Technical Note

On the post-processing of data obtained from cracked components

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Abstract: The use of a simple singular value decomposition (SVD) technique to post-process far-field data from cracked components is discussed. The technique employs the series expansion for stress or displacements ahead of the crack tip, which is available for a wide range of configurations. The use of higher-order terms, necessary for the post-processing, is beneficial for two reasons: (a) it permits the abstraction of the maximum usable amount of information; (b) it gives a better understanding of the fracture mechanics, especially regarding crack tip plasticity and dynamic propagation. Several *numerical* cases are examined and a comparison is made with analytical results, permitting an evaluation to be made of the pure numerical error in the post-processing.

Keywords: post-processing, cracked components, simple singular value decomposition technique, far-field data

NOTATION

a	crack dimension (for an edge crack), half-dimension (for an interior crack)
$K_n^{\mathrm{I}}, K_n^{\mathrm{II}}$	series expansion coefficients
$K_{\mathrm{I}}, K_{\mathrm{II}} = K_0^{\mathrm{I}}, K_0^{\mathrm{II}}$	stress intensity factors
n, n_e	number of unknown terms, number of equations
(r, θ)	polar coordinates
$\boldsymbol{u}$	displacement vector
σ	stress component

1 INTRODUCTION

Providing the classical approximations of small-scale plasticity are fulfilled, the stress field around a crack can be characterized, apart from a region very close to the crack tip,* by the LEFM (linear elastic fracture mechanics) series expansion. Therefore, the classical argument for design purposes is that, in the 'asymptotic region' where propagation may occur, the first terms of the series, K_{I}, K_{II}, dominate the stress field and should consequently characterize the fracture process. A single value of stress in the asymptotic region (where K_{I}, K_{II} are directly proportional to $1/\sqrt{r}$) could be sufficient for the determination of the parameter.

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However, the following questions arise immediately:

1. What are inner and outer limits of the 'asymptotic region'? Is the asymptotic region easily accessible?†
2. Where is the region of best choice for the extraction of reliable data? Where is the 'density' of information highest and could it be found in an automatic way?
3. If data are available in the 'intermediate region' and the 'nominal region', is it possibe to use any of them so as to improve the reliability of the results?

These questions lead directly to a first reason for considering higher-order terms in the expansion, using data in the proximity of the crack tip, but *not only* in the asymptotic region, for determining the values of K_{I}, K_{II} field parameters. A second reason is that higher-order terms are now widely recognized as relevant in certain conditions: the second term in the expansion, called the *T-stress*, quantifies the transverse stress present near the crack apex, affecting significantly the crack tip plasticity behaviour. In dynamically propagating cracks, the local values of stresses

*Two different regions in the proximity of the crack tip can be distinguished: a close *process zone*, where finite deformations and void nucleation occur, and, further away, a region of *elastoplastic* deformations, where the HRR stress field is eventually valid, behaving like $1/r^{1/(n+1)}$, n being the hardening coefficient of the material.
†For example, in recent experimental work (**1**) it has been found that the $1\sqrt{r}$ elastic stress field is valid, for a typical test (an Al 6061 T6 aluminium compact tension specimen), only in the region 100–300 CTOD, which corresponds in that case to about 0.3–0.9 mm.

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depend on the time history of the values of crack tip speed and stress intensity factor, related to higher-order terms in the Williams-type expansion, with respect to time and space. In particular, the second and third terms have proved to be consistent in justifying the scattering in the theoretically predicted relation between K_{I}, K_{II} and propagation speed (**2–4**).

To obtain the desired calculation of the terms of the series for the stress or displacement fields, a form of 'boundary collocation method' will be used [see reference (**5**) for a list of references]. As the classical least-squares fitting procedure becomes easily very ill-conditioned under these typically rapidly varying data, the following are proposed:

(a) always an overdetermined number of conditions;
(b) a singular value decomposition (SVD) procedure to solve the system of equations.

