Projection Schemes in Stochastic Finite Element Analysis*

Prasanth B. Nair

Computational Engineering and Design Group,
University of Southampton, Highfield, Southampton, SO17 1BJ, UK.
P.B.Nair@soton.ac.uk

1 Introduction

In traditional computational mechanics, it is often assumed that the physical properties of the system under consideration are deterministic. This assumption of determinism forms the basis of most mathematical modeling procedures used to formulate partial differential equations (PDEs) governing the system response. In practice, however, some degree of uncertainty in characterizing virtually any engineering system is inevitable. In a structural system, deterministic characterization of the system properties and its environment may not be desirable due to several reasons, including uncertainty in the material properties due to statistically inhomogeneous microstructure, variations in nominal geometry due to manufacturing tolerances, and uncertainty in loading due to the nondeterministic nature of the operating environment. These uncertainties can be modeled within a probabilistic framework, which leads to PDEs with random coefficients and associated boundary and initial conditions governing the system dynamics. It is implicitly assumed here that uncertainty in the PDE coefficients can be described by random variables or random fields that are constructed using experimental data or stochastic micromechanical analysis.

The main focus of the area of computational stochastic mechanics is the development of numerical techniques for solving stochastic PDEs. Over the last decade, much progress has been made in combining well known spatial discretization schemes such as finite elements, finite-differences, spectral methods and boundary elements with random field discretization techniques to solve this class of problems. In particular, there is a wide body of work which deals with the application of the finite element method (FEM) in conjunction with random field discretization techniques to solve stochastic PDEs; see, for example, the research monographs by Ghanem and Spanos [1], and Kleiber and Hien [2], and a comprehensive review of the state-of-the-art edited by Schuëller [3].

By combining conventional spatial discretization schemes with random field discretization techniques, it becomes possible to arrive at finite-dimensional approximations of stochastic PDEs as a system of ordinary differential equations (ODEs) with random coefficients. The system of ODEs can be converted into a system of random algebraic equations using a temporal discretization scheme or a frequency domain transform. In the case of steady-state stochastic PDEs, discretization in space and the random dimension directly leads to a system of random algebraic equations. Hence, efficient numerical schemes for solving random ODEs and algebraic equations are essential tools for tackling problems in computational stochastic mechanics.

* Chapter 21 in *CRC Engineering Design Reliability Handbook*, editors: D. M. Ghiocel, E. Nikolaidis and S. Singhal, CRC Press, Boca Raton, FL, 2004.

Given a system of random ODEs or algebraic equations, the Monte Carlo simulation technique or its variants can be readily applied to approximate the response statistics to an arbitrary degree of accuracy [4, 5]. Simulation techniques are general purpose in scope and hence they are applicable to a wide range of complex problems [6]. In practice, however, this is the method of last resort since the attendant computational cost can be prohibitive for systems modeled with high-fidelity. The perturbation method and the Neumann series offer computationally efficient alternatives and have been popularly applied to compute the first two statistical moments of the response quantities; see, for example, [2, 3, 7, 8, 9, 10]. The major drawback of such local approximation techniques is that the results becomes highly inaccurate when the coefficients of variation of the input random variables are increased. The response surface method is another approximation technique which is usually applied to construct linear and quadratic models of the response quantities as a function of the basic random variables [11]. It is also possible to construct more general nonlinear models by leveraging techniques from the function approximation literature [12, 13]. However, such approximation techniques do not scale well to problems with large number of variables due to the *curse of dimensionality.*[2] An overview of some alternatives to perturbation methods can be found in the recent monograph of Elishakoff and Ren [14].

In this chapter, we shall primarily focus on stochastic finite element analysis techniques which use polynomial chaos expansions and stochastic reduced basis representations coupled with projection schemes. We shall henceforth refer to this class of numerical methods as stochastic subspace projection schemes. Such projection schemes can be considered as a rational extension of existing numerical schemes for solving deterministic algebraic and differential equations. This connection allows for the possibility of leveraging and extending existing results in the literature for rigorous mathematical analysis. It is to be noted here that stochastic subspace projection theory is an evolving area of research. Numerical evidence accumulated so far suggests that methods based on this idea offer significantly better accuracy than perturbation methods [1, 15]. Excellent overviews of more traditional approaches to stochastic finite element analysis based on local approximations and recent developments in this area can be found elsewhere in the literature; see, for example, [2, 3, 16, 17] and the references therein. Before delving into details of stochastic subspace projection schemes, we shall first outline the essential ideas used in these formulations.

The idea of using functional expansion techniques to represent stochastic processes was originally proposed by Wiener in 1938 [18]. Ghanem and Spanos [1] leveraged the notion of polynomial chaos (PC) expansions introduced by Wiener to develop a spectral approach for stochastic finite element analysis. Over the last decade, PC projection schemes have been applied to solve a wide variety of problems in computational stochastic mechanics. To illustrate the basic ideas used in functional expansion approaches (PC expansion can be viewed as a special case of this approach), consider the continuous stochastic operator problem

$$\mathcal{L}(\mathbf{x};\theta)u(\mathbf{x};\theta) = f(\mathbf{x};\theta), \tag{1}$$

where $\mathcal{L}(\mathbf{x};\theta)$ is a stochastic differential operator, i.e., a randomly parameterized differential operator. For simplicity of presentation, consider the case when the operator $\mathcal{L}$ is parameterized in terms of a single random variable $\xi(\theta)$.[3] $f(\mathbf{x};\theta)$ is a random function and $u(\mathbf{x};\theta)$ is the random solution process whose statistics are to be computed. Note here that we use the symbol θ to indicate the dependence of any quantity on a random dimension. For example, for each $\mathbf{x} \in \mathbb{R}^d$, $u(\mathbf{x};\theta) : \Theta \to \mathbb{R}$ is a random variable on a suitable probability space $(\Theta, \mathcal{F}, \Gamma)$, where Θ is the set of elementary events, $\mathcal{F}$ is the σ-algebra associated with Θ and Γ is a probability measure.

[2] The curse of dimensionality arises from the fact that the number of hypercubes required to fill out a compact region of a M-dimensional space grows exponentially with M.

[3] For example, $\mathcal{L}(\mathbf{x};\theta) = \xi(\theta)\nabla^2$, where ∇^2 is the Laplacian operator.

The main idea used in functional expansion techniques is to decompose the solution process $u(\mathbf{x};\theta)$ into separable deterministic and stochastic components by the *ansatz* (i.e., assumed form for a function)

$$u(\mathbf{x};\theta) \approx \hat{u}(\mathbf{x};\theta) = \sum_{i=1}^{m} u_i(\mathbf{x})\varphi_i(\xi(\theta)) \tag{2}$$

where $u_i(\mathbf{x}) \in \mathbb{R}$ is an undetermined deterministic function and $\varphi_i(\xi(\theta)) : \Theta \to \mathbb{R}$ is a known stochastic basis function.[4] Henceforth, for notational convenience, we shall not explicitly indicate the dependence of the random variable ξ on the random dimension θ.

It can be seen from Equation (2) that once the stochastic basis functions are chosen, the solution process boils down to computation of the undetermined functions $u_i(\mathbf{x})$, $i = 1, 2, \ldots, m$. Let us now substitute Equation (2) into the governing operator problem to arrive at the following stochastic residual error function

$$\epsilon(\mathbf{x};\theta) = \mathcal{L}(\mathbf{x};\theta)\sum_{i=1}^{m} u_i(\mathbf{x})\varphi_i(\xi) - f(\mathbf{x};\theta). \tag{3}$$

Equations governing the undetermined functions can now be derived by employing a projection scheme along the random dimension θ. Consider the case when the Galerkin projection scheme is employed, where the stochastic residual error is orthogonalized with respect to the approximating space $\varphi_i(\xi)$. In other words, the inner product of $\epsilon(\mathbf{x};\theta)$ with each basis function is set to zero, i.e.,

$$\sum_{i=1}^{m} \langle \varphi_j(\xi)\mathcal{L}(\mathbf{x};\theta)\varphi_i(\xi)\rangle\, u_i(\mathbf{x}) - \langle \varphi_j(\xi) f(\mathbf{x};\theta)\rangle = 0 \quad \forall j = 1, 2, \ldots, m, \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in the Hilbert space of random variables [21], i.e.,

$$\langle f(\theta)g(\theta)\rangle = \int f(\theta)g(\theta)d\Gamma(\theta) \tag{5}$$

Since $\mathcal{L}(\mathbf{x};\theta)$ is a randomly parameterized differential operator, $\langle \varphi_j(\xi)\mathcal{L}(\mathbf{x};\theta)\varphi_i(\xi)\rangle$ is a deterministic differential operator.[5] Further, $\langle \varphi_j(\xi) f(\mathbf{x};\theta)\rangle$ is a deterministic function. Equation (4) therefore represents a set of m coupled deterministic operator problems which govern $u_i(\mathbf{x})$. Hence, by applying a functional expansion of the form given in Equation (2) in conjunction with the Galerkin projection scheme, we have arrived at a system of coupled deterministic operator problems, thereby increasing the dimensionality of the problem. Since Equation (4) is deterministic, it can be readily solved using conventional numerical techniques such as the FEM and $u_i(\mathbf{x})$ can be computed. Subsequently, it becomes possible to efficiently approximate the complete statistics of $u(\mathbf{x},\theta)$ in the postprocessing phase using Equation (2). In summary, a functional expansion approach for solving stochastic operator problems involves four steps: (i) selection of a suitable set of stochastic basis functions $\varphi_i(\xi)$, (ii) application of a projection scheme along the random dimension θ to arrive at a system of coupled equations, (iii) numerical solution of the coupled system of deterministic equations to compute the undetermined functions $u_i(\mathbf{x})$, and (iv) postprocessing to compute the statistics of interest after substituting the results obtained in the earlier step into the expansion in Equation (2).

An alternative approach can be formulated by first spatially discretizing the continuous governing Equation (1) using any conventional technique such as the FEM. This semi-discretization procedure

[4] In the PC projection scheme of Ghanem and Spanos [1], Hermite polynomials are chosen as basis functions.
[5] For example, when $\mathcal{L}(\mathbf{x};\theta) = \xi\nabla^2$, $\langle \varphi_j(\xi)\mathcal{L}(\mathbf{x};\theta)\varphi_i(\xi)\rangle = \langle \varphi_j(\xi)\xi\varphi_i(\xi)\rangle\nabla^2$.

essentially involves treating the discrete nodal values of the field variable $u(\mathbf{x}, \theta)$ as random variables.[6] This process ultimately leads to a system of random algebraic equations of the form given below

$$\mathbf{R}(\mathbf{u}(\xi); \xi) = 0, \tag{6}$$

where $\mathbf{u}(\xi) \in \mathbb{R}^n$ denotes a random vector composed of the values of the discretized field variable (i.e., the vector solution process) whose statistics are to be computed.

It is possible to apply the functional expansion approach outlined earlier to approximate the random vector $\mathbf{u}(\xi)$ in the following form

$$\hat{\mathbf{u}}(\xi) = \sum_{i=1}^{m} \mathbf{u}_i \varphi_i(\xi), \tag{7}$$

where $\mathbf{u}_i \in \mathbb{R}^n, i = 1, 2, \ldots, m$ are vectors of undetermined coefficients. These unknown vectors can be uniquely computed by substituting Equation (7) into Equation (6) and applying a Galerkin projection scheme along the random dimension θ. This results in the following system of deterministic algebraic equations with increased dimensionality (mn unknowns in comparison to Equation (6) which has only n unknowns)

$$\left\langle \varphi_j(\xi) \mathbf{R} \left(\sum_{i=1}^{m} \mathbf{u}_i \varphi_i(\xi); \xi \right) \right\rangle = 0, \quad \forall j = 1, 2, \ldots, m \tag{8}$$

It can be shown that the preceding system of algebraic equations is equivalent to a spatially discretized version of Equation (4). In other words, the functional expansion approach can be applied either to the continuous form of the stochastic operator equation or its spatially discretized version.

However, since we are dealing here with a discretized problem where it is desired to approximate the random vector $\mathbf{u}(\xi)$ and not the random function $u(\mathbf{x}; \theta)$, it appears more natural to use a stochastic reduced basis approximation of the form

$$\hat{\mathbf{u}}(\xi) = \alpha_1 \boldsymbol{\psi}_1(\xi) + \alpha_2 \boldsymbol{\psi}_2(\xi) + \ldots + \alpha_m \boldsymbol{\psi}_m(\xi) = \boldsymbol{\Psi}(\xi) \boldsymbol{\alpha}, \tag{9}$$

where $\boldsymbol{\Psi}(\xi) = [\boldsymbol{\psi}_1(\xi), \boldsymbol{\psi}_2(\xi), \cdots, \boldsymbol{\psi}_m(\xi)] \in \mathbb{R}^{n \times m}$ denotes a matrix of known stochastic basis vectors and $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \in \mathbb{R}^m$ is a vector of undetermined coefficients. Note that the above representation has only m unknowns, whereas Equation (7) has a total of mn unknowns.

The main idea used here is to employ a rich set of (problem dependant) stochastic basis vectors which ensure that accurate approximations can be obtained for $m <<< n$. This idea of stochastic reduced basis representations was introduced recently in the literature [20, 15] in the context of solving large-scale linear random algebraic system of equations obtained from semi-discretization of stochastic PDEs. In contrast to the functional expansion scheme (where PC basis functions are typically used), the solution process is represented using basis vectors spanning the preconditioned stochastic Krylov subspace.

Substituting Equation (9) into Equation (6) and applying the Galerkin projection scheme, we arrive at the following reduced-order deterministic system of equations to be solved for the m unknown coefficients, $\alpha_1, \alpha_2, \ldots, \alpha_m$

$$\langle \boldsymbol{\Psi}^*(\xi) \mathbf{R}(\boldsymbol{\Psi}(\xi) \boldsymbol{\alpha}; \xi) \rangle = 0, \tag{10}$$

where the superscript * is used to denote the complex conjugate transpose of a vector or matrix (if it is complex), or the transpose (if it is real).

[6] In the context of the FEM applied to static problems in structural mechanics, this means that the nodal displacements are treated as random variables. We shall cover this in more detail in the next section.

Comparing Equation (8) with Equation (10), it can be observed that a key advantage of stochastic reduced basis representations is that the undetermined quantities can be efficiently computed compared to the functional expansion approach outlined earlier. This is because, in the stochastic reduced basis approach, application of the Galerkin projection scheme leads to a reduced-order system of equations. In contrast, the functional expansion scheme leads to system of equations with increased dimensionality. Clearly, the success of this reduced basis approach critically hinges on the choice of stochastic basis vectors. As we shall show later, using basis vectors spanning the preconditioned stochastic Krylov subspace, highly accurate results can be obtained using only a few basis vectors.

