Stochastic meshfree projection scheme for flow in random porous media

Prasanth B. Nair

Computational Engineering and Design Group,
University of Southampton, Highfield, Southampton, SO17 1BJ, UK.
P.B.Nair@soton.ac.uk

Summary. We present a stochastic meshfree projection scheme for analysis of flow in random porous media. The random field describing the log-permeability is discretized using the Karhunen-Loève expansion scheme. We perform spatial discretization of the governing equations by representing the pressure distribution using an expansion in terms of radial basis functions with undetermined random coefficients. Subsequently, we apply a collocation scheme over physical space, which leads to a system of linear random algebraic equations. The algebraic equations are solved by Galerkin projection on to a preconditioned stochastic Krylov subspace. This enables the possibility of accurately estimating the statistics of the pressure distribution at a computational cost only marginally higher than the solution of the deterministic version of the governing equations. We present numerical results for two test cases to illustrate the performance of the proposed approach.

Key words: Random porous media, stochastic diffusion, stochastic projection, meshfree methods.

1 INTRODUCTION

In this paper, we consider the problem of incompressible flow in random porous media. Using the continuity equation and Darcy's law, it can be shown that the steady-state pressure distribution in the domain is governed by the following stochastic diffusion equation [1]

$$\begin{aligned} \nabla\left[\kappa(x;\omega)\nabla p(x;\omega)\right] &= f(x;\omega), \quad (x;\omega) \in \mathcal{D}\times\Omega \\ \mathcal{B}p(x;\omega) &= g(x;\omega), \ (x;\omega) \in \partial\mathcal{D}\times\Omega, \end{aligned} \tag{1}$$

where $\kappa(x;\omega)$ denotes the spatially varying permeability field and $p(x;\omega)$ is pressure. $\mathcal{D}$ is a bounded domain in $\mathbb{R}^d$ and $\partial\mathcal{D}$ denotes its boundary. $\mathcal{B}$ is an operator representing the boundary conditions imposed on $p(x;\omega)$. Ω is a probability space. $f(x;\omega)$, $g(x;\omega)$ and $\kappa(x;\omega)$ are real valued functions on $\mathcal{D}\times\Omega$. Note that we use the symbol ω or $\boldsymbol{\theta}$ to indicate the dependence of any function on a random dimension.

Numerical solution of (1) involves computing the statistics of the pressure distribution $p(x;\omega)$, given the statistics of the permeability κ and the functions f and g. In this probabilistic model, it is implicitly assumed that random field models for κ, f and g are available. For simplicity of presentation, we shall henceforth assume that only the permeability $\kappa(x;\omega)$ is random, i.e., both the source term and boundary conditions are deterministic.

Stochastic approaches to flow and transport in random porous media have been extensively studied over the last two decades; see, for example, the recent monograph by Zhang [1]. A straightforward approach for solving (1) involves the application of Monte Carlo simulation techniques or its variants [2]. In this approach, a deterministic solver based on a standard spatial discretization scheme such as the finite difference or finite element method is first created to solve (1) given a realization of the permeability field κ. This solver can then be run for many realizations of the random field $\kappa^{(1)}, \kappa^{(2)}, \ldots, \kappa^{(Q)}$ to generate the corresponding pressure distributions, say $p^{(1)}, p^{(2)}, \ldots, p^{(Q)}$. Subsequently, the statistics of p can be approximated to an arbitrary degree of accuracy given sufficient number of samples Q. Clearly, the major drawback of simulation techniques is the high computational cost incurred when a fine grid is used in the deterministic solver and the number of samples is large.

Perturbation-based moment equation approaches are computationally cheaper alternatives for approximating the statistics of p; see, for example, [2, 3]. However, the results can be highly inaccurate when the variability in κ is significant [2]. More recently, polynomial chaos projection schemes have been proposed in the literature for solving (1); see, for example, Ghanem [4]. Even though this approach is more accurate than perturbation methods, the computational cost can be prohibitive when the correlation length of $\kappa(x;\omega)$ is small.

