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Instability of thermoelastic contact for two half-planes sliding out-of-plane with contact resistance and frictional heating

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Abstract

Thermoelastic contact is known to show instabilities when the heat transmitted across the interface depends on the pressure, either because of a pressure-dependent thermal contact resistance $R(p)$ or because of frictional heating due to the product of friction coefficient, speed, and pressure, fVp. Recently, the combined effect of pressure-dependent thermal contact resistance and frictional heating has been studied in the context of simple rod models or for a more realistic elastic conducting half-plane sliding against a rigid perfect conductor "wall". Because $R(p)$ introduces a non-linearity even in full contact, the "critical speed" for the uniform pressure solution to be unstable depends not just on material properties, and geometry, but also on the heat flux and on pressure.

Here, the case of two different elastic and conducting half-planes is studied, and frictional heating is shown to produce significant effects on the stability boundaries with respect to the Zhang and Barber (J. Appl. Mech. 57 (1990) 365) corresponding case with no sliding. In particular, frictional heating makes instability possible for a larger range of prescribed temperature drop at the interface including, at sufficiently high speeds, the region of opposite sign of that giving instability in the corresponding static case. The effect of frictional heating is particularly relevant for one material combinations of the Zhang and Barber (J. Appl. Mech. 57 (1990) 365) classification (denominated class b here), as above a certain critical speed, the system is unstable *regardless* of temperature drop at the interface.

Finally, if the system has a prescribed heat flow into one of the materials, the results are similar, except that frictional heating may also become a stabilizing effect, if the resistance function and the material properties satisfy a certain condition.

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1. Introduction

Frictional heat generation in sliding systems gives rise to instabilities for a certain critical product of friction coefficient and sliding speed, fV_{cr}, leading to localization of the contact in hot spots, a phenomenon known as *frictionally-excited thermoelastic instability* (TEI) (Barber, 1969; Dow and Burton, 1972; Bryant et al., 1995). Burton et al. (1973) were the first to use a perturbation method to define an eigenvalue problem for the stability of one the simplest models, i.e. the contact between two sliding half-planes, for which the uniform pressure solution becomes unstable when the critical speed V_{cr} makes the real part of the growth rate of the perturbation to be zero.

Generally, the influence of thermal contact resistance R is neglected in these studies, and either zero resistance (perfect thermal contact) or infinite resistance (open gap) is assumed to occur at the interface, and the critical condition is a function only of geometrical and material characteristics. However, even if there is full contact between the two bodies, there will generally be a thermal contact resistance at the interface because nominally flat surfaces are always rough at the microscopic scale giving rise to constriction resistance (Shlykov and Ganin, 1964; Thomas and Probert, 1970). Also, just the process of sliding of two bodies can lead to temperature differences between the bodies. Early papers (Blok, 1937; Jaeger, 1942), studying the temperature rise in sliding contact assume a single contact patch between two half-spaces at zero temperature at infinity. However, if we consider the half-spaces to represent finite bodies whose dimensions are large compared with the size of the contact area, there is no reason to suppose that the temperatures at infinity will be the same. In essence, the temperature at infinity becomes a measure of what is usually referred to as the 'bulk temperature' in Tribology papers (as distinct from the flash temperature). In two papers, Barber (1967, 1970) made clear the point that in most cases the heat transfer from the distant boundaries of the sliding bodies will cause the bulk temperatures of the two bodies to be different, giving a temperature discontinuity across the interface except in regions of actual contact. The same idea of a temperature jump was introduced in a paper by Ling and Simkins (1963). The problem is complicated by the fact that the heat is not necessarily generated exactly at the interface, but may be generated in surface layers on one side or the other of the constriction resistance. Barber (1967, 1970) made some crude calculations for the partition of frictional heating, which are further developed in Berry (1976), and the context of numerical models by Johansson (1993), and Johansson and Klarbring (1993) and references therein.

From a more mathematical perspective, contact resistance had been postulated also with the intent of solving some spurious non-existence of steady-state solutions for static contact when the heat flows into the material with the lower distortivity. This was shown to arise from the impossibility of a transition between perfect thermal contact and separation with complete insulation (Comninou and Dundurs, 1979). Existence of a steady state is proved by Duvaut (1979) for the general three-dimensional thermoelastic contact problem, and recently extended also in the case of frictional heating for sufficiently small speeds (Andersson et al., 2003), suggesting that for large enough speeds, existence cannot be guaranteed by introducing the resistance, and is an intrinsic feature of thermoelastic contact, probably associated with the possibility of seizure, i.e. the growth of pressure without limit (if displacements are constrained). However, contact resistance gives rise to another source of instability, even in the absence of sliding. For example, Barber (1987) studied the contact of two half-planes and Zhang and Barber (1990) were able to characterize the instability as a function of material properties, showing that the materials can be classified into 5 classes, depending on three dimensionless ratios of the material properties: most cases exhibit instability only for heat flow into the more distortive material; but for some material combinations, instability can occur for either direction of heat flow, and the growth rate can be complex, involving oscillatory growth of the perturbation. In particular, in some conditions it was shown that unique solutions can be unstable, thus justifying also long-term oscillatory behaviors.

More recently, in an attempt to study the interaction of frictional heating and contact resistance for the instability condition in half-plane contact, Afferrante and Ciavarella (2004) studied the case where one of the half-planes is a rigid perfectly conducting wall at fixed temperature. The critical sliding speed, at which the system becomes unstable, was found to decrease with heat flux q_0 if $q_0 < 0$, i.e. heat flow is directed into the more distortive material, the imperfect conductor half-plane, *only in the case* $(R_0 + p_0 R') > 0$, where p_0, R_0 are the values of the pressure and the resistance at the steady state, and R' the gradient of the resistance with pressure. If the temperature drop between the wall and the half-plane, $(T_W - T_H)$, is prescribed, then the critical speed increases with the temperature drop if $(T_W - T_H) < 0$, independently on the sign of $R_0 + p_0 R'$.

Here, we attempt the more general case of two sliding half-planes in the presence of contact resistance. Given the results cited in the previous 2 papers, Afferrante and Ciavarella (2004) and Zhang and Barber (1990), we expect the classification of instability behavior to depend both on the sliding speed and on temperature drop across the interface.

2. Formulation

Consider two half-planes sliding out-of-plane as in Fig. 1. The material constants are K_i, k_i, α_i, μ_i, v_i, for conductivity, diffusivity, coefficient of thermal expansion, elastic shear modulus and Poisson's ratio of material i. Because a contact resistance is postulated at the interface, it is generally necessary to define β, the partitioning parameter of frictional heat generation (see Barber, 1967, 1970; Johansson, 1993; Johansson and Klarbring, 1993 and references therein). Note the existence of a non-zero temperature difference $T_2 - T_1$ will cause the proportion of the frictional heat *flowing* into a given body to differ from the proportion *generated* in that body. Authors in Tribology sometimes confuse these quantities, so it is important to make it clear that our factor is a partition of frictional heat *generation*.