In this chapter, we present the theoretical foundations of projection schemes which employ the PC expansion and the preconditioned stochastic Krylov subspace to approximate the solution of stochastic PDEs. The remainder of this chapter is organized as follows. In the next section we outline the steps involved in stochastic finite element analysis of random media. Section 3 presents the generalized PC expansion scheme which can be applied to represent the solution of stochastic PDEs. Section 4 outlines how the PC expansion scheme can be applied in conjunction with Galerkin projection to stochastic finite element analysis. In Section 5, we present the stochastic Krylov subspace as an alternative to PC expansions and outline some of its theoretical properties. Section 6 presents a number of stochastic reduced basis projection schemes which employ basis vectors spanning the preconditioned stochastic Krylov subspace for solving random algebraic equations. Section 7 outlines some procedures for post-processing the solutions obtained using the projection schemes in order to compute the response statistical moments and conduct reliability assessment studies. Section 8 contains numerical studies on two-dimensional elasticity problems and the relative performance of various projection schemes are compared. Section 9 concludes the chapter and outlines some areas for further investigation.

2 Finite Element Formulations for Random Media

To illustrate the basic steps involved in stochastic finite element analysis of random media, consider a two-dimensional isotropic solid whose Youngs modulus is modeled as a random field, say $h(\mathbf{x};\theta)$. In other words, for each $\mathbf{x} \in \mathbb{R}^2$, h: $\Theta \rightarrow \mathbb{R}$ is a random variable. Since the Youngs modulus is represented by a random field, the elasticity matrix becomes a function of the spatial coordinates and a random dimension, i.e.,

$$\mathbf{D}(\mathbf{x};\theta) = h(\mathbf{x};\theta)\mathbf{D}_0, \tag{11}$$

where $\mathbf{D}_0$ is the deterministic part of the elasticity matrix.

From a practical viewpoint, the elastic properties of a solid may be random due to the intrinsic stochastic inhomogeneity of the microstructure. Such situations may arise, for example, when studying the mechanical behavior of cellular solids such as metallic foams and bone [22], and granular media such as sand and soil [23]. In such cases, it is not sufficient to model the Youngs modulus alone as a random field since that may lead to underestimation of the response variability [24]. A more systematic approach would be to use stochastic homogenization techniques to derive random field models for the terms of the elasticity matrix; see, for example, [24, 25, 26]. Such models can be readily accommodated within the stochastic FEM. Further, it is also possible to include uncertainty in geometric variables, boundary conditions and external loading in the formulation.

2.1 Random Field Discretization

In order to apply the FEM to problems wherein one or more of the physical quantities are modeled as random fields, we need to represent them first by an enumerable set of random variables. For a

detailed exposition of random field modeling, the reader is referred to the chapter by VanMarcke in this handbook. Various discretization techniques are available in the literature for approximating random fields including the mid-point method, shape function methods, optimal linear estimation, weighted integral methods, orthogonal series expansion and the Karhunen-Loève (KL) expansion scheme; see, for example, [31, 28, 30].

Let the correlation function of the random field $h(\mathbf{x};\theta)$ be $R_h(\mathbf{x},\mathbf{y})$. Then, a discretized version of the random field $h(\mathbf{x};\theta)$ can be written in the general form

$$h(\mathbf{x};\theta) = \langle h(\mathbf{x};\theta)\rangle + \sum_{i=1}^{\infty} \xi_i(\theta) h_i(\mathbf{x}), \tag{12}$$

where $\xi_i(\theta), i = 1, 2, \ldots, \infty$ are a set of uncorrelated random variables and $h_i(\mathbf{x})$, $i = 1, 2, \ldots, \infty$ are a set of basis functions used in the random field discretization procedure. $\langle h(\mathbf{x};\theta)\rangle$ denotes the mean of the random field.

When the KL expansion scheme is employed, the ith basis function can be written as $h_i(\mathbf{x}) = \sqrt{\lambda_i}\kappa_i(\mathbf{x})$, where λ_i and $\kappa_i(x)$ are the eigenvalues and the eigenfunctions, respectively, of a Fredholm integral equation of the second kind given below

$$\int_{\mathcal{D}} R_h(\mathbf{x},\mathbf{y})\kappa_i(\mathbf{x})d\mathbf{x} = \lambda_i\kappa_i(\mathbf{y}). \tag{13}$$

Analytical solutions of the above integral eigenvalue problem can be obtained only for a special class of correlation functions (e.g., the exponential correlation function) defined on geometrically simple domains. For more general cases, numerical discretization schemes have to be employed to compute the eigenvalues and eigenfunctions of $R_h(\mathbf{x},\mathbf{y})$; see, for example, [1, 29, 30].

In order to implement the random field discretization scheme computationally, we truncate Equation (12) at the M-th term to arrive at the finite-dimensional approximation

$$h(\mathbf{x};\theta) \approx \langle h(\mathbf{x};\theta)\rangle + \sum_{i=1}^{M} \xi_i(\theta) h_i(\mathbf{x}). \tag{14}$$

If the eigenvalues of the covariance function $R_h(\mathbf{x},\mathbf{y})$ decay rapidly, then only a few number of terms will be required to ensure an accurate representation of the random field. In the limiting case, when the correlation length of the random field field tends to zero, the number of terms M will grow very rapidly towards infinity. In summary, the correlation length of the random field dictates the number of terms (M) required to ensure an accurate finite-dimensional representation.

2.2 Spatial Discretization

Substitution of the discretized random field representation into Equation (11) results in a representation of the elasticity matrix in terms of a finite number of random variables. This sets the stage for the application of the FEM to spatially discretize the governing equations. The starting point for the FEM is the weak form of the governing equations, which is obtained by multiplying the governing equation by a test function and integrating by parts; a detailed overview may be found in any standard text [27]. Subsequently, the domain is divided into a number of elements and the field variables are approximated within each element using a set of shape functions as

$$\tilde{u}(\mathbf{x}) = \sum_{i=1}^{n_e} u_i(\theta) N_i(\mathbf{x}), \tag{15}$$

where N_i denotes the ith shape function and $u_i(\theta)$ can be interpreted as a generalized field variable.

Substituting the above approximation into the weak form of the governing equations, we arrive at expressions for the element stiffness and mass matrices. For the case of linear structural systems, assembly of the element stiffness, mass and damping matrices lead to a system of coupled ODEs with random coefficients of the following form

$$\mathbf{M}(\boldsymbol{\xi})\frac{d^2}{dt^2}\mathbf{u}(\boldsymbol{\xi},t) + \mathbf{C}(\boldsymbol{\xi})\frac{d}{dt}\mathbf{u}(\boldsymbol{\xi},t) + \mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi},t) = \mathbf{f}(\theta,t), \tag{16}$$

where $\mathbf{M}(\boldsymbol{\xi}), \mathbf{C}(\boldsymbol{\xi})$ and $\mathbf{K}(\boldsymbol{\xi}) \in \mathbb{R}^{n\times n}$ denote the system mass, damping and stiffness matrices, respectively. $\mathbf{f}(\theta,t) \in \mathbb{R}^n$ denotes the generalized force vector which can be either deterministic or random, where $t \in \mathbb{R}^+$ refers to time. $\mathbf{u}(\boldsymbol{\xi},t) \in \mathbb{R}^n$ is the random displacement vector whose statistics are to be computed. $\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_M\} \in \mathbb{R}^M$ denotes the set of random variables arising from discretization of the random field representing uncertainty in the Youngs modulus and n is the total number of degrees of freedom (dof).

To illustrate how the coefficient matrices in Equation (16) are computed, consider the case of static response analysis of a two-dimensional elastic solid with random Youngs modulus subject to deterministic loading.

Here, the element stiffness matrix can be written as

$$k^e = \int_{\mathcal{D}_e} \mathbf{B}^T\mathbf{D}(\mathbf{x};\theta)\mathbf{B}d\mathbf{x}, \tag{17}$$

where $\mathbf{B}$ is the strain-displacement matrix and $\mathcal{D}_e$ denotes the domain of the element.

Substituting the discretized version of the elasticity matrix $\mathbf{D}(\mathbf{x};\theta)$ in the preceding equation, we arrive at the following expression for the stochastic element stiffness matrix

$$k^e(\boldsymbol{\theta}) = k_0^e + \sum_{j=1}^{M} k_j^e \xi_j, \tag{18}$$

where

$$k_0^e = \int_{\mathcal{D}_e} \langle h(\mathbf{x},\theta)\rangle \mathbf{B}^T\mathbf{D}_0\mathbf{B}d\mathbf{x}, \tag{19}$$

and

$$k_j^e = \int_{\mathcal{D}_e} h_j(\mathbf{x})\mathbf{B}^T\mathbf{D}_0\mathbf{B}d\mathbf{x}. \tag{20}$$

Standard numerical quadrature schemes can be used to evaluate the integrals in Equations (19) and (20). A detailed discussion of these implementation issues can be found in the literature [1, 31]. Assembly of the element stiffness matrices and application of the specified boundary conditions result in the following system of linear random algebraic equations

$$\left(\mathbf{K}_0 + \sum_{i=1}^{M} \mathbf{K}_i\xi_i\right)\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}, \tag{21}$$

where $\mathbf{K}_0 \in \mathbb{R}^{n\times n}$ and $\mathbf{K}_i \in \mathbb{R}^{n\times n}$ are deterministic matrices and $\mathbf{u}(\boldsymbol{\xi}) \in \mathbb{R}^n$ is the random displacement vector. $\mathbf{f} \in \mathbb{R}^n$ denotes the force vector which we assume to be deterministic for simplicity of presentation. The preceding equation can be rewritten as

$$\left(\sum_{i=0}^{M} \mathbf{K}_i \xi_i\right) \mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}, \tag{22}$$

where $\xi_0 = 1$.

A similar set of equations can also be arrived at, for time-dependent problems, by applying a time-stepping scheme or a frequency domain transform to Equation (16). In the case of linear structural systems, the equations of motion in the frequency domain is a system of complex linear random algebraic equations of the form

$$\left[\mathbf{K}(\boldsymbol{\xi}) - \omega^2 \mathbf{M}(\boldsymbol{\xi}) + \Im\omega \mathbf{C}(\boldsymbol{\xi})\right] \mathbf{u}(\boldsymbol{\xi}, \omega) = \mathbf{f}, \tag{23}$$

where ω is the frequency of excitation and $\Im = \sqrt{-1}$.

It is to be noted here that Equation (22) is strictly valid only when the stiffness matrix is a linear function of $\boldsymbol{\xi}$. For the more general case when uncertainties exist in the material properties as well as the geometric parameters of the system, $\mathbf{K}(\boldsymbol{\xi})$ will not be a linear function of $\boldsymbol{\xi}$. In the sections that follow, we shall outline how the projection schemes developed for Equation (22) can be extended to tackle such general cases.

3 Polynomial Chaos Expansions

The idea of polynomial chaos (PC) representations of stochastic processes was introduced by Wiener [18, 19] as a generalization of Fourier series expansion. More specifically, Wiener used multidimensional Hermite polynomials as basis functions for representing stochastic processes. The basic idea is to project the process under consideration onto a stochastic subspace spanned by a set of complete orthogonal random polynomials. To illustrate the process of constructing PC expansions,

let $\phi_i(\theta), i = 1, 2, \ldots, \infty$ denote a set of polynomials which form an orthogonal basis in $L_2(\Theta, \mathcal{F}, \Gamma)$. Then, a general second-order stochastic process (i.e., a process with finite variance) $h(\theta)$ can be represented as

$$\begin{aligned} h(\theta) = c_0 \Phi_0 + \sum_{i_1=1}^{\infty} c_{i_1} \Phi_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} c_{i_1 i_2} \Phi_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) \\ + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} c_{i_1 i_2 i_3} \Phi_3(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)) + \cdots, \end{aligned} \tag{24}$$

where $\Phi_p(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_p})$ denote the generalized polynomial chaos of order p, which is a tensor product of one-dimensional polynomial basis functions $\phi_i, i = 1, 2, \ldots, p$.

In the original work of Wiener [18], Φ_p is chosen to be a multidimensional Hermite polynomial in terms of a set of uncorrelated Gaussian random variables $\xi_1, \xi_2, \ldots, \xi_p$ which have zero mean and unit variance. The general expression for the Hermite chaos of order p can be written as

$$\Phi_p(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_p}) = (-1)^p \mathrm{e}^{\frac{1}{2}\boldsymbol{\xi}^*\boldsymbol{\xi}} \frac{\partial^p}{\partial \xi_{i_1} \ldots \partial \xi_{i_p}} \left[\mathrm{e}^{-\frac{1}{2}\boldsymbol{\xi}^*\boldsymbol{\xi}}\right]. \tag{25}$$

For example, if Hermite polynomials are used as basis functions, a second-order two-dimensional PC expansion of $h(\theta)$ can be written as

$$h(\theta) = h_0 + h_1 \xi_1 + h_2 \xi_2 + h_3(\xi_1^2 - 1) + h_4 \xi_1 \xi_2 + h_5(\xi_2^2 - 1). \tag{26}$$

It can be seen from the above equation that the first term of the PC expansion represents the mean value of $h(\theta)$ since ξ_1 and ξ_2 are uncorrelated Gaussian random variables with zero-mean and unit variance. Another point worth noting here is that the number of terms in the expansion grows very quickly with the dimension of $\boldsymbol{\xi}$ and the order of the expansion.

More recently, Xiu and Karniadakis [32] proposed a generalized PC approach which employs basis functions from the Askey family of orthogonal polynomials, which form a complete basis in the Hilbert space. The Hermite chaos expansion appears as a special case in this generalized approach which is referred to as Wiener-Askey chaos. The motivation for this generalization arises from the observation that the convergence of Hermite chaos expansions can be far from optimal for non-Gaussian inputs. In such cases, the convergence rate can be improved by replacing Hermite polynomials with other orthogonal polynomials that best represent the input. Table 1 shows alternative basis functions suitable for different distributions of $\boldsymbol{\xi}$. Numerical studies that demonstrate the improvement in convergence due to the use of generalized PC expansions can be found in the literature [32, 33].

Table 1. Choice of Askey polynomials for different random inputs.

