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On non-symmetrical plane contacts

Michele Ciavarella[a,*], Giuseppe Demelio[b]

[a]Department of Engineering Science, Oxford University, Parks Road, Oxford OX1 3PJ, UK
[b]Dipartimento di Progettazione e Produzione Industriale Politecnico di Bari, Viale Japigia 182, 70126 Bari, Italy

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Abstract

Plane elastic contact problems are considered, with particular emphasis on asymmetrical punch profiles, in the case of 'complete', 'partially complete' and 'incomplete' contact. An explicit, analytical solution is presented for the case of a single area of contact where the overlap is described by a generic *spline function*, and examples presented. The interior stress field and strength of the contact, under full or partial slip conditions, are also discussed, and some example shown for representative cases. It is found also that the *direction* of sliding has a significant effect for the strength of non-symmetrical contacts. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In Hertz's theory for contact between elastic bodies [1] surface particles within the contact zone move *normal* to the free surface by an amount equal to the original gap, which is a symmetrical (specifically, quadratic) function with respect to the point of initial contact, and the relative approach of remote points is simply a rigid-body motion in the direction normal to the contact. Apart from the Hertzian case, the *majority* of solutions encountered in the literature assume symmetrical profiles and symmetrical indentation. This greatly simpli fies the solution of the related contact problems. However, non-symmetrical cases may be of considerable interest in the design of machine components, or experimental setups, or in other engineering applications. Even with nominally symmetrical contacts, there is a possible e ffect of a relative rotation, as a result of an applied moment, or of undesired geometrical asymmetry.

*Correspondence address: Department of Mechanical Engineering, University of Southampton, High field, Southampton S017 1BJ, UK.

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Notice that, as we will only consider plane problem, relative rotation is only possible with respect to the out-of-plane axis. Mathematical techniques for this class of problems are well developed (see, for example, [2, ch. 4; 3, ch. 2; 4, ch. 2]), so that a vast range of analytical results is available. Although general solutions in quadrature are well-known, we consider the specific case of a spline gap function, which gives a very general analytical treatment of the single area of contact case. We then show some properties of the resulting contact, which have not been clearly discussed before, in the best of authors' knowledge. Moreover, we address the problem of tangential loading of such contacts, as well as the strength of contact in such conditions, which is reported for representative cases, to show the main effects of non-symmetry.

2. Formulation

A contact problem in general involves two bodies with different profiles. However, without loss of generality, we can formulate it as an elastic indenter (body 1) indenting an elastic half-plane (body 2), as shown in Fig. 1. The elastic properties of indenter and half-plane may be different, but the solution to be developed is mathematically *exact* if there are no shearing tractions present in the normal indentation, which in turn requires absence of interfacial friction, $f = 0$, or that the materials are elastically similar, i.e. [5]

$$\frac{1-2\nu_1}{\mu_1} = \frac{1-2\nu_2}{\mu_2}, \tag{1}$$

where μ_i is the shear modulus and ν_i is the Poisson's ratio of body i. Under this limitation, which is not too restrictive for practical material's combinations, frictional stresses do not affect the pressure distribution, and so the equations of compatibility of normal displacements, and its equivalent one in the tangential direction (here omitted) are *uncoupled*, with the consequence that solutions for the normal and tangential loading may be obtained independently.

Let us define the function $h(x)$, as the amount of overlap if the bodies could freely interpenetrate each other, as

$$h(x) = \alpha x + h_0(x) = \alpha x + C - [f_1(x) - f_2(x)], \tag{2}$$

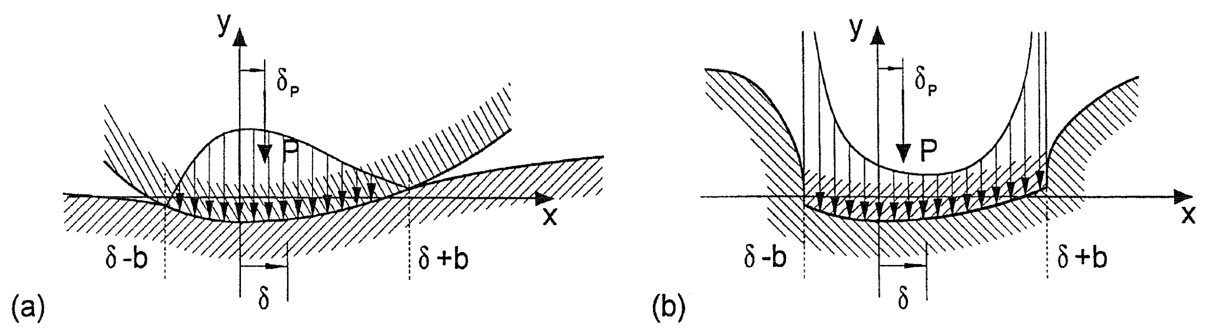

Fig. 1. General non-symmetric plane contact. Coordinate system and symbols used: (a) Incomplete; (b) complete contact.

where $y = f_1(x)$ and $y = f_2(x)$ describe the geometry of the undeformed bodies, C is the approach of two remote points, which characteristically remains undetermined for half-plane elasticity, and α is the anti-clockwise *relative* rotation of the contacting bodies (the function $h_0(x)$ defines the overlap for the non-rotated case).

The integral equation relating the pressure to the normal displacements can be written [4, Section 2.7] in terms of displacements, but it is usually preferable to work with displacement derivatives. Let us now further specialize the problem to a single contact area. In general, the contact area, of dimension $2b$, is not symmetrical relative to the origin; let then us assume that it extends over the interval $-b+\delta \leqslant x \leqslant b+\delta$, where δ is an *offset* with respect to the origin of coordinates $x = 0$. Moreover, we can write

$$\frac{1}{A} h'(\xi+\delta) = \frac{1}{\pi}\int_{-b}^{b} \frac{p(\tau+\delta)\,\mathrm{d}\tau}{\xi-\tau}, \quad -b \leqslant \xi \leqslant b, \tag{3}$$

where we have used the substitution

$$x = \xi + \delta. \tag{4}$$

Eq. (3) is the main integral equation describing behaviour of the system. Here, A is a measure of the composite compliance of the bodies, [1] defined by

$$A = \frac{1+\kappa_1}{4\mu_1} + \frac{1+\kappa_2}{4\mu_2}, \tag{5}$$

where κ is the Kolosov constant, defined as $\kappa = 3 - 4\nu$ under plane strain and $\kappa = (3-\nu)/(1+\nu)$ under plane stress conditions. Moreover, μ_i and ν_i are shear modulus and Poisson's ratio, respectively, of the material i.