The heat flow q_1 is positive if entering into the half-plane 1 ($y > 0$), and

$$q_1 = \frac{T_2 - T_1}{R} + \beta f V p, \tag{1}$$

$$q_1 = -K_1 \left. \frac{\partial T_1}{\partial y} \right|_{y=0} \tag{2}$$

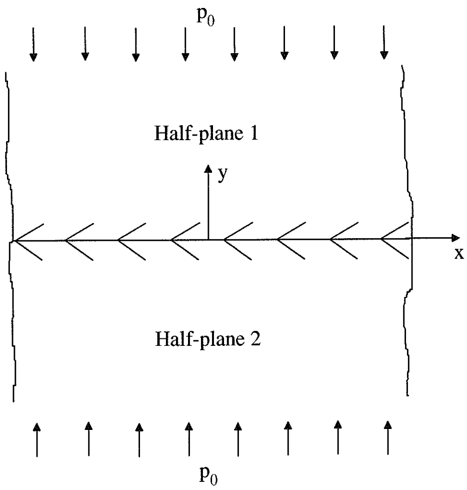

Fig. 1. Half-planes sliding out-of-plane.

for half-plane 1. For half-plane 2, we use the opposite notation (i.e. it is positive if exiting 2),

$$q_2 = \frac{T_2 - T_1}{R} - (1-\beta) f V p, \tag{3}$$

$$q_2 = -K_2 \left.\frac{\partial T_2}{\partial y}\right|_{y=0}, \tag{4}$$

where $T_2 - T_1$ is the temperature drop across the sliding interface. Obviously, q_2 is related to q_1 by the heat balance equation

$$q_1 - q_2 = f V p. \tag{5}$$

The stability of the system to small perturbation (the system being nonlinear) can be investigated by a linear perturbation analysis about the steady state. Therefore, by perturbing, for example, Eq. (1) we can write an equation in terms of $\Delta T = \Delta(T_2 - T_1)$, and using $\Delta R = R' \Delta p$,

$$\Delta q_1 R_0 = \Delta T + [-q_{10} R' + \beta f V (R_0 + p_0 R')] \Delta p, \tag{6}$$

where p_0, q_{10} and R_0 are the steady-state value for the contact pressure, heat flow into half-planes 1 and thermal contact resistance, respectively. Eq. (6) with zero sliding speed reduces to Eq. (27) in Barber (1987) (where q_1 was the heat flow *exchanged* at interface as there was no heat generated there), we note that the stability problems seem to be equivalent by replacing the multipliers of Δp.

Note that, q_1 is only the heat flow entering half-plane 1 and $q_2 = q_1 - f V p$ depends on frictional heating for fixed q_1. An alternative formulation is for a fixed temperature drop across the interface. Rewriting Eq. (6), using Eq. (1) we obtain

$$\Delta q_1 R_0 = \Delta T + \left[-\frac{T_{20} - T_{10}}{R_0} R' + \beta f V R_0 \right] \Delta p, \tag{7}$$

where T_{20} and T_{10} are the steady-state temperatures on the surfaces of the half-planes. We shall prefer this formulation for the first analysis, and we shall return to the case of prescribed heat flux into material 1 in a specific paragraph.

3. Stability analysis

Following Burton et al. (1973), the temperature perturbation in two half-planes can be written as

$$T_i(x, y, t) = A_i e^{bt - a_i y} \cos[mx], \tag{8}$$

where A_i $(i = 1, 2)$ are the arbitrary constants and

$$a_i = \pm \sqrt{m^2 + \frac{b}{k_i}} \tag{9}$$

and the sign + holds for $i = 1$ and the sign − for $i = 2$, because the perturbation must decay away the contact interface.

To obtain the thermoelastic stresses and displacements, following (Barber, 1987), in the half-plane 1 $(y > 0)$

$$\sigma_{yy_{\rm th}}|_{y=0} = -\frac{2\mu_1 \alpha_1 (1 + \nu_1) A_1 m}{(1 - \nu_1)(m + a_1)} e^{bt} \cos[mx], \tag{10}$$

$$u_{y_{\rm th}}|_{y=0} = 0 \tag{11}$$

and similarly, for the half-plane 2 $(y < 0)$

$$\sigma_{yy_{\rm th}}|_{y=0} = -\frac{2\mu_2 \alpha_2 (1 + \nu_2) A_2 m}{(1 - \nu_2)(m - a_2)} e^{bt} \cos[mx], \tag{12}$$

$$u_{y_{\rm th}}|_{y=0} = 0. \tag{13}$$

The full solution for stresses and displacements can be obtained by superposing an isothermal solution corresponding to each half-plane being loaded by a sinusoidal normal traction. Such isothermal solution can be written as

$$\sigma_{yy_{\rm iso}}|_{y=0} = B_i \cos[mx], \tag{14}$$

$$u_{y_{\rm iso}}|_{y=0} = \mp \frac{B_i (1 - \nu_i)}{m \mu_i} \cos[mx]. \tag{15}$$

In the latter equation we take sign − for $i = 1$ and sign + for $i = 2$. It follows that:

$$\sigma_{yy}|_{y=0} = \sigma_{yy_{\rm th}}|_{y=0} + \sigma_{yy_{\rm iso}}|_{y=0}, \tag{16}$$

$$u_y|_{y=0} = u_{y_{\rm th}}|_{y=0} + u_{y_{\rm iso}}|_{y=0}. \tag{17}$$

The boundary conditions at the contact interface ($y=0$) impose

$$u_{y_1}|_{y=0} = u_{y_2}|_{y=0}, \tag{18}$$

$$\sigma_{yy_1}|_{y=0} = \sigma_{yy_2}|_{y=0} \tag{19}$$

and the heat balance condition (5) depends on the pressure p, where

$$p = -\sigma_{yy_1}|_{y=0} = -\sigma_{yy_2}|_{y=0}. \tag{20}$$

Hence, from Eqs. (2) and (4) we write for the heat flow at interface ($y=0$)

$$q_1 = A_1 K_1 a_1 e^{bt} \cos[mx], \tag{21}$$

$$q_2 = A_2 K_2 a_2\, e^{bt} \cos[mx]. \tag{22}$$

Substituting Eqs. (16) and (17) in Eqs. (18)–(22) in Eq. (5) we obtain the following equations:

$$\frac{B_1(1-\nu_1)}{\mu_1} + \frac{B_2(1-\nu_2)}{\mu_2} = 0, \tag{23}$$

$$-\frac{2\mu_1\delta_1 K_1 A_1 m}{(1-\nu_1)(m+a_1)}\, e^{bt} + B_1 = -\frac{2\mu_2\delta_2 K_2 A_2 m}{(1-\nu_2)(m-a_2)}\, e^{bt} + B_2, \tag{24}$$

$$(A_1K_1a_1 - A_2K_2a_2)e^{bt} = fV\left(\frac{2\mu_1\delta_1 A_1 m}{(1-\nu_1)(m+a_1)}\, e^{bt} - B_1\right), \tag{25}$$

where $\delta_i = \alpha_i(1+\nu_i)/K_i$ $(i=1,2)$ is the *distortivity* of material i. Eliminating the constants A_i and B_i, the perturbations can be expressed in terms of the single constant

$$A = \left(K_2a_2 + 4M\,\frac{m\delta_2K_2fV}{m-a_2}\right)A_2 = \left(K_1a_1 - 4M\,\frac{m\delta_1K_1fV}{m+a_1}\right)A_1, \tag{26}$$

where

$$\frac{1}{2M} = (1-\nu_1)/\mu_1 + (1-\nu_2)/\mu_2. \tag{27}$$

Therefore,

$$\Delta T = A\left(\frac{m-a_2}{K_2a_2(m-a_2) + 4Mm\delta_2K_2fV} - \frac{m+a_1}{K_1a_1(m+a_1) - 4Mm\delta_1K_1fV}\right) e^{bt}\cos[mx], \tag{28}$$

$$\Delta q = q_1 = \frac{a_1(m+a_1)}{a_1(m+a_1) - 4Mm\delta_1 fV}\, e^{bt}\cos[mx], \tag{29}$$

$$\Delta p = -\sigma_{yy}|_{y=0} = 4M\left(\frac{\delta_1}{a_1(m+a_1) - 4Mm\delta_1 fV} + \frac{\delta_2}{a_2(m-a_2) + 4Mm\delta_2 fV}\right) Am\, e^{bt}\cos[mx], \tag{30}$$

where $\Delta T = T_2 - T_1$.

Substituting the above equation in Eq. (6), the characteristic equation is obtained as

$$4M[q_{10}R' - \beta f V(R_0 + p_0 R')]m\left(\frac{\delta_1}{a_1(m+a_1) - 4M\delta_1 mfV} + \frac{\delta_2}{a_2(m-a_2) + 4M\delta_2 mfV}\right) + \frac{R_0}{1 - \frac{4Mm\delta_1 fV}{(m+a_1)a_1}} + \frac{1}{K_1 a_1 - \frac{4Mm\delta_1 K_1 fV}{m+a_1}} - \frac{1}{K_2 a_2 + \frac{4Mm\delta_2 K_2 fV}{m-a_2}} = 0. \tag{31}$$

We shall introduce the same dimensionless parameters and notation as Zhang and Barber (1990),

$$R^* = mK_1R_0, \quad r_1 = \frac{k_2}{k_1}, \quad r_2 = \frac{\delta_2}{\delta_1}, \quad r_3 = \frac{K_2}{K_1},$$

$$z = \frac{b}{m^2 k_1}, \quad c_1 = \sqrt{1+z}, \quad c_2 = \sqrt{1 + \frac{z}{r_1}} \tag{32}$$

and additionally define a dimensionless speed

$$\hat{V} = 2MfV\delta_1/m. \tag{33}$$

Note that this dimensionless speed is chosen as to be equal to unity at the critical speed when body 2 is a rigid non-conductor and there is no resistance (Burton et al., 1973). With this notation, Eq. (31) can be rewritten as

$$\widehat{\Delta T} = F(z), \tag{34}$$

where $\widehat{\Delta T}$ is the dimensionless "interface temperature drop" or "heat flux at the interface" (since $R' < 0$, then $\widehat{\Delta T} > 0$ indicates a $T_{20} - T_{10} > 0$), defined by

$$\widehat{\Delta T} \equiv -4M\delta_1 K_1\left(\frac{T_{20} - T_{10}}{R_0}\right) R' \tag{35}$$

and

$$F(z) \equiv \frac{R^* + (1/c_1) + (1/r_3c_2) - c\hat{V}}{(1/c_1(1+c_1)) - (r_2/c_2(1+c_2))} \tag{36}$$

with

$$c \equiv 2\left[\frac{\beta}{c_1(1+c_1)} + \frac{r_2(1-\beta)}{c_2(1+c_2)}\right] R^* + \frac{2}{c_1c_2}\left[\frac{r_2}{1+c_2} + \frac{1/r_3}{1+c_1}\right]. \tag{37}$$

Note $\widehat{\Delta T}$ coincides to the dimensionless heat flux $\hat{Q}$ into material 1 in the static case ($\hat{V}=0$) defined by Zhang and Barber (1990). For $\hat{V}>0$ the heat flux into material 1 depends on $\widehat{\Delta T}$ and an additional term due to the frictional heating

$$[-4M\delta_1 K_1 q_{10} + mK_1\beta\hat{V}p_0]R' = \widehat{\Delta T} \tag{38}$$

Therefore, in exploring how the critical condition depends on frictional heating, we shall consider a given temperature drop at the interface, $\widehat{\Delta T}$. Note that $\widehat{\Delta T}>0$ indicates in the static case $q_{10}>0$, a positive heat flux into material 1, i.e. for $r_2=(\delta_2/\delta_1)<1$ the more distortive material, and for $r_2>1$, the less distortive material. For the sliding case, $\widehat{\Delta T}>0$ is better translated into "higher temperature" of the less distortive material for $r_2<1$, and of the more distortive material, for $r_2>1$.

4. Analysis of the function $F(z)$

To study the stability of the system, it is necessary to analyze the behavior of function (36) which has the form $F(z)=N(z)/D(z)$, and $z=x+iy$ is here the complex variable, representing the growth rate b; the condition $\operatorname{Re} b=0$ defines the critical condition. Next, we shall assume, without loss of generality, that the materials are arranged such that $r_1>1$.

4.1. Real roots

For real z ($y=0$), unstable real roots correspond to solutions of $F(x)=\widehat{\Delta T}$, with $x>0$. The numerator of the right-hand side of Eq. (36) is

$$N(x,\hat{V}) = R^* + \frac{1}{c_1} + \frac{1}{r_3 c_2} - c\hat{V}, \tag{39}$$

i.e. the difference between two positive decreasing functions with limiting behavior

$$N(0,\hat{V}) = \frac{2[1+r_3(1+R^*)] - \{1+r_3 r_2 + r_3[r_2(1-\beta)+\beta]R^*\}\hat{V}}{2r_3}, \tag{40}$$

$$N(\infty,\hat{V}) = R^*. \tag{41}$$

Further, the numerator N is a positive function for

$$\hat{V} < \hat{V}^* = 2\,\frac{1/r_3 + 1 + R^*}{1/r_3 + r_2 + [r_2(1-\beta)+\beta]R^*} \tag{42}$$

and it assumes negative values for small x and positive values for larger x, when $\hat{V}>\hat{V}^*$.