Random variables ($\boldsymbol{\xi}$)	Weiner-Askey Chaos ($\varphi(\boldsymbol{\xi})$)	Support
Gaussian	Hermite-Chaos	$(-\infty, \infty)$
Gamma	Laguerre-Chaos	$[0, \infty]$
Beta	Jacobi-Chaos	$[a, b]$
Uniform	Legendre-Chaos	$[a, b]$

For notational convenience, Equation (24) can be rewritten as

$$h(\theta) = \sum_{i=0}^{\infty} h_i \varphi_i(\boldsymbol{\xi}), \tag{27}$$

where there is a one-to-one correspondence between the functions $\Phi_p(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_p})$ and $\varphi_i(\boldsymbol{\xi})$. Also note here that $\varphi_0 = 1$ and $\langle \varphi_i \rangle = 0$ for $i > 0$. Since, $\varphi_i(\boldsymbol{\xi}), i = 0, 1, 2, \ldots, \infty$ form an orthogonal basis in $L_2(\Theta, \mathcal{F}, \Gamma)$

$$\langle \varphi_i(\boldsymbol{\xi}) \varphi_j(\boldsymbol{\xi}) \rangle = \langle \varphi_i^2(\boldsymbol{\xi}) \rangle \delta_{ij}, \tag{28}$$

where δ_{ij} is the Kronecker delta operator, and $\langle . \rangle$ is the ensemble average operator, i.e.,

$$\langle f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) \rangle = \int f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) W(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{29}$$

where $W(\boldsymbol{\xi})$ is the weight function corresponding to the PC basis.

The weight function is chosen to correspond to the distribution of the elements of $\boldsymbol{\xi}$; see Table 1. For example, when Hermite polynomials are used as basis functions, the weight function is given by the M-dimensional normal distribution

$$W(\boldsymbol{\xi}) = \frac{1}{\sqrt{(2\pi)^M}} \mathrm{e}^{-\frac{1}{2}\boldsymbol{\xi}^* \boldsymbol{\xi}}. \tag{30}$$

Cameron and Martin [34] proved the following result which guarantees that the Hermite chaos expansion converges in a mean square sense for any second-order stochastic process when the number of terms is increased.

Theorem 1: *The Hermite chaos expansion of any (real or complex) functional $h(\boldsymbol{\xi})$ of $L_2(\Theta)$ converges in the $L_2(\Theta)$ sense to $h(\boldsymbol{\xi})$. This means that if $h(\boldsymbol{\xi})$ is a second-order stochastic process, i.e.,*

$$\int |h(\boldsymbol{\xi})|^2 W(\boldsymbol{\xi}) d\boldsymbol{\xi} \; < \; \infty \tag{31}$$

then

$$\int |h(\boldsymbol{\xi}) - \sum_{i=0}^{m} h_i \varphi_i(\boldsymbol{\xi})|^2 W(\boldsymbol{\xi}) d\boldsymbol{\xi} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty, \tag{32}$$

where h_i is the Fourier-Hermite coefficient

$$h_i = \int h(\boldsymbol{\xi}) \varphi_i(\boldsymbol{\xi}) W(\boldsymbol{\xi}) d\boldsymbol{\xi} \tag{33}$$

The Cameron-Martin theorem can be generalized to arrive at the result that expansion of any second-order stochastic process in terms of basis functions from the Weiner-Askey family converges in the L_2 sense [32]. Another standard fact which has been noted earlier in the literature is that the convergence rate of PC expansions is faster than exponential. Further, it has also been shown that the error in the expansion decays as $\mathcal{O}(\frac{1}{(p+1)!})$, where p is the highest order of Hermite polynomials used in the basis. More specifically, Hou et al. [35] presented the following convergence estimate for a one-dimensional PC expansion

$$||h(\xi) - \sum_{i=0}^{m} h_i \varphi_i(\xi)|| \leq \frac{C}{(m+1)!} || \frac{\partial^{m+1} h}{\partial \xi^{m+1}} ||, \tag{34}$$

where C is a constant.

The above estimate can be extended to the case when multidimensional Hermite polynomials are used as basis functions.

4 Polynomial Chaos Projection Schemes

In this section, we outline a weak Galerkin projection scheme which can be used in conjunction with a PC expansion of the response process to solve stochastic PDEs. We consider the case when the stochastic projection scheme is applied to the spatially discretized version of the governing equations. As outlined in the introduction, we can also apply a stochastic projection scheme directly to the continuous form of the governing equations. However, both these approaches ultimately lead to similar sets of equations.

Consider the two-dimensional elasticity problem described in Section 2, where semi-discretization of the governing equations leads to the system of linear random algebraic equations given in Equation (22). In the PC projection scheme of Ghanem and Spanos [1], the random nodal displacements are first expanded using a set of multidimensional Hermite polynomials. This results in the following expansion for the response process

$$\mathbf{u}(\boldsymbol{\xi}) = \sum_{i=0}^{P-1} \mathbf{u}_i \varphi_i(\boldsymbol{\xi}), \tag{35}$$

where $\mathbf{u}_i \in \mathbb{R}^n, i = 0, 1, 2, \ldots, P-1$ are sets of vectors formed from the undetermined coefficients in the PC expansions for each nodal displacement, and $\varphi_i(\boldsymbol{\xi})$ is a set of orthogonal Hermite polynomials in $\boldsymbol{\xi}$. The number of terms in the expansion, P, is given by

$$P = \sum_{k=0}^{p} \frac{(M+k-1)!}{k!(M-1)!}, \tag{36}$$

where p is called the order of the PC expansion, i.e., the highest order of the set of Hermite polynomials $\varphi_i(\boldsymbol{\xi})$.

Substitution of the PC expansion for $\mathbf{u}(\boldsymbol{\xi})$ into the governing random algebraic equations given in Equation (22) gives

$$\left(\sum_{i=0}^{M} \mathbf{K}_i \xi_i\right) \left(\sum_{j=0}^{P-1} \mathbf{u}_j \varphi_j(\boldsymbol{\xi})\right) = \mathbf{f}. \tag{37}$$

As shown by Ghanem and Spanos [1], the undetermined terms in the PC expansion can be uniquely computed by imposing the Galerkin condition, which involves orthogonalizing the stochastic residual error to the approximating subspace as shown below

$$\langle \boldsymbol{\epsilon}(\boldsymbol{\xi}), \varphi_k(\boldsymbol{\xi}) \rangle = 0, \quad k = 0, 1, 2, \ldots, P-1, \tag{38}$$

where the stochastic residual error vector $\boldsymbol{\epsilon}(\boldsymbol{\xi}) \in \mathbb{R}^n$ is given by

$$\boldsymbol{\epsilon}(\boldsymbol{\xi}) = \left(\sum_{i=0}^{M} \mathbf{K}_i \xi_i\right) \left(\sum_{j=0}^{P-1} \mathbf{u}_j \varphi_j(\boldsymbol{\xi})\right) - \mathbf{f}. \tag{39}$$

Substituting Equation (39) in Equation (38), we arrive at the following system of deterministic equations

$$\sum_{i=0}^{M} \sum_{j=0}^{P-1} \mathbf{K}_i \mathbf{u}_j \langle \xi_i \varphi_j \varphi_k \rangle = \langle \varphi_k \mathbf{f} \rangle \qquad k = 0, 1, 2, \ldots, P-1. \tag{40}$$

The above equation can be rewritten in a more compact fashion as

$$\sum_{j=0}^{P-1} \mathbf{K}_{jk} \mathbf{u}_j = \mathbf{f}_k \qquad k = 0, ..., P-1, \tag{41}$$

where

$$\mathbf{K}_{jk} = \sum_{i=0}^{M} \langle \xi_i \varphi_j \varphi_k \rangle \mathbf{K}_i \in \mathbb{R}^{n \times n} \tag{42}$$

and

$$\mathbf{f}_k = \langle \varphi_k \mathbf{f} \rangle \in \mathbb{R}^n. \tag{43}$$

The expectation operations in Equations (42,43) can be readily carried out using the properties of Hermite chaos; see, for example, [1, 31]. Now, expanding the above equation about the subscripts j and k, we arrive at the following system of linear algebraic equations

$$\begin{bmatrix} \mathbf{K}_{0,0} & \mathbf{K}_{0,1} & . & \mathbf{K}_{0,P-1} \\ \mathbf{K}_{1,0} & \mathbf{K}_{1,1} & . & \mathbf{K}_{1,P-1} \\ . & . & . & . \\ . & . & . & . \\ \mathbf{K}_{P-1,0} & \mathbf{K}_{P-1,1} & . & \mathbf{K}_{P-1,P-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ . \\ . \\ \mathbf{u}_{P-1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ . \\ . \\ \mathbf{f}_{P-1} \end{bmatrix}, \tag{44}$$

which is of the form $\tilde{\mathbf{K}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}$, where $\tilde{\mathbf{K}} \in \mathbb{R}^{nP \times nP}$ and $\tilde{\mathbf{u}}, \tilde{\mathbf{f}} \in \mathbb{R}^{nP}$.

Table 2 shows the values of P for different values of p (order of the polynomial chaos) and M (number of terms in the random field discretization). It can be seen that the computational complexity and memory requirements of the PC projection scheme grow rapidly when M and p are increased. The memory requirements can be reduced by precomputing and storing the ensemble average terms of the form $\langle \xi_i \varphi_j \varphi_k \rangle$ along with the matrices $\mathbf{K}_i, i = 0, 1, 2, \ldots, P-1$, instead of storing the matrices $\mathbf{K}_{jk}$ given in Equation (42). Further, the sparsity of the tensor products $\langle \xi_i \varphi_j \varphi_k \rangle$ can also be exploited to accelerate the computations. A detailed overview of numerical schemes which exploit the peculiar structure of Equation (44) can be found in the literature [36, 37]. Anders and Hori [38] proposed the idea of using block Jacobi iteration to solve Equation (44). Here, it is only required to factorize the diagonal blocks of the coefficient matrix $\tilde{\mathbf{K}}$. An attractive feature of this approach is that the solution procedure can be easily parallelized.

Table 2. Values of P for different values of M and p.

M	Order of PC(p)				
	0	1	2	3	4
2	1	3	6	10	15
4	1	5	15	35	70
6	1	7	28	83	210

After solving Equation (44) and substituting the results into Equation (35), we arrive at an explicit expression for the response process. This enables the statistics of the displacements as well as other response quantities of interest to be efficiently computed in the post-processing phase. A more detailed discussion of issues involved in postprocessing the final solution will be presented later.

4.1 Nonlinear Dependence of K on ξ

In this section, we consider the case when the stiffness matrix is a nonlinear function of the basic random variables. This can occur when the Youngs modulus is described by a non-Gaussian distribution or when the geometrical variables are assumed to be uncertain. First, consider the case when uncertainty in the Youngs modulus is described using a lognormal random field, i.e., the Youngs modulus is given by

$$E(\mathbf{x}; \theta) = \exp\left(h(\mathbf{x}; \theta)\right), \tag{45}$$

where $h(\mathbf{x}; \theta)$ is a Gaussian random field.

Discretization of the random field $h(\mathbf{x}; \theta)$ results in the following representation of $E(\mathbf{x}; \theta)$ in terms of a finite number of random variables

$$E(\mathbf{x}; \boldsymbol{\xi}) = \exp\left(\langle h(\mathbf{x}; \theta) \rangle + \sum_{i=1}^{M} \xi_i h_i(\mathbf{x})\right), \tag{46}$$

where $\xi_i, i = 1, 2, \ldots, M$ are a set of uncorrelated Gaussian random variables.

Since $E(\mathbf{x}; \boldsymbol{\xi})$ is a random function, it admits a PC decomposition of the form

$$E(\mathbf{x}; \boldsymbol{\xi}) = \sum_{j=0}^{N} E_i \varphi_i(\boldsymbol{\xi}), \tag{47}$$

where φ_i are multidimensional Hermite polynomials in $\xi_1, \xi_2, \ldots, \xi_M$, and E_i are expansion coefficients.

The expansion coefficients E_i can be computed analytically using the properties of Hermite polynomials [31, 39]. However, note here that there are two levels of approximations. Firstly, the expansion of the random field $h(\mathbf{x};\theta)$ is truncated at the M term. Secondly, only the first N terms in the PC expansion of $E(\mathbf{x};\boldsymbol{\xi})$ are retained.

Using Equation (47), we can now derive an expression for the element stiffness matrix along the lines of the procedure outlined earlier in Section 2.3. Hence, the governing linear random algebraic equations presented earlier in Equations (22) can be rewritten in the more general form

$$\left(\sum_{i=0}^{N} \varphi_i(\boldsymbol{\xi})\mathbf{K}_i\right)\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}. \tag{48}$$

Equation (48) can be readily solved using the PC projection scheme presented earlier. The only major difference is that ensemble averages of the form $\langle \xi_i \varphi_j \varphi_k \rangle$ have to be replaced with $\langle \varphi_i \varphi_j \varphi_k \rangle$. For example, Equation (40) now becomes

$$\sum_{i=0}^{N}\sum_{j=0}^{P-1} \mathbf{K}_i \mathbf{u}_j \langle \varphi_i \varphi_j \varphi_k \rangle = \langle \varphi_k \mathbf{f} \rangle \qquad k = 0, 1, 2, \ldots, P-1. \tag{49}$$

A similar approach can be employed when the stiffness matrix is a general nonlinear function of the random variable vector $\boldsymbol{\xi}$. Here, we need to first compute the PC decomposition of $\mathbf{K}(\boldsymbol{\xi})$ to arrive at a random algebraic system of equations of the form given in Equation (48).

4.2 Remarks

Over the last decade, the PC projection scheme has been successfully applied to solve a wide range of problems in stochastic mechanics, including elasticity problems [1], random vibration [40], soil mechanics [41], transport process in heterogeneous media [42], plasticity problems [43, 38], soil-structure interaction problems [44], fluid dynamics [45], mid-frequency structural dynamics [46], and wave propagation in random media [47]. The Weiner-Askey chaos proposed by Xiu and Karniadakis [32] has recently been applied to solve diffusion problems [54], fluid-structures interaction problems [55], the Navier-Stokes equations [56], and heat transfer problems [57] in the presence of parameter uncertainty. For a more detailed overview of the theoretical foundations of PC projection schemes and related implementation issues, the reader is referred to [48, 50, 49, 51, 53].

More recently, Mathelin and Hussaini [58] presented a stochastic collocation approach which can be employed in conjunction with a PC expansion of the solution process. The main idea is to apply a collocation method in order to collapse the multidimensional summations which appear in the standard Galerkin scheme into a one-dimensional summation. Numerical studies were presented for fluid flow problems to show that the collocation approach is computationally more efficient than the Galerkin projection scheme.

