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Brief Note

A Paradox in Sliding Contact Problems With Friction

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In problems involving the relative sliding to two bodies, the frictional force is taken to oppose the direction of the local relative slip velocity. For a rigid flat punch sliding over a half-plane at any speed, it is shown that the velocities of the half-plane particles near the edges of the punch seem to grow without limit in the same direction as the punch motion. Thus the local relative slip velocity changes sign. This phenomenon leads to a paradox in friction, in the sense that the assumed direction of sliding used for Coulomb friction is opposite that of the resulting slip velocity in the region sufficiently close to each of the edges of the punch. This paradox is not restricted to the case of a rigid punch, as it is due to the deformations in the half-plane over which the pressure is moving. It would therefore occur for any punch shape and elastic constants (including an elastic wedge) for which the applied pressure, moving along the free surface of the half-plane, is singular. The paradox is resolved by using a finite strain analysis of the kinematics for the rigid punch problem and it is expected that finite strain theory would resolve the paradox for a more general contact problem. [DOI: 10.1115/1.1867992]

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Formulation

Consider a rigid flat punch, indenting an elastic half-plane as shown in Fig. 1. From the classical equations of plane-strain elasticity (see, e.g., Barber [1], p. 154) the displacements $\bar u, \bar v$ on the surface of the half-plane are given by

$$\frac{d\bar u}{d\bar x}(\bar x,0) = -\frac{1-\nu}{\pi\mu}\int_S \frac{\bar q(\bar\xi)d\bar\xi}{\bar x-\bar\xi} - \frac{1-2\nu}{2\mu}\bar p(\bar x), \qquad (1)$$

$$\frac{d\bar v}{d\bar x}(\bar x,0) = \frac{1-\nu}{\pi\mu}\int_S \frac{\bar p(\bar\xi)d\bar\xi}{\bar x-\bar\xi} - \frac{1-2\nu}{2\mu}\bar q(\bar x), \qquad (2)$$

where $\bar p, \bar q$ are the pressure and shear, respectively, transmitted at the interface ($\bar p$ is positive in compression), ν is the Poisson's ratio, and μ is the shear modulus.

We initially consider a frictionless punch ($\bar q=0$) that is moving with a velocity U_0 which is much less than any of the wave speeds of the elastic body. Hence inertia effects can be neglected and the steady solution is given in terms of a moving coordinate system (x,y) where

$$x=\bar x - U_0 t, \quad y=\bar y, \quad u(x,y)=\bar u(\bar x,\bar y,t), \quad v(x,y)=\bar v(\bar x,\bar y,t) \qquad (3)$$

The velocity of particles in the elastic material is denoted by U, where

$$U \equiv \frac{d\bar u}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} \Rightarrow U = -U_0\frac{du}{dx}(x,0) \qquad (4)$$

Under frictionless sliding conditions, Eqs. (1) and (4) yield

$$U = 0 \; |x| > a,$$

$$= U_0\frac{1-2\nu}{2\mu}p(x) \to +\infty \; |x| \to a^- \qquad (5)$$

where a is the half-width of the indenter.

Thus the effect of elastic deformation is to produce a velocity under the punch which is a function of position x and is in average very small if the mean pressure is, as we generally expect, much less than the elastic modulus. However, as the pressure becomes singular at the edges, *the speed U near the corners becomes greater than U_0 and, therefore, the local relative slip velocity is in the opposite direction to that of the punch motion.* Hence material points on the half-plane which enter under the leading edge of the punch are forced to move forward of the punch. Similarly material points near the trailing edge have a velocity greater than that of the punch and so, it would seem, never leave from under the punch.

This behavior itself is paradoxical, but is of particular concern as it persists when sliding is accompanied by Coulomb friction.

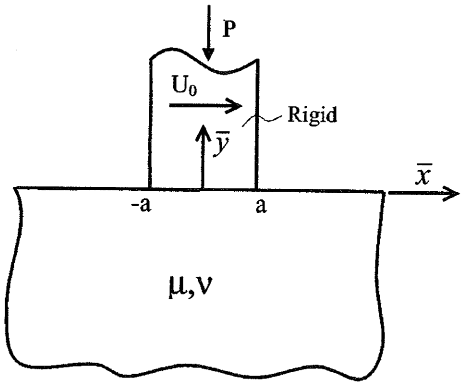

Fig. 1 A flat rigid punch indenting an elastic half-plane

One of the fundamental laws of friction is that the friction force opposes the relative slip velocity. Normally, this direction is inferred a priori, so that if Coulomb's friction law is assumed to hold, we anticipate

$$q(x)=fp(x), \tag{6}$$

where f is the coefficient of friction and Eqs. (1) and (2) reduce to

$$\frac{du}{dx}(x,0)=-\frac{(1-\nu)f}{\pi\mu}\int_{-a}^{a}\frac{p(\xi)d\xi}{x-\xi}-\frac{(1-2\nu)}{2\mu}p(x), \tag{7}$$

$$\frac{dv}{dx}(x,0)=\frac{(1-\nu)}{\pi\mu}\int_{-a}^{a}\frac{p(\xi)d\xi}{x-\xi}-\frac{(1-2\nu)}{2\mu}fp(x). \tag{8}$$

Since the punch is flat, we have

$$\frac{dv}{dx}(x,0)=0, \quad x\in(-a,a) \tag{9}$$

and hence

$$\int_{-a}^{a}\frac{p(\xi)d\xi}{x-\xi}=\frac{(1-2\nu)\pi}{2(1-\nu)}fp(x), \quad x\in(-a,a). \tag{10}$$

