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The thermoelastic Aldo contact model with frictional heating

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Abstract

In the study of the essential features of thermoelastic contact, Comninou and Dundurs (J. Therm. Stresses 3 (1980) 427) devised a simplified model, the so-called "Aldo model", where the full $3D$ body is replaced by a large number of thin rods normal to the interface and insulated between each other, and the system was further reduced to 2 rods by Barber's Conjecture (ASME J. Appl. Mech. 48 (1981) 555). They studied in particular the case of heat flux at the interface driven by temperature differences of the bodies, and opposed by a contact resistance, finding possible multiple and history dependent solutions, depending on the imposed temperature differences.

The Aldo model is here extended to include the presence of frictional heating. It is found that the number of solutions of the problem is still always odd, and Barber's graphical construction and the stability analysis of the previous case with no frictional heating can be extended. For any given imposed temperature difference, a critical speed is found for which the uniform pressure solution becomes non-unique and/or unstable. For one direction of the temperature difference, the uniform pressure solution is non-unique before it becomes unstable. When multiple solutions occur, outermost solutions (those involving only one rod in contact) are always stable.

A full numerical analysis has been performed to explore the transient behaviour of the system, in the case of two rods of different size. In the general case of N rods, Barber's conjecture is shown to hold since there can only be two stable states for all the rods, and the reduction to two rods is always possible, a posteriori.

Keywords: Thermoelastic instability (TEI); Constriction resistance; Thermoelastic contact; Friction

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Nomenclature

A_i	sectional area of rod i [m^2]
b	growth rate [s^{-1}]
E	Young's modulus of rods [N/m^2]
f	frictional coefficient
F	total contact force [N]
k	diffusivity of rods [m^2/s]
K	thermal conductivity of rods [W/m°C]
L	length of the rods [m]
p_i	contact pressure on rod i [N/m^2]
q_i	heat flux in rod i [W/m^2]
R_i	contact resistance of rod i [m^2°C/W]
t	time [s]
T_B, T_0	wall temperature [°C]
T_A	temperature of fixed end of rods [°C]
T_{C_i}	temperature of free end of rod i [°C]
V	sliding speed [m/s]
$\hat{V}$	dimensionless sliding speed
α	thermal expansion coefficient of rods [°C^{-1}]
Δ_i	unrestrained thermal expansion of rod i [m]

1. Introduction

Thermoelastic contact problems show non-existence or non-uniqueness and as a result, instability. This behavior is a consequence of inequalities (non-linearities) regulating the problem at the interface. When more than a stable solution is present, the system shows obviously history-dependence. Problems of this class are found in many applications, one of the most important being sliding systems such as brakes, clutches and seals, where thermoelastic effects manifest themselves as *frictionally-excited thermoelastic instabilities* or TEI (Barber, 1969; Dow and Burton, 1972; Lee and Barber, 1993; Zagrodzki, 1990) and is of critical importance in the design of brakes and clutches (Parker and Marshall, 1948; Kennedy and Ling, 1974). Such systems are unstable if the sliding speed is sufficiently high, or more precisely, for a certain critical product of friction coefficient and speed $(fV)_{\mathrm{crit}}$, the contact pressure perturbations grow, leading to localization of load and heat generation and hence to hot spots at the sliding interface. This behavior is obviously undesirable with respect to the trivial uniform pressure distribution solution, and it is known to cause material damage, wear, noise and frictional vibrations (Lee and Dinwiddie, 1998).

Until recently, although a qualitative connection was sometimes suggested, TEI was treated separately from instabilities in heat conduction across an interface. Apart from fundamental models of tribology, specific applications of this class are duplex tube exchangers (Srinivasan and France, 1985), solidification of a metal against a plane mould (Yigit and Barber, 1994), and many others. Again, non-existence or non-uniqueness of solutions is found, even in the context of a simple one-dimensional model (Barber et al., 1980). The problem of non-existence of the steady-state solution can be resolved by postulating a *constriction resistance* at the interface, a subject of extensive experimental (Clausing and Chao, 1965; Thomas and Probert, 1970) and theoretical (Shlykov and Ganin, 1964; Cooper et al., 1969) investigations, although the effect of sliding is less well known. Existence was proved by Duvaut (1979) for the general three-dimensional thermoelastic contact problem for the special case where the thermal contact resistance varies inversely with the contact pressure. The problem of non-uniqueness, vice versa, is a more profound one and although Duvaut (1979) gave a condition in terms of the constriction resistance parameters, reported experimental measurements show that this condition is unlikely to be met in practice.

Recently (Ciavarella et al., 2003), an attempt has been made to study the connection between TEI and "static" thermoelastic instabilities, in the context of the simple Barber rod model (Barber et al., 1980). There, a single rod is built in a wall and slides with respect to a target wall, from which the displacement in the normal direction is prescribed. It was found that equalities and associated inequalities for the normal pressure p (positive if compressive) and the gap g (positive if measuring separation)

$$\text{contact}: \ p > 0; \ g = 0 \tag{1}$$

$$\text{separation}: \ p = 0; \ g > 0 \tag{2}$$

can be combined into a single equation for a certain functional $\mathscr{F}$, whose zeros give steady state solutions and whose derivative $\mathscr{F}'$ dictates stability. Since the system has prescribed displacement, the pressure may grow without limit, causing *seizure*, similarly to what had been suggested in a shaft rotating in a bearing (Burton and Staph, 1967; Tu and Stein, 1995).

Hence, a critical speed was defined in terms of the limit resistance value R_∞ for large pressures $p \to \infty$, the length of the rod L, and K the thermal conductivity, [1]

$$\hat{V}_\infty \equiv 1 + \frac{L}{KR_\infty}, \tag{3}$$

where the dimensionless speed is defined as

$$\hat{V} = fV\frac{E\alpha L}{2K}, \tag{4}$$

i.e. a function of friction coefficient f, elastic modulus E, and α thermal expansion coefficient. For low sliding speeds $\hat{V} < \hat{V}_\infty$, the results are qualitatively similar to those with no sliding. In particular, the number of steady-state solutions is odd; if the steady-state is unique it is stable and if it is non-unique, stable and unstable solutions alternate, with the outlying solutions being stable. By contrast, for $\hat{V} > \hat{V}_\infty$ either there are no steady-state solutions (non-existence of solution) or the number of steady states

[1] Notice that the original notation in the paper (Ciavarella et al., 2003) was $\hat{V}_0$ instead of $\hat{V}_\infty$ used here.

is even. In the latter case, stable and unstable states again alternate, so that there is always an outlying unstable steady-state. A numerical study for the special case where the resistance function is defined as $R = d + c/p$ showed that when the system has no steady states, the contact pressure grows without limit from *any* initial condition. If it has steady states, but $\hat{V} > \hat{V}_\infty$, the system will either tend to a stable steady state or the contact pressure will increase without limit, depending on the initial condition. In all cases, if $\hat{V} < \hat{V}_\infty$ the system will tend to one of the stable steady states.

It is immediate to extend this model to the case of multiple rods, as if the condition is still maintained in terms of the gap, the rods will remain independent from each other, and therefore the steady state of the system and its stability will simply be given by the independent steady states and stability of each rod. Therefore, a critical speed can be defined for each of the rods and it will be a single value if material properties and the contact resistance function are the same for all rods.

