

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_4 \end{bmatrix} \tag{58}$$

in which $\mathbf{0}$ are zero matrices of the appropriate size, and

$$\mathbf{J}_1 = \mathbf{J}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{J}_3 = \mathbf{J}_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{59}$$

show the zero eigenvalue on the leading diagonal, with a unity element on the superdiagonal indicating the coupling between vectors, within four distinct eigenspaces. The state vector boundary condition, $\mathbf{s}(0)$, on the end $z = 0$ is a linear combination of the eigen- and principal vectors and may be expressed as

$$\mathbf{s}(0) = \mathbf{VC}, \tag{60}$$

where the vector of participation coefficients is

$$\mathbf{C} = [C_1\ C_2 \cdots C_{12}]^{\mathrm{T}}. \tag{61}$$

In order to determine the final solution one needs to determine the matrix exponential, $e^{\mathbf{H}z}$; one of the standard methods [24] employs the Jordan canonical form, together with the inverse of the matrix of eigen- and principle vectors, $\mathbf{V}$. However, the latter is of dimension (6×12), so is obviously not invertible. Instead, partition this matrix as

$$\mathbf{V} = [\mathbf{V}_{13}\ \mathbf{V}_{24}], \tag{62}$$

where

$$\mathbf{V}_{13} = [\bar{\mathbf{s}}_1^{(0)}\quad \bar{\mathbf{s}}_1^{(1)}\quad \bar{\mathbf{s}}_1^{(2)}\quad \bar{\mathbf{s}}_1^{(3)}\quad \bar{\mathbf{s}}_3^{(0)}\quad \bar{\mathbf{s}}_3^{(1)}],\quad \mathbf{V}_{24} = [\bar{\mathbf{s}}_2^{(0)}\quad \bar{\mathbf{s}}_2^{(1)}\quad \bar{\mathbf{s}}_2^{(2)}\quad \bar{\mathbf{s}}_2^{(3)}\quad \bar{\mathbf{s}}_4^{(0)}\quad \bar{\mathbf{s}}_4^{(1)}], \tag{63}$$

are both invertible, and of the same size as $\mathbf{H}$. One then has

$$\mathbf{V}_{13}^{-1}\mathbf{H}\mathbf{V}_{13} = \mathbf{J}_{13} = \begin{bmatrix} \mathbf{J}_1 & 0 \\ 0 & \mathbf{J}_3 \end{bmatrix},\quad \mathbf{V}_{24}^{-1}\mathbf{H}\mathbf{V}_{24} = \mathbf{J}_{24} = \begin{bmatrix} \mathbf{J}_2 & 0 \\ 0 & \mathbf{J}_4 \end{bmatrix} \tag{64}$$

and within the eigenspaces spanned by $\mathbf{V}_{13}$ and $\mathbf{V}_{24}$, one has

$$e^{\mathbf{H}z} = \mathbf{V}_{13}\, e^{\mathbf{J}_{13}z}\, \mathbf{V}_{13}^{-1},\ e^{\mathbf{H}z} = \mathbf{V}_{24}\, e^{\mathbf{J}_{24}z}\, \mathbf{V}_{24}^{-1}. \tag{65}$$

Now $\mathbf{J}_{13}$ and $\mathbf{J}_{24}$ are identical, and their matrix exponential is

$$e^{\mathbf{J}_{13}z} = e^{\mathbf{J}_{24}z} = \begin{bmatrix} 1 & z & z^2/2 & z^3/6 & 0 & 0 \\ 0 & 1 & z & z^2/2 & 0 & 0 \\ 0 & 0 & 1 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{66}$$