From Zhang and Barber (1990) we know that the denominator

$$D(x) = \frac{1}{c_1(1+c_1)} - \frac{r_2}{c_2(1+c_2)} \tag{43}$$

is a monotonically decreasing function of x, with limiting behavior

$$D(0) = (1 - r_2)/2, \tag{44}$$

$$D(\infty) \to (1 - r_1 r_2)/x. \tag{45}$$

Consequently the function $F(x)$ has limiting behavior

$$F(0) = \frac{2[1 + r_3(1 + R^*)] - \{1 + r_3 r_2 + r_3[r_2(1 - \beta) + \beta]R^*\}\hat{V}}{r_3(1 - r_2)} \tag{46}$$

which depends on speed and thermal contact resistance, whereas

$$F(\infty) = \frac{x}{1 - r_1 r_2} \tag{47}$$

is independent on $\hat{V}$ and R^*.

If we plot a graph of $F(x)$ against x, as in Fig. 2, solutions of $F(x) = \widehat{\Delta T}$ correspond to intersections of the curve $F(x)$ and the horizontal line $\widehat{\Delta T}$. The first such intersection will occur at *either* the origin, *or* infinity *or* at some intermediate point. If it occurs at an intermediate point, this must correspond to a local minimum of $F(x)$. Plotting the motion of these real roots on the complex plane, if a root arises because of a minimum in $F(x)$, it appears at some value of $\widehat{\Delta T}$ at a point on the positive real axis and bifurcates into a left-moving and a right-moving real zero as $\widehat{\Delta T}$ increases. The complete behavior of the roots of the characteristic equation must be continuous in the complex plane, so we conclude that at $\widehat{\Delta T}$ a bit lower than those where the double real root appears, there must be a pair of complex conjugate roots that converge on the real axis from either side at the critical condition. The continuous motion of these roots to the bifurcation point must have occurred by crossing the imaginary axis (if they had crossed through the real axis, there would have been already a real zero corresponding to a lower value of $F(0)$ than that at the minimum of the $F(x)$ curve).

A minimum in the $F(x)$ curve might arise if the function $F(x)$ is not monotonic. It is then possible to argue that the curve $F(x)$ has to pass continuously from $N(0)/D(0)$ to $N(\infty)/D(\infty)$, unless $D(x)$ has a zero in $(0, \infty)$, in which case $F(x)$ will pass through plus and minus infinity at the zero. Hence, when a minimum in the $F(x)$ curve arises, above the zero it must either (a) pass through a minimum on the way to $x \to \infty$, (b) decrease (increase for $F(\infty) < 0$) monotonically to a final minimum at infinity. The case (b) would imply infinite growth rate solutions. However, $F(x)$ is unbounded at infinity, so that an infinite $\widehat{\Delta T}$ would be needed to get infinite growth rates. Thus, the only possible case is (a) giving a complex root for the critical condition as explained above. In other words, this argument shows that in appropriate cases, the existence of a minimum in $F(x)$ demands that there be complex roots.

For solutions at the origin, $x = 0$ ($\mathrm{Re}\, b = \mathrm{Im}\, b = 0$) and the critical condition in Eq. (34) is real

$$\widehat{\Delta T}(1 - r_2) = 2[1/r_3 + (1 + R^*)] - \{1/r_3 + r_2 + [r_2(1 - \beta) + \beta]R^*\}\hat{V}. \tag{48}$$

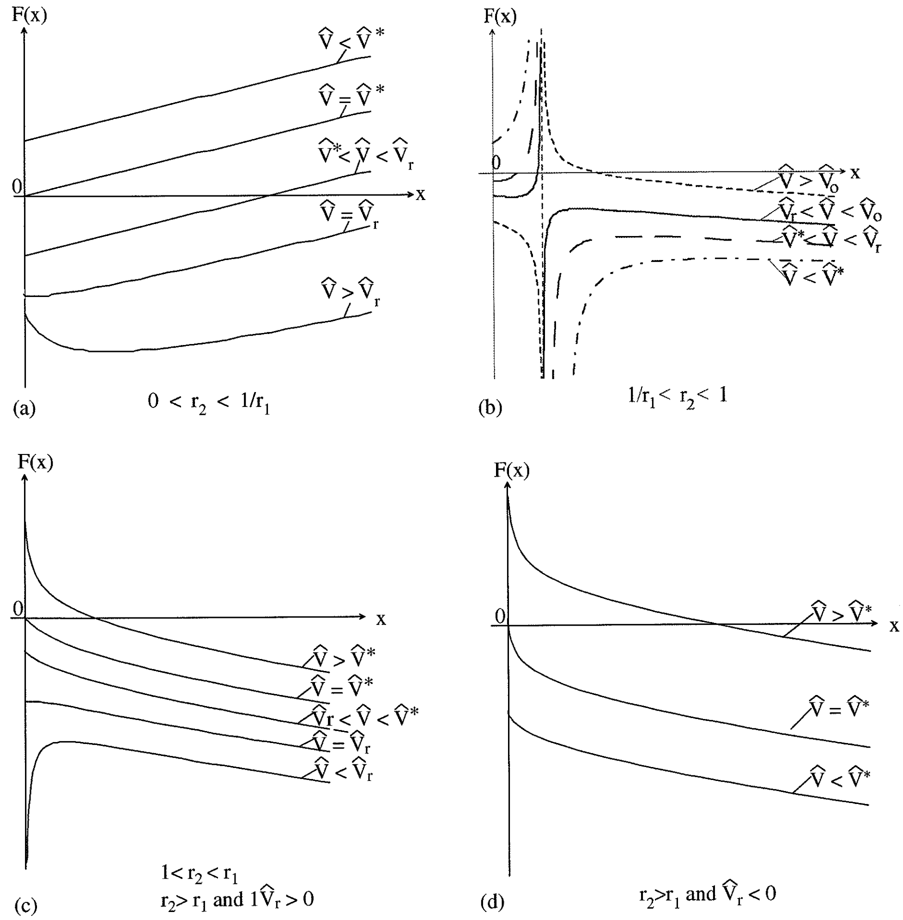

Fig. 2. Possible behaviors of function $F(x)$: (a) $0 < r_2 < 1/r_1$; (b) $1/r_1 < r_2 < 1$; (c) $1 < r_2 < r_1$ and $r_2 > r_1$ with $\hat{V}_r > 0$; and (d) ($r_2 > r_1$ with $\hat{V}_r < 0$).