More interesting is the case where the total force is prescribed, which requires the additional effort described in this paper. The system becomes more complex, because of the coupling dictated by the equilibrium equation. If there is no wear, it is generally believed the system will then tend to an alternate stable steady-state involving a reduced contact area (Burton et al., 1973; Zagrodzki et al., 2001). The model with independent rods has been studied in the simplest case of no frictional heating by Comninou and Dundurs (1980), see Fig. 1, and named "Aldo model". The original intention was to model a full 3D contact problem reducing the solid to a large number of thin rods normal to the interface and insulated between each other. Barber (1981) further conjectured that because they are insulated, the exact distribution of the rods does not matter; moreover, the basic features of the system are found in the case of two rods of different cross sections. For prescribed force for a true system of N rods, there will probably be many steady states: however, for every particular steady state there must be a corresponding particular value of the location of the free (loaded) end of the system of rods. Each of the many rods must be in a state that is possible for a rod

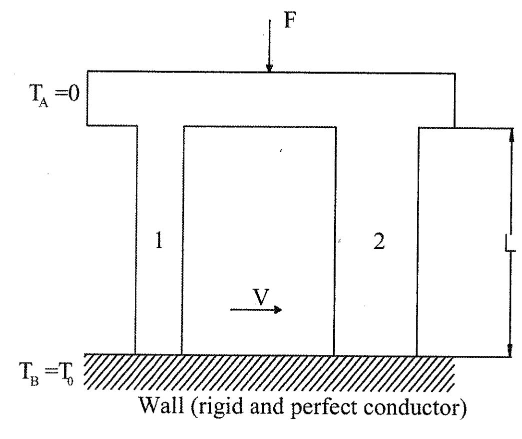


Fig. 1. The Aldo model.

with a fixed end at this location, which is the model studied in Ciavarella et al. (2003). Thus, all the rods for any particular steady state must be in one of three states, and for stability, in one of two states (two different contact pressures or one contact pressure and one separation). Therefore, for a system of a given total area, if the steady state is not uniform pressure, the solution must involve two states, and the only unknown is the partition of areas between the two states. Obviously, since the partition of the area varies with speed, or with a perturbation, it remains to be investigated if the stability of the full system coincides with the stability of the reduced system. Here, we extend the results in Barber (1981) to the case where the rods slide with respect to the rigid wall, thereby introducing the effect of frictional heating.

2. Steady-state solution

The system is generalization of that in Comninou and Dundurs (1980), and in Barber (1981). With reference to Fig. 1, consider two rods maintained at constant temperature ($T_A = 0$) from one end, and pressed against a perfect conductor at temperature $T_B = T_0$, at the other end. For each of the rods, the end temperature of the rods, T_{C_i}, is given by heat flux balance[2]

$$q_i = \frac{T_0 - T_{C_i}}{R_i} + fVp_i = K\frac{T_{C_i}}{L}, \tag{5}$$

$$q_i = -K\left.\frac{\partial T_{C_i}}{\partial y}\right|_{y=L}, \tag{6}$$

and eliminating T_{C_i} from the first equation, we find

$$T_{C_i} = \frac{T_0 + fVp_iR_i}{1 + KR_i/L}. \tag{7}$$

The *unrestrained* thermal expansion of each rod is found by elementary integration as

$$\Delta_i = \frac{\alpha L}{2}T_{C_i} = f_i\left(\frac{\alpha L}{2}R_ip_ifV + u\right), \tag{8}$$

where

$$f_i = \begin{cases} 0; & g_i > 0,\ p_i = 0, \\ \dfrac{L}{KR_i + L}; & g_i = 0,\ p_i > 0 \end{cases} \tag{9}$$

and

$$u = \tfrac{1}{2}\alpha LT_0. \tag{10}$$

[2] Notice that here the heat flux is per unit area, different from total heat flux notation in Barber (1981) and Barber et al. (1980).

Using these results, the equalities and associated inequalities of the contact problem (1,2) result in the following three cases:

- Rod 1 in contact:

$$g_2 = \Delta_1 - p_1 \frac{L}{E} = f_1\left(\frac{\alpha L}{2} R_1 p_1 f V + u\right) - p_1 \frac{L}{E} > 0, \tag{11}$$

$$g_1 = 0; \quad p_1 = \frac{F}{A_1}; \quad p_2 = 0. \tag{12}$$

- Rod 2 in contact:

$$g_1 = \Delta_2 - p_2 \frac{L}{E} = f_2\left(\frac{\alpha L}{2} R_2 p_2 f V + u\right) - p_2 \frac{L}{E} > 0, \tag{13}$$

$$g_2 = 0; \quad p_2 = \frac{F}{A_2}; \quad p_1 = 0. \tag{14}$$

- Both rods in contact:

$$\Delta_1 - \Delta_2 = (p_1 - p_2)\frac{L}{E}, \tag{15}$$

i.e.

$$f_1\left(\frac{\alpha L}{2} R_1 p_1 f V + u\right) - f_2\left(\frac{\alpha L}{2} R_2 p_2 f V + u\right) = (p_1 - p_2)\frac{L}{E} \tag{16}$$

and

$$g_1 = g_2 = 0; \quad p_1, p_2 > 0. \tag{17}$$

We then define $x = \Delta_1 - \Delta_2$ as the difference of the unrestrained thermal expansions. From (11)–(17) we find

$$x = g_2 + p_1 \frac{L}{E}; \; g_2 > 0; \; p_1 = \frac{F}{A_1}; \; p_2 = 0 \tag{18}$$

$$x = (p_1 - p_2)\frac{L}{E}; \; g_1 = g_2 = 0 \tag{19}$$

$$x = -g_1 - p_2 \frac{L}{E}; \; g_1 > 0; \; p_1 = 0; \; p_2 = \frac{F}{A_2} \tag{20}$$

and all equalities and the associated inequalities of the system for the 3 conditions (18)–(20) can be combined into a single equation on a function $\mathscr{F}$

$$\mathscr{F}(x) = x - f_1\left(R_1 p_1 \frac{K}{E}\hat{V} + u\right) + f_2\left(R_2 p_2 \frac{K}{E}\hat{V} + u\right) = 0. \tag{21}$$

3. Stability

The stability of the system can be investigated by performing a linear perturbation analysis about the steady state of (5),

$$\Delta q_i = -\frac{T_0 - T_{C_i}}{R_i^2}\,\Delta R_i - \frac{\Delta T_{C_i}}{R_i} + fV\Delta p_i. \tag{22}$$

We write

$$\Delta R_i = \frac{\partial R_i}{\partial x}\,\Delta x = R_i'\Delta x; \qquad \Delta p_i = \frac{\partial p_i}{\partial x}\,\Delta x = p_i'\Delta x \tag{23}$$

and using (7), (9) and (22)

$$\Delta q_i = -\frac{K}{L}\frac{f_i}{1-f_i}\,\Delta T_{C_i} + \left[fV\left(p_i' - p_i\frac{f_i'}{1-f_i}\right) + T_0\frac{K}{L}\frac{f_i'}{1-f_i}\right]\Delta x. \tag{24}$$

Following Barber (1981) the perturbation in temperature in the rods can be written as

$$\Delta T_{C_i} = B_i\exp\{bt\}\sinh[\lambda y], \tag{25}$$

where b is the growth rate of perturbation and $\lambda = \sqrt{b/k}$. From (6) the perturbation in heat input will assume the following form

$$\Delta q_i = -B_i K\lambda\ \exp\{bt\}\cosh[\lambda L] \tag{26}$$

and

$$\Delta x = \alpha\int_0^L (T_{C_1} - T_{C_2})\,\mathrm{d}y = \alpha\,\frac{(B_1 - B_2)}{\lambda}\exp\{bt\}(\cosh[\lambda L] - 1) \tag{27}$$

from Eq. (25). Substituting (25)–(27) in (24) the following two equations can be obtained:

$$-B_i\frac{K}{L}z^2\cosh[z] = -B_i\frac{K}{L}\frac{f_i}{1-f_i}z\sinh[z] + W_i(B_1 - B_2)(\cosh[z] - 1), \quad i = 1,2, \tag{28}$$

where $z = \lambda L$ and

$$W_i = 2\hat{V}\frac{K}{E}\left(p_i' - p_i\frac{f_i'}{1-f_i}\right) + 2u\frac{K}{L}\frac{f_i'}{1-f_i}, \quad i = 1,2. \tag{29}$$

Eqs. (28) permit elimination of the coefficients B_1 and B_2 by defining

$$C_i = 2\frac{K}{E}\hat{V}\left(\frac{L}{K}p_i'(1-f_i) - \frac{L}{K}p_i f_i'\right) + 2uf_i', \quad i = 1,2 \tag{30}$$

and the characteristic equation is found as

$$\frac{C_1(\cosh[z]-1)}{(f_1-1)z^2\cosh[z] + f_1 z\sinh[z]} - \frac{C_2(\cosh[z]-1)}{(f_2-1)z^2\cosh[z] + f_2 z\sinh[z]} = 1. \tag{31}$$