2.1. Solution

The solution in quadrature for the general 'complete' contact (where the contact area does not change with load), of half-width b [6, 7] is

$$p(\xi+\delta) = -\frac{1}{\pi\sqrt{b^2-\xi^2}}\left[P - \frac{1}{A}\int_{-b}^{b} \frac{h'(\tau+\delta)\sqrt{b^2-\tau^2}}{\tau-\xi}\,\mathrm{d}\tau\right], \quad -b \leqslant \xi \leqslant b, \tag{6}$$

where b, δ are fixed a priori in the 'complete contact' case. If we consider that

$$\int_{-b}^{b} \frac{\sqrt{b^2-\tau^2}}{\tau-\xi}\,\mathrm{d}\tau = (b^2-\xi^2)\int_{-b}^{b} \frac{\mathrm{d}\tau}{\sqrt{b^2-\tau^2}(\tau-\xi)} - \int_{-b}^{b} \frac{\tau\,\mathrm{d}\tau}{\sqrt{b^2-\tau^2}} - \xi\int_{-b}^{b} \frac{\mathrm{d}\tau}{\sqrt{b^2-\tau^2}} = -\pi\xi, \tag{7}$$

[1] This is the notation used in [3], but it is readily conformed to [4] by substituting $A = 2/E^*$ of Johnson's notation.

where we used the circumstance that the two integrals

$$\int_{-b}^{b} \frac{d\tau}{\sqrt{b^2-\tau^2}(\tau-\xi)} = 0, \quad \int_{-b}^{b} \frac{d\tau}{\sqrt{b^2-\tau^2}} = 0 \tag{8}$$

vanish, we obtain, in terms of the function h_0,

$$p(\xi+\delta) = -\frac{1}{\pi\sqrt{b^2-\xi^2}}\left[P + \frac{\alpha}{A}\pi\xi - \frac{1}{A}\int_{-b}^{b} \frac{h_0'(\tau+\delta)\sqrt{b^2-\tau^2}}{\tau-\xi}\,d\tau\right], \quad -b \leqslant \xi \leqslant b. \tag{9}$$

The solution, in this form, merits some observations. It is evident that the three terms that compose the function within square brackets are completely independent from each other: the first depends on the applied load, the second on the rotation, and the third on the function profile itself, before any rotation. Therefore, the pressure distribution will depend on the combination of the three. In problems with linear kinematics and smoothly turning boundaries, the inequalities of the governing contact problem imply that the contact tractions will tend to zero at the edge of the contact region. This can be demonstrated by examining the asymptotic fields at the transition between a region of contact and separation [4, Section 5.1]. This condition of continuity of contact tractions can be used in place of the inequalities in our case of single area of mechanical contact, but it should be emphasised that the inequality formulation is the correct physical statement of the problem and indeed it can be shown that continuity of tractions is a necessary but not sufficient condition in other classes of problem, e.g. with multiple areas of contact or heat transfer.

Starting, for example, from a particular *complete* contact where P is sufficiently high, the pressure tends to

$$p(\xi+\delta) \to -\frac{P}{\pi\sqrt{b^2-\xi^2}}, \quad -b \leqslant \xi \leqslant b, \tag{10}$$

which is the pressure for a flat punch, singular at the contact area edges. Decreasing the load P, the pressure decreases, in particular at the edges; finally, there will be a value for which the pressure is zero at one edge (eventually at both), and this corresponds to the *partially complete* contact case (eventually *incomplete*); continuing to decrease the value of the load, a tensile zone appears at the boundaries, and this clearly contradicts the condition of contact. Therefore, the load at which either $p(b+\delta)=0$ or $p(-b+\delta)=0$, is the lower physically acceptable limit; what happens in reality is that the contact area dimension has to decrease. Finally, note from Eq. (9) that the sole effect that a rotation produces is to add a linear term to the term under square brackets in Eq. (9). The general solution has therefore three special cases: two of partially complete contact (i) bounded pressure at $\xi=-b$, unbounded at $\xi=b$, or (ii) bounded pressure at $\xi=b$, unbounded at $\xi=-b$, and the case of incomplete contact with bounded pressure at $\xi=\pm b$. In case (i), writing $p(\xi=-b)=0$ in Eq. (6), we obtain the value of the load P as

$$0 = P - \frac{1}{A}\int_{-b}^{b} \frac{h'(\tau+\delta)\sqrt{b^2-\tau^2}}{\tau+b}\,d\tau \tag{11}$$

i.e.

$$P = \frac{1}{A}\int_{-b}^{b} h'(\tau+\delta)\sqrt{\frac{b-\tau}{b+\tau}}\,d\tau. \tag{12}$$

Substituting this value of P in Eq. (6),

$$p(\xi+\delta) = -\frac{1}{\pi A\sqrt{b^2-\xi^2}}\int_{-b}^{b}\left[\frac{h'(\tau+\delta)\sqrt{b^2-\tau^2}}{\tau+b}-\frac{h'(\tau+\delta)\sqrt{b^2-\tau^2}}{\tau-\xi}\right]\mathrm{d}\tau \tag{13}$$

and rearranging

$$p(\xi+\delta) = \frac{1}{\pi A}\sqrt{\frac{b+\xi}{b-\xi}}\int_{-b}^{b}\frac{h'(\tau+\delta)}{\tau-\xi}\sqrt{\frac{b-\tau}{b+\tau}}\,\mathrm{d}\tau, \quad -b \leqslant \xi \leqslant b. \tag{14}$$

The dimension of the contact area, for a given load, is fixed by Eq. (12), which gives simultaneously the value of b, δ, as one contact edge is fixed a priori. For case (ii), it is straightforward to obtain analogous equations. Finally, case (iii) is examined in the next section, in more detail for its importance.

2.2. Incomplete contact

Here both $p(\xi=b)$ and $p(\xi=-b)$ are bounded (it may be proved that in particular $p(\xi=\pm b)\to 0$). Starting from the solution of case (i), Eq. (14), and adding the condition that $p(\xi=b)=0$, gives the following 'consistency condition':

$$0 = \int_{-b}^{b}\frac{h'(\tau+\delta)}{\tau-b}\sqrt{\frac{b-\tau}{b+\tau}}\,\mathrm{d}\tau \tag{15}$$

which can be rearranged, upon isolating the e ffect of rotation,

$$\alpha\pi + \int_{-b}^{b}\frac{h_0'(\tau+\delta)\,\mathrm{d}\tau}{\sqrt{b^2-\tau^2}} = 0. \tag{16}$$