In this paper, we present a meshfree stochastic projection scheme for solving (1) which gives significantly more accurate results than perturbation methods, while requiring computational effort only marginally higher than solving a deterministic version of (1) with $\langle\kappa\rangle$, where $\langle . \rangle$ denotes the ensemble average operator. The present approach is based on stochastic reduced basis projection schemes proposed recently in the literature for solving stochastic operator problems in computational mechanics; see, for example, Nair and Keane [6], and Nair [7]. We first discretize the lognormal random field representing uncertainty in $\kappa(x;\omega)$ using the Karhunen-Loève (KL) expansion in conjunction with the polynomial chaos decomposition scheme [5, 7]. For spatial discretization, the random pressure distribution $p(x;\omega)$ is represented using an expansion in terms of radial basis functions (RBF) with random coefficients. A collocation scheme in space is then applied to derive a system of linear random algebraic equations for the coefficients in the RBF expansion. This system of random algebraic equations is then solved by Galerkin projection onto a preconditioned stochastic Krylov subspace [6, 7]. Numerical results are presented to illustrate the performance of the algorithms.

2 STOCHASTIC MESHFREE FORMULATION

We start with the standard assumption that the permeability field has a lognormal distribution, i.e., $\kappa(x;\omega) = \exp(h(x;\omega))$, where $h(x;\omega)$ is a Gaussian random field with correlation function $R_h(x,y)$. Applying the KL expansion scheme [5] to $h(x;\omega)$, we have

$$\kappa(x;\omega) = \exp\left(\langle h(x;\omega)\rangle + \sum_{i=1}^{M} \sqrt{\lambda_i}\phi_i(x)\theta_i(\omega)\right), \tag{2}$$

where λ_i and $\phi_i(x)$ are the eigenvalues and eigenfunctions, respectively, of the correlation function $R_h(x,y)$, and $\theta_i(\omega), i = 1, 2, \ldots, M$ are uncorrelated Gaussian random variables.

Due to the Cameron-Martin theorem [8], a mean-square convergent expansion (also known as the Weiner chaos decomposition [5]) of the permeability field can be written as

$$\kappa(x;\omega) = \sum_{i=0}^{N} \kappa_i(x)\Gamma_i(\boldsymbol{\theta}), \tag{3}$$

where $\Gamma_i(\boldsymbol{\theta}), i = 0, 1, \ldots, N$ are Hermite polynomials in $\boldsymbol{\theta} = \{\theta_1(\omega), \theta_2(\omega), \ldots, \theta_M(\omega)\} \in \mathbb{R}^M$.

$$\left[A_0 + \sum_{i=1}^{N} A_i \Gamma_i(\boldsymbol{\theta}) \right] \boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\xi} - b \perp \psi_j(\boldsymbol{\theta}), \; \forall \, j = 1, 2, \ldots, m. \tag{9}$$

The Galerkin condition results in a deterministic, reduced-order, $m \times m$ linear algebraic system of equations in $\boldsymbol{\xi}$ of the form

$$\left\langle \boldsymbol{\Psi}^T A_0 \boldsymbol{\Psi} + \sum_{i=1}^{N} \Gamma_i(\boldsymbol{\theta}) \boldsymbol{\Psi}^T A_i \boldsymbol{\Psi} \right\rangle \boldsymbol{\xi} = \langle \boldsymbol{\Psi}^T b \rangle. \tag{10}$$

Solving the preceding system of equations and substituting the computed value of $\boldsymbol{\xi}$ into (8), we arrive at an explicit expression for $\boldsymbol{\alpha}$ in terms of the random variables arising from discretization of the permeability field. Hence, using the stochastic reduced basis approximation for $\boldsymbol{\alpha}$ in conjunction with the RBF expansion for $p(x;\omega)$ in (4), the statistics of the pressure distribution can be efficiently computed. For example, the mean pressure can be written as

$$\langle p(x;\omega) \rangle = \sum_{i=1}^{n} \langle \alpha_i(\boldsymbol{\theta}) \rangle K(||x - x_i||), \text{ where } \langle \boldsymbol{\alpha}(\boldsymbol{\theta}) \rangle = \langle \boldsymbol{\Psi} \rangle \boldsymbol{\xi}. \tag{11}$$

Similarly, the covariance function of the pressure distribution can be written as

$$R_p(y,z) = \langle p(y;\omega) p(z;\omega) \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \alpha_i(\boldsymbol{\theta}) \alpha_j(\boldsymbol{\theta}) \rangle K(||y - x_i||) K(||z - x_j||), \tag{12}$$

where $\langle \alpha_i(\boldsymbol{\theta}) \alpha_j(\boldsymbol{\theta}) \rangle$ is the ijth element of the matrix $\langle \boldsymbol{\alpha}(\boldsymbol{\theta}) \boldsymbol{\alpha}(\boldsymbol{\theta})^T \rangle = \langle \boldsymbol{\Psi} \boldsymbol{\xi} \boldsymbol{\xi}^T \boldsymbol{\Psi}^T \rangle$.