The perturbation is unstable only for a root with positive real part, but as shown in Barber (1981), the roots of Eq. (31) satisfy $Re[z] > 0$, only for real roots and if

$$C_1 - C_2 > 2. \tag{32}$$

After relatively trivial algebraic manipulation the above condition can be written, from the definition of the function $\mathscr{F}$ (21) as

$$\frac{\partial \mathscr{F}}{\partial x} > 0. \tag{33}$$

Some insight into the question of uniqueness and stability of solutions can be gained by considering the behavior of $\mathscr{F}$ as $x \to \pm\infty$. In particular, when $x \to +\infty$, the resistance of rod 1 tends necessarily to a limit, $R_1 \to R_{1\,\mathrm{lim}} = R_1(F/A_1)$. Then,

$$\mathscr{F}(x) \to \mathscr{F}_\infty = x - \left(\frac{KLR_{1\,\mathrm{lim}} F/A_1}{E(KR_{1\,\mathrm{lim}} + L)} \hat{V} + \frac{Lu}{KR_{1\,\mathrm{lim}} + L} \right) \tag{34}$$

and $\mathscr{F}_\infty \to +\infty$, at any speed. At the other extreme, when $x \to -\infty$, $R_2 \to R_{2\,\mathrm{lim}} = R_2(F/A_2)$,

$$\mathscr{F}(x) \to \mathscr{F}_{-\infty} = x + \left(\frac{KLR_{2\,\mathrm{lim}} F/A_2}{E(KR_{2\,\mathrm{lim}} + L)} \hat{V} + \frac{Lu}{KR_{2\,\mathrm{lim}} + L} \right) \tag{35}$$

and $\mathscr{F}_{-\infty} \to -\infty$, at any speed.

Therefore, $\mathscr{F}$ is always a continuous function of x extending from $-\infty$ to $+\infty$ and hence there must be an odd number of roots (except in the case of repeated roots). In particular, the external roots will be always stable whereas when there is a single (the trivial) solution, this will be stable. Notice that uniqueness does not necessarily imply stability in all thermoelastic contact problems, see for example the system of two opposed rods of different materials in Zhang and Barber (1993). When there are three or five roots, etc., the solutions at highest and lowest x will be stable, involving either only rod 1 or 2 in contact, and in the case of five roots also the intermediate solution will return stable. Notice that when there are multiple solutions, as the linear stability condition suggests more than one stable solution, the system becomes path dependent, and for a sufficiently large perturbation, may move from one stable solution to the other.

4. Special case $R_i = d + c/p_i$

We know that, due to roughness, a nominally flat surface generally shows an inverse dependence on pressure (Clausing and Chao, 1965; Thomas and Probert, 1970; Shlykov and Ganin, 1964; Cooper et al., 1969). Also, to take into account that frictional heating will be generated somewhere "between" the two bodies, we should certainly avoid frictional heating to flow out of the interface, not making any effect on the rod, when the pressure is very high. Therefore, we suggest a limit finite resistance in the "wall", using the following relation for the contact resistance:

$$R_i = \begin{cases} d + c/p_i; & p_i > 0, \\ \infty; & p_i = 0. \end{cases} \tag{36}$$

For the case of both rods in contact, we can write from (16)

$$\frac{(K/E)\hat{V}}{L + Kd}[Ld(p_1 - p_2) + Lc(f_1 - f_2)] + u(f_1 - f_2) = x \tag{37}$$

which can be rewritten as

$$f_1 - f_2 = \frac{m}{n}\frac{x}{u}, \tag{38}$$

where

$$n(\hat{V}) = \frac{Lc/u}{L + Kd}\frac{K}{E}\hat{V} + 1; \quad m(\hat{V}) = 1 - \frac{Kd}{L + Kd}\hat{V}. \tag{39}$$

This corresponds to the same graphical construction of the static case (Barber, 1981), where steady state solutions are represented by intersections between the curve $f_1 - f_2$ (independent of the temperature difference and speed, but dependent on the total force and the rest of material properties) and various lines where the effects of imposed temperature difference and speed $\hat{V}$ are concentrated (see Fig. 2). The central region of the figures indicates full contact solutions (i.e. both rods are in contact for $-FL/(EA_2) < x < FL/(EA_1)$), whereas the region $x > FL/(EA_1)$ indicates only rod 1 in contact, and obviously the region $x < -FL/(EA_2)$ indicates only rod 2 in contact, according to the definition of the variable x (18)–(20). In particular, the slope of the lines in the full contact area range, depends on m, n. In the range where only one rod is in contact, the curve becomes horizontal at both ends, while the lines change slope in the regions of separation, depending on speed.

For only rod 1 in contact ($g_2 > 0$; $p_2 = 0$; $p_1 = F/A_1$ and $f_2 = 0$), (37) reduces to

$$f_1 = \frac{1}{n}\frac{x}{u} - x_1 \tag{40}$$

whilst for rod 2 in contact ($g_1 > 0$; $p_1 = 0$; $p_2 = F/A_2$ and $f_1 = 0$)

$$-f_2 = \frac{1}{n}\frac{x}{u} + x_2, \tag{41}$$

where

$$x_i = \frac{K}{nE}\hat{V}\frac{Ld/u}{L + Kd}\frac{F}{A_i}, \quad i = 1, 2. \tag{42}$$

In Fig. 2 an example of steady-state solutions of Eqs. (38), (40) and (41) for the case $u > 0$ is shown. The number and location of intersections between the curve and the various lines are clearly indicated in few sample conditions. Various critical speeds can be defined, which we shall specify more precisely later; at present it can be noticed that:

- the number of solutions is always odd, 1, 3 or 5, and changes with increasing $\hat{V}$;
- as the dimensionless sliding speed $\hat{V}$ is increased, for the initial conditions of Fig. 2, a progression from one solution ($\hat{V} < \hat{V}_u$) to three ($\hat{V}_u < \hat{V} < \hat{V}_1$) and, finally, to

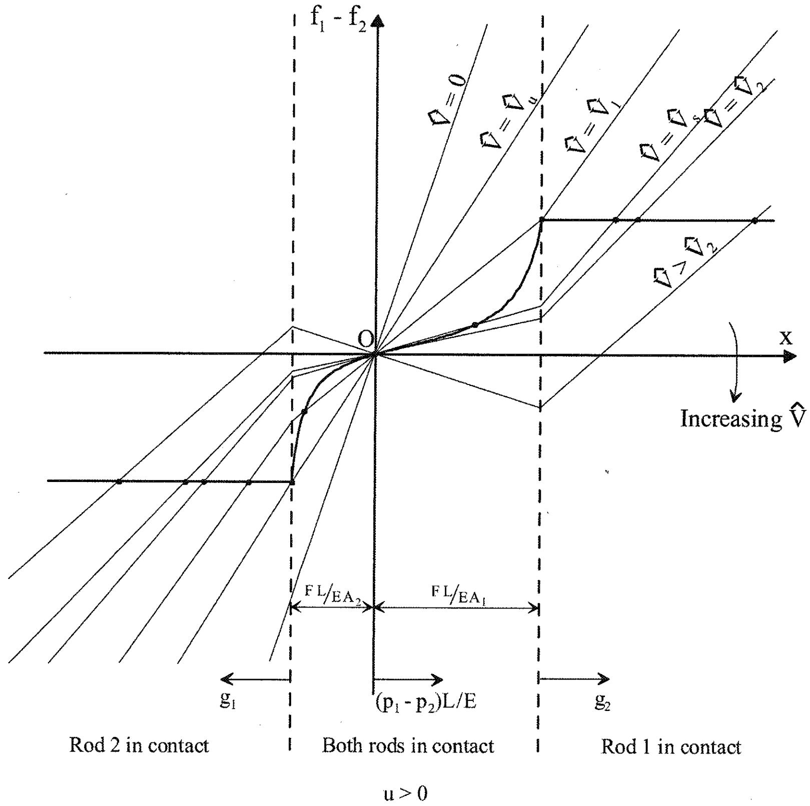
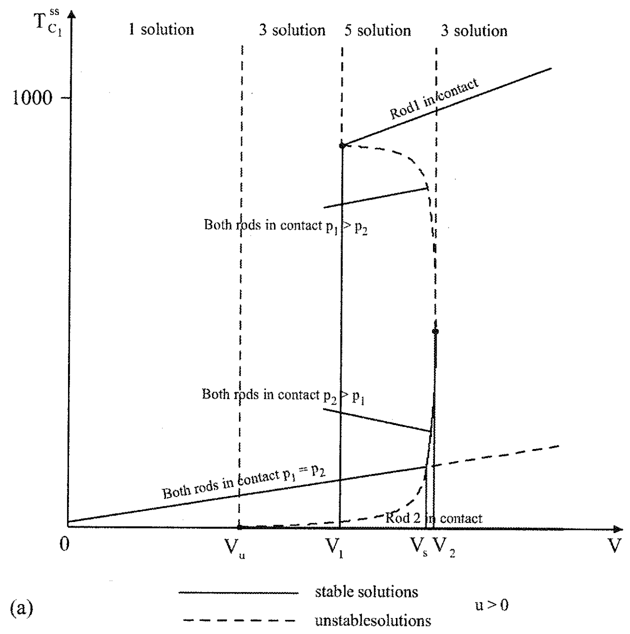


