Tangential Loading of General Three-Dimensional Contacts

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A general three-dimensional contact, between elastically similar half-spaces, is considered. With a fixed normal load, we consider a pure relative tangential translation between the two bodies. We show that, for the case of negligible Poisson's ratio, an exact solution is given by a single component of shearing traction, in the direction of loading. It is well known that, for full sliding conditions, the tangential force must be applied through the center of the pressure distribution. Instead, for a full stick case the tangential force must be applied through the center of the pressure distribution under a rigid flat indenter whose planform is the contact area of the problem under consideration. Finally, for finite friction a partial slip regime has to be introduced. It is shown that this problem corresponds to a difference between the actual normal contact problem, and a corrective problem corresponding to a lower load, but with same rotation of the actual normal indentation. Therefore for a pure translation to occur in the partial slip regime, the point of application of the tangential load must follow the center of the "difference" pressure. The latter also provides a complete solution of the partial slip problem. In particular, the general solution in quadrature is given for the axisymmetric case, where it is also possible to take into account of the effect of Poisson's ratio, as shown in the Appendix.

1 Introduction

Transmission of loads and guiding of components is very often accomplished by mechanical contact, in a variety of possible geometries. It becomes important, then, to quantify traction distributions to improve our understanding of the "strength" of the contact as defined usually by the maximum value of the von Mises parameter. Alternatively, depending on the application, the quantification of fretting damage and surface frictional energy dissipation, as well as crack initiation and crack propagation. Also, in many applications of robotics and, in general, in production and manufacturing, such as positioning of objects and workpieces, it is of paramount importance to precisely quantify the elastic deformations, and the relative displacements induced into bodies. Indeed, it may be desired to produce *pure relative translation* of the two bodies. It is known that, in a general contact in *gross or full sliding*, the condition to obtain pure relative translation is the application of a tangential force through the *centroid* of the pressure, i.e., the point of application of the normal force[2] (MacMillan, 1936). However, the condition for full sliding cannot be reached, other than through an increase of the tangential force from zero. Now, when a tangential load is applied to a general contact between elastic bodies, a complex frictional shearing traction distribution arises to equilibrate the applied load. The contact area is divided into regions of micro-slip and regions of adhesion, according to Coulomb's law. Therefore it is possible that, on applying the tangential force through the centroid of pressure, there is an undesired relative rotation until the full sliding conditions are reached. To investigate this possibility, we need to consider the contact problem in the partial slip regime, evolving from the fully adhesive condition in the presence of normal load only, towards the full sliding conditions, when the tangential load is fully applied.

There is a number of results available in the literature for (frictionless) normal contact problems, at least in the case where the contacting bodies can be approximated as half-spaces. For a circular contact area, that practically requires axisymmetry of the geometry and boundary conditions, a general solution in quadrature is known (see, for example, Scthaerman, 1949). In the case of general shape of the (single) contact area, approximate solutions are known (Fabrikant, 1986), and the normal compliance and relation load-contact area can be also predicted accurately by ad hoc methods (Barber and Billings, 1990). There remains the far more complicated case of multiple contact area, which is of particular importance when rough surfaces are in contact. In this case, recourse to numerical methods to solve the governing equations is in practice not avoidable. Indeed, several solution techniques have been developed (see, for example, Kalker, 1990), although it is not always easy to achieve the desired accuracy, especially regarding localized regions, for example when singularities are expected at the contact area boundary, and in that case they may also change intensity with position (Fabrikant et al., 1988), in the case of sharp corners.

Moving to the case of interest for the present investigation, where a tangential action is applied with a finite friction coefficient, the number of known solutions is largely inferior, even in the simplest case of elastically similar materials. The problem corresponds to the finding of a shearing traction distribution such that there is rigid-body displacement in the entire stick zone, fulfilling Coulomb's law in both stick and slip areas. Cattaneo (1938) solved the three-dimensional contact problem for second-order surfaces (Hertzian contact). The solution is obtained by making an "educated" guess of a distribution of tractions that fulfills the requirements for displacement fields. It is little known that Cattaneo also solved the case of an axisymmetric contact with distance function of fourth degree, in 1947 (Cattaneo, 1947a, 1947b). Later on, Mindlin (1949) independently reobtained the solution of the Hertzian contact, and in Mindlin et al. (1952) and Mindlin and Deresiewicz (1953) also gave solutions to the unloading, cyclic, and oblique loading problems. Deresiewicz (1957) gave the treatment for the oscil-

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[2] This is actually true, a rigori, only in the absence of Poisson's effect, as we will see in the discussion for the general case $\gamma \neq 0$ below.