3 NUMERICAL RESULTS

In this section, we present numerical studies on one- and two-dimensional test problems. The first example considered is the following stochastic differential equation [10]

$$\frac{d}{dx} \kappa(x;\omega) \frac{d}{dx} p(x;\omega) = 0, \quad x \in [0,1], \tag{13}$$

with the boundary conditions $p(0;\omega) = 0$ and $p(1;\omega) = 1$. The random diffusivity has the form $\kappa(x;\omega) = 1 + \epsilon(\omega)x$, where $\epsilon(\omega)$ is a Gaussian random variable with standard deviation σ_ϵ. The exact solution of this problem is given as

$$\begin{aligned} p_e(x;\omega) &= \frac{\log[1 + \epsilon(\omega)x]}{\log[1 + \epsilon(\omega)]}, \quad \text{for } \epsilon(\omega) \neq 0. \\ &= x, \quad \text{for } \epsilon(\omega) = 0. \end{aligned} \tag{14}$$

We use Gaussian RBFs for meshfree approximation with 10 uniformly distributed collocation points. Figure 1 compares the standard deviation of $p(x;\omega)$ computed using the stochastic reduced basis projection schemes with the exact values for $\sigma_\epsilon = 0.1$ and 0.2. In the figure, SRBM1 and SRBM2 denote results obtained using two and three basis vectors, respectively. It can be seen that the statistics computed using SBRM2 agree well with the exact results for this problem.

In the second example, we consider the two-dimensional model problem on the unit square $[0,1] \times [0,1]$ described in [2]. We model the log-permeability as a homogenous Gaussian random field with

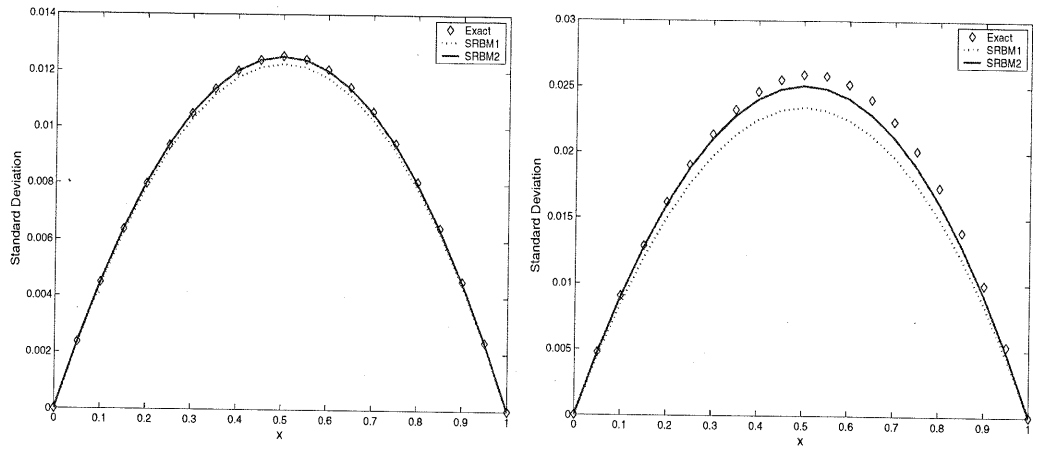

Fig. 1. Comparison of standard deviation of $p(x;\omega)$ computed using stochastic reduced basis methods with exact values for $\sigma_\epsilon = 0.1$ and 0.2.

covariance function $\sigma_h^2 \exp\{-(x-x')/b_1 - (y-y')/b_2\}$. We conducted numerical studies to investigate the performance of the stochastic meshfree approach for $\sigma_h^2 = 0.1$ and 0.2 with $b_1 = b_2 = 1.0$. Ten terms are retained in the KL expansion of the log-permeability field, i.e., $M = 10$. We use Gaussian RBFs for meshfree approximation and an uniformly distributed set of 11×11 collocation points.

Since no exact solutions are available for this problem, we generated benchmark results using Monte Carlo simulation (MCS) by solving (7) exactly for 2000 realizations of $\boldsymbol{\theta}$. Figure 2 compares the standard deviation of pressure along the x direction at $y = 0.5$ computed using stochastic projection schemes and MCS for $\sigma_h^2 = 0.1$ and 0.2. It can be seen that the projection scheme with three basis vectors (SRBM2) shows good agreement with MCS.