Fig. 2. Graphical construction of the problem solutions for different sliding speed, $u > 0$ and $A_2 = 2A_1$.

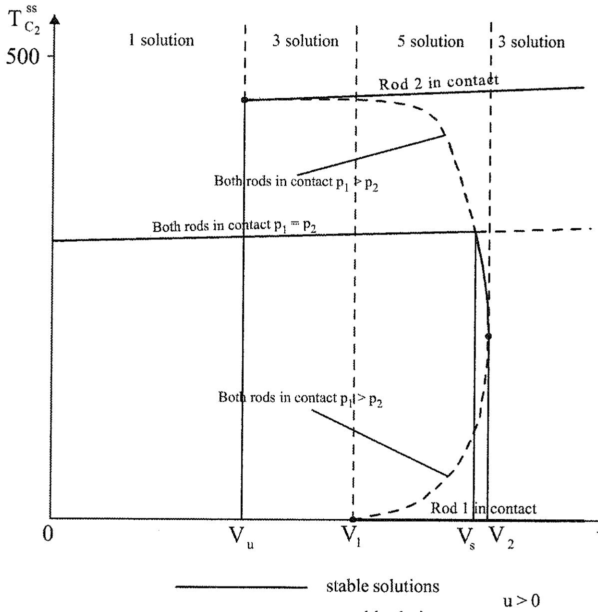
five solutions ($\hat{V}_1 < \hat{V} < \hat{V}_2$) is observed. However, with a further increase in sliding speed, the system passes into a new regime with again three solutions ($\hat{V} > \hat{V}_2$).

Further, according to the stability criterion (33), stable and unstable solutions alternate with increasing x and, hence, the outermost solutions are always stable, whereas the interior solutions are always unstable, *except* when there is only one solution, or when there are 5, in which case the uniform pressure solution becomes stable again: given it is non-unique, for some initial conditions, the system may *not* be reaching the uniform solution, but one of the other two.

In Fig. 3(a)–(b), the dependence of the steady-state temperatures ($T_{C_1}^{ss}$ and $T_{C_2}^{ss}$) of the rods on the sliding speed is shown for $u > 0$. Stable solutions are presented with solid line, whilst unstable ones with dashed line. These figures show that *uniqueness implies stability, but stability does not imply uniqueness*. This is the opposite of Zhang and Barber's result (Zhang and Barber, 1993), where uniqueness did not guarantee stability, so equally stability does not guarantee uniqueness. It is possible therefore that in general, neither uniqueness implies stability nor stability implies uniqueness.



(a)



(b)

Fig. 3. (a) Dependence of the steady-state end temperature $T^{ss}_{C_1}$ of rod 1 on the sliding speed V for $u > 0$; (b) Dependence of the steady-state end temperature $T^{ss}_{C_2}$ of rod 2 on the sliding speed V for $u > 0$.

We shall return to this point in the discussion, to show that this result is not due to the varying contact resistance.

For the system with $u > 0$ there can be either 1, 2 or 3 stable solutions. It is clear that the greatest steady state temperatures are reached for only rod 1 in contact (notice that the scale of temperature in Fig. 3(a) is about two times larger than that of Fig. 3(b)): this was expected, given in our case the area A_2 is twice larger than area A_1. Notice as well that, as long as the pressure does not vary with speed, the resistance being constant, the steady-state temperature increase linearly with speed. It is only when non-uniform full contact solutions appear that they change non-linearly with speed. Obviously, a sudden jump in the temperature is found when the steady state changes from two rods to only 1 in contact.

For $u < 0$ (i.e. for the temperature T_A larger than wall temperature $T_B = T_0$) the behavior of the system changes as shown in Fig. 4 and Fig. 5(a)–(b). In particular, for the initial conditions of Fig. 4, it can be observed that as dimensionless sliding speed $\hat{V}$ is increased the solution number changes from one to three. Then, for $\hat{V} > \hat{V}_s$,

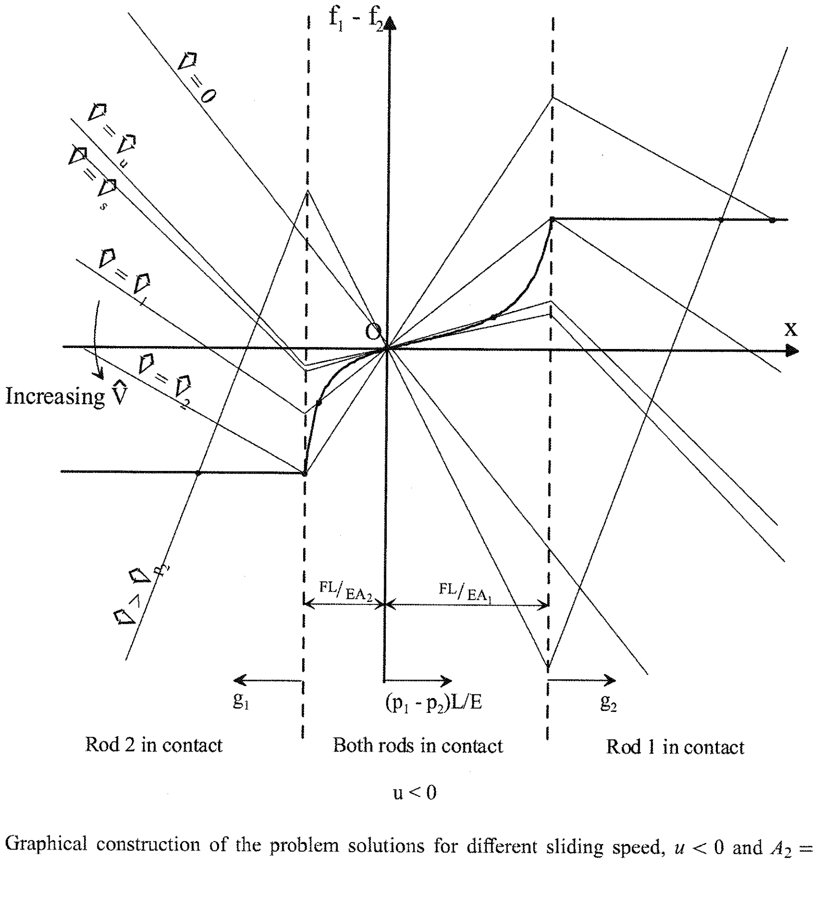


Fig. 4. Graphical construction of the problem solutions for different sliding speed, $u < 0$ and $A_2 = 2A_1$.