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lating tangential forces in elliptical contact area case, while many experimental results by Mindlin himself (Mindlin et al., 1952), and more extensively, by Johnson (1955), were in very good agreement with the theory in two respects: the tangential compliance, and the appearance of annular regions of slip, where wear resulted. Other results are known for twisting problems (Lubkin, 1951; Cattaneo, 1955; Pacelli, 1956; Hetenyi et al., 1958; Hills et al., 1986) in Hertzian contacts; it is well known that, even in the case of pure twist of Hertzian contacts, the equations become quite cumbersome, in particular in the case of elliptical contact (Pacelli, 1956), and therefore we do not address this configuration. In the case of tangential loading with complex stick-slip patterns, however, the computational effort of a fully numerical solution is, even for ad hoc procedure (Kalker, 1990), very intensive, and it is still difficult to go under certain levels of numerical error.

For the shifting partial slip problem, apart from the geometry already solved by Cattaneo, to the best of the author's knowledge, only the case of a cone indenter has been solved to date (Truman et al., 1995). This is probably because the Cattaneo-Mindlin procedure involves the explicit calculation of the displacements, whereas a general method would ideally avoid completely this step, formulating the problem directly in terms of integral equations. In a recent paper (Ciavarella, 1998a, b), the author has treated the case of partial slip in the general *plane* contact between elastic isotropic bodies, showing in particular a connection in any plane partial slip contact problem with two corresponding frictionless plane contact problems, for which solutions in quadrature are known for single, multiple, and periodical contact: this result has permitted new exact solutions to many plane partial slip problems, for which the Cattaneo-Mindlin procedure, involving the explicit calculation of displacements, would be prohibitively complicated.

In this paper it is shown that in the three-dimensional case, for negligible elastically dissimilarity and combined Poisson's ratio, similar results apply, as a single distribution of shearing tractions (i.e., shear only in the direction of the applied force) solves the problem. This permits one to obtain several properties of the tangential loading regime, and provides a means to do actual calculations either analytically or numerically in a much simpler way. It is known that, in the presence of Poisson's effect, even the Cattaneo-Mindlin solution for Hertzian contacts is *approximate*, in that it neglects the presence of a small shearing distribution in the transverse direction (Munisamy et al., 1992). In view of this, attention will be concentrated on contact problems for the case of no Poisson's effect ($\gamma = 0$ as below).

2 Formulation

Let us start from the governing equations for the case of normal indentation. In general, there is coupling between tangential and normal displacements, and therefore between pressure and shearing tractions. In that case, exact analytical solutions are not known even in the simplest geometrical and loading configurations. However, the effect of coupling is generally small (Goodman, 1962), and indeed there is a vast category of situations where the contacting bodies material are of same material, or the materials are elastically similar: specifically, the case where Dundurs' constant, β, is zero

$$\beta = \frac{\mu_2(\kappa_1 - 1) - \mu_1(\kappa_2 - 1)}{\mu_2(\kappa_1 + 1) + \mu_1(\kappa_2 + 1)} = 0, \qquad (1)$$

where κ is the Kolosov's constant, given by $\kappa = (3 - 4\nu)$ as in the case of plane-strain conditions, (ν_i, μ_i are the Poisson's ratio and shear modulus of the material of body i). In this case, the equations relating normal and tangential traction become uncoupled, and the contact pressure can be calculated a priori. Let us write relative surface displacements as $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$. Then, the integral equation relating normal displacements to the pressure is (Johnson, 1985)

$$\frac{2\pi}{A} u_z(x, y) = \iint_S \frac{p(\xi, \eta)}{\rho} d\xi d\eta, \quad (x, y) \in S \qquad (2)$$

where S is the contact area, as yet undetermined, ρ is the distance between field and integration point

$$\rho = \sqrt{(\xi - x)^2 + (\eta - y)^2}, \qquad (3)$$

and A is the "composite compliance" of the bodies[3]

$$A = \left(\frac{1 - \nu_1}{\mu_1} + \frac{1 - \nu_2}{\mu_2}\right). \qquad (4)$$

