Technical Note

Conditions of yield and cyclic plasticity around inclusions

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Abstract: In this paper the stress field in the proximity of a circular (cylindrical) inclusion is considered. The conditions for in-plane plastic flow in the matrix are examined from a classical elasticity solution obtained by Goodier. Elementary cases are considered such as remote loading ranging from pure tensile and pure shear to equibiaxial tension. For proportional loading, it is argued that the upper bound to the shakedown limit is always twice the elastic limit; therefore, within the limits of our assumptions, if the elastic stress concentration for the equivalent stress is greater than *two*, there is a possibility of cyclic plasticity before bulk yielding, which means that possibly an arbitrarily large plastic strain can cumulate with increasingly high risk of exhaustion of ductility and void nucleation or detachment of the interface.

Consequently, conditions under which it is possible to reach *twice* the elastic limit *before* full-scale yielding are shown in the Dundurs plane representing all possible combinations of elastic parameters. Following these lines, it is shown that there is no possibility of cyclic plasticity under remote shear; there is a limited area of the Dundurs plane for tension, including the hole case; finally, in the equibiaxial limiting case, cyclic plasticity is always possible for any combination of elastic properties.

Keywords: yield, cyclic plasticity, inclusions

1 INTRODUCTION

The study of inclusions involves several areas of research, namely the study of polycrystals, metal matrix composites (MMCs) and more general particulate materials [**1**–**4**]. Knowledge of the mechanical properties of an MMC are essential for the design of structural components, in aerospace, automotive and other applications. Typically, the metal matrix has a well-defined plastic behaviour; however, the yielding limit and strain-hardening characteristics of a metal matrix depend on several factors, e.g. ageing treatments, and has been found to influence significantly the strength of an MMC [**1**, **3**]. Often theoretical predictions are made with models based on Eshelby's equivalent inclusion method [**2**, **5**], which basically replaces the material of the inclusion with an equivalently strained material of the same elastic constant as the matrix. This is an easy task because of the simplicity of the stress field inside the inclusion, as discovered by Eshelby. However, this method covers mainly elastic behaviour, whereas here we are interested in finding the conditions of yield and cyclic plasticity in the matrix.

Because of the many parameters involved, it becomes important to simplify the treatment and the presentation of the result by initial assumptions. However, not only is there no study in the literature of the elastoplastic strains in the general case of elastic dissimilarity, but also not even the elastic stress concentration is studied to the best of the present author's knowledge. This note is therefore mainly concerned with the details of the stress field that arises in proximity of an elastic inclusion, with particular reference to study of the elastic limit, shakedown and cyclic plasticity regimes. Cyclic plasticity is the regime above the shakedown limit and below bulk yielding which may lead to the cumulation of large strains around the inclusion, with an increasingly high likelihood of void nucleation or detachment of the matrix–inclusion interface or other damage in the matrix.

Conditions for incipient in-plane flow to arise in the matrix are considered as functions of the elastic constants (for our purposes, the Dundurs reduced dependence on just two constants α and β will be considered). For this purpose, the classical solution firstly developed by Goodier [**1**] and put into a modern form according to the Dundurs α and β parameters (see, for example, references [**7**] to [**9**]) is used,

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and results are shown for the inclusion subject to far-field loads, ranging from pure tension or pure shear to equibiaxial tension. Some interesting properties of the elastic stress field were discussed by Greenwood **[10]** for the limiting case of a hole, namely a relation between the stresses in the absence of a hole, and the stresses in the presence of the hole; this may prove useful to extend the present analysis to the case where the unperturbed stress field is not uniform.

2 FORMULATION

It is well known that the stress state in a homogeneous material undergoing plane deformation (in the xy plane, say) depends on only two elastic parameters (Young's modulus and Poisson's ratio). In general, *in-plane* stresses, i.e. σ_{xx}, σ_{yy} and σ_{xy}, for a bimaterial composite depend on three parameters, whereas displacements depend on all four elastic parameters (for a more complete explanation, see, for example, reference **[11]** or **[12]**). Dundurs **[7]**, however, showed that the in-plane stresses in a perfectly adhesive bimaterial joint undergoing plane deformation and loaded only by tractions on the boundaries depend on only *two* parameters: α and β **[8]**. Next the Dundurs parameters are defined and whether use can be made of this reduced dependence is examined.