However, even when a zero at the origin is possible, it may occur that the derivative of $|F|$ with respect to x at the origin is negative, indicating that zeros cannot reach the real axis through the origin because they occur on the positive real line at lower $|F|$ than those needed to produce a zero at the origin. Hence, the stability boundary will be determined by a real root if the derivative of $|F|$ at $x = 0$ has the same sign as the value of F at $x = 0$. Otherwise, it will be determined by a complex root.

By differentiating Eq. (36) and rearranging, we can show that $F'(0)$ has the same sign as the expression

$$\Gamma(R^*, \hat{V}) = \Gamma_0(R^*) - \hat{V} \times G(R^*), \tag{49}$$

where

$$\Gamma_0(R^*) = 3R^*\left(1-\frac{r_2}{r_1}\right)+3\left(1+\frac{1}{r_3}\right)\left(1-\frac{r_2}{r_1}\right)-2(1-r_2)\left(1+\frac{1}{r_1r_3}\right)$$
$$= 3R^*\left(1-\frac{r_2}{r_1}\right)+\left[2\left(1-\frac{1}{r_1}\right)+\left(1-\frac{r_2}{r_1}\right)\right]\frac{1}{r_3}$$
$$+3r_2\left(1-\frac{1}{r_1}\right)+(1-r_2) \tag{50}$$

and

$$G(R^*) = \frac{3}{2}r_2R^*\left(1-\frac{1}{r_1}\right)+\frac{r_2}{2r_3}\left(3-\frac{1}{r_1}\right)+r_2\left(\frac{1}{2}+r_2\right)$$
$$-\frac{1}{r_1}\left(\frac{1}{r_3}+\frac{3}{2}r_2\right). \tag{51}$$

Note that for $\hat{V}=0$, Eq. (49) becomes $\Gamma(R^*)=\Gamma_0(R^*)$, coinciding with Zhang and Barber (1990).[1] Note also that if $r_2<1$, $\Gamma_0(R^*)$ is always positive and if

$$H = \frac{r_2}{2r_3}\left(3-\frac{1}{r_1}\right)+r_2\left(\frac{1}{2}+r_2\right)-\frac{1}{r_1}\left(\frac{1}{r_3}+\frac{3}{2}r_2\right)>0 \tag{52}$$

then $G(R^*)$ is also positive for all R^* and $\Gamma(R^*,\hat{V})$ will be positive for $\hat{V}<\Gamma_0(R^*)/G(R^*)=\hat{V}_r$, where

$$\hat{V}_r = \frac{2r_1[3+r_3(1+3R^*+2r_2)]-2[2+r_2+3r_2r_3(1+R^*)]}{r_2\{r_1[3+r_3(1+3R^*+2r_2)]-3r_3(1+R^*)-1\}-2} \lesseqgtr 0. \tag{53}$$

If $H<0$, but

$$R^* > \frac{2r_1|H|}{3r_2(r_1-1)} \tag{54}$$

then $G(R^*)$ will be also positive and $\Gamma(R^*,\hat{V})>0$ for $\hat{V}<\hat{V}_r$. Finally, if $R^*<\frac{2r_1|H|}{3r_2(r_1-1)}$, then $\Gamma(R^*,\hat{V})>0$ for all $\hat{V}$.

The possible behaviors of $F(x)$ as a function of speed are shown in Fig. 2. In particular, Fig. 2a considers material range $0<r_2<1/r_1$. In fact, in this range the limit behaviors of $F(z)$ are

$$0<r_2<1/r_1: \begin{cases} \hat{V}<\hat{V}^*: \begin{cases} N(0)>0 \\ D(0)>0 \end{cases} \Rightarrow F(0)>0; \begin{cases} N(\infty)>R^* \\ D(\infty)=0^+ \end{cases} \Rightarrow F(\infty)=+\infty, \\ \hat{V}>\hat{V}^*: \begin{cases} N(0)<0 \\ D(0)>0 \end{cases} \Rightarrow F(0)<0; \begin{cases} N(\infty)>R^* \\ D(\infty)=0^+ \end{cases} \Rightarrow F(\infty)=+\infty. \end{cases}$$

[1] The term $\{2(1-1/r_1)+(1-r_2/r_1)\}$, in Eq. (20) in Zhang and Barber (1990), need to be divided for r_3.

Further, for $\hat{V} < \hat{V}^*$, $F(0)$ and Γ cannot assume opposite sign because the condition $\Gamma < 0$ involves that

$$H > 0 \text{ and } \hat{V} > \hat{V}_r, \text{ or}$$

$$H < 0, \text{ but } R^* > \frac{2r_1|H|}{3r_2(r_1 - 1)} \text{ and } \hat{V} > \hat{V}_r$$

and the speed $\hat{V}_r$ in the range above identified is always larger than $\hat{V}^*$. Therefore, the inequalities $\hat{V} < \hat{V}^*$ and $\hat{V} > \hat{V}_r$ cannot be satisfied at the same time and, hence, for $\hat{V} < \hat{V}^*$ we cannot obtain complex roots because the curve $F(x)$ does not have a minimum.

With similar arguments it is possible to show that the possible behaviors for $F(x)$ in the material ranges $1/r_1 < r_2 < 1$, $1 < r_2 < r_1$ and $r_2 > r_1$ are shown in Fig. 2b–d, respectively.

4.2. Complex roots

We have shown that the existence of a minimum in $F(z)$ demands that there be complex roots. Complex roots in our problem correspond to sinusoids oscillating in time (see the form of the perturbation in (1), where the real part only is implicitly assumed to contribute to the temperature) and this should not be confused with the case of migrating waves occurring in the case of sliding in-plane, for which there may be migration speeds V_1 and V_2 relative to materials 1 and 2, respectively, such that

$$V_1 - V_2 = V,$$

where V is the sliding speed in plane. Thus, the case of independent travelling waves in opposite direction is obtained if the waves move in opposite directions as $V = V_1 - V_2 = 0$, i.e. static contact or out of plane sliding. When there is a travelling wave (migration) (i.e. not two counter moving waves creating a standing wave), the imaginary parts create a phase lag between the expansion, the heat flux and the surface temperature. For the standing wave case, this translates to a phase lag in time with cosine form. In the in-plane case, the phase lags are on opposite sides of $z = 0$ in the two materials because generally the relative migration speeds are opposite, and the two problems are not equivalent.