5 The Stochastic Krylov Subspace

In this section, we present an alternative approach where the solution of Equation (22) is approximated using a stochastic reduced basis representation. The idea of using a stochastic reduced basis representation to solve linear random algebraic equations was proposed recently in Nair [20] and Nair and Keane

[15]. It was shown that highly accurate approximations for the response process can be computed using a set of basis vectors spanning the preconditioned stochastic Krylov subspace. This approach is essentially a stochastic generalization of Krylov subspace methods in the numerical linear algebra literature which have been popularly applied to solve large-scale deterministic linear algebraic equations; see, for example, the text by Saad [60] for a detailed exposition. It also of interest to note that numerical methods based on the Krylov subspace have a history of nearly 50 years of existence, and they continue to be an area of extensive research; see Saad and Van der Vorst [61] for a historical overview.

In the context of linear random algebraic equations, the main idea used here is to approximate the response process $\mathbf{u}(\boldsymbol{\xi})$ using basis vectors spanning the stochastic Krylov subspace defined below

$$\mathcal{K}_m\left(\mathbf{K}(\boldsymbol{\xi}),\mathbf{f}\right) = \text{span}\left\{\mathbf{f},\mathbf{K}(\boldsymbol{\xi})\mathbf{f},\mathbf{K}(\boldsymbol{\xi})^2\mathbf{f},\cdots,\mathbf{K}(\boldsymbol{\xi})^{m-1}\mathbf{f}\right\}. \tag{50}$$

This representation of the response process can be justified by the following theorem [15] which establishes the optimality of the stochastic Krylov subspace for solving Equation (22).

Theorem 2: *If the minimal random polynomial of a nonsingular random square matrix* $\mathbf{K}(\boldsymbol{\xi})$ *has degree* m*, then the solution to* $\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}$ *lies in the stochastic Krylov subspace* $\mathcal{K}_m(\mathbf{K}(\boldsymbol{\xi}),\mathbf{f})$.

The degree of the minimal polynomial (m) of a random matrix depends on the distribution of its eigenvalues. More specifically, the number of basis vectors required to compute accurate approximations depends on the degree of overlap of the pdfs of the eigenvalues of the coefficient matrix $\mathbf{K}(\boldsymbol{\xi})$ [15]. To ensure good approximations using a small number of basis vectors, it is preferable to use a preconditioner. In other words, we premultiply both sides of Equation (22) with the preconditioner $\mathbf{P} \in \mathbb{R}^{n\times n}$ to arrive at the following system of equations

$$\left(\sum_{i=0}^{M}\mathbf{P}\mathbf{K}_i\xi_i\right)\mathbf{u}(\boldsymbol{\xi}) = \mathbf{P}\mathbf{f}, \tag{51}$$

The key idea here is to choose a matrix $\mathbf{P}$ such that the pdfs of the eigenvalues of the random matrix $\mathbf{P}\sum_{i=0}^{M}\mathbf{K}_i\xi_i$ numerically tend to have a high degree of overlap. In [20, 15], the deterministic matrix $\langle\mathbf{K}(\boldsymbol{\xi})\rangle^{-1} = \mathbf{K}_0^{-1}$ is used as the preconditioner.[7] This choice is motivated by the observation that $\mathbf{K}_0^{-1}\mathbf{K}(\boldsymbol{\xi})$ will numerically behave like a matrix with a small number of distinct eigenvalues, particularly when the coefficients of variation of $\xi_i, i = 1,2,\ldots,M$ are small. Note that, in theory, convergence can be guaranteed as long as the preconditioner is invertible. However, by using the preconditioner suggested here, convergence can be significantly accelerated - in other words, it becomes possible to achieve high accuracy using around three to four basis vectors.

A stochastic reduced basis representation of the response process can be written as

$$\hat{\mathbf{u}}(\boldsymbol{\xi}) = \alpha_1\boldsymbol{\psi}_1(\boldsymbol{\xi}) + \alpha_2\boldsymbol{\psi}_2(\boldsymbol{\xi}) + \cdots + \alpha_m\boldsymbol{\psi}_m(\boldsymbol{\xi}) = \boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha}, \tag{52}$$

where $\boldsymbol{\Psi}(\boldsymbol{\xi}) = \{\boldsymbol{\psi}_1(\boldsymbol{\xi}),\boldsymbol{\psi}_2(\boldsymbol{\xi}),\ldots,\boldsymbol{\psi}_m(\boldsymbol{\xi})\} \in \mathbb{R}^{n\times m}$ is a matrix of basis vectors spanning the preconditioned stochastic Krylov subspace $\mathcal{K}_m(\mathbf{K}_0^{-1}\mathbf{K}(\boldsymbol{\xi}),\mathbf{K}_0^{-1}\mathbf{f})$ and $\boldsymbol{\alpha} = \{\alpha_1,\alpha_2,\ldots,\alpha_m\}^T \in \mathbb{R}^m$ is a vector of undetermined coefficients.

The numerical studies conducted by Nair and Keane [15] and Sachdeva et al. [62] suggest that using the first three basis vectors spanning the preconditioned stochastic Krylov subspace, highly accurate

[7] If the matrix $\mathbf{K}(\boldsymbol{\xi}_0)^{-1}$ is used as the preconditioner, then the error in the stochastic reduced basis representation will converge faster to zero near the point $\boldsymbol{\xi}_0$. This feature of SRBMs can be exploited in practice to accurately estimate the statistics of the extremes. For example, in reliability analysis problems $\boldsymbol{\xi}_0$ can be chosen to be the most probable point of failure.

results can be obtained. Using Equation (50), the first three basis vectors spanning $\mathcal{K}_m(\mathbf{K}_0^{-1}\mathbf{K}(\boldsymbol{\xi}), \mathbf{K}_0^{-1}\mathbf{f})$ can be written as

$$\psi_1(\boldsymbol{\xi}) = \mathbf{K}_0^{-1}\mathbf{f} \tag{53}$$

$$\psi_2(\boldsymbol{\xi}) = \mathbf{K}_0^{-1}\mathbf{K}(\boldsymbol{\xi})\psi_1(\boldsymbol{\xi}) \tag{54}$$

$$\psi_3(\boldsymbol{\xi}) = \mathbf{K}_0^{-1}\mathbf{K}(\boldsymbol{\xi})\psi_2(\boldsymbol{\xi}) \tag{55}$$

Since $\mathbf{K}(\boldsymbol{\xi}) = \mathbf{K}_0 + \sum_{i=1}^{M} \xi_i \mathbf{K}_i$, the basis vectors can be compactly rewritten as follows

$$\psi_1(\boldsymbol{\xi}) = \mathbf{u}_0 \tag{56}$$

$$\psi_2(\boldsymbol{\xi}) = \sum_{i=1}^{M} \mathbf{d}_i \xi_i \tag{57}$$

$$\psi_3(\boldsymbol{\xi}) = \sum_{i=1}^{M}\sum_{j=1}^{M} \mathbf{e}_{ij}\xi_i\xi_j \tag{58}$$

where $\mathbf{u}_0 = \mathbf{K}_0^{-1}\mathbf{f}$, $\mathbf{d}_i = \mathbf{K}_0^{-1}\mathbf{K}_i\mathbf{u}_0$ and $\mathbf{e}_{ij} = \mathbf{K}_0^{-1}\mathbf{K}_i\mathbf{d}_j$.

It can be clearly seen from the above expressions that the basis vectors are random polynomials which can be written as explicit functions of $\boldsymbol{\xi}$. Because of the recursive representation of the basis vectors, they can be efficiently computed given the factored form of the preconditioner $\mathbf{K}_0^{-1}$, which is readily available as a byproduct of deterministic analysis of the problem. Another point worth noting is that the basis vectors coincide with the Neumann series[8] when the matrix $\mathbf{K}_0^{-1}$ is chosen to be the preconditioner. However, when a general preconditioner $\mathbf{K}(\boldsymbol{\xi}_0)^{-1}$ is chosen, this observation does not hold true. Further, when the stiffness matrix depends nonlinearly on $\boldsymbol{\xi}$, the basis vectors become nonlinear functions of $\boldsymbol{\xi}$. For such cases, using the general representation of the governing linear random algebraic equations given in Equation (48), the basis vectors can be written in terms of PC basis functions as

$$\psi_2(\boldsymbol{\xi}) = \sum_{i=1}^{N} \mathbf{d}_i \varphi_i(\boldsymbol{\xi}) \tag{59}$$

$$\psi_3(\boldsymbol{\xi}) = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{e}_{ij}\varphi_i(\boldsymbol{\xi})\varphi_j(\boldsymbol{\xi}), \tag{60}$$

where N is the number of terms retained in the PC decomposition of $\mathbf{K}(\boldsymbol{\xi})$.

6 Stochastic Reduced Basis Projection Schemes

In this section, we present some projection schemes which can be employed in conjunction with the stochastic Krylov subspace representation of the solution process. Since the formulations presented in this section lead to reduced-order systems of equations, we shall refer to them as stochastic reduced basis methods (SRBMs).

[8] The Neumann series for the solution of Equation (22) can be written as $(I - \mathbf{K}_0^{-1}\Delta\mathbf{K} + (\mathbf{K}_0^{-1}\Delta\mathbf{K})^2 + \cdots)\mathbf{K}_0^{-1}\mathbf{f}$, where $\Delta\mathbf{K} = \sum_{i=1}^{M} \mathbf{K}_i\xi_i$; see, for example, [1, 16].

6.1 Weak Galerkin Scheme

To compute the vector of undetermined coefficients $\boldsymbol{\alpha}$ using the Galerkin scheme, we first substitute Equation (52) into the governing random algebraic equations given in Equation (22) to arrive at the following stochastic residual error vector

$$\boldsymbol{\epsilon}(\boldsymbol{\xi}) = \left(\sum_{i=0}^{M} \mathbf{K}_i \xi_i\right) \boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha} - \mathbf{f} \in \mathbb{R}^n. \tag{61}$$

If we restrict our attention to self-adjoint stochastic PDEs, the matrices $\mathbf{K}_i$, $i = 0, 1, 2, \ldots, M$ are guaranteed to be symmetric positive definite (which is the case for the static problem described in Section 2). Hence, the undetermined coefficients in Equation (52) can be computed by enforcing the Galerkin condition

$$\sum_{i=0}^{M} \xi_i \mathbf{K}_i \boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha} - \mathbf{f} \perp \boldsymbol{\psi}_j(\boldsymbol{\xi}), \quad \forall \quad j = 1, 2, \ldots, m. \tag{62}$$

The condition in Equation (62) demands that the stochastic residual error vector $\boldsymbol{\epsilon}(\boldsymbol{\xi})$ be made orthogonal to the approximating subspace $\boldsymbol{\Psi}(\boldsymbol{\xi})$. Hence, the Galerkin condition is also referred to as an orthogonal projection scheme. We shall first consider the case when the orthogonality condition is imposed using the definition of inner products in the Hilbert space of random variables; see Equation (5). Here, since only the ensemble average of the random functions $\boldsymbol{\psi}_i^*(\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})$, $i = 1, 2, \ldots, m$ are set to zero, we shall refer to Equation (62) as a weak Galerkin condition. As we shall show later, SRBMs based on a stronger Galerkin condition can also be formulated.

Application of the weak Galerkin condition results in the following reduced-order $m \times m$ deterministic system of equations for $\boldsymbol{\alpha}$

$$\left[\sum_{i=0}^{M} \langle \xi_i \boldsymbol{\Psi}^*(\boldsymbol{\xi}) \mathbf{K}_i \boldsymbol{\Psi}(\boldsymbol{\xi}) \rangle\right] \boldsymbol{\alpha} = \langle \boldsymbol{\Psi}^*(\boldsymbol{\xi}) \mathbf{f} \rangle \,. \tag{63}$$

Since explicit expressions for the stochastic basis vectors are available, the expectation operations required to compute the elements of the reduced-order terms in Equation (63) can be readily carried out. The deterministic reduced-order 3×3 system of equations for the second-order SRBM ($m = 3$) are given below for the case $\xi_i, i = 1, 2, \ldots, M$ are uncorrelated Gaussian random variables. Note that for the sake of compactness, we have used the Einstein repeated index notation; for example, a repeated index i indicates summation with respect to that index over the range $1, 2, \ldots, M$.

$$\begin{bmatrix} \mathbf{u}_0^* \mathbf{K}_0 \mathbf{u}_0 & \langle \xi_i^2 \rangle \mathbf{u}_0^* \mathbf{K}_i \mathbf{d}_i & \langle \xi_i^2 \rangle \mathbf{u}_0^* \mathbf{K}_0 \mathbf{e}_{ii} \\ & \langle \xi_i^2 \rangle \mathbf{d}_i^* \mathbf{K}_0 \mathbf{d}_i & \langle \xi_i \xi_j \xi_k \xi_l \rangle \mathbf{d}_i^* \mathbf{K}_j \mathbf{e}_{kl} \\ \text{sym} & & \langle \xi_i \xi_j \xi_k \xi_l \rangle \mathbf{e}_{ij}^* \mathbf{K}_0 \mathbf{e}_{kl} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_0^* \mathbf{f} \\ 0 \\ \langle \xi_i^2 \rangle \mathbf{e}_{ii}^* \mathbf{f} \end{bmatrix}. \tag{64}$$

Note that the terms involving fourth-order products of the form $\langle \xi_i \xi_j \xi_k \xi_l \rangle$ can be readily computed using the identity

$$\langle \xi_i \xi_j \xi_k \xi_l \rangle = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}, \tag{65}$$

where δ is the Kronecker delta function.

As we shall show later, three or four basis vectors are sufficient to ensure high accuracy. Solving the preceding reduced-order system of equations and substituting the computed value of $\boldsymbol{\alpha}$ in Equation (52), we arrive at an explicit expression for $\mathbf{u}(\boldsymbol{\xi})$. Hence, similar to the PC representation, this enables the application of efficient post-processing techniques to compute the statistics and distribution functions of the response quantities of interest.

Note that when two basis vectors are used, the stochastic reduced basis approximation is of first-order since the second basis vector is a linear function of random variables. Similarly, when three basis vectors are used, the approximation is of second-order.