2 THE WILLIAMS-TYPE SERIES EXPANSION

The stresses $\sigma(r,\theta)$ and the displacement vector $\boldsymbol{u}(r,\theta)$ on the point defined by the polar coordinate system r, θ centred on the crack tip can be obtained for the homogeneous static case, with little algebra, from (**6**)

$$\sigma(r,\theta)=\sum_{n=0}^{\infty}\frac{r^{(n-1)/2}}{\sqrt{(2\pi)}}\left[K_n^{\mathrm{I}}\sigma_n^{\mathrm{I}}(\theta)+K_n^{\mathrm{II}}\sigma_n^{\mathrm{II}}(\theta)\right] \tag{1}$$

and

$$\boldsymbol{u}(r,\theta)=\sum_{n=0}^{\infty}\frac{r^{(n+1)/2}}{\sqrt{(2\pi)}}\left[K_n^{\mathrm{I}}\boldsymbol{u}_n^{\mathrm{I}}(\theta)+K_n^{\mathrm{II}}\boldsymbol{u}_n^{\mathrm{II}}(\theta)\right] \tag{2}$$

where K_n^{I}, K_n^{II}, for $n=0,1,\ldots$, are the coefficients of the series expansion and for $n=0$ are the well-known stress intensity factors (SIF) corresponding to mode I and mode II deformations; $\sigma_n^{\mathrm{I}}(\theta)$ and $\sigma_n^{\mathrm{II}}(\theta)$ and $\boldsymbol{u}_n^{\mathrm{I}}(\theta)$, $\boldsymbol{u}_n^{\mathrm{II}}(\theta)$ are given in Appendix 1. From equations (1) and (2), it is clear that K_1^{II} gives a rigid-body rotation, and hence gives the null stress field, whereas the only regular term to produce a non-zero stress field at the crack tip is K_1^{I}, which for the form of displacement is called the T-stress.

3 SVD PROCEDURE

SVD is the recommended procedure for a general badly conditioned linear least-squares problem, either underdetermined or overdetermined. In the former case, SVD produces a solution whose values are smallest in the least-squares sense; for the latter, the solution is the best least-squares approximation, and in both cases this is what is usually desired [see reference (**7**), Sections 2.6 and 15.4]. Here the problem is to find the coefficients of a series expansion matching some values of the stress and/or displacement at certain positions. If a proper number of conditions is given for the elastic solution to be unique, it is sure also that the expansion will converge, and uniformly, to that solution (**8**). Unfortunately, for an infinite number of coefficients, an infinite number of conditions would be needed. However, the SVD procedure corresponds to a usually successful search for the physically reasonable solution. The engineeristic criterion to be used, then, is convergence of the coefficients upon increasing the collocation points number.

The procedure is implemented in the MATLAB software (**9**) and a synthesis is listed in Appendix 2. A matrix $matr[n_{\mathrm{eq}}\times n]$ is written for the coefficients, directly obtained in terms of the series expansions in Appendix 1, and a vector *vect* from the known values (of stresses and/or displacements), where n_{eq} is given by the number of conditions that are written, whereas the number of unknowns n is equal to the number of terms K_n^{I} and K_n^{II} that need to be extracted (and $n_{\mathrm{eq}}>n$). The parameter *tol* is introduced that gives the lowest value in the diagonal matrix s of the SVD procedure to be considered significant [see reference (**7**), Section 2.6]. Finally, x gives the solution of the problem, i.e. the vector of the coefficients K_n^{I} and K_n^{II}.

4 NUMERICAL EXPERIMENTS

The case of a homogeneous disc with a radial edge crack under isotropic tension (see the sketch in Fig. 1) is considered as the benchmark test, but it is clear that any crack in a homogeneous body surrounded by a sector of material like this can be considered; moreover, the disc is considered to be in a state of isotropic tension, because for this particular loading case the solution is analytical for any geometrical ratio a/r (**10, 11**). The solution is given by

$$\frac{K_{\mathrm{I}}}{K_0}=\frac{2}{K_+}\left(2-\frac{a}{r}\right)^{-3/2};\qquad K_0=\sigma_0\sqrt{a},\ K_+=0.355\,715 \tag{3}$$

where the value of K_+ is correct to six significant figures. Here σ_0 is the applied tension, a is the dimension of the crack and r is the radius of the disc.

4.1 Collocation of stresses at $r=1$

Figure 1 shows the result of collocating $\sigma_r=\sigma_0$ and $\sigma_{r\theta}=0$ at $r=1$. The number of collocation points n_c varies with a ratio of equations to unknowns $\geqslant 2$. It may be appreciated that the error is immediately under 10 per cent, for n_c greater than about 10, even for non-optimal aspect ratios $(0.4<a/r<1.6)$, and becomes less than 2 per cent for $n_c>24$. Also, if the aspect ratio is kept closer to the optimum $a/r=1$, say $0.8<a/r<1.4$, the error is under 0.1 per cent if $n_c>10$.