5. Implications for stability behavior

The above analysis enables us to characterize the stability behavior of the system with respect to the material properties, using the same classification in material classes of Zhang and Barber (1990), and by arranging the materials such that $r_1 > 1$. Recollecting briefly their results for the static case, instability occurs either for (see, Table 1)

1. flow only into the more distortive material (class a) with a real root,
2. flow only into the less distortive (class c with a complex root, or real root for class d and $\Gamma_0 < 0$),

Table 1
Stability behavior for the static case Zhang and Barber (1990)

Parameters range	Inst. cond.	Root	Class
$0 < r_2 < 1/r_1$	$\widehat{\Delta T} > 0$	Re	a
$1/r_1 < r_2 < 1$	$\widehat{\Delta T} > 0$	Re	b
	$\widehat{\Delta T} < 0$	C	
$1 < r_2 < r_1$	$\widehat{\Delta T} < 0$	C	c
$r_2 > r_1$	$\widehat{\Delta T} < 0$	C	d
$r_2 > r_1(\Gamma_0 < 0)$	$\widehat{\Delta T} < 0$	Re	

Table 2
Stability behavior for sliding contact

Parameters range	Speed	Inst. cond.	Root	Class
	$\hat{V} < \hat{V}^*$	$\widehat{\Delta T} > \widehat{\Delta T}_{cr} > 0$	Re	a
$0 < r_2 < 1/r_1$	$\hat{V}^* < \hat{V} < \hat{V}_r$	$\widehat{\Delta T} > \widehat{\Delta T}_{cr} < 0$	Re	
	$\hat{V} > \hat{V}_r$	$\widehat{\Delta T} > \widehat{\Delta T}_{cr} < 0$	C	
	$\hat{V} < \hat{V}^*$	$\widehat{\Delta T} > \widehat{\Delta T}_{1cr} > 0$	Re	b
		$\widehat{\Delta T} < \widehat{\Delta T}_{2cr} < 0$	C	
$1/r_1 < r_2 < 1$	$\hat{V}^* < \hat{V} < \hat{V}_r$	$\widehat{\Delta T} > \widehat{\Delta T}_{1cr} < 0$	Re	
		$\widehat{\Delta T} < \widehat{\Delta T}_{2cr} < 0$	C	
	$\hat{V}_r < \hat{V} < \hat{V}_0$	$\widehat{\Delta T} > \widehat{\Delta T}_{1cr} < 0$	C	
		$\widehat{\Delta T} < \widehat{\Delta T}_{2cr} < 0$	C	
	$\hat{V} > \hat{V}_0$	All $\widehat{\Delta T}$	Re	
	$\hat{V} < \hat{V}_r$	$\widehat{\Delta T} < \widehat{\Delta T}_{cr} < 0$	C	c
$1 < r_2 < r_1$	$\hat{V}_r < \hat{V} < \hat{V}^*$	$\widehat{\Delta T} < \widehat{\Delta T}_{cr} < 0$	Re	
	$\hat{V} > \hat{V}^*$	$\widehat{\Delta T} < \widehat{\Delta T}_{cr} > 0$	Re	
$r_2 > r_1$ with $\hat{V}_r > 0$	Like material class $1 < r_2 < r_1$			d
$r_2 > r_1$ with $\hat{V}_r < 0$	$\hat{V} < \hat{V}^*$	$\widehat{\Delta T} < \widehat{\Delta T}_{cr} < 0$	Re	
	$\hat{V} > \hat{V}^*$	$\widehat{\Delta T} < \widehat{\Delta T}_{cr} > 0$	Re	

3. flow of either signs (class b with a real root for flow into the more distortive material, and a complex root for the other flow sign).

Since most material combinations fall into classes a and b (see, Appendix A), it was concluded that the most likely instability was in direction $\widehat{\Delta T} > 0$.

In the sliding case, $\widehat{\Delta T} > 0$ is translated into higher temperature of the less distortive material for $r_2 < 1$, and of the more distortive material, for $r_2 > 1$. The effect of dimensionless speed $\hat{V}$ is summarized in Table 2, for each material classes (a–d), in

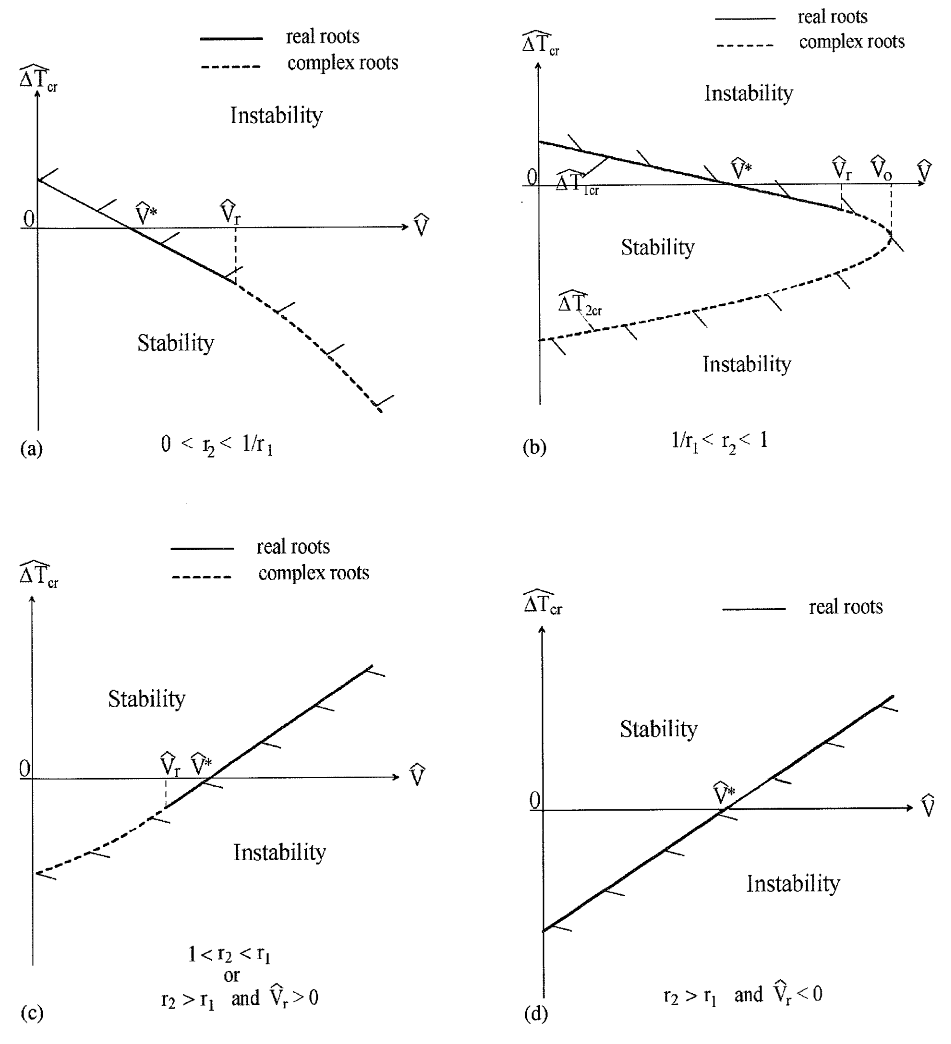

Fig. 3. Possible stability ranges as $\widehat{\Delta T}_{cr}$ as a function of speed $\hat{V}$: (a) $0 < r_2 < 1/r_1$; (b) $1/r_1 < r_2 < 1$; (c) $1 < r_2 < r_1$ and $r_2 > r_1$ with $\hat{V}_r > 0$; and (d) $r_2 > r_1$ with $\hat{V}_r < 0$.