The equation for the load (12) also simpli fies immediately to

$$P = -\frac{1}{A}\int_{-b}^{b}\frac{h_0'(\tau+\delta)\tau\,\mathrm{d}\tau}{\sqrt{b^2-\tau^2}} \tag{17}$$

from which it is evident that in incomplete contacts the rotation aff ects directly the offset δ (16), and only indirectly the load (17). Finally, the solution in quadrature for the pressure (14) can be simplified using again the condition $p(\xi=b)=0$, obtaining

$$p(\xi+\delta)-p(b+\delta) = \frac{1}{\pi A}\sqrt{\frac{b+\xi}{b-\xi}}\int_{-b}^{b}\frac{h'(\tau+\delta)}{\tau-\xi}\sqrt{\frac{b-\tau}{b+\tau}}\left[\frac{1}{\tau-\xi}-\frac{1}{\tau-b}\right]\mathrm{d}\tau \tag{18}$$

and, upon rearranging, we have

$$p(\xi+\delta) = \frac{1}{\pi A}\sqrt{b^2-\xi^2}\int_{-b}^{b}\frac{h_0'(\tau+\delta)\,\mathrm{d}\tau}{\sqrt{b^2-\tau^2}(\tau-\xi)}, \quad -b \leqslant \xi \leqslant b, \tag{19}$$

where the term corresponding to the rotation α in $h'(x)$ cancels out, due to the vanishing integral

$$\int_{-b}^{b}\frac{\mathrm{d}\tau}{\sqrt{b^2-\tau^2}(\tau-\xi)} = 0. \tag{20}$$

2.3. Offset of the load

P is defined as the magnitude of the load, i.e. a scalar quantity; in order to compute the offset of the load δ_P, with respect to $x = 0$ direction, we compute the moment M of the pressure with respect to the same direction

$$M = -\int_{-b+\delta}^{b+\delta} p(t)t\,\mathrm{d}t = -\int_{-b}^{b} p(\tau+\delta)(\tau+\delta)\,\mathrm{d}\tau$$

$$= P\delta - \int_{-b}^{b} p(\tau+\delta)\tau\,\mathrm{d}\tau, \tag{21}$$

where we have used Eq. (4). On taking Eq. (3), multiplying by $\sqrt{b^2-\xi^2}$, and integrating them with respect to ξ from $-b$ to b, we get

$$\frac{1}{\pi}\int_{-b}^{b} p(\tau+\delta)\left[\int_{-b}^{b}\frac{\sqrt{b^2-\xi^2}}{\tau-\xi}\,\mathrm{d}\xi\right]\mathrm{d}\tau = -\frac{\pi}{A}\int_{-b}^{b} h'(\xi+\delta)\sqrt{b^2-\xi^2}\,\mathrm{d}\xi \tag{22}$$

Further, the integral within squared brackets can be evaluated as in Eq. (7), hence

$$\int_{-b}^{b} p(\tau+\delta)\tau\,\mathrm{d}\tau = -\frac{1}{A}\int_{-b}^{b} h'(\xi+\delta)\sqrt{b^2-\xi^2}\,\mathrm{d}\xi \tag{23}$$

so that the offset of the load, δ_P, is

$$\delta_P = \frac{M}{P} = \delta + \frac{\alpha\pi}{2}\frac{b^2}{AP} + \frac{1}{AP}\int_{-b}^{b} h_0'(\tau+\delta)\sqrt{b^2-\tau^2}\,\mathrm{d}\tau \tag{24}$$

which is valid in all cases of contacts. Of course, δ is known a priori in the complete case, so that the relationship $M-\alpha$ is direct. For P we could use Eq. (17). Vice versa, it may be that M is known and so the previous equations allow us to compute the relative rotation α.

3. Spline profile

The case of a spline profile is here considered in the case of a generic complete or incomplete single contact. Let us consider two profiles that are, in general, piecewise parabolic. The function $h(x)$ will then be described by a set of n parabolic functions, so that $h_0'(x)$ in the ith interval $(i = 1, \ldots, n)$ will be given by

$$h_0'(x) = m_i x + D_i, \quad x_i < x < x_{i+1}. \tag{25}$$

The solution for the pressure will be given only for the complete contact case, which includes, as a special case, the incomplete contact. Additional equations for the latter will be given later.

3.1. Complete contact

In the case of *complete* contact the contact area boundaries x_1, x_{n+1} are known a priori, and are $x_1 = -b + \delta$, $x_{n+1} = b + \delta$. Let us reconsider the two piecewise parabolic pro files giving Eq. (25). On imposing the shift (4), we have from Eq. (9) that, for $-b < \xi < b$,

$$p(\xi + \delta) = -\frac{1}{\pi\sqrt{b^2 - \xi^2}}\left[P + \frac{\alpha\pi}{A}\xi - \frac{1}{A}\sum_{i=1}^{n}\int_{x_i-\delta}^{x_{i+1}-\delta}\frac{[m_i(\tau+\delta)+D_i]\sqrt{b^2-\tau^2}}{\tau-\xi}\,\mathrm{d}\tau\right]. \tag{26}$$

Notice that we maintain the symbol δ for the 'offset', but it is clear that the contact area position is known a priori, so that δ is a fixed, given quantity. Let us de fine an integral $I(\xi)$, so that, on adding and subtracting $m_i\xi$, we obtain

$$bI(\xi) = \sum_{i=1}^{n}\int_{x_i-\delta}^{x_{i+1}-\delta}\frac{[m_i(\tau+\delta)+D_i]\sqrt{b^2-\tau^2}}{\tau-\xi}\,\mathrm{d}\tau$$

$$= \sum_{i=1}^{n}\int_{x_i-\delta}^{x_{i+1}-\delta} m_i\sqrt{b^2-\tau^2}\,\mathrm{d}\tau + \sum_{i=1}^{n}[m_i(\xi+\delta)+D_i]\int_{x_i-\delta}^{x_{i+1}-\delta}\frac{\sqrt{b^2-\tau^2}}{\tau-\xi}\,\mathrm{d}\tau. \tag{27}$$

Then, on using the substitutions

$$\sin\varphi = \frac{\xi}{b}, \tag{28}$$

$$\sin\vartheta = \frac{\tau}{b}, \tag{29}$$

and hence for $i = 1, \ldots, n$

$$\sin\varphi_i = \frac{\xi_i}{b} = \frac{x_i - \delta}{b}, \tag{30}$$

so that $\varphi_1 = -\pi/2$, $\varphi_{n+1} = \pi/2$, we then have

$$I(\varphi) = \sum_{i=1}^{n} m_i\int_{\varphi_i}^{\varphi_{i+1}}\cos^2\vartheta\,\mathrm{d}\vartheta + \sum_{i=1}^{n}[m_i(b\sin\varphi+\delta)+D_i]\int_{\varphi_i}^{\varphi_{i+1}}\frac{\cos^2\vartheta}{\sin\vartheta-\sin\varphi}\,\mathrm{d}\vartheta.$$