2.1 The Dundurs parameters

Let μ_1, ν_1 and μ_2, ν_2 be the modulus of rigidity and Poisson's ratio of material 1 and material 2 respectively. The Dundurs parameters are then given by (see, for example, references **[7]** to **[9]** and **[12]**)

$$\alpha = \frac{\mu_2(\kappa_1+1)-\mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)}$$

$$\beta = \frac{\mu_2(\kappa_1-1)-\mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)} \tag{1}$$

where κ is the Kolosov constant: $\kappa_i = (3-\nu_i)/(1+\nu_i)$ in plane stress and $\kappa_i = 3-4\nu_i$ in plane strain, ν_i being Poisson's ratio of material i, and $i = 1, 2$.

This reduction in the dependence on elastic constants turns out to be extremely useful. Note, however, that displacement fields do *not* depend on only two parameters, but on three, namely ν_1, ν_2 and the ratio μ_2/μ_1. Further, the out-of-plane stress depends on the local value of Poisson's ratio, and therefore the correct version of the Tresca or von Mises criterion for plastic yielding requires knowledge of all the constants. As this greatly complicates the possibility of showing general results, and as out-of-plane flow is anyway limited if the loads are in plane, only the possibility of in-plane flow will be considered. This does not introduce any approximation, in the case of the Tresca criterion, when the out-of-plane stress is intermediate between the two in-plane principal stresses, as is often the case;* the equivalent stress is then $\sigma_e = \sigma_1 - \sigma_2$, which is twice the maximum in-plane shear stress τ_{max}, and so, at yield,

$$\tau_{max} = \frac{\sigma_Y}{2} \tag{3}$$

The $\alpha\beta$ plane provides a convenient means of classifying composite materials **[7–9, 12]** and for displaying results that depend on elastic constants, adopted in this paper for all the results (Fig. 1 and the following). It is clear that the magnitudes of α and β describe the degree of mismatch between the materials (the origin represents identical materials). The line $\alpha = \beta$ indicates the cases for which $\mu_1 = \mu_2$. Therefore, to the left of this line, $\mu_1 > \mu_2$, i.e. the matrix is more rigid than the inclusion.

2.2 Elastic solution

The starting point for the solution is to use the well-known result for the stress state induced in an infinite body with a circular inclusion by uniaxial remote tension **[1]**, σ_o:

$$\sigma_{rr} = \frac{\sigma_o}{2}\left\{[1-\cos(2\theta)] + \frac{1}{r^2}\left[-l - m\left(\frac{3}{r^2}-4\right)\cos(2\theta)\right]\right\}$$

$$\sigma_{\theta\theta} = \frac{\sigma_o}{2}\left\{[1+\cos(2\theta)] + \frac{1}{r^2}\left[l + m\frac{3}{r^2}\cos(2\theta)\right]\right\}$$

$$\sigma_{r\theta} = \frac{\sigma_o}{2}\left[1 - m\frac{1}{r^2}\left(\frac{3}{r^2}-2\right)\right]\sin(2\theta) \tag{4}$$

where the radial coordinate has been normalized with respect to the radius of the inclusion, a. Further, rearranging in modern notation the parameters l and m become

$$l = \frac{2\beta}{2\beta-\alpha-1} \tag{5}$$

$$m = \frac{\beta-\alpha}{\beta+1} \tag{6}$$

*The von Mises criterion in general can be written as

$$\tfrac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] = \sigma_Y^2 \tag{2}$$

Considering only in-plane flow is therefore strictly correct for the von Mises criterion only when the out-of-plane stress is exactly the mean value between in-plane stresses, i.e. for incompressible materials, in which case we obtain

$$3\tau_{max}^2 = \sigma_Y^2$$

so that $\sigma_e = \sqrt{3}\tau_{max}$.

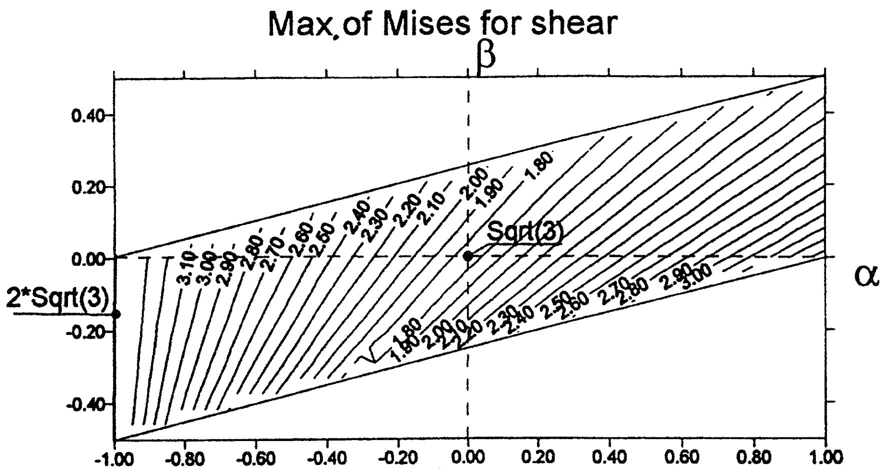

Fig. 1 Results for a stress ratio of −1 in the Dundurs $\alpha\beta$ plane

An analogous solution is available for shear; it can be obtained by superposition, as well as biaxial tension, and all intermediate combinations.