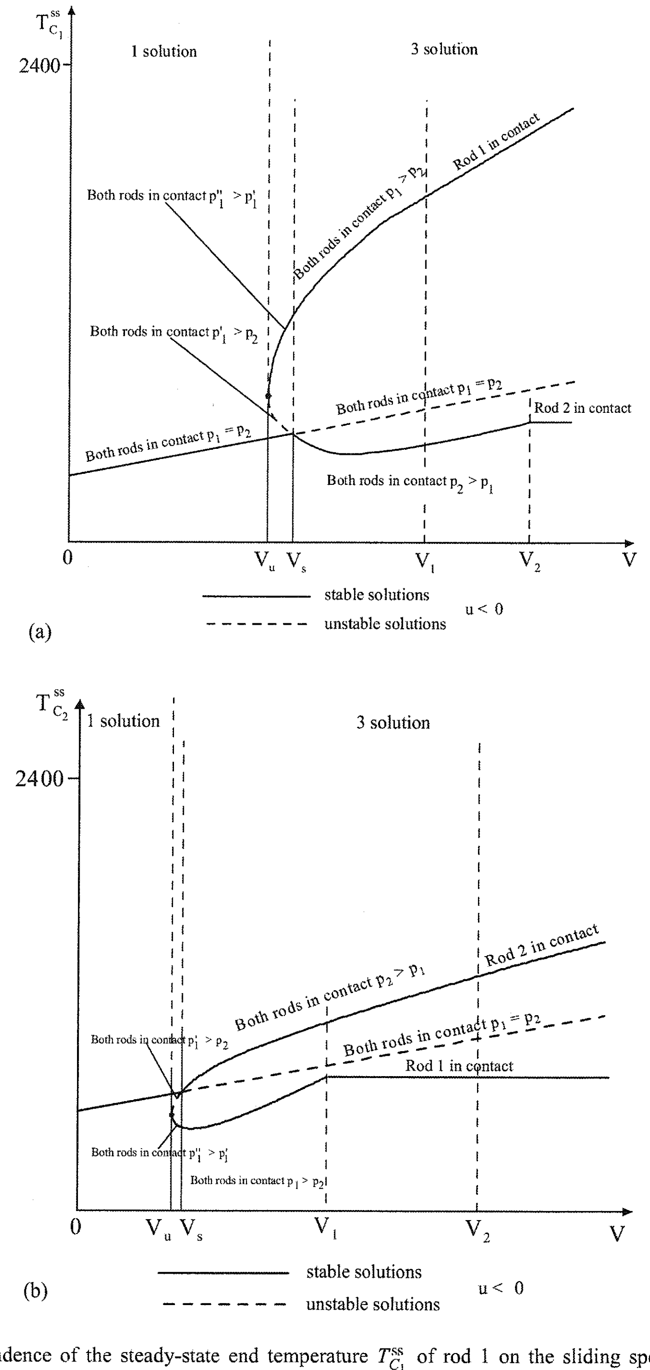


Fig. 5. (a) Dependence of the steady-state end temperature $T_{C_1}^{ss}$ of rod 1 on the sliding speed V for $u < 0$; (b) Dependence of the steady-state end temperature $T_{C_2}^{ss}$ of rod 2 on the sliding speed V for $u < 0$.

it is possible obtain the following stable steady-state solutions:

- with both rods in contact ($p_1 > p_2$ or $p_2 > p_1$), when $\hat{V} < \hat{V}_1$;
- with rod 1 ($p_2 = 0$) or both rods ($p_2 > p_1$) in contact, when $\hat{V}_1 < \hat{V} < \hat{V}_2$;
- with rod 1 ($p_2 = 0$) or rod 2 ($p_1 = 0$) in contact, when $\hat{V} > \hat{V}_2$.

Therefore, the main difference with respect to the previous case $u > 0$ is that there is a wider range of conditions for which non-uniform full contact solutions exist, and indeed in the range $\hat{V}_u < \hat{V} < \hat{V}_s$ there are two full contact solutions, one being the uniform one, and in the range $\hat{V}_s < \hat{V} < \hat{V}_1$ there are two full contact solutions, both involving different pressures in the two rods. Notice also that the only distinct "jump" in the steady state occurs at $\hat{V}_u$ as otherwise, both at $\hat{V}_1$ and $\hat{V}_2$, despite separated solutions appear, they emerge gradually from the full contact non-uniform solutions.

4.1. Critical speeds

The "critical speed" as it is generally intended in the TEI literature, is the speed for which the uniform pressure solution becomes unstable. The closest to this "critical speed" concept here is probably $\hat{V}_s$. However, the classical TEI literature doesn't consider the effect of constriction resistance varying with pressure, and there is no general equivalent method to search for the speed when uniqueness is lost or to examine all the possible steady states of the system. Therefore, we will define various "critical speeds", making clear the meaning of each definition. We shall consider separately the cases $u > 0$ and $u < 0$.

4.1.1. Case $u > 0$

The sliding speed for which the uniqueness of the solution is lost, $\hat{V}_u$, can be easily found by computing (41) with $x = -FL/EA_2$, obtaining immediately that

$$\hat{V}_u = \frac{A_2(cK - uE) + (\mathrm{d}K + L)F}{(cA_2 + \mathrm{d}F)K}. \tag{43}$$

It can be shown that this is the lowest critical condition, defined as the speed for which, for some initial conditions, the system will evolve towards non-uniform solutions, in the form of separated solutions with only rod 2 in contact (for $u < 0$ both rods in contact under different pressures). Therefore, although the system will not necessarily evolve towards these solutions, there is a first bifurcation of the solutions. Notice that $\hat{V}_u$ depends *only* on the size A_2, and therefore systems of various total size but having the same A_2 will show the same critical speed for uniqueness, as this is the speed at which separated solutions with only rod 2 in contact are possible.

A second bifurcation appears when the solution with only rod 1 in contact appears, and precisely at sliding speed $\hat{V}_1$ which can be immediately written by computing (40) with $x = FL/EA_1$, obtaining

$$\hat{V}_1 = \frac{A_1(cK - uE) + (\mathrm{d}K + L)F}{(cA_1 + \mathrm{d}F)K} \tag{44}$$

clearly depending, this time, only on the area A_1 of the rod which remains in contact. Again, the solution with uniform pressure remains stable, so the system can gravitate towards each of the 3 (for $u > 0$) or 2 (for $u < 0$) stable solutions, depending on the initial conditions.

The sliding speed $\hat{V}_2$ corresponds to the appearance of the solution with rod 2 in contact (for $u < 0$) but to the loss of the stable full contact solution with different pressures for $u > 0$. Its expression can be derived resolving the following system:

$$\frac{d(f_1 - f_2)}{dx} = \frac{m}{n}\frac{1}{u},$$

$$f_1 - f_2 = \frac{m}{n}\frac{x}{u}. \tag{45}$$

It is observed that $\hat{V}_2$ separately depends on A_1 and A_2.

Further, a characteristic sliding speed can be defined as that speed for which the uniform solution ($p_1 = p_2 = F/(A_1 + A_2)$) becomes unstable. Such speed is referred to in the Figs. 2–5 as $\hat{V}_s$ and, as already noticed, is close to the standard concept of critical speed $\hat{V}_{cr}$ in the TEI literature. However, initially a full contact solution remains possible, having two different pressures for rods 1 and 2. This speed therefore has a distinctly different character as the steady state does not show a jump in the values of temperature and pressures. Its expression can be obtained by imposing that the straight line mx/nu is tangent to the curve $f_1 - f_2$ in $x = 0$, i.e.

$$\left.\frac{d(f_1 - f_2)}{dx}\right|_{x=0} = \frac{m}{n}\frac{1}{u}. \tag{46}$$

Therefore, the dimensionless critical sliding speed assumes the following form:

$$\hat{V}_s = \frac{[c(A_1 + A_2)K + (L + dK)F]^2 - cuEK(A_1 + A_2)^2}{K^2[c(A_1 + A_2) + dF]^2 + dKLF^2}. \tag{47}$$

Notice that the critical speed depends on the total area $A_{tot} = A_1 + A_2$ of the rods.

Let us now consider the various speeds more in detail. When $F \to 0$, the characteristic speeds $\hat{V}_u$, $\hat{V}_1$ and $\hat{V}_s$ tend to the limit value

$$\hat{V}_0 = 1 - \frac{uE}{cK}, \tag{48}$$

which is negative if

$$u > \frac{cK}{E}, \tag{49}$$

indicating in this case that for those conditions (in particular, at light forces) and for large enough positive temperature difference (i.e. the heat flow is directed into the rods), the system shows always multiple solutions. This conclusion agrees with the general behavior of thermoelastic contact, where multiple solutions are generally observed for this direction of heat flow (Barber, 1987). For example, in Afferrante and Ciavarella (2003), where the problem of an elastic half-plane in sliding contact with a rigid perfect conductor wall is considered, it is demonstrated that when the contact pressure tends to zero the instability conditions is for the heat flow directed into the more distortive material (the half-plane, which in our case is simulated with rods) and greater than a certain threshold.