This integral equation is sufficient to solve the normal contact problem, imposing the appropriate boundary conditions. In particular, the equality (inequality) relating to contact over the area S (noninterpenetration condition exterior to S) is

$$u_z(x, y) = \delta_z - [\theta_x y + \theta_y x] - [f_1(x, y) - f_2(x, y)], \quad (x, y) \in S \qquad (5)$$

$$u_z(x, y) > \delta_z - [\theta_x y + \theta_y x] - [f_1(x, y) - f_2(x, y)], \quad (x, y) \notin S \qquad (6)$$

where $f_1(x, y)$, $f_2(x, y)$ are the functions describing the profiles of the contacting bodies in the undeformed configuration, and (δ_z, θ_x, θ_y) is the given rigid-body motion necessary to bring the two body into contact. Given (δ_z, θ_x, θ_y), the distribution of pressure $p(x, y)$ is determined, from which a resultant load, and moment $\mathbf{M}$ (of components M_x and M_y) with respect to the coordinate system (ξ, η) can be calculated as

$$P = \iint_S p(\xi, \eta) d\xi d\eta, \quad M_x = \iint_S p(\xi, \eta)\xi d\xi d\eta,$$

$$M_y = \iint_S p(\xi, \eta)\eta d\xi d\eta. \qquad (7)$$

Of course, there is a point where the resultant P can be applied without any moment, and it is called the *center of gravity* of the normal pressure distribution (simply, the *centroid* of the pressure), and indicated by N; any coordinate system (ξ_0, η_0) through such a point N will satisfy

$$\iint_S p(\xi_0, \eta_0)\xi_0 d\xi_0 d\eta_0 = 0,$$

$$\iint_S p(\xi_0, \eta_0)\eta_0 d\xi_0 d\eta_0 = 0. \qquad (8)$$

Although there is no exact method to solve the contact problem in general, and in particular for other than circular or elliptical contact areas S[4], remarkable is Barber's method (see Barber et al., 1990) which is based on a variational formulation, using the theorem that the exact shape of the contact area is the one that maximizes the load, for a given indentation; a numerical method is then needed anyway, to find this maximum.

On applying a monotonically increasing tangential load in direction x, there will in general be stick and slip zones. The shearing tractions are strictly related to the limiting value of Coulomb friction (see below). The other two integral equations defining the problem both relate to displacement of particles parallel with the surface, one in the direction of the tangential force, and the other perpendicular to it. They are

[3] Notice that A corresponds to $2/E^*$ in Johnson's notation (Johnson, 1985).

[4] In fact, for circular area, the problem being axisymmetric, a solution in quadrature is possible (Schtaerman, 1949). For elliptical contact area, it is difficult to think of any non-Hertzian problem that produces exactly this shape.

$$\frac{2\pi}{A} u_x(x, y) = \iint_S \frac{q_x(\xi, \eta)}{\rho} d\xi d\eta + \gamma \iint_S \left[\frac{(\xi - x)^2}{\rho^3} q_x(\xi, \eta) - \frac{(\xi - x)(\eta - y)}{\rho^3} q_y(\xi, \eta) \right] d\xi d\eta \quad (9)$$

$$\frac{2\pi}{A} u_y(x, y) = \iint_S \frac{q_y(\xi, \eta)}{\rho} d\xi d\eta + \gamma \iint_S \left[\frac{(\eta - y)^2}{\rho^3} q_y(\xi, \eta) + \frac{(\xi - x)(\eta - y)}{\rho^3} q_x(\xi, \eta) \right] d\xi d\eta \quad (10)$$

for $(x, y) \in S$, where

$$\gamma = \left(\frac{\nu_1}{\mu_1} + \frac{\nu_2}{\mu_2} \right) \Big/ A. \quad (11)$$

It is evident that, even though we are considering the $\beta = 0$ case, there is a second coupling, due to γ, and it is between the distributions of shearing tractions themselves. Indeed, this system of coupled equations gives a solution $[q_x(x, y), q_y(x, y)]$ for any couple of assigned functions $[u_x(x, y), u_y(x, y)]$ in the domain S. Therefore, in the general case, even if we impose, say, $u_y(x, y) = 0$, due to the coupling, it is not necessary that $q_y(x, y) = 0$. Also, again due to the coupling, there is no analogy between the integral equation for the normal displacements (2) and the tangential correspondents (9), (10). Therefore, in order to avoid coupling, and obtain an equation similar to (2), let us consider the case that the composite Poisson's ratio of the bodies is zero[5] $\gamma = 0$: equations in tangential direction simplify to

$$\frac{2\pi}{A} u_x(x, y) = \iint_S \frac{q_x(\xi, \eta)}{\rho} d\xi d\eta \quad (12)$$

$$\frac{2\pi}{A} u_y(x, y) = \iint_S \frac{q_y(\xi, \eta)}{\rho} d\xi d\eta \quad (13)$$

which are two independent problems, each one analogous to a normal contact problem in (2). The total shearing forces are

$$Q_x = \iint_S q_x(\xi, \eta) d\xi d\eta, \quad Q_y = \iint_S q_y(\xi, \eta) d\xi d\eta. \quad (14)$$

The condition we look for is the *pure relative translation.*[6] Therefore, we seek here for conditions under which a simple solution (i.e., with only one component of shearing traction distribution) for the pure relative translation problem is possible. Let us denote with x the direction where the resultant force is applied, i.e., $Q_y = 0$: it is clear that a priori we do not know whether this is also the direction of the relative translation. However, as the shearing tractions in the two bodies are equal and opposite for Newton's third law, the relative tangential displacement of surface particles must be constant within the stick zone, and equal to the imposed rigid translation

[5] For materials having positive Poisson's ratio, this is only possible if both $\nu_1 = \nu_2 = 0$.