6.2 Strong Galerkin Scheme

In this section, we show how the weak Galerkin condition in Equation (62) can be reinterpreted to derive a stronger condition. The main idea is to enforce the condition that, for *each realization* of the random variable vector $\boldsymbol{\xi}$, the residual error vector $\boldsymbol{\epsilon}(\boldsymbol{\xi})$ is orthogonal to the m basis vectors $\boldsymbol{\psi}_1(\boldsymbol{\xi}), \boldsymbol{\psi}_2(\boldsymbol{\xi}), \ldots, \boldsymbol{\psi}_m(\boldsymbol{\xi})$. In other words, the set of random functions $\boldsymbol{\psi}_i^*(\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})$, $i = 1, 2, \ldots, m$, is zero with probability one. This condition, which we refer to as a strong Galerkin condition can be stated as follows

$$P\left[\boldsymbol{\psi}_i^*(\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi}) = 0\right] = 1 \quad \forall \quad i = 1, 2, \ldots, m. \tag{66}$$

It is straightforward to show that the strong Galerkin condition will be satisfied only when $\boldsymbol{\alpha}$ is computed by solving the following reduced-order $m \times m$ *random* algebraic system of equations

$$\left[\sum_{i=0}^{M} \xi_i \boldsymbol{\Psi}^*(\boldsymbol{\xi}) \mathbf{K}_i \boldsymbol{\Psi}(\boldsymbol{\xi})\right] \boldsymbol{\alpha}(\boldsymbol{\xi}) = \boldsymbol{\Psi}^*(\boldsymbol{\xi})\mathbf{f}. \tag{67}$$

As can be seen from the preceding equation, in order to satisfy the strong Galerkin condition, we need to model the undetermined coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m$ as functions of $\boldsymbol{\xi}$, i.e., the stochastic reduced basis approximation in Equation (52) has to be rewritten as

$$\hat{\mathbf{u}}(\boldsymbol{\xi}) = \alpha_1(\boldsymbol{\xi})\boldsymbol{\psi}_1(\boldsymbol{\xi}) + \alpha_2(\boldsymbol{\xi})\boldsymbol{\psi}_2(\boldsymbol{\xi}) + \cdots + \alpha_m(\boldsymbol{\xi})\boldsymbol{\psi}_m(\boldsymbol{\xi}) = \boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha}(\boldsymbol{\xi}). \tag{68}$$

Initial studies presented by Nair and Keane [15] suggest that by using Equation (67) to compute $\boldsymbol{\alpha}$, it is possible to derive higher-order approximations that are significantly more accurate than SRBMs and the PC projection scheme employing the weak Galerkin condition. Unfortunately, since $\boldsymbol{\alpha}$ now becomes a highly nonlinear function of $\boldsymbol{\xi}$, analytical expressions for the response statistics are difficult to obtain.

One way to relax the strong Galerkin condition in Equation (66) is to employ a PC decomposition of the undetermined coefficient vector $\boldsymbol{\alpha}(\boldsymbol{\xi})$, i.e.,

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) = \sum_{i=0}^{P_1-1} \boldsymbol{\alpha}_i \varphi_i(\boldsymbol{\xi}), \tag{69}$$

where $\boldsymbol{\alpha}_i \in \mathbb{R}^m, i = 0, 1, 2, \ldots, P_1 - 1$ are undetermined vectors. In order to compute these undetermined coefficient vectors, we substitute Equation (69) into Equation (67) and apply the Galerkin projection scheme, i.e., the stochastic residual error in satisfying Equation (67) is made orthogonal to the PC basis functions. This results in the following deterministic system of equations

$$\left\langle \varphi_k(\boldsymbol{\xi}) \left[\sum_{i=0}^{M} \xi_i \boldsymbol{\Psi}^*(\boldsymbol{\xi}) \mathbf{K}_i \boldsymbol{\Psi}(\boldsymbol{\xi})\right] \sum_{j=0}^{P_1-1} \boldsymbol{\alpha}_j \varphi_j(\boldsymbol{\xi}) \right\rangle = \langle \varphi_k(\boldsymbol{\xi}) \boldsymbol{\Psi}^*(\boldsymbol{\xi})\mathbf{f} \rangle \ \forall \ k = 0, 1, 2, \ldots, P_1 - 1. \tag{70}$$

The above equation can rewritten as

$$\sum_{i=0}^{M}\sum_{j=0}^{P_1-1}\langle\xi_i\varphi_k(\boldsymbol{\xi})\varphi_j(\boldsymbol{\xi})\boldsymbol{\Psi}^*(\boldsymbol{\xi})\mathbf{K}_i\boldsymbol{\Psi}(\boldsymbol{\xi})\rangle\,\boldsymbol{\alpha}_j=\langle\varphi_k(\boldsymbol{\xi})\boldsymbol{\Psi}^*(\boldsymbol{\xi})\mathbf{f}\rangle\ \ \forall\ k=0,1,2,\ldots,P_1-1. \tag{71}$$

It can be seen that Equation (71) is essentially an $mP_1\times mP_1$ system of deterministic linear algebraic equations. Hence, the strong Galerkin formulation based on PC expansion of $\boldsymbol{\alpha}$ is equivalent to employing the augmented set of basis vectors $[\boldsymbol{\Psi}(\boldsymbol{\xi}),\varphi_1(\boldsymbol{\xi})\boldsymbol{\Psi}(\boldsymbol{\xi}),\ldots,\varphi_{P_1-1}(\boldsymbol{\xi})\boldsymbol{\Psi}(\boldsymbol{\xi})]\in\mathbb{R}^{n\times mP_1}$ in conjunction with the weak Galerkin condition. In comparison, the standard weak Galerkin SRBM presented earlier in Section 6.1 results in an $m\times m$ system of equations. However, since the number of basis vectors $m<<<n$, the preceding system of equations is still significantly smaller than the system of equations arising from the standard PC projection scheme outlined earlier in Section 4.

The strong Galerkin scheme based on PC decomposition of $\boldsymbol{\alpha}$ can also be viewed as one possible way to hybridize the stochastic Krylov subspace representation with the PC projection scheme of Ghanem and Spanos [1]. Another important point worth noting is that the hybrid scheme presented here is a generalization of the weak Galerkin scheme. This is because if we set $P_1=1$ in Equation (69), we recover the weak Galerkin projection scheme presented earlier in Section 6.1. We shall numerically demonstrate later that this hybrid approach gives better results than the weak Galerkin formulation of SRBMs as well as the PC projection scheme.

6.3 Petrov-Galerkin Scheme

An alternative formulation for computing the undetermined coefficients in SRBMs can be derived by employing the Petrov-Galerkin scheme. This is an oblique projection scheme which essentially involves enforcing the condition that the stochastic residual error vector is orthogonal to the subspace $\mathbf{K}(\boldsymbol{\xi})\boldsymbol{\Psi}(\boldsymbol{\xi})$, i.e.,

$$\boldsymbol{\epsilon}(\boldsymbol{\xi})\perp\mathbf{K}(\boldsymbol{\xi})\boldsymbol{\psi}_i(\boldsymbol{\xi})\quad\forall\quad i=1,2,\ldots,m. \tag{72}$$

Note that imposition of the above condition is equivalent to directly minimizing the L_2 norm of the residual error vector $\boldsymbol{\epsilon}(\boldsymbol{\xi})$. Enforcing the Petrov-Galerkin condition using the definition of orthogonality in the Hilbert space of random variables, we arrive at the following reduced-order $m\times m$ deterministic system of equations for $\boldsymbol{\alpha}$

$$\langle\boldsymbol{\Psi}^*(\boldsymbol{\xi})\mathbf{K}^*(\boldsymbol{\xi})\mathbf{K}(\boldsymbol{\xi})\boldsymbol{\Psi}(\boldsymbol{\xi})\rangle\,\boldsymbol{\alpha}=\langle\boldsymbol{\Psi}^*(\boldsymbol{\xi})\mathbf{K}^*(\boldsymbol{\xi})\mathbf{f}\rangle. \tag{73}$$

Using Equation (22), we have

$$\left[\sum_{i=0}^{M}\sum_{j=0}^{M}\langle\xi_i\xi_j\boldsymbol{\Psi}^*(\boldsymbol{\xi})\mathbf{K}_i^*\mathbf{K}_j\boldsymbol{\Psi}(\boldsymbol{\xi})\rangle\right]\boldsymbol{\alpha}=\sum_{i=0}^{M}\langle\xi_i\boldsymbol{\Psi}^*(\boldsymbol{\xi})\mathbf{K}_i^*\mathbf{f}\rangle\,. \tag{74}$$

It can be seen from the preceding equation that in comparison to the weak Galerkin scheme, application of the Petrov-Galerkin condition leads to the requirement of computing higher-order ensemble averages. Note that it is also possible to formulate a strong Petrov-Galerkin condition along the lines of the approach used earlier in the strong Galerkin scheme. Another important point worth noting here is that the Petrov-Galerkin scheme ensures that the L_2 norm of the residual error, $\langle\boldsymbol{\epsilon}^*(\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\rangle$, converges in a mean square sense when the number of basis vectors is increased. As discussed later in Section 6.5, the convergence of the Galerkin scheme can be proved only when $\mathbf{K}(\boldsymbol{\xi})$ is a Hermitian positive definite matrix. Hence, SRBMs based on the Petrov-Galerkin scheme are expected to work well for linear stochastic structural dynamic analysis in the frequency domain, since the dynamic stiffness matrix is non-Hermitian; see [64] for a more detailed discussion of this point.

6.4 Nonlinear Dependence of K on ξ

Let us now apply SRBMs to the general case when $\mathbf{K}$ is a nonlinear function of $\boldsymbol{\xi}$ or a lognormal random field is used to describe uncertainty in the Youngs modulus. For such problems, a PC decomposition of $\mathbf{K}(\boldsymbol{\xi})$ can be used to arrive at a system of linear random algebraic equations of the form given earlier in Equation (48). As outlined earlier in Section 5, for such a representation of the stiffness matrix, the basis vectors spanning the preconditioned stochastic Krylov subspace become functions of the PC basis. Using Equations (59) and (60), the matrix of stochastic basis vectors can be written as

$$\boldsymbol{\Psi}(\boldsymbol{\xi}) = \left[\mathbf{u}_0, \sum_{i=1}^{N} \mathbf{d}_i \varphi_i(\boldsymbol{\xi}), \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{e}_{ij} \varphi_i(\boldsymbol{\xi}) \varphi_j(\boldsymbol{\xi}), \cdots \right] \in \mathbb{R}^{n \times m}, \tag{75}$$

where the terms $\mathbf{d}_i$ and $\mathbf{e}_{ij}$ have been defined earlier in Section 5.

It is conceptually very straightforward to apply any of the projection schemes outlined earlier when the basis vectors are given in this form. However, from an implementation point of view, it may be preferable to first construct a PC decomposition of $\boldsymbol{\Psi}(\boldsymbol{\xi})$. In other words, we compute a PC decomposition of the basis vectors spanning the preconditioned stochastic Krylov subspace. Fortunately, since the basis vectors are polynomials in $\boldsymbol{\xi}$, the expansion coefficients can be calculated analytically. This leads to the following representation of the matrix of basis vectors used in SRBMs.

$$\boldsymbol{\Psi}(\boldsymbol{\xi}) = \sum_{i=0}^{P_2-1} \boldsymbol{\Psi}_i \varphi_i(\boldsymbol{\xi}), \tag{76}$$

where $\boldsymbol{\Psi}_i \in \mathbb{R}^{n \times m}, i = 0, 1, 2, \ldots, P_2 - 1$ are deterministic matrices.

Note that the PC expansion of $\boldsymbol{\Psi}(\boldsymbol{\xi})$ does not lead to an increase in the number of basis vectors. Now, applying the weak Galerkin scheme using the above set of basis vectors to Equation (48), we arrive the following deterministic system of equations

$$\left(\sum_{i=0}^{P_2-1} \sum_{j=0}^{N} \sum_{k=0}^{P_2-1} \langle \varphi_i(\boldsymbol{\xi}) \varphi_j(\boldsymbol{\xi}) \varphi_k(\boldsymbol{\xi}) \rangle \boldsymbol{\Psi}_i^* \mathbf{K}_j \boldsymbol{\Psi}_k \right) \boldsymbol{\alpha} = \sum_{i=0}^{P_2-1} \langle \varphi_i(\boldsymbol{\xi}) \rangle \boldsymbol{\Psi}_i^* \mathbf{f} \tag{77}$$

The above equation is analogous to that presented earlier in Equation (40) in the context of the PC projection scheme. However, in contrast to the PC projection scheme, Equation (77) is a reduced-order $m \times m$ system of equations. Note that the derivation presented here can be extended to the case when the strong Galerkin condition or the Petrov-Galerkin scheme is employed to compute the undetermined coefficients in SRBMs.