terms of stability boundaries and indicating if the root is real (Re), or complex (C). In the former case, the stability boundary is simply given by Eq. (48), and complex roots are obtained as indicated in more details in the next paragraph. The stability boundaries are correspondingly sketched in Fig. 3a ($0 < r_2 < 1/r_1$), Fig. 3b ($1/r_1 < r_2 < 1$), Fig. 3c ($1 < r_2 < r_1$ and $r_2 > r_1$ with $\hat{V}_r > 0$, and finally Fig. 3d ($r_2 > r_1$ with $\hat{V}_r < 0$) as a function of speed.

At speed $\hat{V}^*$, the isothermal condition at the interface becomes critical (it is obviously not critical at zero speed). Speed $\hat{V}_r$ indicates the passage from complex to real roots in one of the branches of the stability boundary, or vice versa, and note that in the complex root regime, the relation between critical temperature drop and speed is nonlinear. Finally, at the speed $\hat{V}_0$, speed for which the numerator and the denominator of function $F(z)$ pass through zero in a same point, the materials of class b show critical conditions for all applied temperature differences. It is clear that the classification of the behaviors for the static case is not altered much for speeds below $\hat{V} < \hat{V}^*$, but the critical condition may change from real to complex. Above this speed, a change of sign in one of the critical values occurs, and therefore even a temperature drop in the opposite direction of that causing instability in the static case, may cause instability.

6. Parametric study of complex roots

Complex roots appear when there is a minimum of the function and the actual range for instability needs direct complex root computation. A simple strategy to obtain results is to use a parametric investigation. In particular, if the characteristic Eq. (34) is rewritten as

$$\hat{V}[R^*\Omega_1 + \Omega_2] - R^* - \Omega_3 - \widehat{\Delta T}\Omega_4 = 0, \tag{55}$$

where

$$\Omega_1 = \frac{\beta}{c_1(1+c_1)} + \frac{r_2(1-\beta)}{c_2(1+c_2)}; \quad \Omega_2 = \frac{1}{c_1 c_2}\left(\frac{r_2}{1+c_2} + \frac{1/r_3}{1+c_1}\right), \tag{56}$$

$$\Omega_3 = \frac{1}{c_1} + \frac{1}{r_3 c_2}; \quad \Omega_4 = \frac{r_2}{c_2(1+c_2)} - \frac{1}{c_1(1+c_1)}. \tag{57}$$

Taking real and imaginary part of Eq. (55), by fixing material properties and R^*, we define, for a given complex z, parametrically the two real quantities $\hat{V}$, $\widehat{\Delta T}$. Starting from eliminating $\widehat{\Delta T}$, we obtain

$$\widehat{\Delta T} = \frac{\frac{R^*\operatorname{Im}\Omega_1 + \operatorname{Im}\Omega_2}{R^*\operatorname{Re}\Omega_1 + \operatorname{Re}\Omega_2}(R^* + \operatorname{Re}\Omega_3) - \operatorname{Im}\Omega_3}{\operatorname{Im}\Omega_4 - \operatorname{Re}\Omega_4\frac{R^*\operatorname{Im}\Omega_1 + \operatorname{Im}\Omega_2}{R^*\operatorname{Re}\Omega_1 + \operatorname{Re}\Omega_2}} \tag{58}$$

and substituting back,

$$\hat{V} = \frac{R^* + \operatorname{Re}\Omega_3 + \widehat{\Delta T}\operatorname{Re}\Omega_4}{R^*\operatorname{Re}\Omega_1 + \operatorname{Re}\Omega_2}.$$

This analysis can be used for purely imaginary z in order to explore the stability boundaries for complex roots, or for a general complex z if growth rate in unstable regime are desired. The analysis was repeated to find the possible conditions of Table 2 and in Fig. 3.

7. Prescribed heat flux into one material

In the single half-plane sliding against a perfect conductor (Afferrante and Ciavarella, 2004), it was found that at pressures for which $R(p_0) + p_0 R'(p_0) < 0$, frictional heating becomes a stabilizing effect (for a given total heat flux transmitted). Vice versa, heat flow makes always the system more unstable when directed into the conducting material (i.e. with higher distortivity). In the discussion so far, we have been concerned with the case of prescribed $\widehat{\Delta T}$, as this was simpler than the case with prescribed heat into one body.[2] Using Eq. (1), we define a dimensionless heat flux into material 1 as[3]

$$\hat{Q}_{10} = -4M\delta_1 K_1 q_{10} R' = \widehat{\Delta T} - \beta \hat{V} \widehat{R'} \tag{60}$$

with

$$\widehat{R'} = mK_1 p_0 R'. \tag{61}$$

For a given $\hat{Q}_{10}$, the critical conditions found in terms of $\widehat{\Delta T}$, cannot be immediately translated. In fact, suppose we are in class a materials, for the condition given by real root, then instability is given for $\widehat{\Delta T}$ higher than that given in Eq. (48), and using Eq. (60),

$$\hat{Q}_{10} > 2\frac{1/r_3 + (1+R^*)}{(1-r_2)} - \left(\frac{1/r_3 + r_2 + [r_2(1-\beta) + \beta]R^*}{(1-r_2)} + \beta\widehat{R'}\right)\hat{V}. \tag{62}$$

Hence, it is clear that frictional heating may be a stabilizing factor if the term under parenthesis multiplying the speed is negative, i.e.

$$\widehat{R'} < -\frac{1/r_3 + r_2 + [r_2(1-\beta) + \beta]R^*}{(1-r_2)\beta} \quad \text{with } r_2 < 1 \tag{63}$$

which replaces the corresponding $p_0 R' < -R_0$ found in the case of the half-plane sliding against a perfect conductor (Afferrante and Ciavarella, 2004), which is immediately re-obtained for $r_2 = 0, r_3 = \infty$. Since $R' < 0$, inequality (63) defines clearly a range of special cases for which frictional heating has indeed a stabilizing effect.