Using this result to eliminate the integral in (7), we obtain

$$\frac{du}{dx}(x,0)=-\frac{(1-2\nu)(1+f^2)}{2\mu}p(x), \quad x\in(-a,a) \tag{11}$$

The contact pressure $p(x)$ must be positive and $\nu\leq 0.5$, so we conclude that du/dx is always negative under the punch (except for an incompressible material, in which case it vanishes), giving a positive value of the particle velocity U in Eq. (4) which *becomes unbounded in a region near the punch corners (thus $U>U_0$) and hence where the local relative sliding motion is opposed to that of the punch motion.*

The paradox would continue to occur in the case of two elastic materials, for those combinations of punch wedge angles and material constants for which the pressure induced is singular (Dundurs and Lee, [2]; Gdoutos and Theocaris, [3]). Note that the elastic displacement in an elastic punch is constant in the frame of reference moving with it, and accordingly would not affect the relative sliding velocity.

Finite Strain Kinematics Analysis

Consider now X,Y as material coordinates in the reference state, and x,y the spatial coordinates of the same points in the deformed state. A deformation state can be represented as $x=x(X,Y,t)$, whereas displacements can be written as $u=x-X$ and $v=y-Y$. Hence, the *stretch ratio* λ (ratio of length in the deformed state to the length in the reference state) of a material filament $(dX,0)$ can be given as

$$\lambda=\sqrt{\left(\frac{dx}{dX}\right)^2+\left(\frac{dy}{dX}\right)^2} \tag{12}$$

We observe that $\lambda>0$.

Consider the flat rigid indenter as maintained on the plane $y=y_0$ (<0) moving at speed U_0 in the $+X$ direction. The material to be indented and also any apparatus which supports it along some plane $Y=y_1$ ($>y_0$) is to be considered translationally invariant relative to the X coordinate. Then, if a steady state solution exists to the problem, the displacement u must have the form

$$u=u(X-U_0t,Y) \tag{13}$$

and hence the velocity in the $+X$ direction of material points on the surface of the body is

$$\frac{du}{dt}=-U_0\frac{du}{dX}=-U_0\left(\frac{dx}{dX}-1\right) \tag{14}$$

When there is contact with the flat face of the indentor, $y=y_0=const$ and, therefore, $dy/dX=0$. Hence, the stretch ratio defined by (12) reduces to

$$\lambda=\frac{dx}{dX} \tag{15}$$

and thus the velocity of material points along the contact with the indentor is

$$\frac{du}{dt}=-U_0(\lambda-1) \tag{16}$$

This in turn gives the slip velocity s as

$$s\equiv U_0-\frac{du}{dt}=\lambda U_0>0 \tag{17}$$

and a negative value for s can never occur since the stretch ratio cannot be negative. The paradox happens in regions where the infinitesimal strain solution predicts compressive strains so large that λ is predicted (impossibly) to be negative. Had we evaluated it, we would have done so by using

$$\varepsilon_{xx}=\frac{du}{dX}=\lambda-1 \tag{18}$$

along the contact zone under the indentor. That means that the strain ε_{xx} can be no more negative than -1 (since λ has to be positive), whereas we fail this test when we approach the singularity at the corner of the indenter.

Discussion

This paradox would occur also in elastodynamics. In fact below the Rayleigh wave speed (c_R) the solution to any contact problem is the same as the corresponding quasi-static problem with a reduced modulus (which goes to zero at c_R), [4]. The elastodynamic solution for a normal point force moving at constant speed over the surface of an half-plane becomes resonant at the Rayleigh wave speed (c_R) and above that speed a downward force produces an upward displacement.[1] Hence, for elastodynamic problems involving, for example, a rigid punch sliding over an elastodynamic half-plane, the solution behaves as the static indentation of an elastic half-plane of reduced modulus (which approaches zero at the Rayleigh wave speed). However, it is still true that a compressive strain of magnitude greater than unity is needed to produce the paradox (although the required normal pressure is reduced).

[1]This phenomenon leads to a paradoxical behavior of its own, i.e., the Craggs–Roberts paradox [5,6].

Therefore, the paradox considered in this paper is even more likely to occur, as strains can be arbitrarily large with any pressure, provided we are close enough to c_R.

The finite strain kinematics analysis shows that the paradox disappears when the correct kinematics is used. In this case it is a reasonable engineering solution to use the infinitesimal theory with the assumption that slip is always in the original direction of sliding, because the paradox occurs only in very small regions in which the infinitesimal theory is unrealistic.

There is, however, a class of problems where we see some doubt as to the proper formulation using infinitesimal theory. For the moving punch, we propose to use the direction of the punch motion to determine the relative slip velocity, i.e., we ignore the velocity reversal due to the singularity. Now suppose that the punch is stationary and is subjected to an incoming wave. The direction of the particle motion beneath the punch governs the slip direction and the singularity does not produce a slip reversal, due to (4), because $U_0=0$. However, the imposed motion itself may be sufficient to give slip reversal in some regions. Now consider the case in which the punch is given a small velocity. According to (4) any finite (no matter how small) velocity will produce slip, near the moving singularities, in the opposite direction as that due to the punch motion alone. In this case it is unclear as to whether to ignore the effect of the velocity reversal, as we did for the moving punch without the incoming wave, or to include its effect as we suggest when the punch is perfectly stationary.

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