The values at convergence have been checked for the case $a/r=1$ with reference (**12**). Table 1 shows the first 10 coefficients obtained with 360 uniformly spaced collocation

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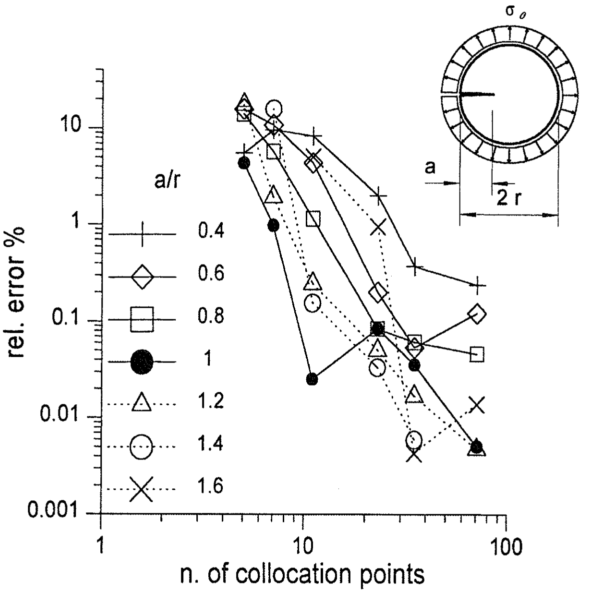

Fig. 1 Convergence of K_I with collocation of stresses at the boundary as a function of the number n_c of collocation points; for $n_c < 35$ the number of terms in the series expansion $n = n_c$ (ratio of equations to unknowns is 2); for $n_c > 35$, n is held constant, $n = 35$ (ratio of equations to unknowns is greater than 2)

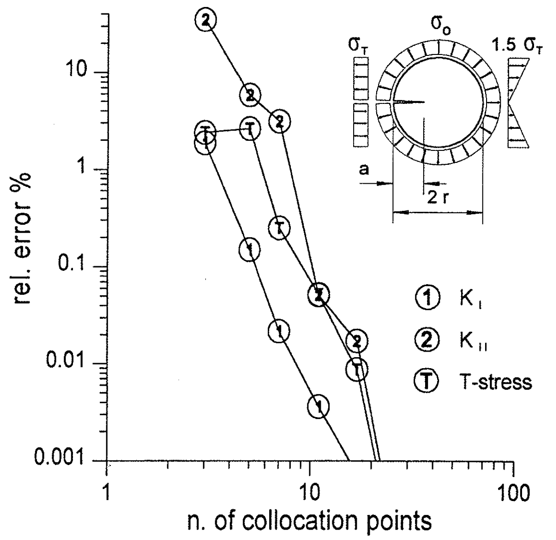

Fig. 2 Convergence of K_I, K_{II} and T-stress with collocation of displacements at the boundary as a function of the number n_c of collocation points (the number of terms is chosen as $n \approx \frac{3}{4} n_c$, so that the ratio of equations to unknowns is 4/3), uniformly spaced along $a/r = 1$

points with the number, n, of terms equal to 100 and a tolerance parameter in the SVD procedure (see procedure 1) equal to $tol = 1.0E - 4$. The comparison with the two references (where only the significant digits given by the authors are shown) shows that agreement is excellent.

4.2 Collocation of displacements at $r = 1$

To check the capability of keeping separated information due to symmetrical and antisymmetric contributions, the case of Fig. 2 was investigated, where the boundary conditions of the previous case are superposed to a self-equilibrated antisymmetric load condition. Symmetrical coefficients are obviously the same as for the previous analysis, whereas antisymmetric coefficients, for which there is no analytical solution, are compared with those obtained with values converged to six significant digits. The interior displacement field is generated using coefficients obtained from the previous experiments, accurate to six digits. The overdetermination ratio (ratio of equations to unknowns) has been varied between 1 and 4, but the best results are obtained for a ratio of about 4/3. The results (Fig. 2) therefore show a better behaviour with respect to the stress collocation of Fig. 1 (with n_c greater than 10, the error is less than 1 per cent).