2.3 Elastic limit, shakedown limit and cyclic plasticity

For a material which possesses an elastic–ideally plastic stress–strain curve, i.e. no work hardening, the only material characteristic to add to the elastic description is k, the yield stress in shear (or, equivalently, σ_Y that was introduced earlier, the yield stress in tension). Now consider that, as a result of the presence of the inclusion, there is a stress concentration. In the presense of a crack, the stress concentration would be infinite, but linear elastic fracture mechanics could classify the effect of the crack by means of a stress intensity factor, which depends on the nominal stress and the size of the crack, and decide then what kind of damage could occur in the material. In this case, no points of singular stress exist, so that the elastic limit is finite; however, it is clear that the concentration factor is not dependent on size. In other words, in the idealized model of assigned values for the stress field at remote boundaries, there is a corresponding stress concentration which is not dependent on the actual size of the inclusion. This is true of course also for the case of the crack, where the stress is infinite independently of the size of the crack, but there the theory has found the concept of the stress intensity factor to quantify the severity of the concentration induced. Here, if the onset of plasticity is considered as the limiting condition, it is clear that this is not dependent on the dimension of the inclusion but only on the remote state of stress (for a given combination of materials and the given shape of the inclusion). In other words, there is no damage at all if this limit is not exceeded. In general, however, once the elastic limit is exceeded, yielding occurs in the region close to the point of maximum stress, and the location and dimension of this region are clearly dependent on the dimension of the inclusion (the radius, in the circular case), the dependence being obviously linear for geometrical similarity. However, this region is not necessarily affected entirely by plastic 'damage'; if cyclic loading, which is common in most applications, is considered, as soon as the region of plastic deformations is entered, beneficial residual stresses, which are usually of opposite sign to the applied load, produce *shakedown*. The *shakedown limit* is actually defined as the largest applied load for which the residual stresses summed with the applied stresses lie exactly on the yield surface at the most highly stressed point, therefore providing the limit load for which a steady state elastic state exists. It is well known that this process is to be considered possible with elastic–perfectly plastic materials such as those here under consideration, and for which Melan's theorem applies to find a lower bound to the shakedown limit [**13**]. If a self-equilibrated distribution of residual stresses can be found that satisfies the limit shakedown condition, then shakedown will occur at least to that limit. Of course, if the loading is increased above the shakedown limit and cycled, the material may exhibit cyclic plasticity, if full yielding does not occur first.

The lower bound proves difficult to find. However, an upper bound is almost invariably obtained immediately. Indeed, the maximum effective stress always occurs at a point on the interface between matrix and inclusion. Then, a reverse plasticity mechanism where equal and opposite increments of plastic strain occur at the two extremes of the cycle will yield an upper bound which corresponds to when the difference between these extremes is equal to twice the yield. Therefore, it can almost always be said that, above twice the elastic limit, elastic shakedown is exceeded, and a damage mechanism acts, by either cyclic plasticity or ratchetting. As these mechanisms can be responsible for cumulation of strains, it is therefore important to show the conditions under which it is possible to reach twice the elastic limit before full yielding.

In Figs 1 to 5, the maximum in-plane tangential stress is shown, rescaled as an equivalent von Mises parameter. Specifically, the parameter indicated in the figures is

$$\text{Maximum of von Mises parameter} = \sqrt{3}\tau_{\max} = \sqrt{3}\left|\frac{\sigma_1 - \sigma_2}{2}\right|_{\max} \tag{7}$$