[6] Things become much more complicated when a relative tangential rotation is present. In the simplest case of "pure spin," only for the Hertzian circular and elliptical contacts a solution is known, although the results are not in closed form (Lubkin, 1951; Cattaneo, 1955; Pacelli, 1956; Hetenyi et al., 1958; Hills et al., 1986).

$$u_x(x, y) = \delta_x; \quad u_y(x, y) = \delta_y, \quad (x, y) \in S_{stick} \quad (15)$$

where δ_y is expected to be zero under conditions of symmetry. Moreover, Coulomb's law requires that the shearing traction must be less than the limiting value in the stick zone, i.e.,

$$|\mathbf{q}(x, y)| < fp(x, y), \quad (x, y) \in S_{stick}. \quad (16)$$

Within the slip zones the shearing traction is limited by friction again, i.e.,

$$|\mathbf{q}(x, y)| = fp(x, y), \quad (x, y) \in S_{slip} \quad (17)$$

where the vectorial notation is necessary because, in general, $\mathbf{q}(x, y)$ has two components. The shear traction must always oppose the direction of relative change in the direction of slip, i.e. under monotonic tangential loading,

$$\frac{\mathbf{q}(x, y)}{|\mathbf{q}(x, y)|} + \frac{\partial}{\partial t} \frac{\mathbf{u}_t(x, y)}{|\mathbf{u}_t(x, y)|} = 0, \quad (x, y) \in S_{slip}, \quad (18)$$

where $\mathbf{u}_t = \mathbf{u}_x + \mathbf{u}_y$ is the relative tangential displacement, and the variable time t is introduced, but it is clear that we are considering a quasi-static formulation. The above equations provide the framework for solving the problem.

3 Solution

As in the normal loading phase there was no tendency for surface particles to slip, the initial stick zone envelopes the entire contact. A monotonically increasing tangential remote relative displacement will therefore give rise to a *receding* problem, according to Dundurs' classification (Dundurs, 1975), and therefore there is no need to follow the entire loading path, and we can solve directly for any particular value of δ_x. We only need to distinguish between the limit full stick conditions, and finite friction conditions.

3.1 Full Stick Conditions. We shall assume that the shearing distribution will have act only in the direction of the tangential displacement. Writing the boundary condition (15) in the integral equation (12), we have

$$\delta_x = \frac{A}{2\pi} \iint_S \frac{q_x(\xi, \eta)}{\rho} d\xi d\eta \quad (x, y) \in S. \quad (19)$$

Equation (13) is identically satisfied, as $u_y(x, y)$ depends only on $q_y(x, y)$, which is assumed to be zero. Integral Eq. (19) then, is formally equivalent to the problem for a *rigid flat* punch of planform S, indenting the half-space with a rigid vertical displacement. Therefore, q_x is proportional to the distribution of pressure, which is singular along the boundary of S, of the cited normal indentation problem; this implies also that a pure translation solution is possible only if the line of application of the resultant Q_x is through the center of the *rigid flat* punch of planform S pressure distribution, that we call the tangential center T (in the full stick conditions). Moreover, this solution will satisfy the problem *exactly.* We can therefore invoke *uniqueness* of the elastic solution to prove that there will be no other solution of the tangential shift problem, perhaps with a self-equilibrated distribution q_y of shearing tractions. It is interesting to study in more detail the tangential center T; we have proved that, if the direction of the tangential force Q_x does not pass through this point, there will be relative twist of the contacting bodies, because, for uniqueness, the conditions $u_x(x, y) = \delta_x$; $u_y(x, y) = 0$ cannot be satisfied in the entire contact area by any other distribution of shearing tractions than the given q_x distribution. It is clear that for a fairly general profile, for which the shape of the contact area S changes continuously increasing the load, and consequently so does the tangential center T, for each value of the normal load the set of possible direction of tangential load for which there is pure translation is the set of lines intersecting in such point T. Only on condition of symmetry the direction of application of Q_x for pure translation will not change. To note that the result applies equally to connected, as well as multiply connected areas S.