The end condition can also be decomposed into the two combined eigenspaces, as

$$\mathbf{s}_{13}(0) = \mathbf{V}_{13}\mathbf{C}_{13} \quad \text{and} \quad \mathbf{s}_{24}(0) = \mathbf{V}_{24}\mathbf{C}_{24} \tag{67}$$

with participation coefficients

$$\mathbf{C}_{13} = [C_1\ C_2\ C_3\ C_4\ C_9\ C_{10}]^{\mathrm{T}} \quad \text{and} \quad \mathbf{C}_{24} = [C_5\ C_6\ C_7\ C_8\ C_{11}\ C_{12}]^{\mathrm{T}}; \tag{68}$$

the formal solutions become

$$\mathbf{s}_{13}(z) = e^{\mathbf{H}z}\mathbf{s}_{13}(0) = \mathbf{V}_{13}e^{\mathbf{J}_{13}z}\mathbf{V}_{13}^{-1}\mathbf{V}_{13}\mathbf{C}_{13} = \mathbf{V}_{13}e^{\mathbf{J}_{13}z}\mathbf{C}_{13} \tag{69}$$

or in expanded form

$$\mathbf{s}_{13}(z) = \begin{bmatrix} \bar{\mathbf{s}}_1^{(0)} & \bar{\mathbf{s}}_1^{(1)} & \bar{\mathbf{s}}_1^{(2)} & \bar{\mathbf{s}}_1^{(3)} & \bar{\mathbf{s}}_3^{(0)} & \bar{\mathbf{s}}_3^{(1)} \end{bmatrix} \begin{bmatrix} C_1 + C_2 z + C_3 z^2/2 + C_4 z^3/6 \\ C_2 + C_3 z + C_4 z^2/2 \\ C_3 + C_4 z \\ C_4 \\ C_9 + C_{10} z \\ C_{10} \end{bmatrix} \tag{70}$$

or

$$\begin{aligned} \mathbf{s}_{13}(z) &= (C_1 + C_2 z + C_3 z^2/2 + C_4 z^3/6)\bar{\mathbf{s}}_1^{(0)} + (C_2 + C_3 z + C_4 z^2/2)\bar{\mathbf{s}}_1^{(1)} \\ &\quad + (C_3 + C_4 z)\bar{\mathbf{s}}_1^{(2)} + C_4 \bar{\mathbf{s}}_1^{(3)} + (C_9 + C_{10} z)\bar{\mathbf{s}}_3^{(0)} + C_{10}\bar{\mathbf{s}}_3^{(1)}. \end{aligned} \tag{71}$$

This can be interpreted as follows: suppose one has a cantilevered beam of length L, with a shearing force Q in the xz-plane, only, on the end $z=0$ (implying that the origin is located at the cross-section having the maximum deflection and slope); all participation coefficients are set equal to zero with the exception of C_4, which may be set equal to unity as the principal vector describing shear already contains the arbitrary constant a which can be related to Q. The state vector at generic cross-section z is then

$$\mathbf{s}(z) = z^3\bar{\mathbf{s}}_1^{(0)}/6 + z^2\bar{\mathbf{s}}_1^{(1)}/2 + z\bar{\mathbf{s}}_1^{(2)} + \bar{\mathbf{s}}_1^{(3)}, \tag{72}$$

where the state vector consists of the shearing force vector, a bending moment which increases linearly with the axial coordinate, z, and rotation and transverse displacement vectors which increase as z^2 and z^3, respectively.

More conventionally, one might locate the origin at the fixed end, where the rotation and transverse displacement vectors are equal to zero, and there are reactions of shearing force Q and bending moment $M = -QL$. The non-zero participation coefficients may be set as $C_4 = 1$, and $C_3 = -L$, and the state vector at generic cross-section z is then

$$\mathbf{s}(z) = z^3\bar{\mathbf{s}}_1^{(0)}/6 + z^2(\bar{\mathbf{s}}_1^{(1)} - L\bar{\mathbf{s}}_1^{(0)})/2 + z(\bar{\mathbf{s}}_1^{(2)} - L\bar{\mathbf{s}}_1^{(1)}) + \bar{\mathbf{s}}_1^{(3)} - L\bar{\mathbf{s}}_1^{(2)}, \tag{73}$$

which is in agreement with the expression given by Zhong and Xu [4].