where σ_1 and σ_2 are the in-plane principal stresses; this is shown for several cases of remote loading. Note that one component of the applied remote stress is kept constant and equal to +1, whereas the other component of remote stress varies between −1 and +1. In Fig. 1 is shown the maximum of the von Mises parameter for the stress ratio $\sigma_1/\sigma_2 = -1$, i.e. in pure shear. The $\alpha = \beta = 0$ case (when the inclusion has the same properties as the matrix) indicates a maximum of $\sqrt{3}$ and for the hole limiting case (the vertical line of the far bottom left of the Dundurs parallelogram) a maximum of $2\sqrt{3}$ which means that there is no possibility of cyclic plasticity before the full yielding of the specimen. For a less negative stress ratio (in Fig. 2, $\sigma_1/\sigma_2 = -0.5$), the maximum of the von Mises parameter is lower than that for the $\alpha = \beta = 0$ case; here it is about 1.3, but for the hole limiting case it is higher than twice this value, namely about 3.03 in this particular case. This indicates that there is an entire region in the Dundurs parallelogram with a possibility of cyclic plasticity behaviour (below full yielding), and it is the area exterior to the contour 2.6 in the diagram, including the hole limiting case. Increasing the stress ratio further, $\sigma_1/\sigma_2 = 0$, i.e. the pure tensile case (Fig. 3), is reached; the maximum of the von Mises parameter is $\sqrt{3}/2$ for the $\alpha = \beta = 0$ case, and for the hole limiting case it is about 2.6. Therefore, the region with a possibility of cyclic plasticity behaviour,

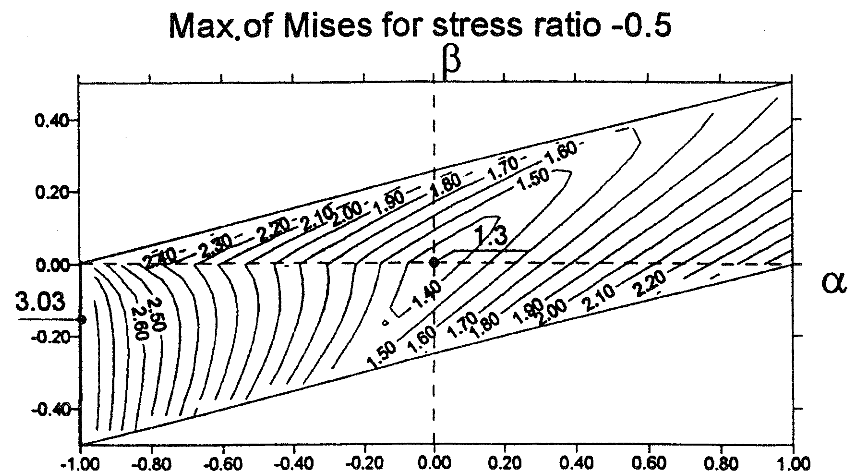

Fig. 2 Results for a stress ratio of −0.5 in the Dundurs $\alpha\beta$ plane

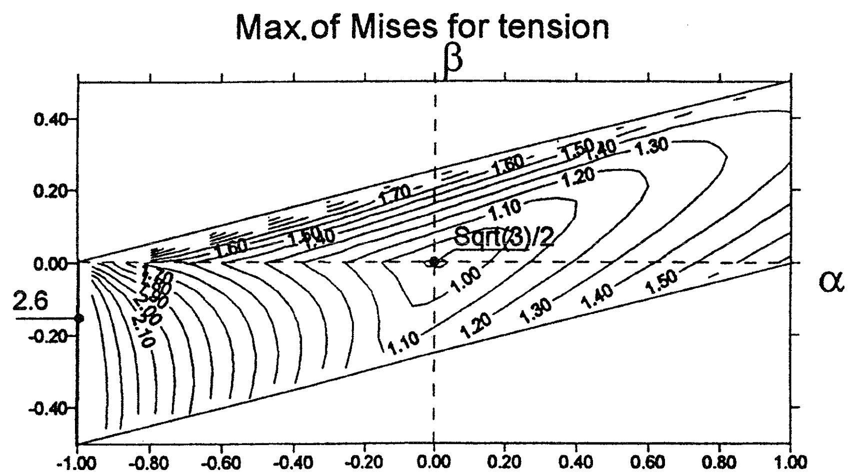

Fig. 3 Results for a stress ratio of 0 in the Dundurs $\alpha\beta$ plane

exterior to the contour $\sqrt{3}$ in the diagram, is larger, including the hole limiting case, and also a region close to the upper line of the parallelogram. With a higher stress ratio, $\sigma_1/\sigma_2 = 0.5$ (Fig. 4), the maximum of the von Mises parameter becomes very small for the $\alpha = \beta = 0$ case; in this particular case it is 0.433, and also for the hole limiting case it decreases to 2.165. The cyclic plasticity region is now a large area around the origin, including also some cases of rigid inclusions (the $\alpha = 1$ line), close to the limit of $\nu_1 = 0$ (the point $\alpha = 1, \beta = 0.5$). Finally, in the limiting case of equibiaxial tension (Fig. 5) for which the solution is clearly axisymmetric, and the value of the maximum von Mises parameter is clearly attained around the entire circle, the $\alpha = \beta = 0$ case gives a zero von Mises parameter, in agreement with the fact that there is no distortion in this case, and therefore the cyclic plasticity region is now the entire parallelogram.