At the other extreme, of very large force ($F \to \infty$), we get for all the characteristic speeds the limit

$$\hat{V}_\infty = 1 + \frac{L}{\mathrm{d}K} \tag{50}$$

which is certainly positive and does not depend on temperature difference, and coincides with the most important critical speed for the single rod system studied in (Ciavarella et al., 2003) for imposed gap. Also, the above speed is similar to the critical speed found for constant constriction resistance of a half-plane in sliding contact with a rigid wall (Afferrante and Ciavarella, 2003), provided as scale length one considers the rods length instead of the wave number of the perturbation.

Finally, if we define the parameter

$$\gamma = \frac{\hat{V}_0}{\hat{V}_\infty} = \frac{1 - uE/cK}{1 + L/\mathrm{d}K}, \tag{51}$$

we can write that for $\gamma < 1$, i.e. $u > -Lc/\mathrm{d}E$, including all cases with $u > 0$, increasing the force F stabilizes the system (critical speeds increasing with F).

This is illustrated in Fig. 6(a), where the variation of characteristic speeds $\hat{V}_u$, $\hat{V}_1$ and $\hat{V}_s$ with the force F are plotted for $u > 0$. Notice that when $u > cK/E$ the limit speed $\hat{V}_0$ is negative and the system shows always multiple solutions at light forces. Further, it is interesting to observe that for any finite force F, the sliding speeds $\hat{V}_1$ and $\hat{V}_2$ again tend to the limit $\hat{V}_\infty$ when $A_1/A_2 \to 0$ (small separation zone).

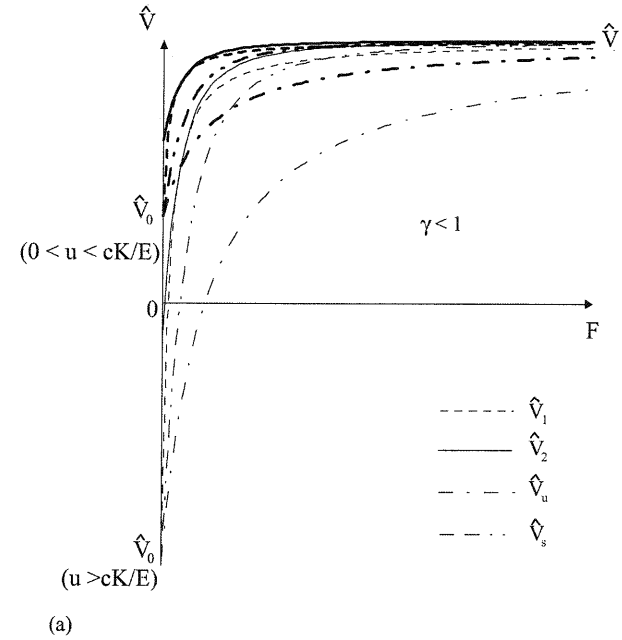
4.1.2. Case $u < 0$

This case is somewhat counterintuitive, as it is not expected that the temperature is lower where frictional heating is produced. The main difference is that the expressions for $\hat{V}_u$ and $\hat{V}_2$ change. In other words, $\hat{V}_u$ ($u < 0$) depends on both areas, and vice versa obviously $\hat{V}_2$ ($u < 0$) depends *only* on the size A_2. When $F \to 0$, the limit for $\hat{V}_2$, $\hat{V}_1$ and $\hat{V}_s$ is still $\hat{V}_0$, which is never negative for $u < 0$, indicating that at light forces and for negative temperature difference (i.e. the heat flow is directed out of the rods), the system always has unique solution for low speeds.

Further, for the above defined parameter γ, we can write that:

- for $\gamma < 1$, i.e. $-Lc/\mathrm{d}E < u < 0$, increasing the force F stabilizes the system (critical speeds increasing with F);
- for $\gamma > 1$, i.e. $u < -Lc/\mathrm{d}E$, increasing the force F destabilizes the system (critical speeds decreasing with F).

This is illustrated in Fig. 6(b), where the variation of characteristic speeds $\hat{V}_u$, $\hat{V}_1$ and $\hat{V}_s$ with the force F are plotted for $\gamma < 1$ and $\gamma > 1$.



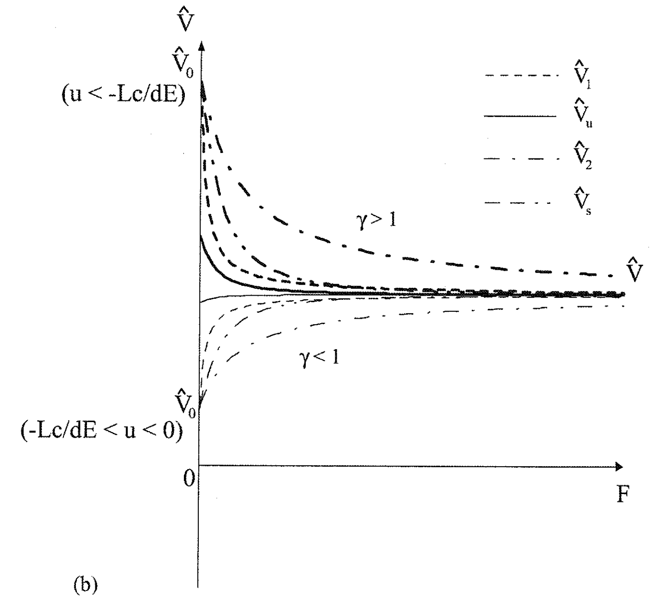


Fig. 6. (a) Dependence of characteristic speeds $\hat{V}_u$, $\hat{V}_s$, $\hat{V}_1$ and $\hat{V}_2$ on the applied force F, for $u > 0$; (b) Dependence of characteristic speeds $\hat{V}_u$, $\hat{V}_s$, $\hat{V}_1$ and $\hat{V}_2$ on the applied force F, for $u < 0$.

Table 1
Values of parameters used in the numerical analysis

α	k	K	E	c	d	f	F	A_1	A_2	L
2.4×10^{-3}	1	1	1	1	100	1	1	1	2	1

5. Numerical simulation

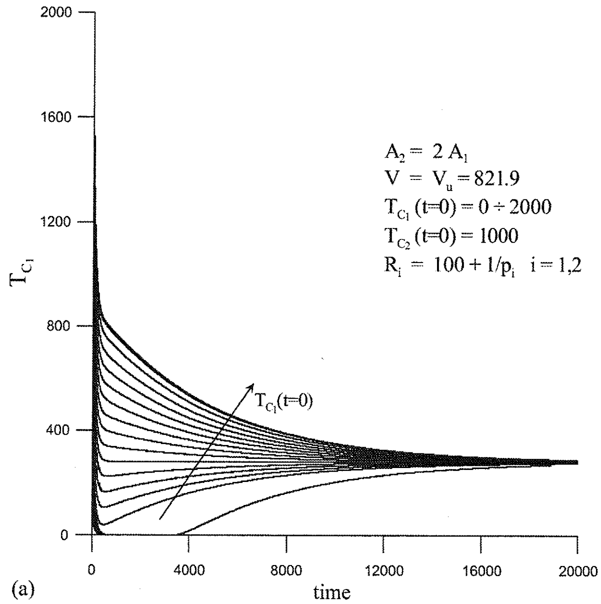
A companion numerical analysis confirms the results outlined in the previous sections. In particular, both rods were divided into 100 elements of equal length and, as initial conditions, a linear variation of the rod's temperatures was considered with $T_A=0$ and 21 evenly spaced values of T_{C_1} $(t=0)$ between 0 and 2000 and T_{C_2} $(t=0)=1000$. The numerical analysis was performed for $u > 0$ and the example $A_2=2A_1$ and the transient evolution of the end temperatures of rods was plotted. The values of parameters used in the numerical simulation are listed in Table 1.