6.5 Theoretical Analysis of Convergence

The Galerkin scheme has a number of interesting properties since it is an orthogonal projection scheme. Consider the case when the coefficient matrix $\mathbf{K}(\boldsymbol{\xi})$ is Hermitian positive definite. Further, let $\mathcal{K}_m$ denote the stochastic subspace spanned by the set of orthogonal[9] basis vectors $\psi_1(\boldsymbol{\xi}), \psi_2(\boldsymbol{\xi}), \ldots, \psi_m(\boldsymbol{\xi})$. Let us first rewrite the weak Galerkin condition given in Equation (62) as

[9] Note that a orthogonal set of basis vectors spanning the preconditioned stochastic Krylov subspace can be computed using the stochastic version of Arnoldi's method presented in Nair [20]

$$\langle \{\mathbf{K}(\boldsymbol{\xi})\hat{\mathbf{u}}(\boldsymbol{\xi}) - \mathbf{f}\}^* \boldsymbol{\psi}_i(\boldsymbol{\xi}) \rangle = 0, \quad \forall \quad \boldsymbol{\psi}_i(\boldsymbol{\xi}) \in \mathcal{K}_m. \tag{78}$$

Substituting $\mathbf{f} = \mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi})$, the above condition becomes

$$\langle \{\hat{\mathbf{u}}(\boldsymbol{\xi}) - \mathbf{u}(\boldsymbol{\xi})\}^* \mathbf{K}^*(\boldsymbol{\xi}) \boldsymbol{\psi}_i(\boldsymbol{\xi}) \rangle = 0, \quad \forall \quad \boldsymbol{\psi}_i(\boldsymbol{\xi}) \in \mathcal{K}_m. \tag{79}$$

From the preceding equation, it can be seen that the weak Galerkin scheme ensures that the difference between the exact and the approximate solution is $\mathbf{K}$-orthogonal to the approximating space $\mathcal{K}_m$. Now using the elementary properties of orthogonal projectors [60], it can be shown that Equation (79) is the necessary and sufficient condition for $\hat{\mathbf{u}}(\boldsymbol{\xi})$ to be a minimizer of the error function $\langle \{\hat{\mathbf{u}}(\boldsymbol{\xi}) - \mathbf{u}(\boldsymbol{\xi})\}^* \mathbf{K} \{\hat{\mathbf{u}}(\boldsymbol{\xi}) - \mathbf{u}(\boldsymbol{\xi})\} \rangle$. This result which establishes the optimality of the weak Galerkin scheme can be stated as follows [15]

Theorem 3: *Let* $\hat{\mathbf{u}}(\boldsymbol{\xi}) = \boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha}$ *be a stochastic reduced basis approximation to the solution of* $\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}$ *, where* $\mathbf{K}(\boldsymbol{\xi}) \in \mathbb{R}^{n \times n}$ *is a random Hermitian positive definite matrix,* $\mathbf{u}(\boldsymbol{\xi}), \mathbf{f} \in \mathbb{R}^n$ *are random vectors,* $\boldsymbol{\Psi}(\boldsymbol{\xi}) \in \mathbb{R}^{n \times m}$ *is a matrix of stochastic basis vectors, and* $\boldsymbol{\alpha} \in \mathbb{R}^m$ *is a vector of undetermined coefficients. If the coefficient vector* $\boldsymbol{\alpha}$ *is computed by imposing the condition* $\mathbf{K}(\boldsymbol{\xi})\boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha} - \mathbf{f} \perp \boldsymbol{\Psi}(\boldsymbol{\xi})$*, then the following deterministic error function is minimized.*

$$\Delta_m = \langle \{\mathbf{u}(\boldsymbol{\xi}) - \hat{\mathbf{u}}(\boldsymbol{\xi})\}^* \mathbf{K}(\boldsymbol{\xi}) \{\mathbf{u}(\boldsymbol{\xi}) - \hat{\mathbf{u}}(\boldsymbol{\xi})\} \rangle \,, \tag{80}$$

where Δ_m *denotes the* $\mathbf{K}$*-norm of the error.*

An important corollary of Theorem 3 is that the weak Galerkin projection scheme used here ensures that $\{\mathbf{u}(\boldsymbol{\xi}) - \hat{\mathbf{u}}(\boldsymbol{\xi})\}^* \mathbf{K}(\boldsymbol{\xi}) \{\mathbf{u}(\boldsymbol{\xi}) - \hat{\mathbf{u}}(\boldsymbol{\xi})\}$ converges in a mean-square sense when the number of basis vectors is increased, i.e., $\Delta_{m+1} \leq \Delta_m$, where Δ_{m+1} and Δ_m denote the $\mathbf{K}$-norm of the error in the solution computed using $m+1$ and m basis vectors, respectively. Hence, it also follows that application of the strong Galerkin condition results in a lower value for the $\mathbf{K}$-norm of the error compared to the weak Galerkin scheme. This is because, as shown earlier in Section 6.2, application of the strong Galerkin condition is equivalent to using an augmented set of basis vectors. A similar set of results can be established for the L_2 norm of the residual for non-Hermitian coefficient matrices when the Petrov-Galerkin scheme outlined in Section 6.3 is employed.

It is also worth noting that the $\mathbf{K}$-norm of the error can be interpreted as an energy norm. This has an important practical ramification for stochastic structural systems. Since SRBMs employing the Galerkin scheme minimize the $\mathbf{K}$-norm of the error, the results are bound to be more accurate for those dof which contain most strain energy. This is a very useful property since in many practical applications such as reliability analysis, we are primarily interested in ensuring good approximations for the highly stressed regions of the structure.

7 Post-Processing Techniques

In this section, we briefly outline how the final expressions for the solution process obtained using various projection schemes can be post-processed to compute the statistics of interest.

7.1 Statistical Moments and Distributions

By virtue of the orthogonality of the basis functions in the PC representation, it becomes possible to derive explicit expressions for the mean and variance of the response once the undetermined coefficient

vectors in the PC expansion are computed using a projection scheme. The mean of the response is simply given by $\mathbf{u}_0$ since by definition $\langle\varphi_0\rangle = 1$ and $\langle\varphi_i(\boldsymbol{\xi})\rangle = 0$ for $i > 0$. The response covariance matrix can be computed as

$$\mathrm{Cov}[\mathbf{u}(\boldsymbol{\xi}), \mathbf{u}(\boldsymbol{\xi})] = \sum_{i=1}^{P-1} \langle\varphi_i^2\rangle \mathbf{u}_i \mathbf{u}_i^* \tag{81}$$

In the case of SRBMs employing the weak Galerkin condition, we ultimately arrive at a random polynomial for the response process in the form

$$\hat{\mathbf{u}}(\boldsymbol{\xi}) = \sum_{i=0}^{m} \alpha_i \boldsymbol{\psi}_i(\boldsymbol{\xi}). \tag{82}$$

Since the coefficients α_i are deterministic scalars, the mean and covariance of the response can be expressed in terms of the statistics of the basis vectors as follows

$$\langle\hat{\mathbf{u}}(\boldsymbol{\xi})\rangle = \sum_{i=0}^{m} \alpha_i \langle\boldsymbol{\psi}_i(\boldsymbol{\xi})\rangle, \tag{83}$$

and

$$\mathrm{Cov}[\mathbf{u}(\boldsymbol{\xi}), \mathbf{u}(\boldsymbol{\xi})] = \langle\boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha}\boldsymbol{\alpha}^*\boldsymbol{\Psi}^*(\boldsymbol{\xi})\rangle = \sum_{i=0}^{m}\sum_{j=0}^{m} \alpha_i\alpha_j\langle\boldsymbol{\psi}_i(\boldsymbol{\xi})\boldsymbol{\psi}_j^*(\boldsymbol{\xi})\rangle. \tag{84}$$

Similarly, the mean and covariance of the response can also be computed for SRBMs employing the strong Galerkin condition and the Petrov-Galerkin scheme. Note that the expectation operations in the preceding equations can be analytically carried out using the joint statistics of ξ_i; see, for example, McCullagh [65] and chapter 4 of Ghanem and Spanos [1] for a detailed exposition on statistical analysis of random polynomials. It is also possible to derive expressions for the statistical moments of quantities such as stress and strain, since they are functions of the random displacements.

7.2 Reliability Analysis

Since all the projection schemes considered here lead to explicit expressions for the solution process, its complete statistical characterization becomes computationally feasible. For example, we can employ simulation schemes in conjunction with kernel density estimation techniques [66] to approximate the pdf of the response quantities given the final expression for the solution process in terms of a random polynomial. Alternatively, the wide body of numerical methods developed for reliability analysis such as the first-order reliability method (FORM), second-order reliability method (SORM) (see, for example, the chapter on reliability analysis by Der Kiureghian in this handbook) and importance sampling (see, for example, the chapter by Ghiocel and Grigoriu on Monte Carlo simulation in this handbook) can be applied to the explicit expressions for the solution process obtained using any of the projection schemes.

Sudret and Der Kiureghian [31, 59] have conducted a detailed investigation into the application of PC projection schemes to reliability analysis. They suggested the use of FORM to postprocess the expression obtained using the PC projection scheme to compute the probability of failure. Further, it was shown that a sensitivity analysis approach can be used to efficiently compute the pdf of the response and derivatives of the reliability index. These procedures are also applicable to SRBMs.

8 Numerical Examples

In this section, we apply SRBMs and PC projection schemes to compute the response statistics and pdfs for two example problems. The results obtained are benchmarked against those obtained using Monte Carlo simulation (MCS) with a sample size of 250,000. For all the problems considered, we use the following two-dimensional exponential correlation function to represent uncertainty in the random Youngs modulus.

$$R(\mathbf{x},\mathbf{y}) = \exp\left(-\frac{|x_1 - x_2|}{b_1} - \frac{|y_1 - y_2|}{b_2}\right), \tag{85}$$

where b_1 and b_2 are the correlation lengths of the random field.

Recollect that in the first-order, second-order and third-order SRBMs, we use two, three, and four basis vectors, respectively. Henceforth, we shall refer to the first-order SRBM, second-order SRBM, and third-order SRBM as SRBMI, SRBMII and SRBMIII, respectively. The first-order PC, second-order PC, and third-order PC schemes are referred to as PCI, PCII and PCIII, respectively. We also present results for the strong Galerkin approach presented in Section 6.2, when the undetermined coefficients in the stochastic reduced basis representation are represented using a first-order PC expansion. These hybrid formulations are referred to as SRBMIPCI (where a first-order PC expansion is used in Equation (69) and two basis vectors are used) and SRBMIIPCI (where a first-order PC expansion is used in Equation (69) and three basis vectors are used). For comparison, we also show results obtained using the first- and second-order perturbation method, referred to as PERI and PERII, respectively.

The numerical studies presented here were conducted using the 'SSFEM toolbox' developed by Sudret and Der Kiureghian [31, 59]. This toolbox contains routines for discretizing random fields, stochastic finite element formulation based on four-noded quadrilateral elements and an implementation of the PC expansion scheme for solving random algebraic equations. Routines implementing SRBMs were integrated with this toolbox to enable a systematic comparison of the various projection schemes considered. Note that all the problems considered here were spatially discretized using four noded quadrilateral elements with two dof per node. Further, the random field representing uncertainty in the Youngs modulus is discretized using the KL expansion scheme.

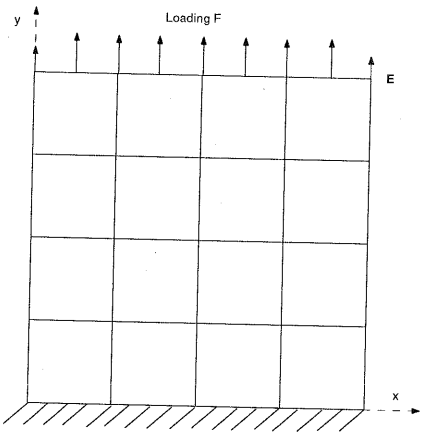

Fig. 1. Schematic of static plate analysis problem.

8.1 Two-dimensional Thin Plate

The first problem considered is a thin square plate of unit length clamped at one edge and subjected to uniform inplane tension at the opposite edge (taken from [1]). The plate is discretized into 16 square elements as shown in Figure 1, which leads to a total of 50 dof. The external loads are assumed to be deterministic and of unit magnitude. The Youngs modulus of the plate is modeled as a two-dimensional Gaussian random field with the exponential correlation model given in Equation (85) with $b_1 = b_2 = 1$. The random field is discretized using the KL expansion scheme and four terms are retained, i.e., $M = 4$.

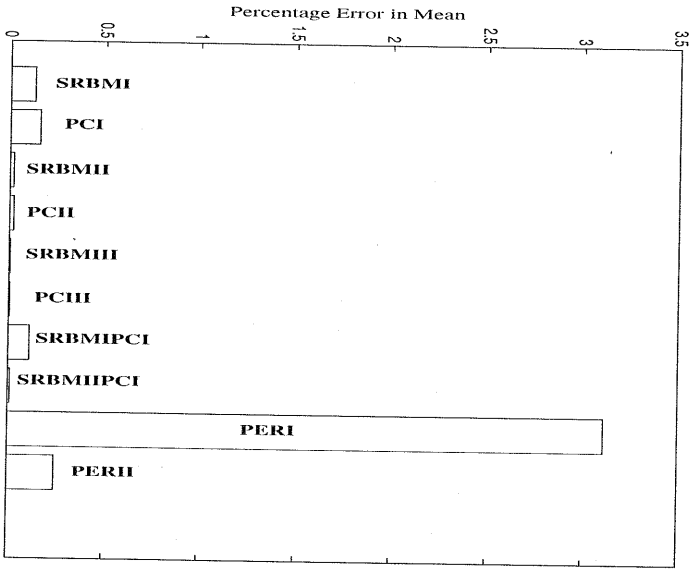

Fig. 2. Percentage error in mean displacement at point 'E' for the plate problem.

Numerical studies were conducted to analyze the performance of the stochastic subspace projection schemes when the coefficient of variation of the random Youngs modulus is increased. Typical trends in the results are presented here for the case when the coefficient of variation is 0.2. The percentage errors in the mean and standard deviation of the displacement at point 'E' (the top right hand corner of the plate shown in Figure 1) are shown in Figures 2 and 3. It can be seen from the results that the errors in the projection schemes reduce rapidly when the number of terms in the expansion is increased. The performance of SRBMs is comparable to PC projection schemes of the same order. This is impressive considering the fact that SRBMs solve a reduced-order system of equations, whereas the PC projection scheme involves the solution of a system of equations with increased dimensionality.

It can also be observed from Figures 2 and 3 that the hybrid schemes motivated by the strong Galerkin condition give better results than the weak Galerkin scheme, i.e., SRBMIPCI is better than SRBMI and SRBMIIPCI is better than SRBMII. In fact, the performance of SRBMIIPCI is comparable to SRBMIII and PCIII. The results obtained using PERII are comparable to SRBMI and PCI. However, as is well known, the rate of convergence of perturbation methods is very low compared to the projection schemes used here.

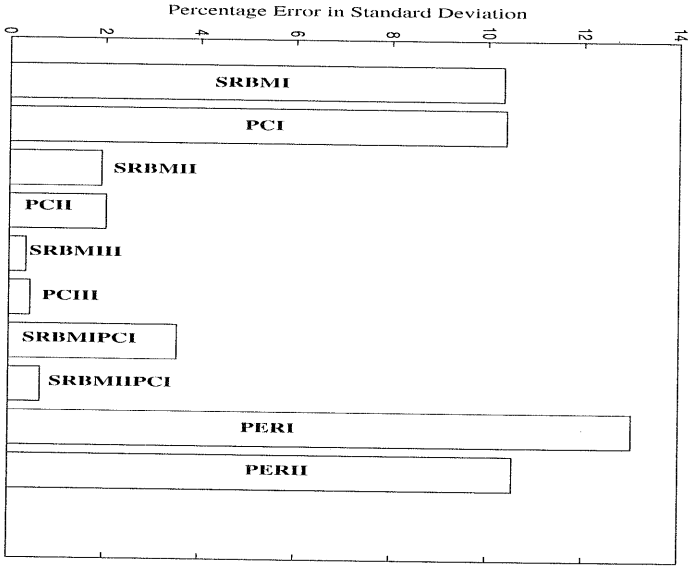

Fig. 3. Percentage error in standard deviation of displacement at point 'E' for the plate problem.