Summarizing, we have, in dimensionless form,

$$p(\varphi)\frac{b}{P} = -\frac{1}{\pi\cos\varphi}\left[1 + \frac{b}{AP}(\alpha\pi\sin\varphi - I(\varphi))\right], \tag{31}$$

where the non-elementary integral is solvable with standard handbooks or symbolic software like Mathematica v.3.0 [7] giving

$$I(\varphi)=\frac{b}{2}\sum_{i=1}^{n} m_i\left(\Delta\varphi_i+\frac{1}{2}\Delta\sin 2\varphi_i\right)+\sum_{i=1}^{n}\left[m_i(b\sin\varphi+\delta)+D_i\right]$$

$$\times\left[\Delta\cos\varphi_i-\Delta\varphi_i\sin\varphi+\cos\varphi\ln\left|\frac{\cos\frac{\varphi+\varphi_i}{2}\ \sin\frac{\varphi-\varphi_{i+1}}{2}}{\cos\frac{\varphi+\varphi_{i+1}}{2}\ \sin\frac{\varphi-\varphi_i}{2}}\right|\right],\tag{32}$$

where $\Delta\varphi_i=\varphi_{i+1}-\varphi_i$, $\Delta\cos\varphi_i=\cos\varphi_{i+1}-\cos\varphi_i$, $\Delta\sin 2\varphi_i=\sin 2\varphi_{i+1}-\sin 2\varphi_i$.

In order to obtain the moment with respect to the contact area centre, M_0, we compute from Eq. (34)

$$AM_0=\frac{\alpha\pi}{2}b^2+\sum_{i=1}^{n}\int_{x_i-\delta}^{x_{i+1}-\delta}\left[m_i(\tau+\delta)+D_i\right]\sqrt{b^2-\tau^2}\,\mathrm{d}\tau$$

$$=\frac{\alpha\pi}{2}b^2+b^2\sum_{i=1}^{n}(D_i+m_i\delta)\int_{\varphi_i}^{\varphi_{i+1}}\cos^2\varphi\,\mathrm{d}\varphi-b^3\sum_{i=1}^{n}m_i\int_{\varphi_i}^{\varphi_{i+1}}\cos^2\varphi\,\mathrm{d}\cos\varphi\tag{33}$$

from which

$$\frac{AM_0}{b^2}=\frac{\alpha\pi}{2}+\frac{1}{2}\sum_{i=1}^{n}(D_i+m_i\delta)\left[\Delta\varphi_i+\frac{\Delta\sin 2\varphi_i}{2}\right]-\frac{b}{3}\sum_{i=1}^{n}m_i\Delta\cos^3\varphi_i.\tag{34}$$

3.2. Incomplete contact

In this case, as already stated, the contact area boundaries x_1, x_{n+1} are not known a priori, but we can consider the contact area to be positioned again in the region $x_1=-b+\delta$, $x_{n+1}=b+\delta$, with δ unknown. The difference here from the previous case of complete contact is that, for an assigned load and moment, there is only one dimension of the contact area and one value for the offset which gives the correct zero pressure at each contact boundary. [2] From the '*consistency condition*' (16) it follows that

$$\alpha\pi+\sum_{i=1}^{n}\int_{x_i-\delta}^{x_{i+1}-\delta}\frac{m_i(\tau+\delta)+D_i}{\sqrt{b^2-\tau^2}}\,\mathrm{d}\tau=0.\tag{35}$$

Hence,

$$\alpha\pi+\sum_{i=1}^{n}m_i\int_{x_i-\delta}^{x_{i+1}-\delta}\frac{\tau\mathrm{d}\tau}{\sqrt{b^2-\tau^2}}+\sum_{i=1}^{n}D_i\int_{x_i-\delta}^{x_{i+1}-\delta}\frac{\mathrm{d}\tau}{\sqrt{b^2-\tau^2}}+\delta\sum_{i=1}^{n}m_i\int_{x_i-\delta}^{x_{i+1}-\delta}\frac{\mathrm{d}\tau}{\sqrt{b^2-\tau^2}}=0.\tag{36}$$

[2] The situation is clearly reversible, i.e. for an assigned contact area dimension and offset, there is only one value for the applied load and moment.

which gives, on solving the elementary integrals using the transformation (29) and de finition (30), the following implicit relation for δ:

$$\alpha\pi - b\sum_{i=1}^{n} m_i \Delta\cos\varphi_i + \sum_{i=1}^{n} D_i \Delta\varphi_i + \delta\sum_{i=1}^{n} m_i\Delta\varphi_i = 0, \tag{37}$$

where $\Delta\varphi_i = \varphi_{i+1} - \varphi_i$, and $\Delta\cos\varphi_i = \cos\varphi_{i+1} - \cos\varphi_i$.

From Eq. (17), it follows that

$$P = -\frac{1}{A}\sum_{i=1}^{n}\int_{x_i-\delta}^{x_{i+1}-\delta} \frac{m_i(\tau+\delta)+D_i}{\sqrt{b^2-\tau^2}}\,\tau\,\mathrm{d}\tau \tag{38}$$

so that

$$-AP = \sum_{i=1}^{n} m_i \int_{x_i-\delta}^{x_{i+1}-\delta} \frac{\tau^2\,\mathrm{d}\tau}{\sqrt{b^2-\tau^2}} + \delta\sum_{i=1}^{n} m_i \int_{x_i-\delta}^{x_{i+1}-\delta} \frac{\tau\,\mathrm{d}\tau}{\sqrt{b^2-\tau^2}} + \sum_{i=1}^{n} D_i \int_{x_i-\delta}^{x_{i+1}-\delta} \frac{\tau\,\mathrm{d}\tau}{\sqrt{b^2-\tau^2}}. \tag{39}$$

On solving the elementary integrals we find

$$-\frac{AP}{b} = \frac{b}{2}\sum_{i=1}^{n} m_i\left(\Delta\varphi_i - \frac{\Delta\sin 2\varphi_i}{2}\right) - \delta\sum_{i=1}^{n} m_i\,\Delta\cos\varphi_i - \sum_{i=1}^{n} D_i\Delta\cos\varphi_i, \tag{40}$$

where $\Delta\sin 2\varphi_i = \sin 2\varphi_{i+1} - \sin 2\varphi_i$. The expression for the pressure is obtained from the formulation appropriate to the complete contact case with the proper load (17). In other words, the incomplete contact case may, in these respects, be regarded as a special case of the complete contact, where the o ffset is known, and the load takes a prescribed value.

4. Examples

4.1. Incomplete contacts

As the general solution (31) is somewhat complicated, it is of some interest to derive simpler expressions for cases of special geometry; these are shown in Fig. 2, and we will display also the principal effects of the asymmetry on the contact mechanics.