5 DISCUSSION AND CONCLUSION

The use of a simple SVD collocation technique to the complete Williams-type expansion of the stress and the displacement fields of a crack has been analysed, considering either stresses or displacement boundary data (or a combination of the two). This is proposed as a tool for the post-processing of data in a fast, automatic and reliable way. Using displacement data at a constant distance around the crack tip (equal to the crack dimension, $a/r = 1$) and with a ratio of equations to unknowns of 4/3, an error was obtained of under 0.1 per cent with 10 collocation points. This is the numerical error associated with the post-processing procedure only: depending on the source of the data, other kinds of error

Table 1 $K_n^I/(\sigma_0 a^{(1-n)/2})$ with a convergence to six significant digits, obtained using 360 uniformly spaced collocation points at $a/r = 1$ (Fig. 1), compared with values in references (11) and (12). Number, n, of terms used = 100; $tol = 1.0E{-}4$

n	Present	Reference (12)	Reference (11)	n	Present	Reference (12)
0	5.622 52	5.623	5.622 47	5	−0.153 202	−0.15
1	2.376 48	2.377	—	6	0.126 648	—
2	−9.208 00	−9.208	—	7	0.051 5203	—
3	0.892 847	0.891	—	8	−0.066 6865	—
4	−0.297 936	−0.30	—	9	−0.021 3146	—

can be present, and the total error is not given by the linear superposition. It is outside the scope of this note to give a detailed analysis of all possible situations, because the kinds of error depend on which numerical or experimental method is used to obtain the data. However, a check on the convergence must always be done, and the character of SVD ensures that the procedure will not blow up unless the data are completely unreliable. The technique is particularly efficient in the case where a large number of analyses is required with a large number of data to be processed, like in the case of full-field techniques in dynamically propagating cracks, where the manual fitting of asymptotic data would lead to a time consuming analysis which was prone to error.

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APPENDIX 1

Series expansion for a crack in a homogeneous domain

Here the functions giving the variations with θ in equations (1) and (2) are given. Firstly, as regards the stress field (1):

$$\sigma_n^{\mathrm{I}}(\theta)=\begin{Bmatrix}\sigma_{xxn}^{\mathrm{I}} & (\theta)\\ \sigma_{yyn}^{\mathrm{I}} & (\theta)\\ \sigma_{xyn}^{\mathrm{I}} & (\theta)\end{Bmatrix}=\begin{Bmatrix}\cos\frac{n-1}{2}\theta-\frac{n-1}{2}\sin\theta\sin\frac{n-3}{2}\theta\\ \cos\frac{n-1}{2}\theta+\frac{n-1}{2}\sin\theta\sin\frac{n-3}{2}\theta\\ -\frac{n-1}{2}\sin\theta\cos\frac{n-3}{2}\theta\end{Bmatrix}\qquad n=0,2,4,\ldots \tag{4}$$

$$\sigma_n^{\mathrm{I}}(\theta)=\begin{Bmatrix}\sigma_{xxn}^{\mathrm{I}} & (\theta)\\ \sigma_{yyn}^{\mathrm{I}} & (\theta)\\ \sigma_{xyn}^{\mathrm{I}} & (\theta)\end{Bmatrix}=\begin{Bmatrix}2\cos\frac{n-1}{2}\theta-\frac{n-1}{2}\sin\theta\sin\frac{n-3}{2}\theta\\ \frac{n-1}{2}\sin\theta\sin\frac{n-3}{2}\theta\\ -\sin\frac{n-1}{2}\theta-\frac{n-1}{2}\sin\theta\cos\frac{n-3}{2}\theta\end{Bmatrix}\qquad n=1,3,5,\ldots \tag{5}$$

$$\sigma_n^{\mathrm{II}}(\theta)=\begin{Bmatrix}\sigma_{xxn}^{\mathrm{II}} & (\theta)\\ \sigma_{yyn}^{\mathrm{II}} & (\theta)\\ \sigma_{xyn}^{\mathrm{II}} & (\theta)\end{Bmatrix}=\begin{Bmatrix}2\sin\frac{n-1}{2}\theta+\frac{n-1}{2}\sin\theta\cos\frac{n-3}{2}\theta\\ -\frac{n-1}{2}\sin\theta\cos\frac{n-3}{2}\theta\\ \cos\frac{n-1}{2}\theta-\frac{n-1}{2}\sin\theta\sin\frac{n-3}{2}\theta\end{Bmatrix}\qquad n=0,2,4,\ldots \tag{6}$$