Again, no method exists to provide a solution in the general case of S. This aside, the analogy is sufficient to validate the assumption that full stick implies a singular distribution of shearing traction, that cannot be sustained by a *finite* friction coefficient. Therefore, the mathematical problem of full stick is practically non very meaningful in this context, unless we consider the case where the bodies are welded together, i.e., we have an external crack problem with the singularity giving a stress intensity factor.

3.2 Partial Slip Conditions. Let us then move to a partial slip regime, that arises if friction is finite. We assume a shearing traction distribution given by the sum of two components, a full sliding term, and a correction, as

$$q_x(x, y) = fp(x, y) - q_x^*(x, y), \quad (x, y) \in S \qquad (20)$$

where $q_x^*(x, y) = 0$ in the region of slip, i.e., for $(x, y) \in S_{\text{slip}}$, whereas it is at yet undetermined in the stick region. Writing again the integral equation for relative displacement in the tangential direction (12),

$$\delta_x = \frac{A}{2\pi} \iint_S \frac{fp(\xi, \eta)}{\rho} d\xi d\eta - \frac{A}{2\pi} \iint_{S_{\text{stick}}} \frac{q_x^*(x, y)}{\rho} d\xi d\eta, \quad (x, y) \in S \qquad (21)$$

whereas Eq. (13) is, again, identically satisfied. Now, as the normal loading equation (2) continues to hold, it follows that $q_x^*(x, y)$ is the solution of the integral equation

$$\frac{A}{2\pi} \iint_{S_{\text{stick}}} \frac{q_x^*(x, y)/f}{\rho} d\xi d\eta = u_z(x, y) - \delta_x/f, \quad (x, y) \in S_{\text{stick}} \qquad (22)$$

which can be recognized as being of the same kind as the original equation involving the contact pressure (2), for a lower value of the vertical rigid-body displacement, giving the resulting "contact" area (i.e., the stick area) to be a subregion of the actual contact area. In particular, it is important to note that the rotation is the one fixed by the actual normal contact problem, i.e., by Eq. (5).

The analogy is so far only partly proved, in that we need to check that the inequalities are automatically satisfied by this analogy. Inequality (6) corresponds to the condition that the relative tangential displacement, for the chosen sign of the tangential load, has to be positive: this results from a theorem in normal contact which states that there is no point at which an increase of normal force causes a decrease in the local contact pressure (Barber, 1992, Section 25.3.2). Equation (13) proves that the direction of slip in the slip zones is satisfied, as $u_y = 0$, whilst the definition of $q_x^* = 0$ satisfies implicitly the limiting conditions of Coulomb's law in the slip zones. As the solution found is the exact solution to the problem, and the solution is unique, there cannot be a solution with $q_y(x, y) \neq 0$ in any point of the contact area. Thus, any partial slip solution for a contact problem, for elastically similar materials, and of any shape (simply or multi-connected) of contact area, can be completed immediately, providing the corresponding normal load is itself known.

Let us now consider the implications on the direction of the force Q_x. First, consider that the full sliding component, with tangential force fP, clearly passes through the center of the normal pressure distribution N, in the direction x. Then, we define a center of corrective pressures by T^*. The conditions for determining the centre of the shearing tractions in the actual partial slip tangential problem are given by elementary vector analysis. In particular, if the distance in the y-direction between N and T^* is d^*, then it is immediately possible to find the distance d in the y-direction between N and T, which is

$$d = \left(1 - \left|\frac{Q_x}{fP}\right|\right) d^* \leq d^* \qquad (23)$$

and in particular at the first stages of application of Q_x, i.e., when $Q_x/fP \simeq 0$, it is $d \simeq d^*$ but d^* is expected to be small, as the corrective problem is only slightly different from the actual normal contact. At the other limit, when $Q_x/fP \simeq 1$, close to the full sliding, even if d^* is large, $d \simeq 0$, as expected because the corrective problem becomes of vanishing small load. Therefore, if we look for pure translation *without* moving the direction of application of the tangential load, we can distinguish between three situations, depending on what happens to the contact on unloading the normal load with the fixed rotation of Eqs. (5), (6):

1 T^* does not move—i.e., it rests coincident with N. In this case, all directions passing through this point give possible x-direction of application of tangential load that give pure translation solutions.

2 T^* moves along a line. In this case, only this line gives a possible direction of tangential loading that gives a pure translation.

3 T^* moves along a curve. There is no translation solution, for any direction of application of Q_x.

It is clear that, in general, it is possible to obtain pure translation in any direction, just following the path of the point T, i.e., applying the tangential load at distance d varying according to Eq. (22).