4.1.1. A tilted wedge-shaped punch

The case of a wedge-shaped punch indenting a half-plane is derived from (25) on considering $n = 2$, $m_1 = m_2 = 0$, and $x_2 = 0$; also, put $D_1 = -D_2 = D$. The expression for δ can be obtained in closed form in this case from (37). On putting $\sin\varphi_\delta = \sin\varphi_2 = \delta/b$, we obtain

$$\alpha\pi + D\left(\varphi_\delta + \frac{\pi}{2}\right) - D\left(\frac{\pi}{2} - \varphi_\delta\right) = 0 \tag{41}$$

from which δ is obtained as

$$\frac{\delta}{b} = \sin\varphi_\delta = -\sin\frac{\pi\alpha}{2D}. \tag{42}$$

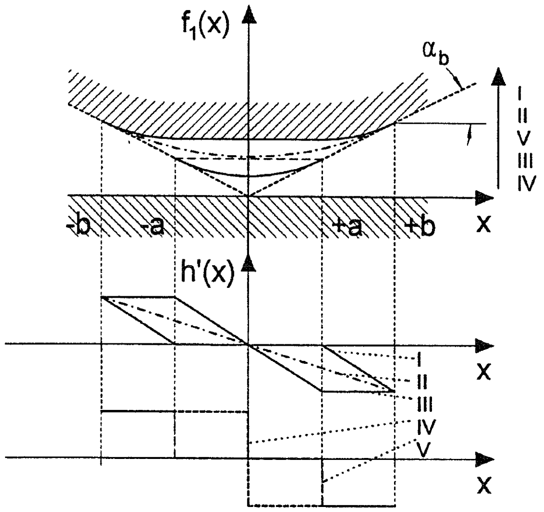

Fig. 2. Example problems. Profiles and $h'(x)$ function for the indenters: (I) flat with rounded corners; (II) parabolic (Hertzian); (III) rounded wedge; (IV) wedge; (V) truncated wedge.

The load is obtained from Eq. (40) as

$$\frac{AP}{b} = 2D\cos\varphi_\delta. \tag{43}$$

Finally, the expression for the pressure (31) gives, after some algebra,

$$p(\varphi)\frac{b}{P} = \frac{1}{\pi\cos\varphi_\delta}\ln\frac{\left|\tan\frac{\varphi_\delta}{2} + \tan\frac{\varphi}{2}\right|}{\left|\tan\frac{\varphi_\delta}{2}\tan\frac{\varphi}{2} + 1\right|}, \tag{44}$$

where φ is defined as usual by Eq. (28). The pressure distribution as a function of the imposed relative rotation, is shown in Fig. 3a, together with the pro file. The rotation α is non-dimensionalized with respect to $\alpha_b = |h'(b)|$, i.e. the external angle of the wedge-shaped punch in symmetrical position. The values shown are $\alpha/\alpha_b = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$.

4.1.2. A tilted Hertzian indenter

The classical case of a Hertzian punch (symmetrical parabola) indenting a half-plane, is obtained from Eq. (25) considering $n = 1$, $m_1 = m$ and $D_1 = 0$, and from the general solution (31) it follows for the pressure distribution

$$p(\varphi)\frac{\pi A}{b} = \pi m\cos\varphi, \tag{45}$$

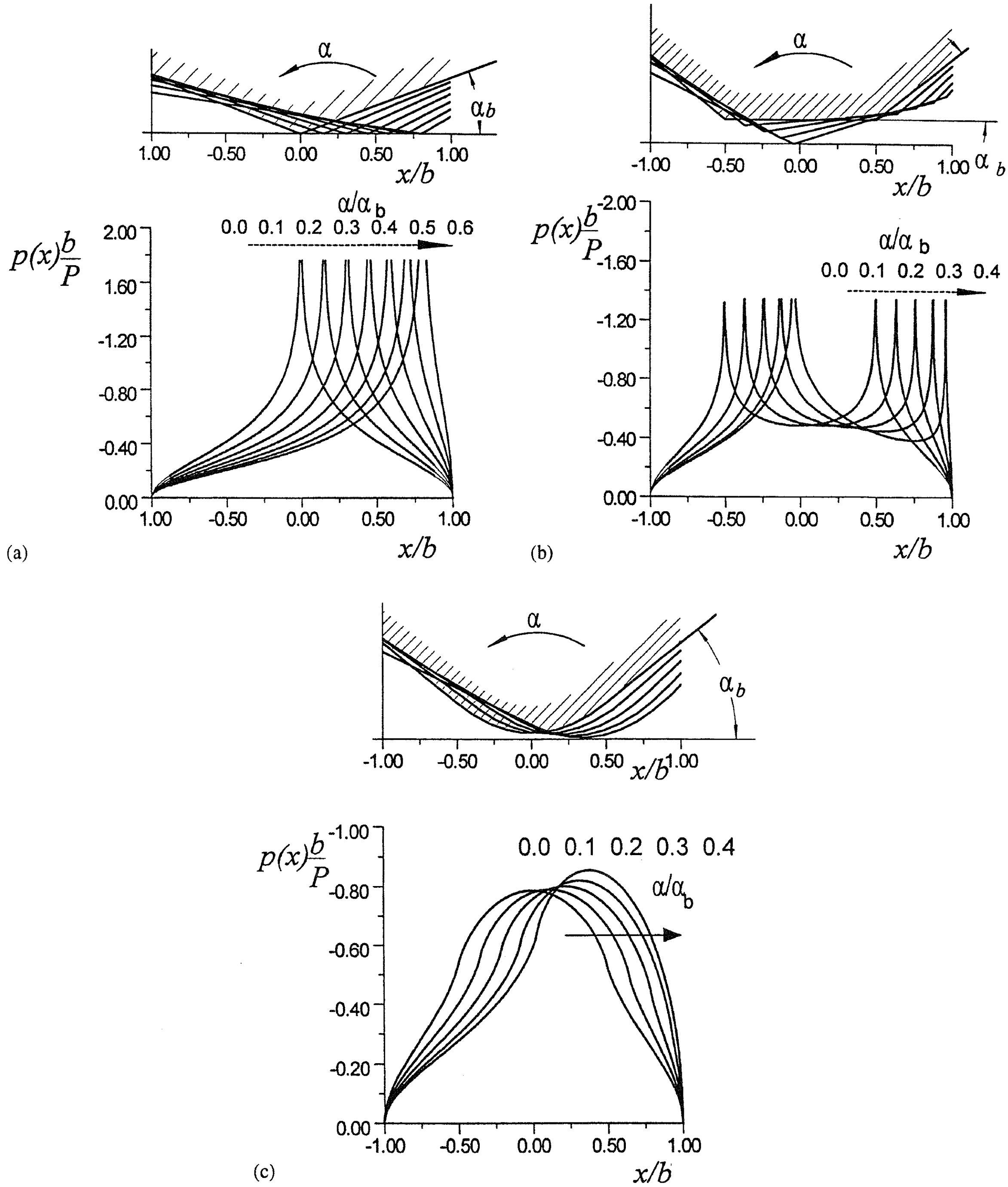

Fig. 3. Non-dimensional pressures as a function of rotation: (a) for a wedge-shaped indenter; (b) a truncated wedge-shaped indenter; (c) for a wedge with rounded apex indenter; (d) for a flat with rounded corners indenter ($a/b = 0.5$).

whereas the load is given by Eq. (40)

$$-\frac{AP}{b} = \pi\frac{b}{2}m \tag{46}$$

so that, dividing the two, the well-known result

$$p(x)\frac{b}{P} = -\frac{2}{\pi}\sqrt{1-\left(\frac{x}{b}\right)^2}. \tag{47}$$

The case where a relative rotation appears, is obtained on considering $D = D_1 \neq 0$, but it does not alter the result for the pressure distribution shape.