For the second combined eigenspace, the formal solution is

$$\mathbf{s}_{24}(z) = \mathrm{e}^{\mathbf{H}z}\mathbf{s}_{24}(0) = \mathbf{V}_{24}\mathrm{e}^{\mathbf{J}_{24}z}\mathbf{C}_{24}. \tag{74}$$

9. The decay eigenproblem for the elastic strip

While the state-space approach can completely resolve the transmission problem for *any* cross-section, the Saint-Venant warping and flexure functions being the only indeterminates, the decay eigenproblem (associated with Saint-Venant's principle) appears restricted to amenable cross-sections in general, and the state space approach appears to offer an advantage only in the case of the two-dimensional strip problem. (At least the present author cannot see how it can extend to a three-dimensional problem such as the rod of circular cross-section; indeed this may be why Xu et al. [8] quoted the Papkovitch–Neuber solution for the decay problem, from Ref. [9].)

With this in mind, consider the two-dimensional elastic semi-infinite strip occupying the space

$$0 \le x \le 2c, \; 0 \le z \le \infty, \tag{75}$$

where the z-axis has been chosen deliberately to coincide with the lower edge of the strip. Returning to Eq. (9), if one deletes the rows and columns pertaining to displacement v and shearing stress τ_{yz}, and drop all terms involving $\partial/\partial y$, one obtains the governing equations in state space form for the plane strain problem; however, the concept of a cross-sectional state vector is more natural in the context of plane stress setting, which can be obtained in the usual way [25] by replacing E by $E'(1+2v')/(1+v')^2$, and v by $v'/(1+v')$ and then dropping the primes. In this way, one finds

$$\frac{\partial}{\partial z}\begin{bmatrix} u \\ w \\ \tau_{xz} \\ \sigma_z \end{bmatrix} = \begin{bmatrix} 0 & -\partial/\partial x & 1/G & 0 \\ -v\partial/\partial x & 0 & 0 & (1-v^2)/E \\ -E\partial^2/\partial x^2 & 0 & 0 & -v\partial/\partial x \\ 0 & 0 & -\partial/\partial x & 0 \end{bmatrix}\begin{bmatrix} u \\ w \\ \tau_{xz} \\ \sigma_z \end{bmatrix} \tag{76}$$

or compactly

$$\frac{\partial \mathbf{s}}{\partial z} = \mathbf{Hs}. \tag{77}$$

Separate variables by setting $\mathbf{s}(x,z) = \bar{\mathbf{s}}(x)\mathrm{e}^{kz}$ which leads to the characteristic equation

$$\det[\mathbf{H} - k\mathbf{I}] = \left(\frac{\partial^2}{\partial x^2} + k^2\right)^2 = 0 \tag{78}$$

suggesting that the eigenvalue $k=0$ has a multiplicity of at least four, for vectors in which the x-dependence is lower than second order; in fact for a planar structure, it is six. However, for a decay eigenmode $k \neq 0$, so one has the double root $\partial/\partial x = \pm \mathrm{i}k, \mathrm{i} = \sqrt{-1}$, and choosing the positive sign, and substituting back into $\mathbf{H}$, gives

$$\mathbf{H} = \begin{bmatrix} 0 & -\mathrm{i}k & 1/G & 0 \\ -v\mathrm{i}k & 0 & 0 & (1-v^2)/E \\ Ek^2 & 0 & 0 & -v\mathrm{i}k \\ 0 & 0 & -\mathrm{i}k & 0 \end{bmatrix}. \tag{79}$$

Again, the formal solution is written as

$$\mathbf{s}(x,z) = \mathrm{e}^{\mathbf{H}z}\mathbf{s}(0), \tag{80}$$

where now [24] the matrix exponential $\mathbf{M} = e^{\mathbf{H}z}$ may be calculated as the inverse Laplace transform of the resolvent matrix $\mathbf{R} = (s\mathbf{I} - \mathbf{H})^{-1}$, where s is the Laplace variable; $\mathbf{R}$ has elements[1] such as $R_{11} = (s^3 + \nu k^2 s)/(s^2 - k^2)^2$, and when one performs the inverse Laplace transformation, this term yields the expression

$$M_{11} = \frac{e^{kz}}{2}\left(1 + \frac{(1+\nu)}{2}kz\right) + \frac{e^{-kz}}{2}\left(1 - \frac{(1+\nu)}{2}kz\right). \tag{81}$$