Moreover, several elementary properties can be deduced:

- Flat regions are either entirely in full stick or are in full slip conditions, as in unloading they either stay in contact, or loose contact simultaneously. In the limiting case of an entirely flat contact, the behavior is full stick/full slip.
- The points that loose contact first in the normal unloading, with the fixed rotation of the actual normal contact problem, are the first to slip when the tangential force is imposed.
- The tangential approach can be calculated from the superposition of the two normal approaches of the normal contact problems, as

$$\frac{\alpha_t}{f} = \alpha_n - \alpha_n^* \qquad (24)$$

- The shape of the stick zone is the one that maximizes the corrective load for a given corrective normal approach, therefore the shape of the slip area is the one that minimize the full sliding component.
- If the indenter profile is symmetrical and self-similar, the stick zone is similar to the initial shape, and the corrective solution is always of the same functional form of the normal pressure in the contact area.

4 Examples

This simple result permits new solutions to be found immediately from know solutions of normal contact problems. Also, numerical methods for simple frictionless normal contact problems, permit the solution of partial slip problems, with evident advantages in terms of computational time. Some solutions are worked out in the following paragraphs.

4.1 Axisymmetric Contact. In this case the solution for the normal contact problem of any continuous gap function $z(r) = f_1(x, y) - f_2(x, y)$ is known in quadrature (Scthaerman, 1949). The simplest cases have been already worked out, namely second and fourth-order contacts (Cattaneo, 1938, 1947a, 1947b), and the cone indenter (Truman, 1995). In the general case, supposing $p(a)$ is bounded (the so-called *incomplete contact conditions*),

$$p(r) = -\frac{2}{\pi A}\int_r^a \frac{F'(s)ds}{\sqrt{s^2 - r^2}}, \quad 0 \le r \le a, \qquad (25)$$

where

$$F(r) = \alpha_n - r\int_0^r \frac{z'(t)dt}{\sqrt{r^2 - t^2}}, \quad 0 \le r \le a \qquad (26)$$

and the approach of remote points in normal direction α_n is given by

$$\alpha_n = a\int_0^a \frac{z'(t)dt}{\sqrt{a^2 - t^2}}. \qquad (27)$$

The equilibrium condition between the applied load and the pressure distribution can be written as

$$P = \int_0^a 2\pi r p(r)dr = \frac{4}{A}\int_0^a \frac{z'(t)t^2dt}{\sqrt{a^2 - t^2}}. \qquad (28)$$

These equations translate immediately into partial slip solution (in the Cattaneo-Mindlin approximate sense), and so

- corrective shear

$$q^*(r)/f = -\frac{2}{\pi A}\int_r^c \frac{F'(s)ds}{\sqrt{s^2 - r^2}}, \quad 0 \le r \le c, \qquad (29)$$

- corrective approach of remote points

$$\alpha_n^* = c\int_0^c \frac{z'(t)dt}{\sqrt{c^2 - t^2}}. \qquad (30)$$

- relation dimension of stick area and corrective load

$$Q^*/f = \frac{4}{A}\int_0^c \frac{z'(t)t^2dt}{\sqrt{c^2 - t^2}}. \qquad (31)$$

In the Appendix this solution is also extended to the case with Poisson's effect, and in the paper (Ciavarella, 1998c), the case of rounded punches like a flat with rounded corners or a cone with rounded tip are discussed in detail.

4.2 General Simply Connected Area. There is no *exact* analytical solution for any contact of general simply connected area S, i.e., different from circular or elliptical shape. Therefore, we can only make use of numerical solutions. In particular, for a full stick problem, we need a solution for the complete contact for a rigid flat indenter of planform S, whereas for the partial slip conditions we need the solution of normal contact for indenters of smooth profile, and for two values of the normal load. For the former, Fabrikant provides an approximate analytical solution (Fabrikant, 1986), whereas for the latter, Barber and Billings (1990) give interesting methods to compute the relation load-penetration, the latter being an integral property. These methods can be used immediately to compute relation load-displacement in the tangential direction in the partial slip problem. Also, Conway et al. (1968) provide a series of numerical results for the relationship between load and penetration for flat punches of arbitrary cross section which are of immediate interest now in terms of the tangential contact problems.

Although having made recourse to a numerical approach, it is still advantageous when solving problems with $\gamma \neq 0$, the use of the superposition of normal contact problems, although this will give only a rough approximate solution (the approximation involved is not only the Cattaneo-Mindlin one, the problem being nonaxisymmetric), as the complete investigation requires the solution of a coupled system of two integral equations.