However, for the case of real root for class c materials, for which $r_2 > 1$, instability is given for $\widehat{\Delta T}$ higher than that given in Eq. (48), and using Eq. (60), and changing signs,

$$\hat{Q}_{10} < 2\frac{1/r_3 + (1+R^*)}{(1-r_2)} - \left(\frac{1/r_3 + r_2 + [r_2(1-\beta) + \beta]R^*}{(1-r_2)} + \beta\widehat{R'}\right)\hat{V} \tag{64}$$

and normally the effect of frictional heating is to enlarge the region of instability, unless

$$\widehat{R'} > \frac{1/r_3 + r_2 + [r_2(1-\beta) + \beta]R^*}{(r_2-1)\beta} \quad \text{with } r_2 > 1$$

[2] Note that for "prescribed" we do not really intend to artificially imposing a temperature drop at the interface nor a heat flux into one material, as neither conditions is simple in practise. A more realistic approach could be to consider prescribed temperatures of the two bodies (or heat fluxes) at some distance from the interface, but because of the difficulties in half-plane elasticity, the present choice is the only viable alternative.

[3] Note that for $V = 0$, $\hat{Q}$, $\hat{Q}_{10}$ and $\widehat{\Delta T}$, all coincide

$$\hat{Q} = \hat{Q}_{10} = -4M\delta_1 K_1 q_{10} R' = \widehat{\Delta T}. \tag{59}$$

which is impossible, given $R' < 0$. It is not practical to explore if there is a larger set of possible conditions making frictional heating a stabilizing effect than those indicated in Eq. (63), particularly when the roots are complex, but it remains only a special set of combinations of materials and resistance function.

8. Effect of heat generation factor β

In the formulation of the problem, we introduced the parameter β, indicating the proportion of frictional heat which is generated in material 1, $\beta f V p$, i.e. $(1-\beta) f V p$ is the complementary part entering body 2. This parameter essentially depends on the exact mechanism generating frictional heat, most likely depending on the roughness and plastic behavior characteristics of the contacting bodies at asperity level. It is very hard to establish the value β unless there are experimental data available, and therefore it is convenient to explore its effect in the entire range of possible values (0–1).

The parameter β of frictional heat generation appears in the expression for c in Eq. (34) and hence affects the stability of the system. In Fig. 4a the effect of β on the stability of the system is shown for class a materials' combinations, $r_2 < 1/r_1$. In this range the increase of β, i.e. the increase of heat generated in the more distortive material, makes the system more unstable and this effect is emphasized for high speeds.

In Fig. 4a the case of class b, i.e. when $1/r_1 < r_2 < 1$, is treated, showing that β has opposite effects on the critical values $\widehat{\Delta T}_{1\mathrm{cr}}$ and $\widehat{\Delta T}_{2\mathrm{cr}}$. When β grows, the region of instability becomes larger with respect to $\widehat{\Delta T}_{1\mathrm{cr}}$, smaller with respect to $\widehat{\Delta T}_{2\mathrm{cr}}$.

For $r_2 > 1$, the increase of β reduces the region of instability. Therefore, as for $r_2 < 1/r_1$, the increase of heat generated in the more distortive material, makes the system more unstable. In Fig. 4c is shown this effect for $r_2 > r_1$ and $\hat{V}_r < 0$ (class d). In the remaining case (class c) the effect is similar.

9. Practical considerations

In the classical literature on TEI, the occurrence of instability is associated with undesired formation of hot spots and localizations of damage, which give rise to a series of problems in brake/clutch systems. Given the standard analysis does not consider resistance at the interface, the TEI critical speed is a well-defined quantity only dependent on geometry and materials. It is generally concluded that undesirable TEI effects can be minimized by increasing the thermal conductivity of the friction material. This has the effect of increasing the migration speed of the disturbance with respect to the good conductor, hence reducing the magnitude of the thermoelastic distortions. In automotive disk brakes, two commonly used friction material categories are non-asbestos organic composites (NAOs) and semi-metallic composites. Semi-metallics have significantly higher conductivity than NAOs because of the metal content. Design experience shows that they are indeed less prone to hot spots and the associated hot judder. Despite some remarkable success in comparing experimental evidence of TEI to the predicted features of unstable modes, there is still some quantitative difference

Table 4
Type of behavior for different material combinations

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
	b	*b*	*b*	*b*	*b*	*a*	*b*	*a*	*a*	*a*	*b*	*b*	*b*	*a*	*a*	*a*	*a*	*a*	*a*	1
		b	*b*	*b*	*b*	*c*	*c*	*b*	*c*	*b*	*b*	*b*	*b*	*b*	*c*	*a*	*a*	*c*	*b*	2
			c	*b*	*a*	*b*	*a*	*b*	*b*	*b*	*b*	*b*	*a*	*b*	*b*	*d*	*c*	*b*	*b*	3
				a	*a*	*b*	*d*	*b*	*b*	*b*	*b*	*b*	*a*	*b*	*b*	*d*	*c*	*b*	*b*	4
					a	*c*	*c*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*c*	*a*	*d*	*b*	*b*	5
						b	*a*	*b*	*b*	*b*	*b*	*b*	*a*	*b*	*b*	*c*	*b*	*b*	*b*	6
							c	*a*	*d*	*a*	*a*	*a*	*c*	*a*	*c*	*a*	*a*	*b*	*b*	7
								b	*c*	*b*	*b*	*b*	*c*	*b*	*c*	*d*	*d*	*c*	*b*	8
									b	*a*	*c*	*d*	*b*	*a*	*a*	*a*	*a*	*a*	*b*	9
										b	*a*	*a*	*b*	*a*	*c*	*a*	*a*	*b*	*b*	10
											a	*a*	*b*	*a*	*d*	*a*	*a*	*c*	*b*	11
												a	*b*	*a*	*a*	*a*	*a*	*a*	*b*	12
													b	*a*	*a*	*a*	*a*	*a*	*b*	13
														b	*c*	*d*	*d*	*b*	*b*	14
															a	*a*	*a*	*a*	*a*	15
																a	*a*	*a*	*a*	16
																	a	*a*	*a*	17
																		a	*a*	18
																			a	19

material (classes a, c and d). For material class b the region of instability is only shifted and above $\hat{V}_0$ the system remains unstable (both $\hat{V}_r$ and $\hat{V}_0$ do not depend on the parameter β).

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Appendix. Material combinations

In the analysis, the combinations of materials introduced by Zhang and Barber (1990) were used, where material properties are specified in Table 3. The resulting behaviour is shown in Table 4, where the classification in four classes (a, b, c, d) is used in Tables 1 and 2.

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