Now the presence of the secular terms $ze^{\pm kz}$ indicates that the manifestation of the double root $\partial/\partial x = ik$ has been assigned to the axial z-direction, which has been shown in Section 6 above, to be impossible. Moreover, the approach gives no information on the possible values of k; thus at first sight, the state space approach appears to offer no obvious advantage.

Instead, define a *transverse* (rather than *cross-sectional*) state vector

$$\mathbf{s}' = [u \quad w \quad \tau_{xz} \quad \sigma_x]^{\mathrm{T}} \tag{82}$$

and recast the governing equations in terms of this vector to give

$$\frac{\partial \mathbf{s}'}{\partial x} = \mathbf{H}'\mathbf{s}', \tag{83}$$

where the new system matrix $\mathbf{H}'$ is identical to $\mathbf{H}$, except that $\partial/\partial z$ replaces $\partial/\partial x$. Now set $\mathbf{s}'(x,z) = \bar{\mathbf{s}}'(z)e^{k'x}$, which leads to the characteristic equation

$$\det[\mathbf{H}' - k'\mathbf{I}] = \left(\frac{\partial^2}{\partial z^2} + k'^2\right)^2 = 0 \tag{84}$$

and again the double roots $\partial/\partial z = \pm ik'$; thus one may conclude that the resolvent matrix $\mathbf{R}' = (s\mathbf{I} - \mathbf{H}')^{-1}$ for this second eigenproblem is identical to that already presented in the Appendix, Eq. (A.1); moreover, the matrix exponential $\mathbf{M}' = e^{\mathbf{H}'x}$ is identical to $\mathbf{M}$ in Eq. (A.2), but with the variable x replacing z, and k' replacing k. Also note that the spatial variation in the x-direction, $\exp(k'x)$, implies a variation in the z-direction of $\exp(ik'z) = \exp(kz)$, where $ik' = k$.

Again the formal solution may be written as

$$\mathbf{s}'(x,z) = e^{\mathbf{H}'x}\mathbf{s}'(0). \tag{85}$$

Now the traction-free boundary condition on the lower edge of the strip, $x=0$, implies that the *initial* state vector is $\mathbf{s}'(0) = [u(0) \quad w(0) \quad 0 \quad 0]^{\mathrm{T}}$, so the above becomes, in more detail

$$\mathbf{s}'(x) = \begin{bmatrix} u(x) \\ w(x) \\ \tau_{xz}(x) \\ \sigma_x(x) \end{bmatrix} = \begin{bmatrix} M'_{11}\ u(0) + M'_{12}\ w(0) \\ M'_{21}\ u(0) + M'_{22}\ w(0) \\ M'_{31}\ u(0) + M'_{32}\ w(0) \\ M'_{41}\ u(0) + M'_{42}\ w(0) \end{bmatrix}. \tag{86}$$

[1]The complete matrix $\mathbf{R}$ is given in the Appendix, together with its inverse Laplace transform.

But one also has traction-free conditions on the upper edge of the strip, that is $\tau_{xz}(2c) = \sigma_x(2c) = 0$, giving

$$\begin{bmatrix} M'_{31} & M'_{32} \\ M'_{41} & M'_{42} \end{bmatrix}_{x=2c} \begin{bmatrix} u(0) \\ w(0) \end{bmatrix} = \mathbf{0}; \tag{87}$$

since the displacement components are not zero on the lower edge, the determinant must be equal to zero, which leads to the eigenequation