A range of systems and initial conditions were considered and in all cases the transient behaviour confirmed the conclusions of the stability analysis. For example, in Fig. 7(a)–(b) the evolution of end temperature of rods is presented for $\hat{V} = \hat{V}_u$ (sliding speed for which the uniqueness is lost). Notice as the system evolves toward the steady-state solution with only rod 2 in contact ($T_{C_1}^{ss} = 0$ and $T_{C_2}^{ss} = 416.67$) for initial conditions with low values of T_{C_1} $(t = 0)$, and toward the uniform solution for large values of T_{C_1} $(t = 0)$ $(T_{C_1}^{ss} = T_{C_2}^{ss} = 280.95)$.

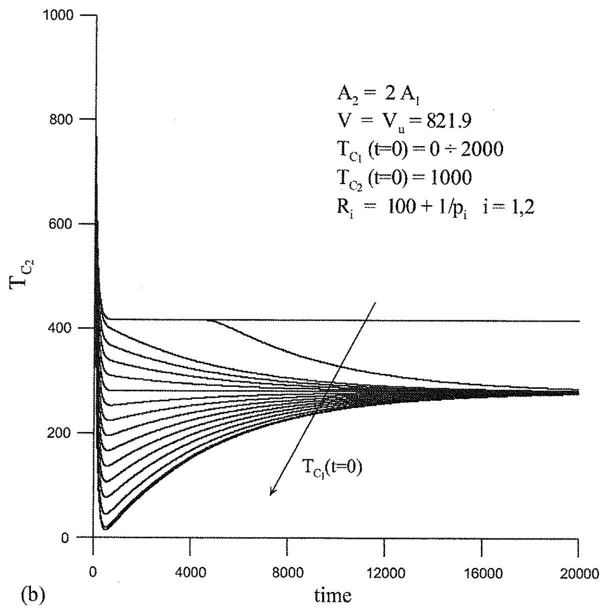
Fig. 8 (a)–(b), where a sliding speed $\hat{V} > \hat{V}_2$ was chosen corresponding to the case with unstable uniform solution, shows that for low initial values of T_{C_1}, the system evolves towards the solution with rod 2 in contact ($T_{C_1}^{ss}=0$ and $T_{C_2}^{ss}=628.64$), whereas for high initial values it tends to solution with rod 1 in contact ($T_{C_1}^{ss} = 1247.55$ and $T_{C_2}^{ss} = 0$). The system has also an unstable uniform steady-state solution. Starting from an initial condition close to this state, initially the deviation is slower, but ultimately the system tends to solutions with only one rod in contact.

6. Discussion

The Aldo model is certainly very idealized with respect to actual contact systems of engineering interest such as brake or clutch systems. However, since the behaviour of thermoelastic contact is very rich, and the classical theorems of existence and uniqueness do not apply, the scope of the present paper was to investigate general features of existence, uniqueness and stability of the system. In the previous paper on gap-prescribed boundary conditions (Ciavarella et al., 2003), which applies also trivially for a system of N rods, it was shown that the most relevant critical speed was defined at the limit value of the contact resistance at high pressures, $\hat{V}_\infty = 1 + L/\mathrm{d}K$, and above this speed *seizure* was possible, a condition corresponding not just to the mathematical "loss of existence", but in practice to possibly very serious damage. In the present paper, it has emerged that for force controlled conditions, where existence



(a)



(b)

Fig. 7. (a) Transient evolution of end temperature T_{C_1} for $u > 0$, $\hat{V} = \hat{V}_u$ and different initial conditions; (b) Transient evolution of end temperature T_{C_2} for $u > 0$, $\hat{V} = \hat{V}_u$ and different initial conditions.

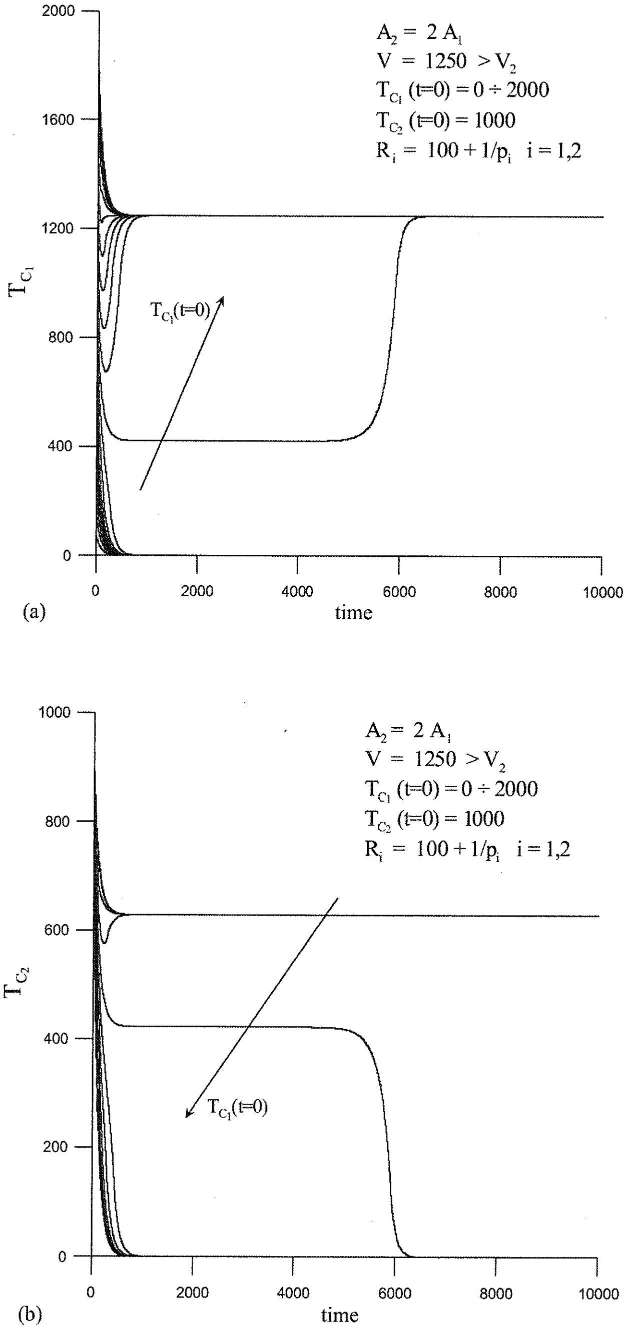


Fig. 8. (a) Transient evolution of end temperature T_{C_1} for $u > 0$, $\hat{V} > \hat{V}_2$ and different initial conditions; (b) Transient evolution of end temperature T_{C_2} for $u > 0$, $\hat{V} > \hat{V}_2$ and different initial conditions.

is guaranteed (and seizure is not possible), the most interesting general result is that stability does not imply uniqueness. Classical TEI investigations are based on the assumption that for low speed, since the stability of the uniform pressure solution is guaranteed, this is the obvious steady state: also, since the variation of contact resistance with pressure is neglected, under these full contact conditions, linearity of the system holds, and because the contact is recessive, this is true even in the separated regime (Joachim-Ajao and Barber, 1998). Therefore, the usual TEI analysis is to find the critical speed of the system as a function of the material properties and the geometry with efficient analytical (Yi et al., 1999) or eigenvalue numerical methods (Yi et al., 2000). In the present paper, it has emerged that other solutions may be possible below the critical speed, and this is not due to the functional form of the contact resistance. In fact, for constant contact resistance in the contact regime (maintaining infinite resistance for separation condition), the general stability analysis holds true, but since the function $f_1 - f_2$ is not continuous, the number of roots is not directly related to the stability of them (hence we can have two solutions and both may be stable). However, any discontinuity in the function has to occur at the separation points, and therefore any solution near the two extremes follows the usual stability analysis, and in particular separate solutions are stable. It is fairly easy to show that the condition of both rods in contact (37) leads to $p_1 - p_2 = 0$, and therefore, any rod in the system can either have contact at one level of pressure, or separation. Therefore, the system with two rods identifies even better an arbitrary system of rods, since only the partition of contact and separation area needs to be adjusted according to the state.