4.1.3. Other cases and comparison with Hertzian

Fig. 3b–d describe respectively the case of a truncated wedge, a wedge with rounded apex, and a flat punch with rounded corners, for the particular geometric ratio $a/b = 0.5$ (see Fig. 2). As before, we indicate the effect of the rotation, through the ratio α/α_b, where α_b indicates the external angle of the profile corresponding to the extreme of the contact area, in the symmetrical configuration.

Let us move, now, to a discussion of the variation of macroscopic quantities with the rotation α. In the Hertzian case, it appears immediately that the offset of the contact area centre, δ^H, and the offset δ_P^H of the load, are given by

$$\delta^H = \delta_P^H = -\frac{D}{m} \tag{48}$$

Notice, furthermore, that the presence of the offset δ in the Hertzian case does not affect the variation of the load P. In practice, the Hertzian case corresponds correctly to the cylinder (or a parabolic idealization), whose profile evidently does *not* change with rotation. As the coefficient D represents the rotation of the profile, the eventual effects of the rotation on δ and e, are here non-dimensionalized with respect to the relevant values of the Hertzian case, to show the differences clearly. Figs. 4a–c show the variation of the load P, δ and δ_P as a function of the ratio α/α_b. For comparison purposes, also, the quantities are non-dimensionalized as $(AP/b)\alpha_b$ for the load, δ/δ^H for the offset, and $(\delta - \delta_P)/\delta^H$ where δ^H is the Hertzian case. Obviously when $\alpha = 0$, the offset in the Hertzian case is $\delta^H = 0$.

In the case of a truncated or rounded wedge, the load for a fixed value of contact dimension has a variation with α which is just shifted with respect to the wedge case. Further, in the case of a flat punch with rounded wedges, the load grows with α, for a given dimension of the contact area. These results are obtained, as already stated, for a particular set of geometries (in particular, the ratio a/b shown in Fig. 2 is 0.5).

4.2. Complete contacts

For a complete contact there is freedom to impose a load and either a moment or a rotation, for any choice of offset and contact area dimension, as the last two are given geometrical values. Therefore, the number of possible configurations is impractically high to display, and we give an account of only the cases shown in Fig. 2, for which we plot in Fig. 5 the function $I(\varphi)$. As the load varies, it is possible to compute the superposition of the three contributions to the pressure from Eq. (9), where the effect of a rotation is shown to be just a linear contribution to the term under square brackets in Eq. (9).

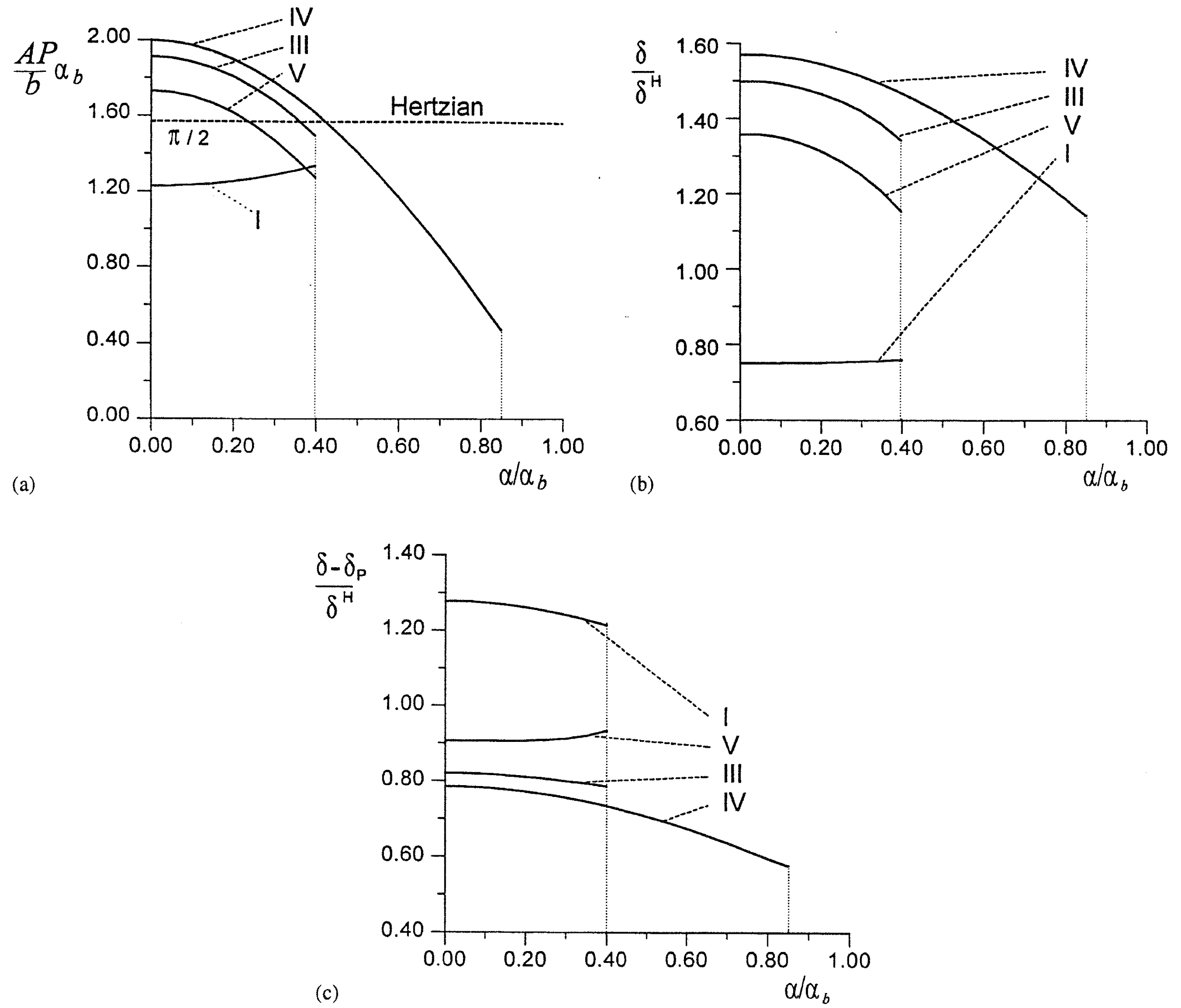

Fig. 4. (a) Non-dimensional load; (b) Offset of the contact area; (c) Offset of the load (with respect to contact area centre) as a function of rotation for the example geometries ($a/b = 0.5$).