3 CONCLUSION

The elastic stress concentration (specifically, the maximum in-plane tangential stress) in a plane problem for a material containing an inclusion of different elastic properties has been studied. It is clear that the presence of the stress concentration implies that the yielding limit is reached always (near the interface) before the stress level needed to reach the onset of plasticity in the bulk of the matrix. However, as in the case of proportional loading it is argued that an upper bound of the shakedown limit is twice the elastic limit; if the stress concentration is lower than two, no significant plastic damage can occur due to the presence of the inclusion. Given that the stress concentration (from the elastic solution) does not depend on the dimension of the inclusion but only on the pair of materials of matrix inclusion, it is possible to draw general conclusions when

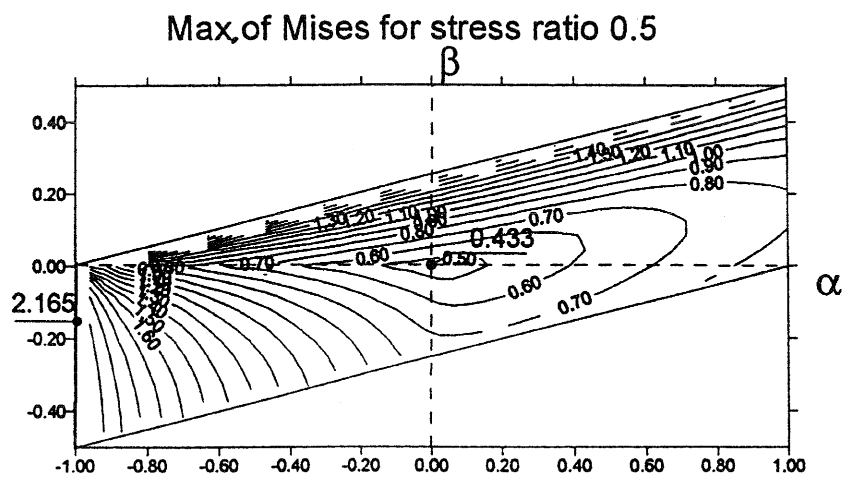

Fig. 4 Results for a stress ratio of 0.5 in the Dundurs $\alpha\beta$ plane

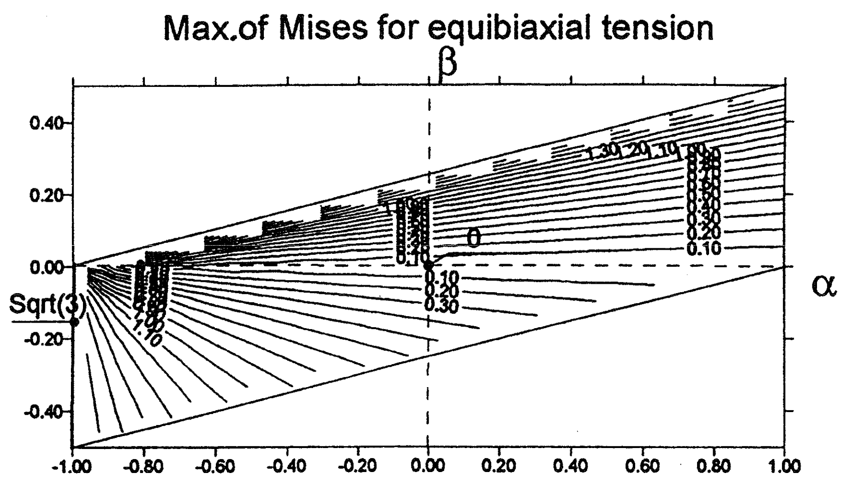

Fig. 5 Results for a stress ratio of 1 in the Dundurs $\alpha\beta$ plane

considering an MMC. In other words, for a given matrix, there are materials (for the inclusion) which, under a certain remote loading, do not cause any possibility of plastic damage, in the sense of possible cyclic plasticity. A series of plots have therefore been shown in the Dundurs plane to cover the entire range of loading conditions and material combinations for the case of a circular inclusion. In particular, no possibility of cyclic plasticity under remote shear has been found, whereas any combination of elastic properties in the equibiaxial limiting case may lead to the cyclic plasticity regime; in intermediate cases, there is a limited area on the Dundurs plane where cyclic plasticity is possible, generally close to the hole limiting case, but also to a region near the upper line of the Dundurs parallelogram.

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