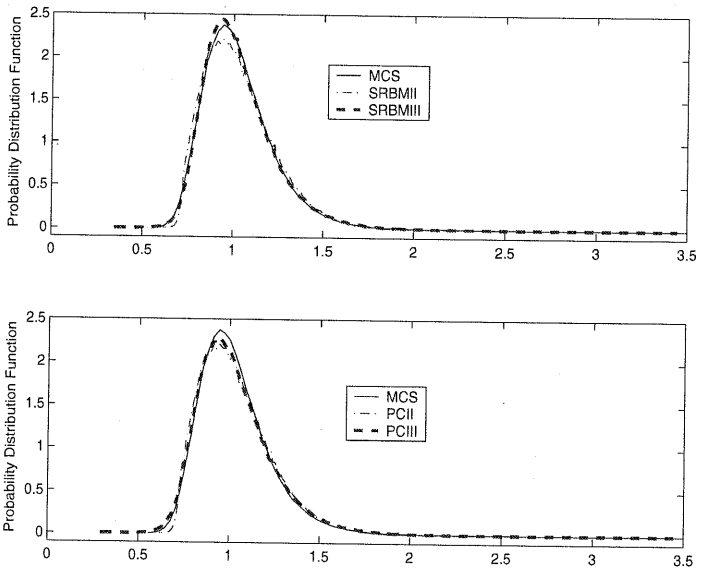

Fig. 4. Comparison of displacement pdf at point 'E' using second- and third-order SRBMs and PC projection schemes.

The pdfs of the displacement at point 'E' obtained using second- and third-order projection schemes are shown in Figure 4. The displacement pdfs for the various projection schemes were calculated by

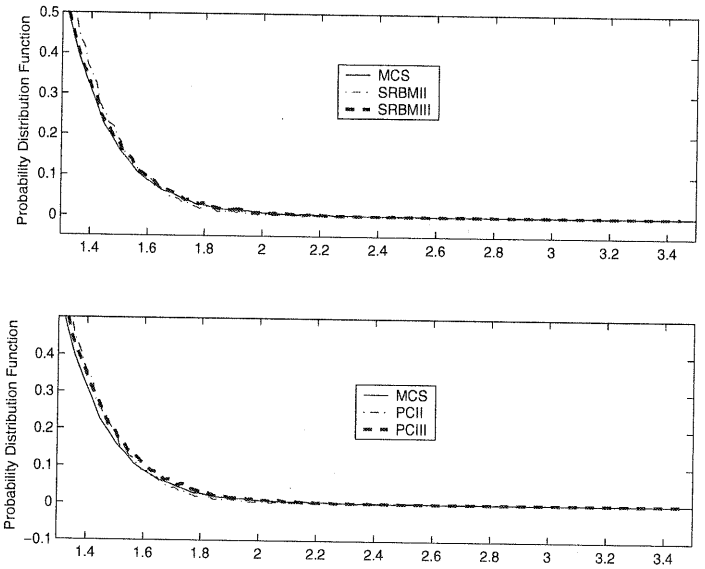

Fig. 5. Comparison of tail of displacement pdf at point 'E' using second- and third-order SRBMs and PC projection schemes.

applying a simulation scheme with a sample size of 250,000 to the obtained expressions for the response process. The accuracy of the approximations to the tail of the pdf are shown in Figure 5. These figures indeed confirm that the performance of SRBMs is comparable to the PC projection schemes. A more detailed comparison between the performance of SRBMs and PC projection schemes can be found in [62, 63].

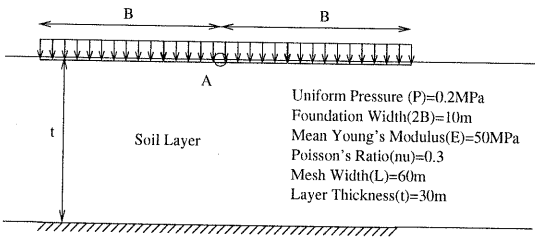

Fig. 6. Schematic for soil foundation problem.

8.2 Analysis of Foundation on Randomly Heterogeneous Soil

Next, we consider a geotechnical problem which involves settlement analysis of a foundation on a randomly heterogeneous soil. This problem has been studied earlier by Sudret and Kiureghian [31, 59] in the context of reliability analysis. Consider an elastic soil layer of thickness t lying on a rigid stratum

as shown in Figure 6. An uniform pressure P is applied over a length $2B$ of the free surface. The soil is modeled as an elastic linear isotropic material and plane strain analysis is carried out. Exploiting symmetry considerations, only one half of the structure is modeled by finite elements. The values of the soil parameters used in the computations are given in Figure 6.

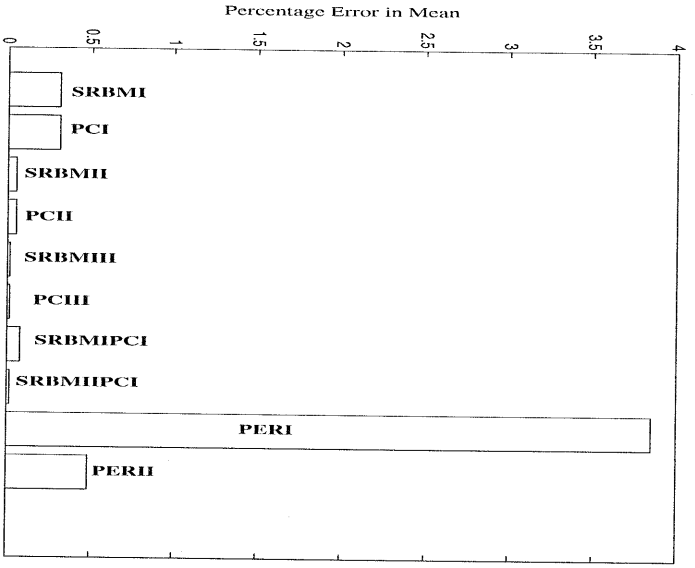

Fig. 7. Percentage error in mean displacement at point 'A' for the foundation problem.

The Youngs modulus of the soil is assumed to be a homogeneous Gaussian random field. The Youngs modulus is considered to vary only in the vertical direction and hence a one dimensional random field with exponential correlation function (see Eq. (85)) is employed. The correlation length b_2 is set at $30m$. The KL expansion scheme was used to discretize this random field and four terms are retained in the expansion, i.e., $M = 4$. An optimal mesh with 99 nodes and 80 elements developed by Sudret and Kiureghian [59] was used for the analysis.

Numerical studies were conducted to analyze the performance of the projection schemes when the coefficient of variation of the random Youngs modulus is set at 0.2. Results are presented here for the statistics of the vertical displacement at point 'A', which is the center of the foundation as shown in Figure 6. The percentage errors in the mean and standard deviation of the displacement at point 'A' are shown in Figures 7 and 8. It can be seen that the trends in the results are similar to those obtained earlier for the plate problem.

The pdfs of the displacement at point 'A' computed using second- and third-order projection schemes are shown in Figure 9. Figure 10 shows how well the stochastic subspace projection schemes approximate the tail of the pdf.

It can be observed from the results that the performance of SRBMs is comparable to PC projection schemes of equivalent order. In comparison, both the first- and second-order perturbation methods perform worse than the stochastic subspace projection schemes. The idea of hybrid schemes motivated by the strong Galerkin condition outlined earlier in Section 6.2 appear to work very well. For example,

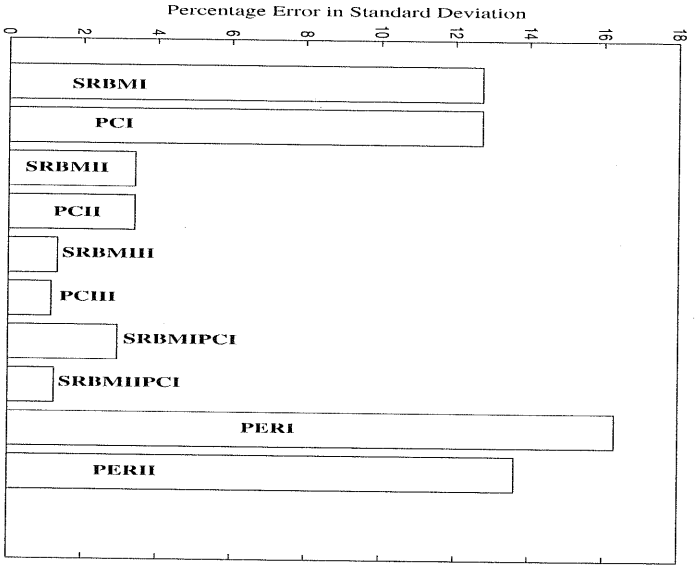

Fig. 8. Percentage error in standard deviation of displacement at point 'A' for the foundation problem.

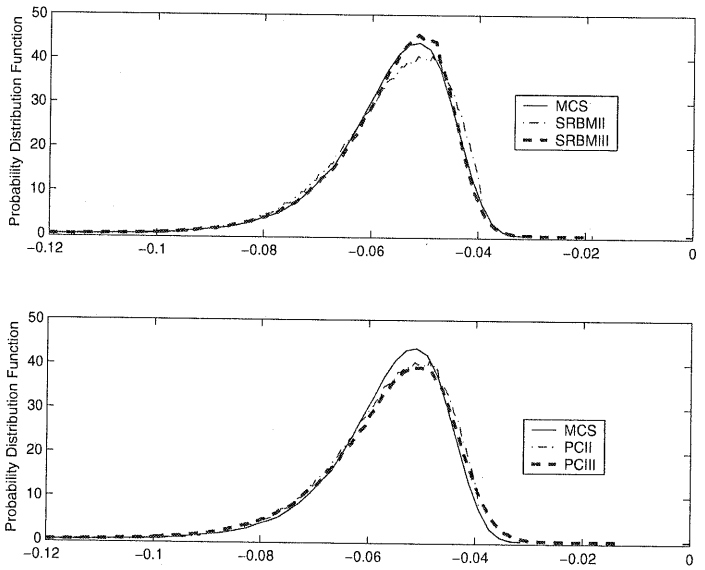

Fig. 9. Comparison of displacement pdf at point 'A' using second- and third-order SRBMs and PC projection schemes for the foundation problem.

both SRBMII and SRBMIIPCI use the first three basis vectors spanning the preconditioned stochastic Krylov subspace. The only difference is that, in SRBMIIPCI, we use a first-order PC expansion of

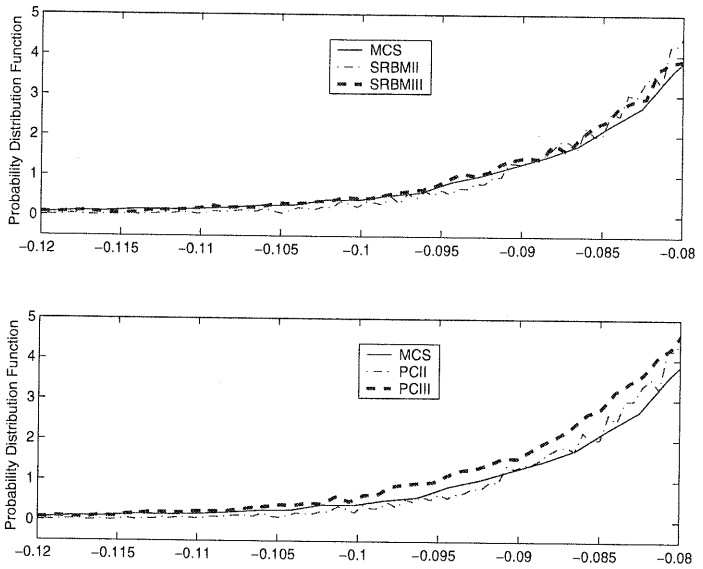

Fig. 10. Comparison of tail of displacement pdf at point 'A' using second- and third-order SRBMs and PC projection schemes for the foundation problem.

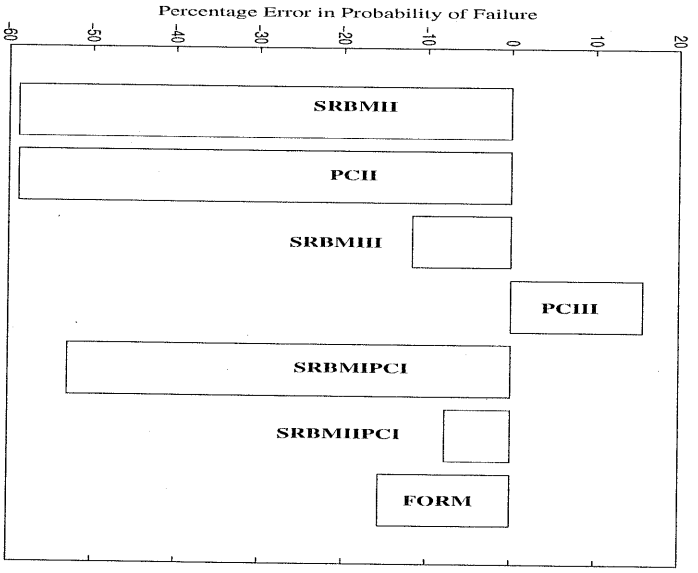

Fig. 11. Comparison of Various Projection Schemes and FORM for Reliability Assessment

the vector of undetermined coefficients $\boldsymbol{\alpha}$; see Equation (69). Since four terms are used in the KL expansion ($M = 4$) and three basis vectors are employed ($m = 3$), it follows that $P_1 = 5$ in Equation

(69). Hence, the SRBMIIPCI formulation leads to a 15×15 deterministic system of equations. In contrast, SRBMII solves a 3×3 system of equations. However, the results obtained using SRBMIIPCI is comparable to PCIII and better than that obtained using PCI and PCII. For reference, PCI, PCII and PCIII projection schemes involve the solution of a 845×845, 2535×2535, and 5915×5915 system of equations, respectively. This suggests that SRBMIIPCI is capable of providing results of better accuracy than PCII at a significantly lower computational cost.

Finally, we applied the projection schemes to reliability analysis of the foundation, when the limit-state function is defined as

$$g(\boldsymbol{\xi}) = u_{\max} - u_A(\boldsymbol{\xi}), \tag{86}$$

where $u_A(\boldsymbol{\xi})$ is the displacement of the foundation at point 'A' and $u_{\max}$ is an admissible threshold which is set to 10 cm.