5. Tangential loading and partial slip

In many contacts arising in engineering components, such as bolted joints, splines, or the dovetail roots of turbine blades, a partial slip regime arises in the contact area, where the tangential force or displacement component varies with time, generally cyclically, whereas the normal load is nearly constant. The case of partial slip with constant normal load can be solved nearly as easily, as shown in recent papers by the first author [8, 9]. Briefly, the shear can be obtained by correcting the full sliding traction component in the stick zone, by

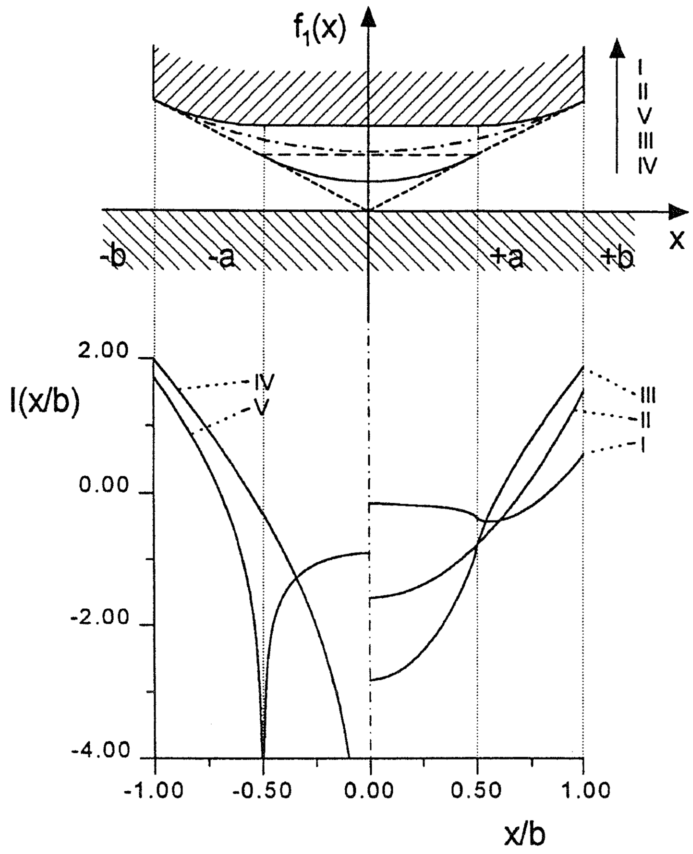

Fig. 5. Function $I(\varphi)$ for complete contacts of the example geometries ($a/b = 0.5$).

Cattaneo's superposition

$$q(x) = \begin{cases} fp(x) - q^*(x), & x \in S_{stick}, \\ fp(x), & x \in S_{slip}, \end{cases} \tag{49}$$

solving a corrective normal *incomplete* contact problem for a reduced load but 'fixed' rotation. Finally, notice that unloading and reloading in tangential direction can also be considered easily, by using appropriate corrections, and indeed it may also be proved that, if wear occurs in the slip areas, the profile evolves such that in the steady state only the originally adhesive region is in contact.

6. Strength of the contact

Although the determination of the pressure and shearing traction distributions is the first step in analysing the contact, it is the stress field induced in the interior of the bodies which determines the strength of the contact, and the relevant quantities for mechanical design.

The most powerful method for the stress field calculation in plane problems is the use of the Muskelishvili potential, $\Phi(z)$, from which all stress components, as well as displacement fields, can be derived from the well-known relations [3, Section 5.2].

$$\frac{\sigma_{xx}+\sigma_{yy}}{2} = 2\,\mathrm{Re}\,\Phi(z),$$

$$\frac{\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy}}{2} = (\bar{z}-z)\Phi'(z) - \bar{\Phi}(z) - \Phi(z), \tag{50}$$

$$2\mu\left(\frac{\partial u_x}{\partial x} + i\frac{\partial u_y}{\partial y}\right) = (\bar{z}-z)\bar{\Phi}'(\bar{z}) - \Phi(\bar{z}) + \kappa\Phi(z).$$

The potential can be calculated from the pressure, for example in the condition of full sliding [3, section 3.1) as[3]

$$\Phi(z) = \frac{1-\mathrm{i}f}{2\pi\mathrm{i}}\int_{-1}^{1}\frac{p(\xi)}{\xi - z}\,\mathrm{d}\xi, \tag{51}$$

where f is the coefficient of friction. The case of partial slip, being related to a normal contact problem, is readily covered by introducing a corrective potential, $\Phi^*(z)$. Analogously, the case of tangential unloading and cyclic loading requires just a correct series of similar corrections.

6.1. Incomplete contacts

Here the strength of material approach is particularly valuable, as the stress state is well defined, and the point of severest stress may occur sub-surface. An analytical solution for the integral (51) is often impossible, and a general approach, to be preferred, is the use of a Chebyshev expansion for the pressure. In the case of an incomplete contact,

$$p(x) = -\sqrt{1-x^2}\sum_{n=0}^{\infty} a_n U_n(x), \tag{52}$$

where $U_n(x)$ are second kind Chebyshev polynomials; the potential (51) can be integrated in closed form [2, Section 2.3] as

$$\Phi(z) = -\frac{\mathrm{i}+f}{2}\sum_{n=0}^{\infty} a_n R_{n+1}(z) \tag{53}$$

where $R_n(z) = [z-(z^2-1)^{1/2}]^n$, $R_0(z) = 1$.

For reasons of space limitation, we cannot investigate all possible configurations of the contact, as the number of parameters would be prohibitively high (geometrical configuration, rotation,

[3] Here and in the following we are considering the coordinate system to be centred in the contact area centre, and the variable non-dimensionalized by the semi-contact width, b.

friction coefficient, plus normal load if the contact is complete, and tangential load if we are operating in the partial slip regime). However, the most relevant factors are here outlined. For the class of incomplete contacts, apart from the well-known Hertzian parabolic pro file and the wedge geometry (see the monograph by Hills et al. [3], more complicated geometries have recently received a detailed investigation, such as a wedge with rounded apex [11], a flat punch with rounded corners [12], all for the symmetrical case only, whereas it is now straightforward to include the non-symmetrical case.