$$k'^2(\mathrm{e}^{4k'c} + \mathrm{e}^{-4k'c} - 16k'^2c^2 - 2) = 0, \tag{88}$$

which may be expressed in the more familiar form

$$k'^2(\sinh 2k'c - 2k'c)(\sinh 2k'c + 2k'c) = 0. \tag{89}$$

This is identical to the well-known Papkovitch–Fadle eigenequation [5, see Article 23], if one replaces $\mathrm{i}k'$ by k. The smallest roots are $k'c = \pm 1.1254 \pm 2.1062\mathrm{i}$, and $k'c = \pm 1.3843 \pm 3.7488\mathrm{i}$ for the positive and negative signs within Eq. (89), respectively. The slowest decay of self-equilibrated end loading on the end $z = 0$, as anticipated by Saint-Venant's principle, is then according to

$$\exp(kz) = \exp((-2.1062 \pm 1.1254\mathrm{i})z/c). \tag{90}$$

Having determined the possible decay rates from the eigenequation (89), one can back-substitute to determine the initial state (eigen) vector; thus the first row of matrix Eq. (87) may be written as

$$[M'_{31}]_{x=2c}u(0) + [M'_{32}]_{x=2c}w(0) = 0 \tag{91}$$

from which one finds

$$w(0) = \left[\frac{2k'c\cosh 2k'c + \sinh 2k'c}{\mathrm{i}2k'c\sinh 2k'c}\right]u(0). \tag{92}$$

For simplicity, set $u(0) = 1$, when the lower edge state vector may be written as

$$\mathbf{s}'(0) = \left[1 \quad \frac{2k'c\cosh 2k'c + \sinh 2k'c}{\mathrm{i}2k'c\sinh 2k'c} \quad 0 \quad 0\right]^{\mathrm{T}}. \tag{93}$$

The displacement components may then be found from Eq. (86), together with the shearing stress τ_{xz}, while the two direct stresses can be calculated from the displacement components.

10. Conclusion

In this paper, the linear elasticity of a prismatic rod or beam has been considered from a state space point of view. Since elasticity may be regarded as a classical field of study, one might imagine that there are no new insights to be gleaned; indeed there seems little point in deriving from first principles the governing equations, which are already well known, and need only be rearranged into the requisite form. On the other hand, the approach treats displacement and stress components on an equal footing, which obviates the need to employ strain compatibility equations. It unifies the treatment of transmission and decay problems, and benefits from the machinery of an eigenvalue problem; these attributes have previously been shown to provide an elegant resolution of the so-called *wedge paradox*, so it is clear that this alternative viewpoint has some definite advantages.

Employing the machinery of an eigenvalue problem, it has been shown that degenerate modes occur only for the zero (transmitting) eigenvalues—repeating decay eigenvalues cannot lead to a non-trivial Jordan canonical form; thus the non-zero eigenvalue degenerate modes considered by Zhong in Section 4.8.3 of his monograph [26], cannot exist for a prismatic structure.

The transmission modes pertaining to the multiple zero eigenvalue can be completely resolved, subject to determination of the Saint-Venant warping and flexure functions for a specific cross-section; the approach is logical and does not rely on the semi-inverse method.

The decay problem for the plane elastic strip can also be treated very simply using the present approach. Determination of the matrix exponential, using Laplace transformation of the system matrix, provides an elegant way of obtaining the well-known Papkovitch–Fadle eigenequations which describe the decay rates; unexpectedly, this requires a second state space formulation in the *transverse direction*. However, the scope for extension of the state space approach to the decay problem for prismatic rods of other cross-section appears severely limited: exact solutions are known for very few cross-sections, such as the circle, both solid and hollow, and it is not apparent that the present method is capable of replicating even these known results. Overall, the main advantage of the approach is that one can draw on the extensive literature on system theory, and it is likely that this will prove beneficial in the control of dynamic problems of continuum elastic structures.