$$\sigma_n^{\mathrm{II}}(\theta)=\begin{Bmatrix}\sigma_{xxn}^{\mathrm{II}} & (\theta)\\ \sigma_{yyn}^{\mathrm{II}} & (\theta)\\ \sigma_{xyn}^{\mathrm{II}} & (\theta)\end{Bmatrix}=\begin{Bmatrix}\sin\frac{n-1}{2}\theta+\frac{n-1}{2}\sin\theta\cos\frac{n-3}{2}\theta\\ \sin\frac{n-1}{2}\theta-\frac{n-1}{2}\sin\theta\cos\frac{n-3}{2}\theta\\ -\frac{n-1}{2}\sin\theta\sin\frac{n-3}{2}\theta\end{Bmatrix}\qquad n=1,3,5,\ldots \tag{7}$$

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With respect to the displacements field (2):

$$\boldsymbol{u}_n^{\mathrm{I}}(\theta) = \begin{Bmatrix} u_{xn}^{\mathrm{I}}(\theta) \\ u_{yn}^{\mathrm{I}}(\theta) \end{Bmatrix} = \frac{1}{2\mu(n+1)} \times \begin{Bmatrix} -(\kappa - 1)\cos\frac{n+1}{2}\theta + (n+1)\sin\theta\sin\frac{n-1}{2}\theta \\ (\kappa + 1)\sin\frac{n+1}{2}\theta - (n+1)\sin\theta\cos\frac{n-1}{2}\theta \end{Bmatrix} \quad n = 0, 2, 4, \ldots \quad (8)$$

$$\boldsymbol{u}_n^{\mathrm{I}}(\theta) = \begin{Bmatrix} u_{xn}^{\mathrm{I}}(\theta) \\ u_{yn}^{\mathrm{I}}(\theta) \end{Bmatrix} = \frac{1}{2\mu(n+1)} \times \begin{Bmatrix} (\kappa + 1)\cos\frac{n+1}{2}\theta - (n+1)\sin\theta\sin\frac{n-1}{2}\theta \\ (\kappa - 1)\sin\frac{n+1}{2}\theta - (n+1)\sin\theta\cos\frac{n-1}{2}\theta \end{Bmatrix} \quad n = 1, 3, 5, \ldots \quad (9)$$

$$\boldsymbol{u}_n^{\mathrm{II}}(\theta) = \begin{Bmatrix} u_{xn}^{\mathrm{II}}(\theta) \\ u_{yn}^{\mathrm{II}}(\theta) \end{Bmatrix} = \frac{1}{2\mu(n+1)} \times \begin{Bmatrix} (\kappa + 1)\sin\frac{n+1}{2}\theta + (n+1)\sin\theta\cos\frac{n-1}{2}\theta \\ -(\kappa - 1)\cos\frac{n+1}{2}\theta - (n+1)\sin\theta\sin\frac{n-1}{2}\theta \end{Bmatrix} \quad n = 0, 2, 4, \ldots \quad (10)$$

$$\boldsymbol{u}_n^{\mathrm{II}}(\theta) = \begin{Bmatrix} u_{xn}^{\mathrm{II}}(\theta) \\ u_{yn}^{\mathrm{II}}(\theta) \end{Bmatrix} = \frac{1}{2\mu(n+1)} \times \begin{Bmatrix} (\kappa - 1)\sin\frac{n+1}{2}\theta + (n+1)\sin\theta\cos\frac{n-1}{2}\theta \\ -(\kappa + 1)\cos\frac{n+1}{2}\theta - (n+1)\sin\theta\sin\frac{n-1}{2}\theta \end{Bmatrix} \quad n = 1, 3, 5, \ldots \quad (11)$$

where μ is the shear modulus of the material, κ is the Kolosov constant, equal to $\kappa = 3 - 4\nu$ under plane strain conditions, where ν is the Poisson ratio of the material.

APPENDIX 2

A MATLAB procedure to implement the SVD solution of the system

```
build matr and vect
...
[u, s, v] = svd(matr, 0);
for i = 1 : size(matr, 2)
  if s(i, i) ≥ tol
  s(i, i) = 0;
  end
  if s(i, i) < tol
  s(i, i) = 1/s(i, i);
  end
end
x = v * s * u' * vect
```