Indeed, Figs. 6 and 7 show some results for sliding contacts, in terms of the non-dimensional elastic limit $P/(bk)$, where k is yield limit in pure shear, i.e. according to von Mises 's criterion. It should be borne in mind that for non-symmetrical pressure distributions, the direction of sliding does have an in fluence in terms of the maximum of Mises parameter, that is on the strength of the contact in the elastic regime. The figures in particular consider two geometrical con figurations: (a) a wedge with rounded apex indenter, and (b) a flat with rounded corners indenter both with $a/b = 0.5$. Fig. 6 refers to one direction of sliding, Fig. 7 to the opposite one. It may be noticed that, in the case of rounded wedge, although some sensible e ffect was found in the pressure distribution, the effect on the strength of the contact is marginal, in either directions (Figs. 6a and 7a), although it is more sensible in the reverse direction (Fig. 7a). In the case of the flat punch, vice versa, the e ffect is not negligible, particularly in the direction indicated in Fig. 7b. Indeed, in the direct direction (Fig. 6b) the effect of frictional shearing tractions is *beneficial* for the rotated con figurations examined, with respect to the normal indentation values, over the range of friction coe fficient $f = 0 - 0.25$.

6.2. Complete contacts

For complete contacts, the singularity at the corners of the contact area is the only parameter that can usefully be determined, according to elasticity theory, and treated with the principles of

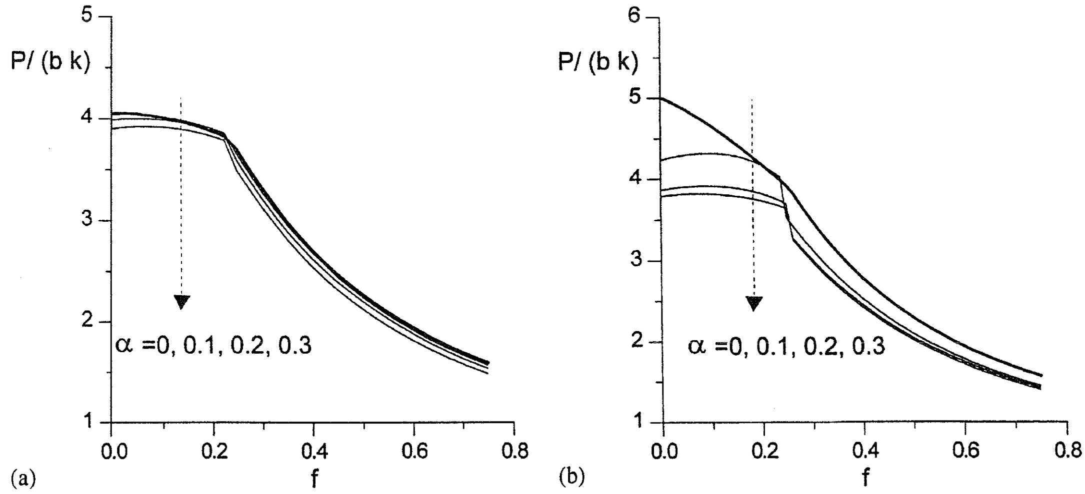

Fig. 6. Strength of the contact: non-dimensional elastic limit P/bk, where k is yield limit in pure shear: (a) for a wedge with rounded apex indenter; (b) for a flat with rounded corners indenter ($a/b = 0.5$).

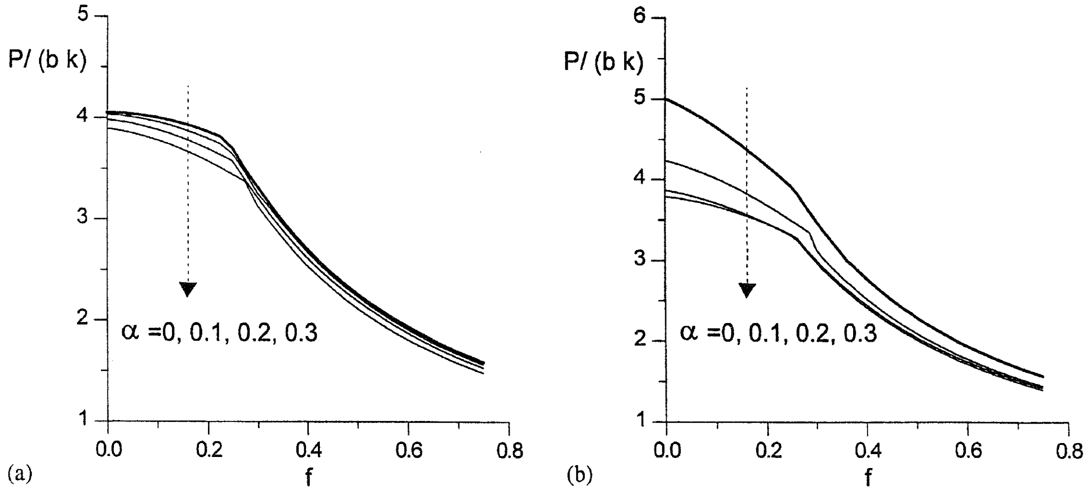

Fig. 7. As in Fig. 6, with opposite direction of sliding.

Linear Elastic Fracture Mechanics (LEFM). Therefore, although it may not be universally accepted, the only approach to quantify the severity of the stress field and therefore attempt to classify different conditions is to define stress intensity factors, to be compared to the material 's characteristic plane strain toughness, K_{Ic}, as

$$K_I(-b) = \lim_{\varphi \to -\pi/2} p(\varphi)\cos\varphi, \quad K_I(b) = \lim_{\varphi \to \pi/2} p(\varphi)\cos\varphi, \tag{54}$$

where as usual, φ is defined in Eq. (30). It follows that, for the case of a single contact area in the spline gap function,

$$K_I(-b)\frac{b}{P} = -\frac{1}{\pi}\left[1 - \frac{b}{AP} I\left(-\frac{\pi}{2}\right)\right], \quad K_I(b)\frac{b}{P} = -\frac{1}{\pi}\left[1 - \frac{b}{AP} I\left(\frac{\pi}{2}\right)\right]. \tag{55}$$

On the other hand, if for some reason the stress field is necessary for computational purposes, [4] it is possible to proceed as in the previous paragraph, although it is better to use another expansion for the pressure, viz.

$$p(x) = -\frac{\sum_{n=0}^{\infty} a_n T_n(x)}{\sqrt{1-x^2}} \tag{56}$$

and this time the potential (51) is given by [2, Section 2.3]

$$\Phi(z) = \frac{i+f}{2} \sum_{n=0}^{\infty} a_n G_{n-1}(z), \tag{57}$$

where $G_{n-1}(z) = -R_n(z)/(z^2-1)^{1/2}$.

[4]For example, a distributed dislocation technique to solve a crack problem in the vicinity of the contact requires the knowledge of the relevant components of the stress field in the *absence* of the crack.

7. Conclusions

The non-symmetrical contact problem between elastic half-planes has been considered for the general case of a spline gap function; the calculation of the o ffset of the contact area centre, as well as relationship between applied moment and relative rotation has been shown; several examples illustrated.

It is found that, although the e ffect on the pressure distribution is often visible, the corresponding effect on the strength of the contact is not necessarily as high as expected, and depends very much on the actual direction of sliding.

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