5 Discussion

We know that if $\gamma \neq 0$, in the case of Hertzian contacts, a solution of the kind just developed, i.e., with the shearing distribution acting in the direction of the tangential force *only*, was given by Cattaneo (1938) and Mindlin (1949) for the Hertzian three-dimensional case, although they did not notice that the solution is approximate, in the sense that there is a small *disalignment* between relative tangential displacements and shearing tractions.[7] The question, therefore, is whether there are more general cases where this solution can be developed under this Cattaneo-Mindlin approximation.

The most important cases are the plane contact, solved in Ciavarella (1998a, b), and the general axisymmetric single contact case, which is shown in Ciavarella (1998c), and the main results are briefly reported in the Appendix. This result is used in Ciavarella (1998c) for the solution of the flat and conical rounded indenters in great detail. More general cases are not only unlikely to be solved, but are also of limited interest as the surfaces required to produce such general smooth contact will be of rather special form, and the shape is also likely to change with the load.

6 Conclusions

Partial slip three-dimensional contact has been considered with new general results, in the framework of contact of bodies that can be elastically approximated as half-spaces. For the solution with a single component of shearing traction, to satisfy the problem exactly, Dundurs' constant β and combined Poisson's ratio γ have to be zero. In this case, simple properties and results can be obtained. In particular, the general solution for axisymmetric contacts has been given. The more general case has been discussed, and the effect of Poisson's ratio is also included rigorously in the general axisymmetric case.

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References

Barber, J. R., 1992, *Elasticity*, Kluwer, Dordrecht, The Netherlands.

Barber, J. R., and Billings, D. A., 1990, "An approximate solution for the contact area and elastic compliance of a smooth punch of arbitrary shape," *Int. J. Mech. Sci.*, Vol. 32, No. 12, pp. 991–997.

Cattaneo, C., 1938, "Sul contatto di due corpi elastici: distribuzione locale degli sforzi," *Rendiconti dell'Accademia Nazionale dei Lincei*, Vol. 27, pp. 342–348, 434–436, 474–478.

Cattaneo, C., 1947a, "Teoria del contatto elastico in seconda approssimazione," *Rendiconti di Matematica e delle sue applicazioni, Serie V*, Vol. 6, pp. 504–512.

Cattaneo, C., 1947b, "Teoria del contatto elastico in seconda approssimazione: Compressione obliqua," *Rendiconti Seminario Facolta' Scienze Università di Cagliari*, Vol. 17, pp. 13–28.

Cattaneo, C., 1955, "Compressione e torsione nel contatto tra corpi elastici di forma qualunque," *Ann. Scuola Norm. Sup. Pisa, Serie III*, Vol. 9, pp. 23–42.

Ciavarella, M., 1998a, "The Generalized Cattaneo Partial Slip Plane Contact Problem. I—Theory," *Int. J. Solids and Struct.*, Vol. 35, No. 18, pp. 2349–2362.

Ciavarella, M., 1998b, "The Generalized Cattaneo Partial Slip Plane Contact Problem. II—Examples," *Int. J. Solids and Struct.*, Vol. 35, No. 18, pp. 2363–2378.

[7] The exact solution implies a slightly noncircular stick zone, and requires a numerical nonaxisymmetrical solution, but the effort to do so is not justified by the small correction achieved (Munisamy et al., 1994). In the more general case of dissimilar elastic materials ($\beta \neq 0$), pressure distribution and shearing tractions in normal and a fortiori tangential problem will present a complicated pattern, and the stick-slip boundary a complicated shape, very much dependent on the actual loading path. This clearly requires an intensive numerical work.

Ciavarella, M., 1998c, "Indentation by nominally Flat or Conical Indenters with rounded Corners," *Int. J. Solids Struct.*, submitted for publication.

Conway, H. D., and Farnham, K. A., 1968, "The relationship between load and penetration for a rigid, flat-ended punch of arbitrary cross section," *Int. J. Engg. Sci.*, Vol. 6, pp. 489–496.

Deresiewicz, H., 1957, "Oblique Contact of Nonspherical Bodies," ASME Journal of Applied Mechanics, Vol. 24, pp. 623–624.

Dundurs, J., 1975, "Properties of elastic bodies in contact," *Mechanics of Contact Between Deformable Bodies*, A. D. de Pater and J. J. Kalker eds., Delft University Press, Delft, The Netherlands.

Fabrikant, V. I., 1986, "Flat punch of arbitrary shape on an elastic half-shape," *Int. J. Engng. Sci.*, Vol. 24, pp. 1731–1738.

Fabrikant, V. I., and Sankar, T. S., 1988, "Singularities at angular points in elastic contact problems," *Comm. in Appl. Num. Meth.* Vol. 4, No. 2, pp. 173–178.