Turning to the issue of stability, putting $c = 0$ in the general expression of $\hat{V}_s$ (47) we obtain

$$\hat{V}_s = \hat{V}_\infty = 1 + \frac{L}{\mathrm{d}K}, \tag{52}$$

which is obviously independent on F and does not depend on temperature difference. The stability of the uniform pressure steady state solution for $R = const$ is obviously independent on the pressure. However, the uniform pressure solution may coexist with stable separate solutions. For example, if we assume $A_2 > A_1$, the speed at which uniqueness is lost indicates the speed at which the solution with rod 2 in contact appears,

$$\hat{V}_u = \frac{-uEA_2}{\mathrm{d}FK} + \hat{V}_\infty = \hat{V}_1 \tag{53}$$

and this speed decreases with increasing A_2 (for $u > 0$) and in particular if

$$A_2 > \frac{F}{uE}(\mathrm{d}K + L) \tag{54}$$

then $\hat{V}_u < 0$. Solutions with only rod 2 in contact may exist well before the stability boundary of the uniform pressure is passed. Notice that $\hat{V}_u$ is highest when $A_2 = 0$ which obviously gives

$$\hat{V}_u = \hat{V}_\infty = \hat{V}_1 = 1 + \frac{L}{\mathrm{d}K}. \tag{55}$$

Moving to the case $u < 0$ the expressions for $\hat{V}_u$ and $\hat{V}_2$ change

$$\hat{V}_u\ (u < 0) = \hat{V}_\infty = 1 + \frac{L}{\mathrm{d}K}, \tag{56}$$

$$\hat{V}_2\ (u < 0) = \frac{-uEA_2}{\mathrm{d}FK} + \hat{V}_\infty \tag{57}$$

and this time the separate solution and uniqueness boundaries coincide.

7. Conclusions

This paper has extended the Aldo model to the case of frictional sliding. It has emerged that when the solution is unique (the uniform pressure solution), it is always stable. Vice versa, the stability of the uniform pressure solution, does not imply its uniqueness. Also, by increasing the sliding speed, more steady states emerge (involving separation) and eventually the uniform pressure solution becomes also unstable. A graphical construction permits the evolution of the solutions and their number as a function of the speed to be followed. Various characteristic speeds have been defined, $\hat{V}_u$, $\hat{V}_1$, $\hat{V}_2$, $\hat{V}_s$. All speeds depend generally non-linearly on the parameters, including the force, and the imposed temperature difference which is implied in u.

The speed $\hat{V}_u$ corresponds to loss of uniqueness, where non-uniform solutions appear, either in the form of separated solutions (for $u > 0$), or in the form of full contact solutions involving different pressures in the rods ($u < 0$). Also, the sliding speed $\hat{V}_1$ corresponds to the appearance of the separated solution with only rod 1 in contact, whereas $\hat{V}_2$ corresponds to the separation of rod 2 (for $u < 0$) but to the loss of the full contact solution for $u > 0$. Finally, $\hat{V}_s$ is the speed for which the uniform solution becomes unstable, and is closest to the critical speed as intended in standard TEI literature. Notice also that for the most realistic case of $u > 0$, increasing the force tends to make the system more unstable, as the characteristic speeds decrease and tend to the critical speed $\hat{V}_\infty$ already defined in the previous paper (Ciavarella et al., 2003) for the single rod with imposed displacement. There, $\hat{V}_\infty$ indicated the possibility of seizure, here is the minimal critical speed for the system. Finally, the case of N rods remains to be further investigated, and in particular the number of possible solutions, how the partition of the area in the various states varies with speed, and the general stability of the full system.

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References

Afferrante, L., Ciavarella, M., 2003. Frictionally excited thermoelastic instability (TEI) in the presence of contact resistance. J. Strain Anal. Eng. Des., submitted.

Barber, J.R., 1969. Thermoelastic instabilities in the sliding of conforming solids. Proceedings of the Royal Society of London, Vol. A312, pp. 381–394.

Barber, J.R., 1981. Stability of Thermoelastic Contact for the Aldo Model. ASME J. Appl. Mech. 48, 555–558.

Barber, J.R., 1987. Stability of thermoelastic contact. International Conference on Tribology, Institution of Mechanical Engineers, London, pp. 981–986.

Barber, J.R., Dundurs, J., Comninou, M., 1980. Stability considerations in thermoelastic contact. ASME J. Appl. Mech. 47, 871–874.

Burton, R.A., Staph, H.E., 1967. Thermally activated seizure of angular contact bearings. ASLE Trans. 10, 408–417.

Burton, R.A., Kilaparti, S.R., Nerlikar, V., 1973. A limiting stationary configuration with partially contacting surfaces. Wear 24, 199–206.

Ciavarella, M., Johansson, L., Afferrante, L., Klarbring, A., Barber, J.R., 2003. Interaction of thermal contact resistance and frictional heating in thermoelastic instability. Int. J. Solids Struct. (in press available on line).

Clausing, A.M., Chao, B.T., 1965. Thermal contact resistance in a vacuum environment. ASME J. Heat Transfer 87, 243–251.

Comninou, M., Dundurs, J., 1980. On the possibility if History Dependence and Instabilities in Thermoelastic contact. J. Therm. Stresses 3, 427–433.

Cooper, M.G., Mikic, B.B., Yovanovich, M.M., 1969. Thermal contact conductance. Int. J. Heat Mass Transfer 12, 279–300.

Dow, T.A., Burton, R.A., 1972. Thermoelastic instability of sliding contact in the absence of wear. Wear 19, 315–328.

Duvaut, G., 1979. Free boundary problem connected with thermoelasticity and unilateral contact. Free Boundary Problems, Vol. 11, Pavia.

Joachim-Ajao, D., Barber, J.R., 1998. Effect of material properties in certain thermoelastic contact problems. ASME J. Appl. Mech. 65, 889–893.

Kennedy, F.E., Ling, F.F., 1974. A thermal, thermoelastic and wear simulation of a high energy sliding contact problem. ASME J. Lubr. Technol. 96, 497–507.

Lee, K., Barber, J.R., 1993. Frictionally-excited thermoelastic instability in automotive disk brakes. ASME J. Tribol. 115, 607–614.

Lee, K., Dinwiddie, R.B., 1998. Conditions of frictional contact in disk brakes and their effects on brake judder. SAE 980598.

Parker, R.C., Marshall, P.R., 1948. The measurement of the temperature of sliding surfaces, with particular reference to railway blocks. Proceedings of the Institute of Mechanical Engineering, Vol. 158, pp. 209–229.

Shlykov, Yu.P., Ganin, Ye.A., 1964. Thermal resistance of metallic contacts. Int. J. Heat Mass Transfer 7, 921–929.

Srinivasan, M.G., France, D.M., 1985. Non-uniqueness in steady-state heat transfer in prestressed duplex tubes—analysis and case history. ASME J. Appl. Mech. 52, 257–262.

Thomas, T.R., Probert, S.D., 1970. Thermal contact resistance: the directional effect and other problems. Int. J. Heat Mass Transfer 13, 789–807.

Tu, J.F., Stein, J.L., 1995. Online preload monitoring for antifriction bearings of high speed machine tools. ASME J. Dyn. System 117, 43–53.

Yi, Y.-B., Du, S., Barber, J.R., Fash, J.W., 1999. Effect of geometry on thermoelastic instability in disk brakes and clutches. ASME J. Tribol. 121, 661–666.

Yi, Y.-B., Barber, J.R., Zagrodzki, P., 2000. Eigenvalue solution of thermoelastic instability problems using Fourier reduction. Proceedings of the Royal Society 456 (2003), 2799–2821.

Yigit, F., Barber, J.R., 1994. Effect of Stefan number on thermoelastic instability in unidirectional solidification. Int. J. Mech. Sci. 36, 707–723.

Zagrodzki, P., 1990. Analysis of thermomechanical phenomena in multidisk clutches and brakes. Wear 140, 291–308.

Zagrodzki, P., Lam, K.B., Al-Bahkali, E., Barber, J.R., 2001. Nonlinear transient behaviour of a sliding system with frictionally excited thermoelastic instability. ASME J. Tribol. 123, 699–708.

Zhang, R., Barber, J.R., 1993. Transient thermoelastic contact and stability of two thin-walled cylinders. J. Therm. Stresses 16, 31–54.