The percentage errors in the probability of failure (P_f) computed using various methods are shown in Figure 11. P_f computed using MCS with sample size 250,000 is considered to be the benchmark result against which all methods are compared. The same sample size was also used for all the projection schemes. The percentage error in P_f using FORM has been taken from Sudret and Kiureghian [59] and is shown here for the sake of comparison. It can be seen that the performance of the third-order projection schemes is comparable to FORM. The hybrid scheme SRBMIIPCI gives the best result for this problem. This suggests that the hybrid scheme based on the strong Galerkin condition gives good approximations for the tail of the pdf. Note that none of the projection schemes make use of the most probable point (MPP) of failure. Further, it should be pointed out here that the performance of SRBMs in reliability analysis can be potentially improved further by using the inverse of the stiffness matrix computed at the MPP as a preconditioner.

9 Concluding Remarks and Future Directions

In this chapter, we presented an overview of the theoretical foundations of stochastic subspace projection schemes in the context of stochastic finite element analysis. Attention was focused on polynomial chaos (PC) expansion schemes, and stochastic reduced basis methods (SRBMs) that employ basis vectors spanning the preconditioned stochastic Krylov subspace. Numerical studies were presented for two-dimensional elasticity problems to compare the relative accuracies of the projection schemes. The results suggest that SRBMs based on the weak Galerkin condition give results which are comparable in accuracy to standard PC projection schemes, while incurring significantly lower computational cost. Further, it appears that the idea of hybridizing the preconditioned stochastic Krylov subspace representation with a PC decomposition of the undetermined coefficients can be used to develop highly accurate and efficient computational schemes for solving linear random algebraic equations. Hybrid projection schemes based on this idea are expected to be not only more accurate than PC projection schemes but also orders of magnitude faster, particularly for large-scale problems with many random variables.

In principle, the projection schemes developed for linear random algebraic equations can serve as the essential backbone for solving nonlinear algebraic equations. For example, the projection scheme outlined in the introduction for the discrete case could be applied irrespective of whether the problem is linear or nonlinear. However, very few papers on the application of stochastic subspace projection schemes to complex nonlinear systems exist in the literature. In comparison, perturbation methods can be readily applied to static and dynamic analysis of nonlinear stochastic systems when the coefficients of variation of the random variables are small; see, for example, [67] and the references therein. For a detailed exposition of the issues involved in applying PC projection schemes to nonlinear stochastic PDEs, the reader is referred to [43, 38, 45, 51, 52]. However, SRBMs are yet to be applied to nonlinear

problems. It is hoped that the ideas developed in the context of the PC projection approach can be leveraged to design efficient stochastic reduced basis projection schemes for nonlinear problems.

It is worth noting here that all the projection schemes presented in this chapter were based on the implicit assumption that a complete probabilistic characterization of the system uncertainties is available. Hence, further research is required to investigate how projection schemes can be employed to compute spectral- and probability-distribution-free upper bounds on various probabilistic indicators of the response of stochastic systems [68]. Further work is also required to extend projection schemes to cases where a combination of probabilistic and possibilistic models are used for representing uncertainty. These theoretical developments can potentially enable the application of projection schemes to problems where insufficient experimental data is available to properly quantify the probability distributions of the system parameters.

Acknowledgments

This research was supported by a grant from the Faculty of Engineering, Science and Mathematics at the University of Southampton. The author thanks Sachin Sachdeva for his assistance in producing the results presented here.

References

1. R. Ghanem, P. Spanos, Stochastic Finite Elements: A Spectral Approach, Springer-Verlag, 1991.
2. M. Kleiber, T.D. Hien, The Stochastic Finite Element Method: Basic Perturbation Technique and Computer Implementation, Chichester: John Wiley, 1992.
3. G.I. Schueller (editor), Special issue - A state-of-the-art report on computational stochastic mechanics, Prob. Engrg. Mech. 12 (1997) 197-321.
4. M. Shinozuka, C.M. Jan, Digital simulation of random processes and its applications, J. Sound Vibr., 25 (1972).
5. J.E. Hurtado, A.H. Barbat, Monte carlo techniques in computational stochastic mechanics, Arch. Comput. Methods Engrg. 5 (1998) 3-29.
6. R.Y. Rubinstein, Simulation and the Monte Carlo Method, Wiley, New York, 1981.
7. F. Yamazaki, M. Shinozuka, G. Dasgupta, Neumann expansion for stochastic finite element analysis, J. Engrg. Mech. 114 (1988), 1335-1355.
8. T.D. Hien, M. Kleiber, Stochastic finite element modelling in linear transient heat transfer, Comput. Meth. Appl. Mech. Engrg., 144 (1997) 111-124.
9. M. Kaminski, Stochastic second-order perturbation approach to the stress-based finite element method, Int. J. Solids Struct., 38 (2001) 3831-3852.
10. A. Haldar, S. Mahadevan, Reliability Assessment using Stochastic Finite Element Analysis, John Wiley and Sons, 2000.
11. M.R. Rajashekhar, B.R. Ellingwood, A new look at the response surface method for reliability analysis, Structural Safety, 12 (1993) 205-220.
12. P.B. Nair, A. Choudhury, A.J. Keane, A Bayesian framework for uncertainty analysis using deterministic black-box simulation codes, AIAA Paper 2001-1676, April 2001.
13. P.B. Nair, A. Choudhury, A.J. Keane, Some greedy algorithms for sparse regression and classification with Mercer kernels, J. Maching Learning Research, 3 (2002) 781-801.
14. I. Elishakoff, Y.J. Ren, Finite Element Methods for Structures With Large Stochastic Variations, Oxford University Press, Oxford, 2003.
15. P.B. Nair, A.J. Keane, Stochastic reduced basis methods, AIAA J. 40 (2002) 1653-1664.

16. H.G. Matthies, C.E. Brenner, C.G. Bucher, C.G. Soares, Uncertainties in probabilistic numerical analysis of structures and solids - stochastic finite elements, Structural Safety, 19 (1997) 283-336.
17. G. Falsone, N. Impollonia, A new approach for the stochastic analysis of finite element modelled structures with uncertain parameters, Comput. Methods Appl. Mech. Engrg. 191 (2002) 5067-5085.
18. N. Wiener, The homogeneous chaos, Amer. J. Math. 60 (1938) 897-936.
19. N. Wiener, Nonlinear Problems in Random Theory, MIT Press, 1958.
20. P.B. Nair, On the theoretical foundations of stochastic reduced basis methods, AIAA Paper 2001-1677, April 2001.
21. M. Loeve, Probability Theory, Fourth Edition, Springer, 1977.
22. L.J. Gibson, M.F. Ashby, Cellular Solids: Structure and Properties, Second Edition, Cambridge University Press, Cambridge, U.K, 1997.
23. K. Alzebdeh, M. Ostoja-Starzewski, On a spring-network model and effective elastic moduli of granular materials, J. Appl. Mech. 66 (1999) 172-180.
24. L. Huyse, M.A. Maes, Random field modeling of elastic properties using homogenization, J. Engrg. Mech. 127 (2001) 27-36.
25. M. Ostoja-Starzewski, Micromechanics as a basis of continuum random fields, Appl. Mech. Rev. 47 (1994) 221-230.
26. M. Ostoja-Starzewski, X. Wang, Stochastic finite elements as a bridge between random material microstructure and global response, Comp. Meth. Appl. Mech. Eng. 168 (1999) 35-49.
27. O.C. Zienkiewicz, R.L. Taylor, Finite Element Method: Volume 1, Butterworth Heinemann, London, 2000.
28. C.-C. Li, A. Der Kiureghian, Optimal discretization of random fields, J. Engrg. Mech., ASCE, 119 (1993) 1136-1154.
29. K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, Cambridge, 1997.
30. S.P. Huang, S.T. Quek, K.K. Phoon, Convergence study of the truncated Karhunen-Loeve expansion for simulation of stochastic processes, Int. J. Numer. Meth. Engrg. 52 (2001) 1029-1043.
31. B. Sudret, A. Der Kiureghian, Stochastic Finite Elements and Reliability: a state-of-the-art Report, Technical Report No. UCB/SEMM-2000/08, University of California, Berkeley, 173 p.
32. D. Xiu, G.E. Karniadakis, The Wiener-Askey polynomial chaos for stochastic differential equations, SIAM J. of Sci. Comput. 24 (2002) 619-644.
33. D. Lucor, D. Xiu, C.-H. Su, G.E. Karniadakis, Predictability and uncertainty in CFD, Int. J. Num. Meth. Fluids, 43 (2003) 483-505.
34. R.H. Cameron, W.T. Martin, The orthogonal development of nonlinear functionals in series of Fourier-Hermite functions, Ann. Math. 48 (1947) 385-392.
35. T.Y. Hou, H. Kim, B. Rozovskii, H.-M. Zhou, Wiener chaos expansions and numerical solutions of randomly forced equations of fluid mechanics, in preparation.
36. R.G. Ghanem, R. Kruger, Numerical solution of spectral stochastic finite element systems, Comput. Meth. Appl. Mech. Engrg. 129 (1996) 289-303.
37. M. Pellissetti, R. Ghanem, Iterative solution of systems of linear equations arising in the context of stochastic finite elements, Adv. Engrg. Softw. 31 (2000) 607-616.
38. M. Anders, M. Hori, Three-dimensional stochastic finite element method for elasto-plastic bodies, Int. J. Numer. Meth. Engrg. 51 (2001) 449-478.
39. R. Ghanem, The nonlinear Gaussian spectrum of log-normal stochastic processes and variables, J. Appl. Mech. 66 (1999) 964-973.
40. R. Ghanem, P. Spanos, A stochastic Galerkin expansion for nonlinear random vibration analysis, Prob. Engrg. Mech. 8 (1993) 255-264.
41. R. Ghanem, V. Brzkala, Stochastic finite element analysis of randomly layered media, J. Engrg. Mech. 122 (1996) 361-369.
42. R. Ghanem, Probabilistic characterization of transport in heterogeneous media. Comput. Meth. Appl. Mech. Engrg. 158 (1998) 199-220.
43. M. Anders, M. Hori, Stochastic finite element method for elasto-plastic body, Int. J. Numer. Meth. Engrg. 46 (1999) 1897-1916.

44. D. Ghiocel, R. Ghanem, Stochastic finite element analysis of seismic soil-structure interaction, J. Engrg. Mech. 128 (2002) 66-77.
45. O. Le Maitre, O. Knio, H. Najm, R. Ghanem, A stochastic projection method for fluid flow: Basic formulation, J. Comput. Phy. 173 (2001) 481-511.
46. A. Sarkar, R. Ghanem, Mid-frequency structural dynamics with parameter uncertainty, Comput. Meth. Appl. Mech. Engrg. 191 (2002) 5499-5513.
47. G.D. Manolis, C.Z. Karakostas, A Green's function method to SH-wave motion in random continuum, Engrg. Anal. Boundary Elements. 27 (2003) 93-100.
48. R. Ghanem, Ingredients for a general purpose stochastic finite elements implementation, Comp. Meth. Appl. Mech. Engrg. 168 (1999) 19-34.
49. M.K. Deb, I.M. Babuska, J.T. Oden, Solution of stochastic partial differential equations using Galerkin finite element techniques," Comput. Meth. Appl. Mech. Engrg. 190 (2001) 6359-6372.
50. H.G. Matthies, C.G. Bucher, Finite element for stochastic media problems, Comput. Meth. Appl. Mech. Engrg, 168 (1999) 3-17.
51. H.G. Matthies, A. Keese, Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations, Informatikbericht Nr.: 2003-08, Insitute of Scientific Computing, Technical University Braunschweig, Brunswick, Germany, also submitted to Comput. Meth. Appl. Mech. Engrg, (2003).
52. A. Keese, H.G. Matthies, Numerical methods and Smolyak quadrature for nonlinear stochastic partial differential equations, Informatikbericht Nr.: 2003-05, Insitute of Scientific Computing, Technical University Braunschweig, Brunswick, Germany, May 2003, also submitted to SIAM J. Sci. Comput.
53. A. Keese, A review of recent developments in the numerical solution of stochastic partial differential equations (stochastic finite elements), Informatikbericht Nr.: 2003-06, Insitute of Scientific Computing, Technical University Braunschweig, Brunswick, Germany, October 2003.
54. D. Xiu, G.E. Karniadakis, Modeling uncertainty in steady-state diffusion problems via generalized polynomial chaos, Comput. Meth. Appl. Mech. Engrg. 191 (2002) 4927-4948.
55. D. Xiu, G.E. Karniadakis, Stochastic modeling of flow-structure interactions using generalized polynomial chaos, J. Fluids Engrg. 124 (2002) 51-59.
56. D. Xiu, G.E. Karniadakis, Modeling uncertainty in flow simulations via generalized polynomial chaos, J. Comput. Phy. 187 (2003) 137-167.
57. D. Xiu, G.E. Karniadakis, A new stochastic approach to transient heat conduction modeling with uncertainty, Int. J. Heat Mass Transfer, 46 (2003) 4681-4693.
58. L. Mathelin, M.Y. Hussaini, A stochastic collocation algorithm for uncertainty analysis, NASA/CR-2003-212153, February 2003.
59. B. Sudret, A. Der Kiureghian, Comparison of finite element reliability methods, Prob. Engrg. Mech. 17 (2003) 337-348.
60. Y. Saad, Iterative Methods for Sparse Linear Systems, Second Edition, SIAM, Philadelphia, 2003.
61. Y. Saad, H.A. Van der Vorst, Iterative Solution of Linear Systems in the 20-th Century, J. Comp. and Appl. Math. 123 (2000) 1-33.
62. S.K. Sachdeva, P.B. Nair, A.J. Keane, Comparative study of projection schemes for stochastic finite element analysis, Comput. Meth. Appl. Mech. Engrg. (2003) submitted for review.
63. S.K. Sachdeva, P.B. Nair, A.J. Keane, Higher-order projection schemes for stochastic finite element analysis, in preparation.
64. M.T. Bah, P.B. Nair, A. Bhaskar, A.J. Keane, Forced response analysis of mistuned bladed disks: a stochastic reduced basis approach, J. Sound Vibr. 263 (2003) 377-397.
65. P. McCullagh, Tensor Methods in Statistics, Chapman and Hall, New York, 1987.
66. A.J. Izenman, Recent developments in nonparametric density estimation, J. Am. Stat. Assoc. 86 (1991) 205-224.
67. N. Impollonia, G. Muscolino, Static and dynamic analysis of non-linear uncertain structures, Meccanica 37 (2002) 179-192.
68. G. Deodatis, L. Graham-Brady, R. Micaletti, A hierarchy of upper bounds on the response of stochastic systems with large variation of their properties: random field case, Prob. Engrg. Mech. 18 (2003) 365-375.