Goodman, L. E., 1962, "Contact Stress Analysis of Normally Loaded Rough Spheres," ASME Journal of Applied Mechanics, Vol. 29, pp. 515–522.

Hetenyi, M., and McDonald, P. H., 1958, "Contact Stresses Under Combined Pressure and Twist," ASME Journal of Applied Mechanics, Vol. 25, pp. 396–401.

Hills, D. A., Nowell, D., and Sackfield, A., 1993, *Mechanics of Elastic Contacts*, Butterworth-Heinemann, Oxford.

Hills, D. A., and Sackfield, A., 1986, "The Stress Field Induced by a Twisting Sphere," ASME Journal of Applied Mechanics, Vol. 54, pp. 8–14.

Johnson, K. L., 1955, "Surface interaction between elastically loaded bodies under tangential forces," *Proc. R. Soc.*, Lond., Vol. A230, pp. 531–548.

Johnson, K. L., 1985, *Contact Mechanics*, Cambridge University Press, Cambridge, UK.

Kalker, J. J., 1990, *Three-dimensional elastic bodies in rolling contact*, Kluwer, Dordrecht, The Netherlands.

Lubkin, J. L., 1951, "The Torsion of Elastic Spheres in Contact," ASME Journal of Applied Mechanics, Vol. 73, pp. 183–187.

MacMillan, W. D., 1936, *Dynamics of rigid bodies*, McGraw-Hill, New York.

Mindlin, R. D., 1949, "Compliance of Elastic Bodies in Contact," ASME Journal of Applied Mechanics, Vol. 71, pp. 259–268.

Mindlin, R. D., Mason, W. P., Osmer, T. F., and Dereciewicz, H., 1952, "Effects of an oscillatory tangential force on the contact surfaces of elastic spheres," *Proc. 1st Nat. Congress of Applied Mechanics*, Vol. 203–208.

Mindlin, R. D., and Dereciewicz, H., 1953, "Elastic Spheres in Contact Under Varying Oblique Forces," ASME Journal of Applied Mechanics, Vol. 75, pp. 327–344.

Munisamy, R. L., Hills, D. A., and Nowell, D., 1994, "Static Axisymmetrical Herzian Contacts Subject to Shearing Forces," ASME Journal of Applied Mechanics, Vol. 61, pp. 278–283.

Pacelli, M., 1956, "Contatto con attrito tra due corpi elastici di forma qualunque: compressione e torsione," *Ann. Scuola Norm. Sup. Pisa, Serie III*, Vol. 10, pp. 155–184.

Scthaerman, I. Ya., 1949, *Contact Problem of the Theory of Elasticity*, Moscow, Leningrad: Gostekhteoretizdat (available from the British Library in a English translation by Foreign Technology Div., FTD-MT-24-61-70, 1970).

Truman, C. E., Sackfield, A., and Hills, D. A., 1995, "Contact mechanics of wedge and cone indenters," *Int. J. Mech. Sci.*, Vol. 37, No. 3, pp. 261–275.

APPENDIX

The Axisymmetric Case With $\gamma \neq 0$

In (Ciavarella, 1998c) it is proved that in the general case of an axisymmetrical problem, a solution of the kind just developed can be obtained. Here the proof is not repeated, but the main results are given, extending therefore what was obtained in the main part of the paper. It is true that the case $\gamma = 0$ is powerfully proved in general terms, and therefore a numerical solution of any partial slip problem can now be developed easily as the superposition of far simpler frictionless contact, but regarding strictly analytical results, only axisymmetrical problems may be of practical interest. The solution of the problem for $\gamma \neq 0$, following what obtained in Ciavarella (1998c), is

$$\frac{2\pi}{A}\iint_{S_{stick}} \frac{q_x^*(r')/f}{\rho}\,dS = u_z(r) - \frac{\delta_x}{f\left(1+\frac{\gamma}{2}\right)},\qquad 0 \le r \le c \quad (32)$$

where c is the size of the circular stick area Sstick, r' is the radial coordinate of the integration point; the field point has radial coordinate r; and the distance between them is ρ. The latter integral equation means that with the corrective part, $q_x^*(r')/f$ can be found from a normal contact problem of the kind (2) for a *reduced* indentation. Also inequalities in the slip area translate to the no-interpenetration condition of the corrective contact problem.

All the relevant quantities can be obtained from the superposition of the normal load solutions, using the corrective load defined by $Q_x^* = fP - Q_x$. The approach in the tangential direction can be related to the superposition of approaches in normal directions through the factor $(1 + (\gamma/2))$, i.e.,

$$\frac{\alpha_t}{f} = \left(1+\frac{\gamma}{2}\right)(\alpha_n - \alpha_n^*). \quad (33)$$