Appendix

The resolvent matrix $\mathbf{R}$ has elements

$$R_{11} = \frac{s^3 + \nu k^2 s}{(s^2 - k^2)^2},\ R_{12} = \frac{-\mathrm{i}k(s^2 + \nu k^2)}{(s^2 - k^2)^2},\ R_{13} = \frac{(1+\nu)}{E}\frac{[2s^2 + (1-\nu)k^2]}{(s^2 - k^2)^2},$$

$$R_{14} = -\frac{\mathrm{i}k(1+\nu)^2}{E}\frac{s}{(s^2 - k^2)^2},$$

$$R_{21} = -\mathrm{i}k\frac{(\nu s^2 + k^2)}{(s^2 - k^2)^2},\ R_{22} = \frac{s(s^2 - (2+\nu)k^2)}{(s^2 - k^2)^2},\ R_{23} = R_{14},\ R_{24} = \frac{(1+\nu)}{E}\frac{[(1-\nu)s^2 - 2k^2]}{(s^2 - k^2)^2},$$

$$R_{31} = Ek^2\frac{s^2}{(s^2 - k^2)^2},\ R_{32} = -\mathrm{i}Ek^3\frac{s}{(s^2 - k^2)^2},\ R_{33} = R_{11},\ R_{34} = R_{21},$$

$$R_{41} = R_{32},\ R_{42} = \frac{-Ek^4}{(s^2 - k^2)^2},\ R_{43} = R_{12},\ R_{44} = R_{22}. \tag{A.1}$$

The inverse Laplace transform of $\mathbf{R}$ is the matrix exponential $\mathbf{M} = \mathrm{e}^{\mathbf{H}z}$ and has elements

$$M_{11} = \frac{\mathrm{e}^{kz}}{2}\left(1 + \frac{(1+v)}{2}kz\right) + \frac{\mathrm{e}^{-kz}}{2}\left(1 - \frac{(1+v)}{2}kz\right),$$

$$M_{12} = \frac{\mathrm{ie}^{kz}}{4}(-1 + v - (1+v)kz) + \frac{\mathrm{ie}^{-kz}}{4}(1 - v - (1+v)kz),$$

$$M_{13} = \frac{(1+v)\mathrm{e}^{kz}}{4Ek}(3 - v + (1+v)kz) + \frac{(1+v)\mathrm{e}^{-kz}}{4Ek}(-3 + v + (1+v)kz),$$

$$M_{14} = -\frac{\mathrm{i}(1+v)^2kz\mathrm{e}^{kz}}{4Ek} + \frac{\mathrm{i}(1+v)^2kz\mathrm{e}^{-kz}}{4Ek},$$

$$M_{21} = \frac{\mathrm{ie}^{kz}}{4}(1 - v - (1+v)kz) + \frac{\mathrm{ie}^{-kz}}{4}(-1 + v - (1+v)kz),$$

$$M_{22} = \frac{\mathrm{e}^{kz}}{2}\left(1 - \frac{(1+v)}{2}kz\right) + \frac{\mathrm{e}^{-kz}}{2}\left(1 + \frac{(1+v)}{2}kz\right), \quad M_{23} = M_{14},$$

$$M_{24} = \frac{(1+v)\mathrm{e}^{kz}}{4Ek}(3 - v - (1+v)kz) + \frac{(1+v)\mathrm{e}^{-kz}}{4Ek}(-3 + v - (1+v)kz),$$

$$M_{31} = \frac{Ek}{4}\mathrm{e}^{kz}(1 + kz) - \frac{Ek}{4}\mathrm{e}^{-kz}(1 - kz), \quad M_{32} = \frac{-\mathrm{i}Ek}{4}kz\mathrm{e}^{kz} + \frac{\mathrm{i}Ek}{4}kz\mathrm{e}^{-kz}, \quad M_{33} = M_{11}, \quad M_{34} = M_{21},$$

$$M_{41} = M_{32}, \quad M_{42} = \frac{Ek}{4}\mathrm{e}^{kz}(1 - kz) - \frac{Ek}{4}\mathrm{e}^{-kz}(1 + kz), \quad M_{43} = M_{12}, \quad M_{44} = M_{22}. \qquad \text{(A.2)}$$

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