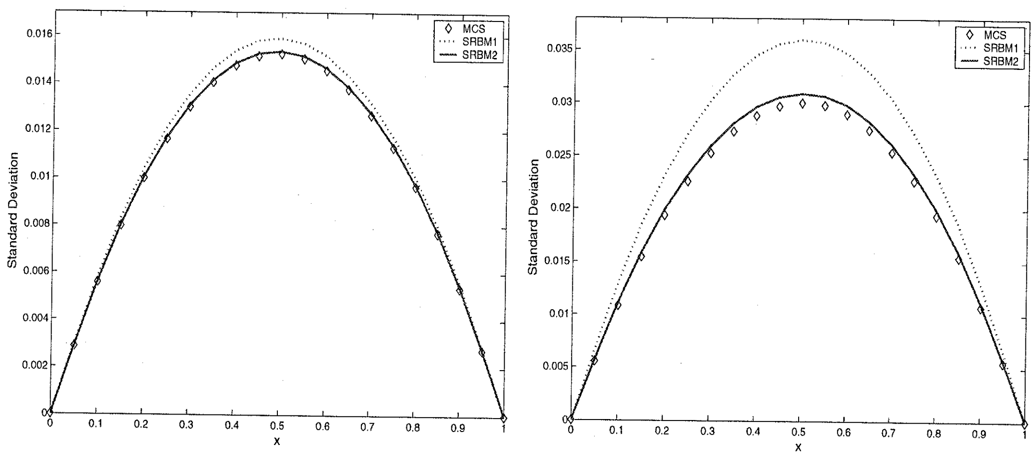

Fig. 2. Comparison of standard deviation of pressure along x direction at $y = 0.5$ computed using MCS and stochastic projection schemes for $\sigma_h^2 = 0.1$ and 0.2.

4 CONCLUDING REMARKS

In this paper, we presented a stochastic meshfree approach for studying flow in random porous media. Numerical studies on model problems indicate that the proposed approach gives accurate results using few stochastic basis vectors ($m << n$). Hence, the computational cost of the stochastic projection scheme is only marginally higher than the solution of a deterministic version of the governing equation with $\langle \kappa(x;\omega) \rangle$.

Further studies are required to investigate the performance of the projection scheme for problems with smaller correlation length and higher variability. In such cases, a few hundred terms will result from the Karhunen-Loève expansion and a dense set of collocation points would be required to resolve the pressure distribution. It is expected that such problems can be solved with high accuracy using a larger number of stochastic basis vectors.

References

1. Zhang, D., *Stochastic Methods for Flow in Porous Media: Coping with Uncertainties*, Academic Press, 2002.
2. Li, L., Tchelepi, H.A., and Zhang. D., "Perturbation-based Moment Equation Approach for Flow in Heterogeneous Porous Media: Applicability Range and Analysis of Higher-Order Terms" *Journal of Computational Physics*, Vol. 188, 2003, pp. 296-317.
3. Tartakovsky, D.M., and Guadagnini, A., "Prior Mapping for Nonlinear Flow in Random Environments," *Physical Review E*, Vol. 64, 2001, pp. 035302:1-4.
4. Ghanem, R., "Scale of Fluctuation and the Propagation of Uncertainty in Random Porous Media," *Water Resources Research*, Vol. 34, 1998, pp. 2123.
5. Ghanem, R., and Spanos, P.D., *Stochastic Finite Elements: A Spectral Approach*, Springer, 1991.
6. Nair, P.B., and Keane, A.J., "Stochastic Reduced Basis Methods," *AIAA Journal*, Vol. 40, 2002, pp. 1653-1664
7. Nair, P.B., "Projection Schemes in Stochastic Finite Element Analysis," *CRC Engineering Design Reliability Handbook*, editors: D.M. Ghiocel and E. Nikolaidis, CRC Press, Boca Raton, FL, 2004.
8. Cameron, R.H., and Martin, W. T., "The Orthogonal Development of Nonlinear Functionals in Series of Fourier-Hermite Functions," *Annals of Mathematics*, Vol. 48, 1947, pp. 385-392.
9. Hon, Y. C., and Schaback, R., "On Unsymmetric Collocation by Radial Basis Functions," *Applied Mathematics and Computation*, Vol. 119, 2001, pp. 177-186.
10. Xiu, C., and Karniadakis, G. E., "Modeling Uncertainty in Steady-State Diffusion Problems via Generalized Polynomial Chaos," *Computer Methods in Applied Mechanics and Engineering*, Vol. 191, 2002, pp. 4927-4948.